Volatility Interpolation

Preliminary Version
March 2010

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Intro

Local volatility models such as Dupire (1994), Andersen and Andreasen (1999), JPMorgan (1999), and Andreasen and Huge (2010) ideally require a full continuum in expiry and strike of arbitrage consistent European option prices as input. In practice, we, of course, only observe a discrete set of option prices. In this paper, we show how a variant of the fully implicit finite difference method can be applied to efficiently interpolate and extrapolate a discrete set of option quotes to an arbitrage consistent full continuous surface in expiry and strike. In a numerical example we show how the model can be fitted to all quoted prices in the SX5E option market (12 expiries, each with roughly 10 strikes) in approximately 0.05s of CPU time.

Discrete Expiries

Given a time grid of expiries $0 = t_0 < t_1 < \ldots$ and a set of volatility functions $\{\vartheta_i(k)\}_{i=0,1,\ldots}$ we construct European option prices for all the discrete expiries, by recursively solving the *forward* system:

$$[1 - \frac{1}{2}\Delta t_i \vartheta_i(k)^2 \frac{\partial^2}{\partial k^2}]c(t_{i+1}, k) = c(t_i, k) \quad , c(0,k) = (s(0) - k)^+ \quad , i = 0,1,\ldots \qquad (1)$$

where $\Delta t_i = t_{i+1} - t_i$.

If we discretise the strike space $k_j = k_0 + j\Delta k$, $j = 0,1,\ldots,n$ and replace the differential operator by the difference operator, we get the following finite difference scheme

$$\begin{aligned} &[1 - \frac{1}{2}\Delta t_i \vartheta_i(k)^2 \delta_{kk}]c(t_{i+1}, k) = c(t_i, k) \quad , c(0,k) = (s(0) - k)^+ \quad , i = 0,1,\ldots \\ &\delta_{kk} f(k) = \frac{1}{\Delta k^2}(f(k - \Delta k) - 2f(k) + f(k + \Delta k)) \end{aligned} \qquad (2)$$

The system (2) can be solved by recursively solving tridiagonal matrix systems. One can thus view the system (1) is a one-step per expiry implicit (finite-difference) discretisation of the Dupire (1994) forward equation

$$0 = -\frac{\partial c}{\partial t} + \frac{1}{2}\sigma(t,k)^2 \frac{\partial^2 c}{\partial k^2} \qquad (3)$$

For a set of discrete option quotes $\{\hat{c}(t_i, k_{ij})\}$, the system (1) can be bootstrapped forward expiry by expiry to find piecewise constant functions

$$\vartheta_i(k) = a_{ij} \quad , b_{i,j-1} < k \le b_{ij} \qquad (4)$$

that minimize the pricing error in (1). I.e. we solve the optimisation problems

$$\inf_{\vartheta_i(\cdot)} \sum_j (c(t_i, k_{ij}) - \hat{c}(t_i, k_{ij}))^2 , \qquad (5)$$

sequentially for $i = 1, 2, \ldots$. The point here is that for each iteration in (5) only *one* tridiagonal matrix system (2) needs to be solved.

Filling the Gaps

The system (1) translates the local volatility functions into arbitrage consistent prices for a discrete set of expiries but it does not directly specify the option prices between the expiries. We fill the gaps by letting the option prices between two expiries be constructed according to

$$[1 - \frac{1}{2}(t - t_i)\vartheta_i(k)^2 \frac{\partial^2}{\partial k^2}]c(t,k) = c(t_i,k) \quad , t \in]t_i, t_{i+1}[\tag{6}$$

Note that we do not step from expiry to expiry that lie between the original expiries; for all expiries $t \in]t_i, t_{i+1}[$, we step from t_i.

Absence of Arbitrage and Stability

Carr (2008) shows that the option prices generated by (1) are consistent with the underlying being a local variance gamma process. From this or from straight calculation we have that (6) can be written as

$$c(t,k) = \int_0^\infty \frac{1}{t - t_i} e^{-u/(t - t_i)} g(u,k) du \quad , t > t_i \tag{7}$$

where $g(u,k)$ is the solution to

$$\begin{aligned} &0 = -\frac{\partial g}{\partial u} + \frac{1}{2}\vartheta(k)^2 \frac{\partial^2 g}{\partial k^2} \quad , u > 0 \\ &g(0,k) = c(t_i,k) \end{aligned} \tag{8}$$

In Appendix B we use this to show that the option prices generated by (1) and (6) are consistent with absence of arbitrage, i.e. that $c_t(t,k) \geq 0, c_{kk}(t,k) \geq 0$ for all (t,k).

For the discrete space case we note that with the additional (absorbing) boundary conditions $c_{kk}(t,k_0) = c_{kk}(t,k_n) = 0$, (2) can be written as

$$Ac(t_{i+1}) = c(t_i) \tag{9}$$

where A is the tri-diagonal matrix

$$A = \begin{bmatrix} 1 & 0 & & & & \\ -z_1 & 1+2z_1 & -z_1 & & & \\ & -z_2 & 1+2z_2 & -z_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -z_{n-1} & 1+2z_{n-1} & -z_{n-1} \\ & & & & 0 & 1 \end{bmatrix} \tag{10}$$

$$z_j = \frac{1}{2}\frac{\Delta t}{\Delta k^2}\vartheta_i(k_j)^2$$

The tri-diagonal matrix A is diagonally dominant with positive diagonal and negative off-diagonals. Nabben (1999) shows that for this type of matrix

$$A^{-1} \geq 0 \tag{11}$$

This implies that the discrete system (2) is stable. As we also have

$$A^{-1}\iota = \iota \quad ,\iota = (1,\ldots,1)', \tag{12}$$

we can further conclude that the discrete system (2) is arbitrage-free. This also holds if the spacing is non-equidistant.

If the problem is formulated in logarithmic space $x = \ln k$, as would often be the case, then the discrete system (2) becomes

$$\begin{aligned} &[1-\frac{1}{2}\Delta t_i \vartheta_i(x)^2(\delta_{xx}-\delta_x)]c(t_{i+1},x) = c(t_i,x) \quad ,c(0,x) = (s(0)-e^x)^+ \quad ,i = 0,1,\ldots \\ &\delta_x f(x) = \frac{1}{2\Delta x}(f(x+\Delta x) - f(x-\Delta x)) \\ &\delta_{xx} f(x) = \frac{1}{\Delta x^2}(f(x-\Delta x) - 2f(x) + f(x+\Delta x)) \end{aligned} \tag{13}$$

It follows that the system is stable if $\Delta x = \ln(k_{j+1}/k_j) \leq 2$, not a constraint that will be breached in any practical application.

As shown in Appendix B, (1) and (6) can be slightly generalised by introducing a deterministic time-change $T(t)$:

$$[1-\frac{1}{2}(T(t)-t_i)\vartheta_i(k)^2\frac{\partial^2}{\partial k^2}]c(t,k) = c(t_i,k) \quad ,t \in]t_i,t_{i+1}] \tag{14}$$

where $T(t_i) = t_i$ and $T'(t) > 0$. In this case, the local volatility function (3) consistent with the model is given by

$$\sigma(t,k)^2 = 2\frac{c_t(t,k)}{c_{kk}(t,k)} = \vartheta_i(k)^2[T'(t) + (T(t)-t_i)\frac{\partial \ln c_{kk}(t,k)}{\partial t}] \qquad (15)$$

The introduction of the time-change gives a handle on the interpolation in the expiry direction. For example, a choice of a piecewise cubic functions $T(t)$ can be used to ensure that implied volatility is roughly linear in expiry.

Algorithm

In summary: a discrete set of European option quotes is interpolated into a full continuous surface of arbitrage consistent option quotes by

Step 1: For each expiry solve an optimisation problem (6) for a piecewise constant volatility function with as many levels as target strikes at the particular expiry. Each iteration involves one update of (1) and is equivalent to one time step in a fully implicit finite difference solver.

Step 2: For expiries between the original expiries, the volatility functions from step 1 is used in conjunction with (7) to generate option prices for all strikes.

Note that step 2 does not involve any iteration. The process of the time stepping is shown in Figure 1.

Numerical Example

In this section we consider fitting the model to the SX5E equity option market. The number of expiries is 12 with up to 15 strikes per expiry. The target data is given in Table 1. We choose to fit a log-normal version of the model based on finite difference solution with 200 grid points. The local volatility function is set up to be piecewise linear with as many levels as calibration strikes per expiry. The model fits to the option prices in approximately 0.05s of CPU time on a standard PC. The average number of iterations is 86 per expiry. Table 2 shows the calibration accuracy for the target options. The standard deviation of the error is 0.03% in implied Black volatility.

After the model has been calibrated we use (6) to compute option prices for all expiries and strikes and deduce the local volatility from the option prices using (3). Figure 2 shows the resulting local volatility surface. We note that the local volatility surface has no poles. So, as expected, the model produces arbitrage consistent European option prices for all expiries and strikes.

References

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Appendix A: Tables and Figures

Table 1: SX5E Implied Volatility Quotes

k\t	0.025	0.101	0.197	0.274	0.523	0.772	1.769	2.267	2.784	3.781	4.778	5.774
51.31%									33.66%	32.91%		
58.64%									31.78%	31.29%	30.08%	
65.97%									30.19%	29.76%	29.75%	
73.30%									28.63%	28.48%	28.48%	
76.97%				32.62%	30.79%	30.01%	28.43%					
80.63%				30.58%	29.36%	28.76%	27.53%	27.13%	27.11%	27.11%	27.22%	28.09%
84.30%				28.87%	27.98%	27.50%	26.66%					
86.13%	33.65%											
87.96%	32.16%	29.06%	27.64%	27.17%	26.63%	26.37%	25.75%	25.55%	25.80%	25.85%	26.11%	26.93%
89.79%	30.43%	27.97%	26.72%									
91.63%	28.80%	26.90%	25.78%	25.57%	25.31%	25.19%	24.97%					
93.46%	27.24%	25.90%	24.89%									
95.29%	25.86%	24.88%	24.05%	24.07%	24.04%	24.11%	24.18%	24.10%	24.48%	24.69%	25.01%	25.84%
97.12%	24.66%	23.90%	23.29%									
98.96%	23.58%	23.00%	22.53%	22.69%	22.84%	22.99%	23.47%					
100.79%	22.47%	22.13%	21.84%									
102.62%	21.59%	21.40%	21.23%	21.42%	21.73%	21.98%	22.83%	22.75%	23.22%	23.84%	23.92%	24.86%
104.45%	20.91%	20.76%	20.69%									
106.29%	20.56%	20.24%	20.25%	20.39%	20.74%	21.04%	22.13%					
108.12%	20.45%	19.82%	19.84%									
109.95%	20.25%	19.59%	19.44%	19.62%	19.88%	20.22%	21.51%	21.61%	22.19%	22.69%	23.05%	23.99%
111.78%	19.33%	19.29%	19.20%									
113.62%				19.02%	19.14%	19.50%	20.91%					
117.28%				18.85%	18.54%	18.88%	20.39%	20.58%	21.22%	21.86%	22.23%	23.21%
120.95%				18.67%	18.11%	18.39%	19.90%					
124.61%				18.71%	17.85%	17.93%	19.45%		20.54%	21.03%	21.64%	22.51%
131.94%									19.88%	20.54%	21.05%	21.90%
139.27%									19.30%	20.02%	20.54%	21.35%
146.60%									18.49%	19.64%	20.12%	

Table 1 shows implied Black volatilities for European options on the SX5E index. Expiries range from two weeks to a bit less than 6 years and strikes range from 50% to 146% of current spot of 2772.70. Data is as of March 1, 2010.

Table 2: SX5E Calibration Accuracy

k\t	0.025	0.101	0.197	0.274	0.523	0.772	1.769	2.267	2.784	3.781	4.778	5.774
51.31%									0.00%	0.00%		
58.64%									0.00%	-0.02%	0.08%	
65.97%									0.00%	0.02%	-0.23%	
73.30%									0.00%	-0.02%	0.05%	
76.97%				-0.02%	-0.01%	0.00%	0.00%					
80.63%				-0.02%	-0.01%	0.00%	0.01%	0.00%	0.00%	0.01%	0.06%	0.00%
84.30%				0.00%	0.00%	0.00%	-0.02%					
86.13%	0.01%											
87.96%	-0.07%	-0.05%	0.01%	0.02%	0.01%	-0.01%	0.01%	0.00%	0.00%	-0.01%	-0.02%	0.00%
89.79%	0.02%	0.01%	0.00%									
91.63%	0.01%	0.01%	0.00%	0.02%	0.01%	0.00%	-0.01%					
93.46%	-0.02%	-0.02%	0.00%									
95.29%	0.00%	0.00%	0.01%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.01%	-0.01%	0.00%
97.12%	0.02%	0.01%	-0.01%									
98.96%	-0.01%	-0.01%	0.00%	0.00%	0.00%	0.00%	0.00%					
100.79%	0.01%	0.00%	0.00%									
102.62%	0.01%	-0.01%	0.00%	0.00%	0.00%	-0.01%	-0.01%	0.00%	0.00%	-0.03%	0.00%	0.00%
104.45%	0.01%	0.00%	0.02%									
106.29%	-0.06%	-0.01%	0.00%	0.01%	0.00%	0.03%	0.01%					
108.12%	0.00%	0.00%	-0.02%									
109.95%	-0.10%	-0.09%	0.00%	-0.02%	0.00%	0.02%	-0.01%	0.00%	-0.01%	0.02%	-0.02%	0.00%
111.78%	-0.02%	0.03%	-0.04%									
113.62%				0.03%	0.00%	-0.01%	0.00%					
117.28%				-0.03%	0.00%	0.01%	0.00%	0.00%	0.02%	-0.02%	0.00%	0.00%
120.95%				0.01%	0.00%	-0.02%	0.00%					
124.61%				0.00%	0.02%	0.07%	0.02%		-0.03%	0.02%	-0.02%	0.00%
131.94%									0.00%	-0.05%	0.01%	0.00%
139.27%									0.00%	0.01%	-0.01%	-0.01%
146.60%									0.02%	-0.01%	0.00%	

Table 2 shows the difference between the model and the target in implied Black volatilities for European options on the SX5E index. Data is as of March 1, 2010.

Figure 1: Model Timeline

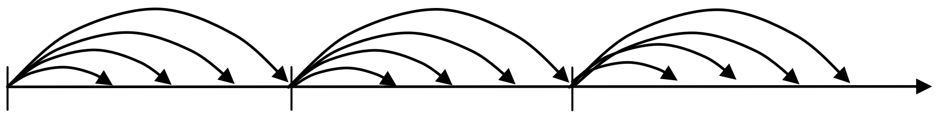

Figure 2: Local Volatility derived from Model Option Prices

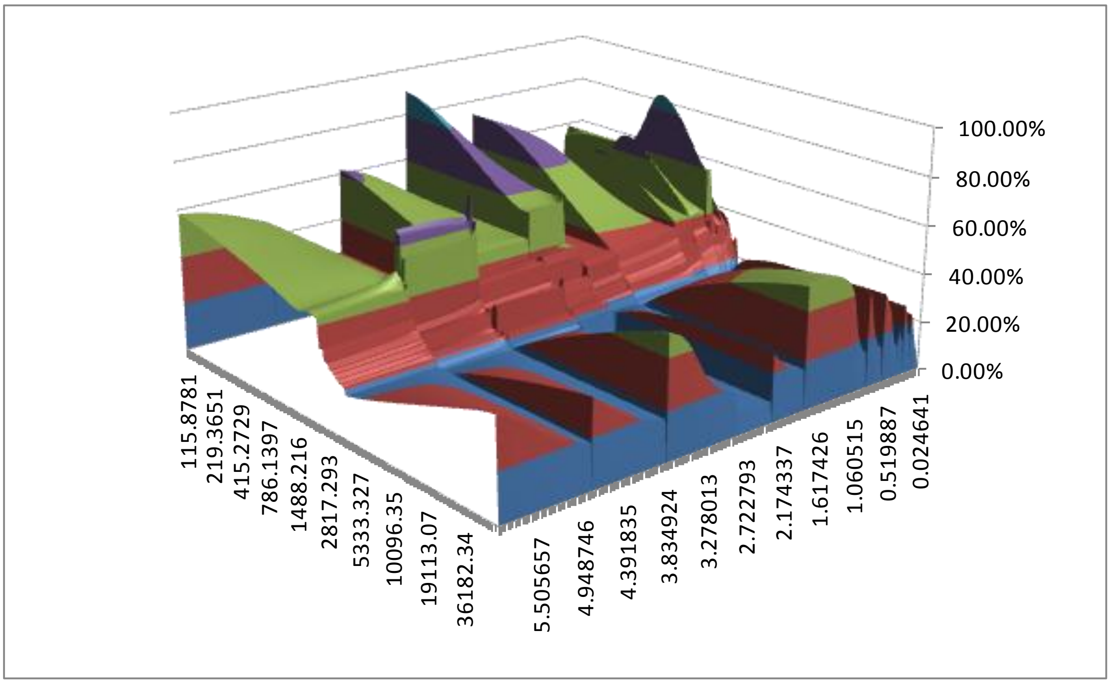

Figure 2 shows the local volatility surface in the model after it has been fitted to the SX5E market. Data is as of March 1, 2010.

Appendix B: Technical Results

Proposition 1: Absence of Arbitrage
The surface of option prices constructed by the recursive schemes (1) and (6) is consistent with absence of arbitrage, i.e.

$$\begin{aligned} c_t(t,k) &\geq 0 \\ c_{kk}(t,k) &\geq 0 \end{aligned} \tag{B1}$$

for all (t,k).

Proof of Proposition 1
Consider option prices generated by the forward equation

$$0 = -\frac{\partial g}{\partial t} + \frac{1}{2}\vartheta(k)^2 \frac{\partial^2 g}{\partial k^2} \tag{B2}$$

given which is solved forward in time t given the initial boundary condition $g(0,k)$.

As also noted in Andreasen (1996), (B2) can also be seen as the *backward* equation for

$$g(t,k) = E^k[g(0,k(0)) \,|\, k(t) = k] \tag{B3}$$

where k follows the process

$$dk(t) = \vartheta(k(t))dZ(t) \tag{B4}$$

and Z is a Brownian motion running backwards in time. The mapping $g(0,\cdot) \mapsto g(t,\cdot)$ given by (B2) thus defines a *positive linear functional* in the sense that

$$g(0,\cdot) \geq 0 \Rightarrow g(t,\cdot) \geq 0 \tag{B5}$$

Further, differentiating (B2) twice with respect to k yields the forward equation for $p = g_{kk}$:

$$\begin{aligned} 0 &= -\frac{\partial p}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial k^2}[\vartheta(k)^2 p] \\ p(0,k) &= g_{kk}(0,k) = \int g_{kk}(0,l)\delta(k-l)dl \end{aligned} \tag{B6}$$

Equation (B3) is equivalent to the Fokker-Planck equation for the process

$$dx(t) = \vartheta(x(t))dW(t) \tag{B7}$$

where W is a standard Brownian motion. From this we conclude that (B2) preserves convexity:

$$g_{kk}(0,\cdot) \geq 0 \Rightarrow g_{kk}(t,\cdot) \geq 0 \qquad \text{(B8)}$$

Let $T(u)$ be a strictly increasing function. Define the (Laplace) transform of the option prices by

$$h(u,k) = \int_0^\infty \frac{1}{T(u)} e^{-t/T(u)} g(t,k) dt \qquad \text{(B9)}$$

Multiplying (B2) by $e^{-t/T(u)}$ and integrating in t yields

$$[1 - \frac{1}{2} T(u) \vartheta(k)^2 \frac{\partial^2}{\partial k^2}] h(u,k) = g(0,k) \qquad \text{(B10)}$$

From (B5) and (B8) we conclude that (B10) defines a positive linear functional that preserves convexity.

Differentiating (B10) with respect to u yields

$$[1 - \frac{1}{2} T(u) \vartheta(k)^2 \frac{\partial^2}{\partial k^2}] h_u(u,k) = \frac{1}{2} T'(u) \vartheta(k)^2 h_{kk}(u,k) \qquad \text{(B11)}$$

Using that (B10) is a positive linear functional that preserves convexity we have that if $g(0,\cdot)$ is convex then

$$h_u(u,k) \geq 0 \qquad \text{(B12)}$$

for all (u,k).

We conclude that the option prices constructed by (1) and (6) are consistent with absence of arbitrage.