Stochastic Volatility for Real

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Abstract

We combine the classical ideas of separable volatility structures in the HJM framework with the latest techniques for calibration of stochastic volatility models and create an efficient multi-factor term structure model with stochastic volatility.

Introduction

In the early 1990s, Cheyette (1992) and others introduce a separable volatility specification of the general Heath, Jarrow, and Morton (1992) model. Contrary to general HJM and Libor market models, this specification allows for Markov representation of the full yield curve in a low number of state variables. In this paper we present a class of separable volatility structure yield curve models that incorporate stochastic volatility to match the volatility smile as observed in the vanilla interest rate options markets. We combine this with recent ideas for approximation of stochastic volatility model with time-dependent parameters by Piterbarg (2005) to yield fast and efficient calibration of the model.

The first sections of the paper consider the notion of so-called "true" stochastic volatility, separable volatility structures in HJM models, and stochastic volatility models for vanilla swaptions and caps. We then introduce our model specification and describe how cap and swaption prices can be approximated in the model. Calibration techniques and numerical examples are considered. The final sections of the paper consider pricing in our model by Monte-Carlo simulations and finite difference solution.

True Stochastic Volatility

So-called *true stochastic volatility* yield curve models are models that have the property that they prescribe moves in the volatility of rates that can not directly be inferred from the shape or the level of the yield curve. I.e. if $P(t,T)$ is the time t price of a zero-coupon bond maturing at time T, and we assume that all zero-coupon bond prices evolve according to

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt + a(t,T)'dW(t) \qquad (1)$$

where W is a vector Brownian motion under the risk-neutral measure, r is the continuously compounded short rate, and a some family of vector processes, then a true stochastic volatility model has the property that there exists some factor z and at least one maturity U, so that

$$\frac{\partial a(t,U)}{\partial z} \neq 0 \qquad (2a)$$

and

$$\frac{\partial P(t,T)}{\partial z} = 0 \qquad (2b)$$

for all T.

Traditional short rate models are very difficult to formulate as true stochastic volatility models. Consider, for example, the model by Fong and Vasicek (1991):

$$dr(t) = \kappa(\theta - r(t))dt + \sqrt{v(t)}dW_1(t)$$
$$dv(t) = \beta(\alpha - v(t))dt + \varepsilon\sqrt{v(t)}dW_2(t)$$
$$dW_1(t) \cdot dW_2(t) = \rho dt$$

where $\kappa, \theta, \beta, \alpha, \varepsilon, \rho$ are constants, and W_1 and W_2 Brownian motions under the risk-neutral measure. In this model we have

$$\begin{aligned} P(t,T) &= E_t[e^{-\int_t^T r(u)du}] \\ &= E[e^{-\int_t^T r(u)du} \mid r(t), v(t)] \\ &\equiv P(t,T; r(t), v(t)) \end{aligned}$$

So the bond price becomes a function of two stochastic variables. Hence, we can invert the system and infer the level of both the short rate and the short rate volatility from any two points on the yield curve. Thus, the model is not a true stochastic volatility model. This is also the case for the Longstaff and Schwartz (1992) model and other early attempts to produce stochastic volatility yield curve model.

In fact, without going into the complicated technical details, this is also the case for any attempt to formulate a stochastic volatility yield curve model in the context of the Markov functional approach by Hunt, Kennedy, and Pellser (1998).

So, as observed by Andreasen, Dufresne and Shi (1994), the most straightforward way of formulating a stochastic volatility yield curve model is to use the HJM approach, or equivalently the Libor market model approach, and directly specify the stochastic nature of the bond or forward rate volatility structure.

Let the time t continuously compounded forward rate for deposit over the interval $[T, T + dT]$ be given by

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}$$

Heath, Jarrow, and Morton (1992) show that any arbitrage-free term structure model with continuous evolution of the yield curve has to satisfy

$$df(t,T) = \sigma(t,T)'(\int_t^T \sigma(t,s)ds)dt + \sigma(t,T)'dW(t) \qquad (3)$$

where W is a vector Brownian motion under the risk-neutral measure and $\{\sigma(t,T)\}_{t \le T}$ is a family of vector processes. The relation between the bond price volatility and the forward rate volatility is

$$a(t,T) = -\int_t^T \sigma(t,s)ds$$

In this model approach we see that it is easy to specify a volatility structure satisfying (2). We could for example set $\sigma(t,T) = c \cdot \sqrt{z(t)}$ for some constant c and some stochastic factor z.

Separable Volatility

The HJM approach prescribes a very straightforward way of specifying an arbitrage free term structure model that automatically fits the initial term structure: all one needs to do is to specify the forward rate volatility structure $\{\sigma(t,T)\}_{t\le T}$.

However, the problem with this modeling approach is that the resulting model is not generally Markov in a limited number of state variables. In general, the HJM model approach requires us to use the full forward curve as a state variable to close (3) as a Markov system. This is independent of the dimension of the driving Brownian motion and it is even the case if the forward rate volatility structure is deterministic. So when we simulate the model (3) we generally need to carry forward all points on the forward curve. Hence, the computational effort of simulation of the model (3) grows at a quadratic rate in the time horizon. Similarly, if we attempt to approximate the process (3) with a discrete process, the resulting tree will be non-recombining and thus have a number of nodes that grow at an exponential rate in the number of time steps, or the time horizon.

However, Cheyette (1992), Babbs (1992), Jamshidian (1991), and Ritchken and Sankarasubramaniam (1993) independently find that if we restrict ourselves to a volatility structure for the forward rates that are *separable* in the sense that there exist a deterministic vector function g on $\mathbb{R}^k$ and a matrix process h on $\mathbb{R}^{k\times k}$, so that

$$\sigma(t,T)' = g(T)' h(t) \tag{4}$$

then a Markov representation of the dynamics of the yield curve, involving (a minimum of) $k + k\cdot(k+1)/2$ state variables, emerges.

Without loss of generality the model can in this case we formulated as

$$dX(t) = (Y(t)\iota - I_{\kappa(t)}X(t))dt + \eta(t)dW(t) \quad , X(0) = 0$$
$$dY(t) = (\eta(t)\eta(t)' - I_{\kappa(t)}Y(t) - Y(t)I_{\kappa(t)})dt \quad , Y(0) = 0$$
$$P(t,T) = \frac{P(0,T)}{P(0,t)} e^{-G(t,T)'X(t) - \frac{1}{2}G(t,T)'Y(t)G(t,T)}$$
$$G(t,T) = \int_t^T g(t,s)ds \quad , g(t,T) = (e^{-\int_t^T \kappa_1(u)du}, \ldots, e^{-\int_t^T \kappa_k(u)du})'$$
$$\iota = (1,\ldots,1)' \quad , X, G, g, \kappa, W \in \mathbb{R}^k \quad , Y, \eta \in \mathbb{R}^{k\times k} \quad , I_a = \mathrm{diag}(a_1,\ldots,a_k) \in \mathbb{R}^{k\times k}$$

(5)

In the context of the separable formulation (4) we have

$$g(T) = g(0,T) \quad ,h(t) = I_{g(t)}^{-1}\eta(t)$$

The first k state variables, the elements of X, can be interpreted as yield curve factors that pertubate the forward curve and are directly associated with the driving Brownian motions, whereas the remaining $k \cdot (k+1)/2$ state variables, the elements of the symmetric matrix Y, can be seen as "convexity" terms that have to be carried along to keep the model arbitrage free.

For $k = 1$ and η deterministic, Y becomes deterministic and we obtain the general Gaussian model, i.e. a Vasicek (1977) model with time-dependent parameters. This lead Jamshidian (1991) and Babbs (1992) to denoting the separable volatility specification as respectively "quasi" and "pseudo" Gaussian models.

The potential computational saving in using this type of model rather than the general HJM approach is considerable. If we for example consider the case of pricing a 30 year structure with quarterly fixings and payments by simulation, the general HJM or LMM approach will require the evolution of at least 120 points on the yield curve, whereas a 4 factor version of the separable model requires the evolution of maximum 14 state variables.

For the one-dimensional case, $k = 1$, finite difference solution is viable and it is most often a more efficient numerical solution method than Monte-Carlo simulation. We will discuss this later in the paper. Finite difference solution of simpler versions of the model are also considered in Andreasen (2000) and Andersen and Andreasen (2002).

It is worth noting that if we let $\kappa_1, \ldots, \kappa_k$ be constants, then

$$\sigma(t, t+\tau)' = \sum_{i=1}^{k} e^{-\kappa_i \tau} \eta_i(t) \to \int e^{-\kappa\tau} \eta_\kappa(t) d\kappa$$

for $k \to \infty$ and an appropriately chosen sequence $\kappa_1, \kappa_2, \ldots$. So the model (5) can be seen as a representation of the forward rate volatility structure on a (discrete) basis of exponential functions. The function $\kappa \mapsto \eta_\kappa(t)$ can thus be viewed as the inverse Laplace transform of the forward rate volatility structure in the tenor dimension: $\tau \mapsto \sigma(t, t+\tau)$.

Stochastic Volatility Processes

The most popular stochastic volatility model for caps and swaptions appears to be the SABR model by Hagan et al (2002) where the volatility is specified as a geometric Brownian motion that has some correlation with the underlying forward swap rate. This model is quite difficult to work with in the context of full yield curve models, for a number of reasons. Firstly, the SABR model does not incorporate mean-reversion in volatility which means that when the model is fitted to observed cap and swaption prices the implied volatility of volatility parameter most often turn out to be decreasing with expiry of the underlying option. This in turn implies that a full dynamic version of the SABR model would have to exhibit even steeper decreasing forward volatility of volatility. Secondly, in many implementations of the SABR model the correlation between volatility and underlying rate are quite different for different expiries and tenors. Non-zero correlation is technically quite difficult to handle in a full yield curve model and potentially time-varying correlation is of course even more complicated. Thirdly, as the SABR model has no closed-form for European option prices, it is typically implemented for European option pricing by expansion techniques whose accuracy deteriorates for longer expiries. This may have limited practical importance if the SABR model is only used for European option pricing, but our scope is to price general path dependent instruments so we need our European option pricing to be consistent with the actual specified dynamics.

Instead we follow Andersen and Andreasen (2002) and use the following model as our basis for developing a full yield curve model with stochastic volatility

$$\begin{aligned} dS(t) &= \lambda\sqrt{z(t)}[mS(t)+(1-m)S(0)]dW_2^A(t) \\ dz(t) &= \beta(1-z(t))dt+\varepsilon\sqrt{z(t)}dW_2^A(t) \\ dW_1^A \cdot dW_2^A &= 0 \end{aligned} \tag{6}$$

where W^A is a Brownian motion under annuity measure, i.e. the martingale measure with the annuity $A(t)=\sum_{i=1}^{n}\delta_i P(t,t_i)$, $\delta_i = t_i - t_{i-1}$ is the day count fraction, $S(t)=(P(t,t_0)-P(t,t_n))/A(t)$ is the forward par swap rate under consideration, and all the parameters $\lambda, m, \varepsilon, \beta$ are constants. We note that the swap rate is a martingale under the annuity measure.

In terms of the implied Black-Scholes volatility smile, the level is controlled by λ. As correlation between the swap rate and the volatility is assumed to be zero, the slope of the smile is fully controlled by the m parameter. The smile becomes increasingly negatively sloped as m is decreased. Sub-normal skews, corresponding to $m<0$, are possible with the note of caution that S is restricted from above by $\frac{1-m}{m}S(0)$ when m is negative. Increasing the volatility of local variance, ε, increases the curvature of the smile. Increasing the speed of mean-reversion, β, increases the rate at which the curvature of the smile decays with expiry.

The model is essentially a "shifted" Heston (1991) model, so it allows an analytic solution based on numerical inversion of Fourier transform:

$$\begin{aligned} E^A[(S(t)-K)^+] &= \frac{1}{m}[S(0)-\frac{K'}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{(-ik+1/2)\ln S(0)/K'}}{k^2+1/4}q(k)dk] \\ K' &= mK+(1-m)S(0) \end{aligned} \tag{7}$$

where

$$q(k)=e^{\mathrm{A}(k)-(k^2+1/4)\mathrm{B}(k)m^2\lambda^2 z(0)}$$

$$\mathrm{A}(k)=-\frac{\beta}{\varepsilon^2}[\psi_+ t+2\ln\frac{\psi_- +\psi_+ e^{-\zeta t}}{2\zeta}]$$

$$\mathrm{B}(k)=\frac{1-e^{-\zeta t}}{\psi_- +\psi_+ e^{-\zeta t}}$$

$$\psi_\pm=\mp\beta+\zeta$$

$$\zeta=\sqrt{k^2\varepsilon^2++\beta^2+\varepsilon^2/4}$$

This representation can also be found in Lipton (2002) and Lewis (2001) and avoids the numerical instability of the representation in the original Heston (1991) paper.

This model gives a good fit to observed cap and swaption prices with reasonably stable parameters across expiries and tenors. An example of the fitted m,ε parameters is given in Table 1.

In our experience, the implied skew and smile parameters m and ε are quite stable over time, so in practice only the volatility level parameter λ needs to be updated on a regular basis, say daily or weekly. To illustrate this, Figure 1 shows the deviations in terms of implied Black volatility from end-of-month Totem consensus quotes for EUR swaptions for strikes ranging from 5% to 95% delta, for two different models (6). The first model had its m and ε grids fitted the month before whereas the second was fitted on the particular date. In both cases we set the λ so that the model fit the at-the-money Totem levels. Expiries range from 6m to 20y and tenors from 1y to 30y. In total 912 swaptions were priced. We see that both models for the most part agree with the Totem consensus quotes within +/-0.25% in Black-Scholes implied volatility for all strikes.

Model Specification

Andersen and Andreasen (2002) suggest a Libor market model with stochastic volatility which is extended by Piterbarg (2003) to allow for a time and tenor dependent local volatility skew parameter. The motivation for this is that if we consider implied parameters of the model (6) as in Table 1, we typically see that the skew parameter m is fairly constant across expiry but it tends to decrease with tenor. On the other hand, the implied ε parameter appears to be fairly constant across both expiry as well as tenor, at least for expiries over 1 year.

If we use continuously compounded rates rather than discrete rates as model primitives, the Piterbarg model can be formulated as

$$df(t,T)=\sqrt{z(t)}[m(t,T)f(t,T)+(1-m(t,T))f(0,T)]\lambda(t,T)\rho(t,T)'dW(t)+O(dt)$$

$$dz(t)=\beta(1-z(t))dt+\sqrt{z(t)}\varepsilon(t)dZ(t)$$

$$\lambda(t,T)\in\mathbb{R},\ \rho(t,T)\in\mathbb{R}^k,\ \|\rho(t,T)\|=1$$

$$Z(t)\in\mathbb{R},\ dZ(t)\cdot dW(t)=0$$

(8)

where m,λ and ρ are deterministic functions of time and maturity, ε is a deterministic function of time and β is a constant. We note that $m=0$ corresponds to a normal model whereas $m=1$ corresponds to a log-normal model.

Fix k tenors $\tau_1,\ldots,\tau_k$. For the corresponding forward rates we have

$$
\begin{aligned}
&dF(t)=\sqrt{z(t)}[I_{M(t)}I_{F(t)}+(I-I_{M(t)})I_{F(0)}]I_{\lambda(t)}R(t)dW(t)+O(dt)\\
&F(t)=(f(t,t+\tau_1),\ldots,f(t,t+\tau_n))'\in\mathbb{R}^k\\
&M(t)=(m(t,t+\tau_1),\ldots,m(t,t+\tau_n))'\in\mathbb{R}^k\\
&I_{\lambda(t)}=\text{Diag}(\lambda(t,t+\tau_1),\ldots,\lambda(t,t+\tau_n))\in\mathbb{R}^{k\times k}\\
&R(t)=(\rho(t,t+\tau_1)';\ldots;\rho(t,t+\tau_1)')\in\mathbb{R}^{k\times k}
\end{aligned}
\tag{9}
$$

Here RR' is the instantaneous correlation matrix for the k forward rates.

Under the separable volatility specification in (5) we have

$$
\begin{aligned}
&dF(t)=\Gamma(t)\eta(t)dW(t)+O(dt)\\
&\Gamma(t)=\begin{bmatrix} g(t,t+\tau_1)' \\ \vdots \\ g(t,t+\tau_k)' \end{bmatrix}\in\mathbb{R}^{k\times k}
\end{aligned}
\tag{10}
$$

Equating diffusion terms of (9) and (10) yields

$$
\begin{aligned}
&\Gamma(t)\eta(t)=\sqrt{z(t)}[I_{M(t)}I_{F(t)}+(I-I_{M(t)})I_{F(0)}]I_{\lambda(t)}R(t)\\
&\Downarrow\\
&\eta(t)=\sqrt{z(t)}\Gamma(t)^{-1}[I_{M(t)}I_{F(t)}+(I-I_{M(t)})I_{F(0)}]I_{\lambda(t)}R(t)
\end{aligned}
\tag{11a}
$$

with

$$
dz(t)=\beta(1-z(t))dt+\varepsilon(t)\sqrt{z(t)}dZ(t)\quad ,dZ\cdot dW=0 \tag{11b}
$$

The volatility specification (11) in combination with (5) defines our model.

In most cases we choose constant $\kappa_1,\ldots,\kappa_k$ as well as a constant correlation structure RR'. The latter is typically estimated from historical time series data of the yield curve. In this case, the model primitives that need to be set by calibration to swaption and cap prices are the following parameters:

a. The forward rate volatility structure, i.e. λ for all times t and the tenors $\tau_1,\ldots,\tau_k$.

b. The forward rate skew structure, i.e. m for all times t and the tenors $\tau_1,\ldots,\tau_k$.

c. The forward volatility of volatility, i.e. ε for all times t.

In terms of the implied Black-Scholes volatility smiles for swaptions and caplets, the first parameter controls the absolute level, the second the slope (skew), and the third the curvature (smile).

We see that the model, at least in principle, has the ability to exactly fit the volatility level and slope for all expiries along k tenors, whereas the curvature can only be fitted exactly for one tenor. In practice, though, our calibration will most often be on a best fit basis.

For the one-factor case, $k=1$, we do, however, often choose to go for an exact fit to a specific strip of swaptions or caplets. In this case we often specify the model a bit differently, namely

$$\eta(t)=\sqrt{z(t)}[m(t)S(t)+(1-m(t))S(0)]\lambda(t) \tag{13}$$

where λ, m are now scalar functions of time and S is a par swap rate referring to different swap periods over the time horizon. If we for example choose to fit the model to the strip of 1x29, 2x28, … , 29x1 swaption smiles, we let S be the 1x29 par swap rate for times between year 0 and 1, 2x28 par swap rate for times between year 1 and 2, … , 29x1 par swap rate for times between year 28 and 29.

Swaption Pricing

For efficient calibration of the model closed-form pricing of caps and swaptions is essential. In this section we describe an accurate (near) closed-form approximation.

Using Ito's lemma and the fact that the swap rate S is a martingale under the annuity measure we get

$$dS(t)=S_X(t)'\eta(t)dW^A(t) \tag{14}$$

where we let subscripts denote partial derivatives, i.e. $S_X=(\partial S/\partial X_1,\ldots,\partial S/\partial X_k)'$. Given fixed mean reversion coefficients $\kappa_1,\ldots,\kappa_k$ this derivative can be computed in closed-form by combining (14) with the bond price formula in (5).

Our approximation goes in two steps:

A. Approximate the SDE (14) by the model

$$\begin{aligned} d\overline{S}(t) &= \sqrt{\overline{z}(t)}\overline{\lambda}(t)[\overline{m}(t)\overline{S}(t)+(1-\overline{m}(t))S(0)]dW_1^A(t) \\ d\overline{z}(t) &= \beta(1-\overline{z}(t))dt+\varepsilon(t)\sqrt{\overline{z}(t)}dW_2^A(t) \end{aligned} \tag{15}$$

where all parameters are time-dependent.

B. Approximate the SDE (15) by the time homogeneous model

$$d\hat{S}(t) = \sqrt{z(t)}\hat{\lambda}[\hat{m}\hat{S}(t) + (1-\hat{m}(t))S(0)]dW_1^A(t)$$
$$d\hat{z}(t) = \beta(1-\hat{z}(t))dt + \hat{\varepsilon}\sqrt{\hat{z}(t)}dW_2^A(t) \tag{16}$$

where all parameters are constant.

Approximation A essentially involves finding time dependent parameters $\bar{\lambda}, \bar{m}$ so that the diffusion in (14) is approximated by the diffusion in (15), i.e.

$$z\bar{\lambda}^2[\bar{m}S + (1-\bar{m})S(0)]^2 \approx \|S_X{}'\eta\|^2 \tag{17}$$

Equating levels in (15) at $X(t) = Y(t) = 0$ yields

$$\bar{\lambda}(t)^2 = \frac{1}{S(0)^2}\| S_X(t)'\Gamma(t)^{-1} I_{F(0)} I_\lambda R \|^2_{X=Y=0} \tag{18}$$

Differentiating (17) with respect to X_i at $X(t) = Y(t) = 0$ yields

$$(\bar{\lambda}(t)^2 S(0)[S_{X_i}(t)]_{X=Y=0}) \cdot \bar{m}(t) = [\frac{1}{z(t)}\frac{\partial}{\partial X_i}\| S_X(t)'\eta(t)\|^2]_{X=Y=0} \tag{19}$$

for $i = 1,\ldots,k$. Due to the form of η, the right hand side of (19) is independent of $z(t)$, so (19) forms k linear equations in $\bar{m}$. We solve these by regression:

$$\bar{m}(t) = \frac{\sum_{i=1}^{k}[S_{X_i}(t)\frac{\partial}{\partial X_i}\| z(t)^{-1}S_X(t)'\eta(t)\|^2]_{X=Y=0}}{\bar{\lambda}(t)^2 S(0)\sum_{i=1}^{k}[S_{X_i}(t)^2]_{X=Y=0}} \tag{20}$$

All quantities in (18) and (20) can be computed in closed-form using the zero coupon bond price formula in (5).

It be should be noted that this approximation can be slightly refined by evaluating (18) and (20) along levels of X, Y corresponding to approximate expected levels of X, Y under the annuity measure of the swaption under consideration.

Approximation B involves finding constant parameters so that the model (16) produces option prices that are close to those of (15) with parameters given by (18) and (20). We use the methodology suggested by Piterbarg (2005). The exact details are quite complicated and are omitted here for space considerations, but the main point is that the technique is both very quick and accurate. Computationally, the method only relies on numerical solution of one Ricatti ODE per swaption pricing -- all remaining calculations are done in closed-form. Relative to direct solution of (15) by numerical inversion of Fourier transform as suggested in Andersen and Andreasen (2002), this techniques is much faster and only marginally less accurate.

Calibration

We start by fixing $\kappa_1,\ldots,\kappa_k$, the correlation structure for the forward rates RR', and a set of tenors $\tau_1,\ldots,\tau_k$ of the model. We further fix a time grid $0=t_0<t_1<\ldots$ of expiries and a set of tenors $\{\mathrm{T}_j\}$ corresponding to the swaption smiles that we wish to calibrate the model to. We assume that we have fitted parameters $\tilde{\lambda}_{hj},\tilde{m}_{hj},\tilde{\varepsilon}_{hj}$ of the model (6) for these expiries (h) and tenors (j) of the calibration swaptions, as in Table 1.

We let the model (5) and (11) be parameterised by

$$\lambda(t,t+\tau_i)=\lambda_{hi},\; m(t,t+\tau_i)=m_{hi},\; \varepsilon(t)=\varepsilon_h$$

for $t_{h-1}<t\le t_h$. We use approximation A and B to give us constant parameters $\hat{\lambda}_{hj},\hat{m}_{hj},\hat{\varepsilon}_{hj}$ for each swaption. We now calibrate the model by bootstrapping, i.e. we solve the optimisation problems

$$\min_{\{\lambda_{hi},m_{hi},\varepsilon_h\}_{i=1,\ldots,k}} \gamma_\lambda\sum_j(\hat{\lambda}_{hj}-\tilde{\lambda}_{hj})^2+\gamma_m\sum_j(\hat{m}_{hj}-\tilde{m}_{hj})^2+\gamma_\varepsilon\sum_j(\hat{\varepsilon}_{hj}-\tilde{\varepsilon}_{hj})^2$$

sequentially for $h=1,2,\ldots$. Here $\gamma_\lambda,\gamma_m,\gamma_\varepsilon$ are weights for balancing the different objectives against each other. Most often we calibrate the model in a sequence where only one of the weights $\gamma_\lambda,\gamma_m,\gamma_\varepsilon$ is non-zero at the time.

As an example of this consider simultaneous calibration of a 4-factor model of the type specified in (5) and (11) to all the EUR cap and swaption implied volatility smiles of 19 expiries ranging from 6m to 20y and 8 tenors ranging from 6m to 30y. The implied volatility smiles are parameterised by the parameters in Table 1. We set

$$(\kappa_1,\kappa_2,\kappa_3,\kappa_4)=(0.015,0.15,0.30,1.20)$$
$$(\tau_1,\tau_2,\tau_3,\tau_4)=(\text{6m},\text{2y},\text{10y},\text{30y})$$

and use a correlation matrix estimated for historical time series data of forward rate curves.

The resulting model parameters are shown in Table 2. We see that the forward skew parameters, m_i, are decreasing more sharply in tenor than the corresponding "term" skew parameters shown in Table 1. This is consistent with the findings in Piterbarg (2003). There does not appear to be a clear trend over time in any of the calibrated parameters. However, there is more noise in the calibrated forward skew parameters than in the Piterbarg (2003) case. This is probably due to the fact that we make no attempts to smooth our calibrated parameters in the time dimension. The calibration takes about 5 seconds of CPU time.

The error of such a calibration can be split in two, first there is the error from the fact that a 4-factor model will not be able to exactly match the smiles of 8 tenors. We show this error by pricing swaptions and caps for all the calibration expiries and tenors by use of the approximation A and B, and comparing the resulting prices to those of the target model. The strikes chosen correspond to 5% to 95% Delta in Black-Scholes terms. All in all we price 1024 caplets and swaptions. We call this error "pure calibration error" and it is shown in Figure 2. We see that the pure calibration error is within +/-0.25% in Black-Scholes volatility terms in most of the range.

What actually counts, however, is of course what the error is when the model is simulated. We call this "total calibration error" and the result of pricing up all the calibration swaptions by simulation is shown in Figure 3. We see that the total calibration error is within +/-0.40% in Black-Scholes volatility terms in most of the range.

In summary: a 4-factor version of the model can simultaneously fit market prices of caps and swaptions for all strikes (5%-95% Delta), expiries (6m-20y), and tenors (6m-30y), within a tolerance of 0.40% in implied Black volatility terms. Moreover, the calibration only takes about 5 seconds of CPU time.

Monte-Carlo Simulation

Strictly speaking, SDEs of the type defined by (5) and (11) can be explosive. To avoid this problem we follow Heath, Jarrow, and Morton (1992) and simply replace $f(t, t+\tau_i)$ in (11) with

$$\tilde{f}(t, t+\tau_i) = \max(f(0, t+\tau_i) - c, \min(f(t, t+\tau_i), f(0, t+\tau_i) + c)) \qquad (19)$$

where c is some constant.

Due to the fact that the natural domain for the stochastic volatility factor z is $\{z \geq 0\}$, straightforward Euler discretization of the SDE for z is going to exhibit very poor convergence as we decrease the time-steps $\Delta t \to 0$. Instead, we prefer to use the following (local) log-normal discretization

$$z(t+\Delta t) = \overline{z} e^{-\frac{1}{2}v^2 + v \cdot N(0,1)}$$

where we choose $\overline{z}, v$ so that the log-normal approximation matches the two first conditional moments of $z(t_{h+1})$ given $z(t_h)$, i.e.

$$\overline{z} = 1 + e^{-\beta \Delta t}(z(t) - 1)$$
$$v^2 = \ln[1 + \overline{z}^{-2}\{\frac{\varepsilon^2}{2\beta}(1 - e^{-\beta \Delta t}) + \frac{\varepsilon^2}{\beta}(z(t) - 1)(e^{-\beta \Delta t} - e^{-2\beta \Delta t})\}]$$

We combine this with standard Euler discretisation of X, Y. With typical parameter values, accurate pricing can be obtained with monthly or quarterly time stepping.

The strength of the separable volatility structure relative to the general HJM or LMM specification is the speed in simulation of the model. To illustrate this we perform simulation of vanilla swaps with monthly rate reset in two models: a LMM with 4 factors and our separable model also with 4 factors. The resulting CPU times are reported in Table 3. We see that in the LMM the computational time increases roughly with the square of the simulation horizon whereas it is linear for the separable model. Table 3 as well as our experience indicate that one can obtain computational savings of up to a factor 10 for longer dated structures with the separable model relative to the LMM.

Finite Difference Solution

For the one-factor model, $k = 1$, finite difference solution is an efficient alternative to Monte-Carlo simulation. The associated pricing PDE can be written as

$$
\begin{aligned}
0 &= \frac{\partial V}{\partial t} + [D_x + D_y + D_z]V \\
D_x &= -\frac{r}{3} + (-\kappa x + y)\frac{\partial}{\partial x} + \frac{1}{2}\eta^2 \frac{\partial^2}{\partial x^2} \\
D_y &= -\frac{r}{3} + (\eta^2 - 2\kappa y)\frac{\partial}{\partial y} \\
D_z &= -\frac{r}{3} + \beta(1-z)\frac{\partial}{\partial z} + \frac{1}{2}\varepsilon^2 z \frac{\partial^2}{\partial z^2}
\end{aligned}
$$

We use an alternating direction implicit (ADI) scheme, see Mitchell and Griffiths (1980), that splits the solution over each time step into three steps

$$
\begin{aligned}
&[\frac{1}{\Delta t} - \frac{1}{2}D_x]V(t + \frac{2}{3}\Delta t) = [\frac{1}{\Delta t} + \frac{1}{2}D_x + D_y + D_z]V(t + \Delta t) \\
&[\frac{1}{\Delta t} - \frac{1}{2}D_y]V(t + \frac{1}{3}\Delta t) = \frac{1}{\Delta t}V(t + \frac{2}{3}\Delta t) - \frac{1}{2}D_y V(t + \Delta t) \qquad (20) \\
&[\frac{1}{\Delta t} - \frac{1}{2}D_z]V(t) = \frac{1}{\Delta t}V(t + \frac{1}{3}\Delta t) - \frac{1}{2}D_z V(t + \Delta t)
\end{aligned}
$$

where $V(t)$ is to be interpreted as 3-dimensional tensors of values at time t.

We use the standard 3-point discretization for D_x and D_z, but for D_y we use a 5-point discretization for the first derivative. This gives higher accuracy in the y dimension, $O(\Delta y^4)$, and enables us to get away with relatively few y-steps, say 10. The disadvantage of the 5-point discretization is that the workload increases at a rate higher than the $O(\Delta y^{-1})$ of a 3-point scheme but we find that is worth it in this particular case.

Square root processes like (11b) with high volatility and low mean reversion and therefore high probability of hitting $z = 0$ can be tricky to solve numerically. Linear discretization of the z axis according to the standard deviation of z at maturity leads to very few points in the interval $[0,1]$ relative to the number of points between 1 and the upper bound of z. Attempting to solve this problem by transforming the state variable introduces infinite drift for the transformed variable at $z = 0$ and this is therefore not a recommendable route. Instead we choose to discretize z according to

$z_j = O(j^2)$. This means that we get lower asymptotic accuracy than $O(\Delta z^2)$ but this does not seem to be a problem in practice.

In summary, we have a scheme with the following properties:

- Uniform von Neuman stability.
- Accuracy of $O(\Delta t^2 + \Delta x^2 + \Delta y^4 + \Delta z^p) \quad , p < 2$
- Workload of $O(\Delta t^{-1} \cdot \Delta x^{-1} \cdot \Delta y^{-q} \cdot \Delta z^{-1}) \quad , q > 1$.

In practice a 30 year Bermuda swaption is accurately priced on a grid of dimensions $50 \times 100 \times 10 \times 15$ ($t \times x \times y \times z$) steps and this takes about 3 seconds of CPU time.

Conclusion

We have presented a class of stochastic volatility yield curve models with quick and accurate calibration and significantly quicker Monte-Carlo simulation than general HJM or Libor market models. A one-factor version of the model can be implemented with finite difference solution and can thus be used as an alternative to the standard one-factor models for day-to-day management of large portfolios of interest rate exotics.

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Table 1: Skew and Smile Parameters fitted to EUR Cap and Swaption Prices

m	6m	1y	2y	5y	10y	15y	20y	30y
6m	0.75	0.75	0.62	0.50	0.44	0.38	0.34	0.29
1y	0.75	0.65	0.55	0.46	0.38	0.32	0.30	0.27
3y	0.68	0.58	0.49	0.37	0.31	0.26	0.25	0.22
5y	0.65	0.52	0.43	0.32	0.27	0.23	0.21	0.17
10y	0.58	0.45	0.38	0.27	0.23	0.21	0.17	0.14
15y	0.48	0.36	0.31	0.26	0.20	0.18	0.15	0.13
ε	6m	1y	2y	5y	10y	15y	20y	30y
6m	1.15	1.13	1.13	1.18	1.29	1.29	1.30	1.27
1y	1.15	1.01	1.05	1.06	1.11	1.14	1.13	1.12
3y	1.05	0.93	0.91	0.93	0.94	0.93	0.94	0.93
5y	0.95	0.88	0.88	0.86	0.85	0.86	0.86	0.85
10y	0.86	0.89	0.89	0.87	0.84	0.84	0.84	0.83
15y	1.03	1.00	0.97	0.94	0.91	0.90	0.91	0.89

Table 1 reports best fit m and ε parameters to observed cap and swaption prices for $\beta = 0.05$. Expiries in the rows and tenors in the columns. The currency is EUR and the parameters were estimated from Totem consensus prices end of a particular month in 2004.

Table 2: Parameters of a Calibrated 4-Factor Model

t	$\lambda_1(t)$	$\lambda_2(t)$	$\lambda_3(t)$	$\lambda_4(t)$	$m_1(t)$	$m_2(t)$	$m_3(t)$	$m_4(t)$	$\varepsilon(t)$
0.5	0.2120	0.2452	0.1063	0.0907	0.74	0.47	-0.28	-0.44	1.22
1	0.2242	0.2198	0.1262	0.1017	0.82	0.32	-0.05	-0.56	0.97
2	0.2266	0.2038	0.1325	0.0992	0.92	0.33	-0.01	-0.74	0.95
3	0.2464	0.1869	0.1336	0.0873	0.89	0.30	0.01	-1.01	0.80
4	0.2602	0.1906	0.1276	0.1021	0.89	0.26	0.01	-0.84	0.82
5	0.2693	0.1664	0.1278	0.0784	0.89	0.22	0.05	-1.65	0.75
6	0.2954	0.1678	0.1246	0.0872	0.83	0.24	0.06	-1.10	0.86
7	0.2798	0.1490	0.1268	0.0805	0.87	0.26	0.08	-1.07	0.85
8	0.3064	0.1496	0.1185	0.0635	0.80	0.22	0.10	-1.48	0.85
9	0.3078	0.1335	0.1207	0.0482	0.80	0.25	0.11	-1.84	0.84
10	0.3040	0.1183	0.1172	0.0410	0.81	0.25	0.13	-1.88	0.84
11	0.2934	0.1162	0.1278	0.0416	0.82	0.26	0.13	-2.01	0.97
12	0.2719	0.1148	0.1219	0.0602	0.87	0.26	0.16	-0.99	0.99
13	0.2541	0.0917	0.1276	0.0519	0.90	0.35	0.16	-1.01	1.02
14	0.1891	0.0798	0.1236	0.0658	1.40	0.37	0.20	-0.50	1.04
15	0.1574	0.0529	0.1254	0.0707	1.83	0.70	0.20	-0.36	1.06
16	0.1671	0.1009	0.1410	0.0695	1.73	0.19	0.20	-1.42	0.94
17	0.1705	0.1013	0.1413	0.0734	1.77	0.35	0.14	-0.89	0.94
18	0.1690	0.0893	0.1492	0.0623	1.97	0.45	0.12	-1.01	0.94
20	0.1147	0.0768	0.1554	0.0631	2.03	1.70	-0.11	0.39	0.94

Table 2 reports the resulting parameters when calibrating a 4-factor model to the EUR swaption and cap data of Table 1.

Table 3: CPU Times for Simulation in LMM and in Separable HJM

maturity	lmm	hjm
5y	02.12	01.14
10y	07.20	02.22
15y	15.19	03.33
20y	26.21	04.46
25y	40.27	05.53
30y	55.13	06.56

CPU times in seconds for simulation of 5y, …, 30y vanilla interest rate swaps with monthly reset in a 4 factor Libor Market Model and our 4-factor separable volatility structure HJM model.

Figure 1: This and Last Month's Models against Totem Quotes

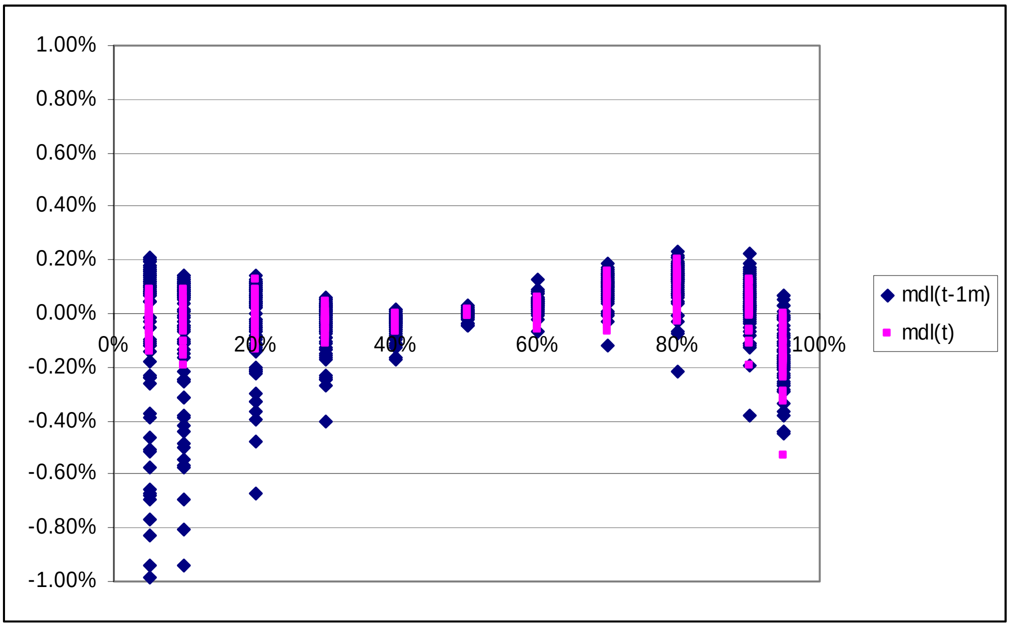

Figure 1 shows the deviations from Totem consensus quotes for EUR swaption in terms of implied Black volatility for two models. One that had its m, ε parameters fitted the month before the other had its parameters fitted this month. Expiries range from 6m to 20y and tenors range from 1y to 30y -- all in all 912 swaptions. All data is as of a particular end of month in 2004.

Figure 2: Pure Calibration Error

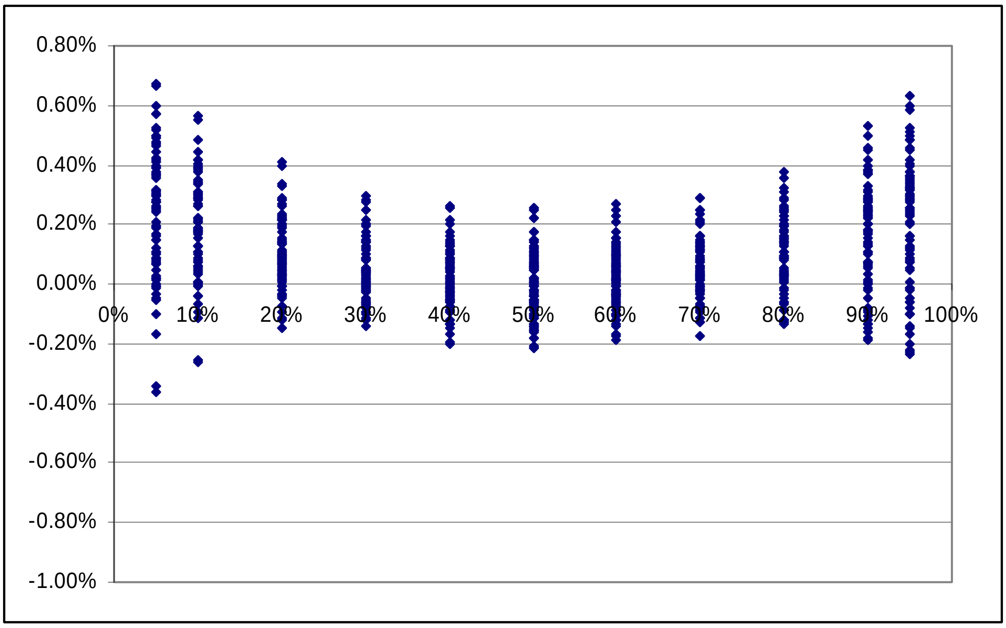

Figure 2 shows the pure calibration error in terms of implied Black-Scholes volatility when calibrating a 4-factor model to the full EUR cap and swaption market. We depict the difference between the yield curve model and the target when we price caps and swaptions under our approximations A and B. Expiries range from 6m to 20y, tenors from 6m to 30y, and strikes from 5% to 95% in terms of Black-Scholes delta -- all in all 1024 caplets and swaptions.

Figure 3: Total Calibration Error

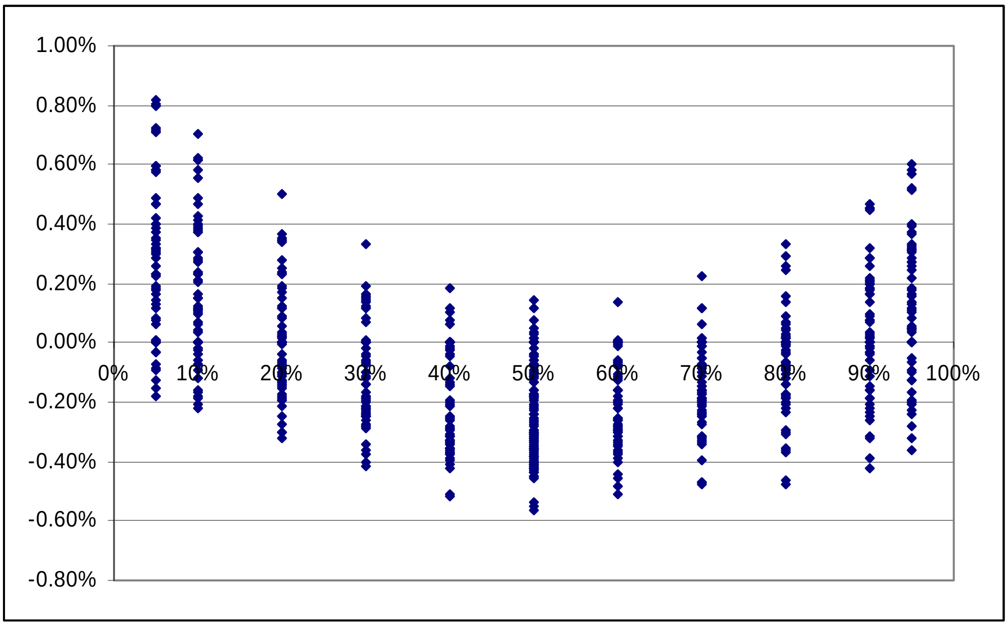

Figure 3 shows the total calibration error in terms of implied Black-Scholes volatility when calibrating a 4-factor model to the full EUR market. We depict the difference between the yield curve model and the target when we price caps and swaptions by simulation. 131,072 simulations were used making the simulation error roughly of the order of 0.10% in terms of implied Black-Scholes volatility. Expiries range from 6m to 20y, tenors range from 6m to 30y, and strikes range from 5% to 95% Black-Scholes delta -- all in all 1024 caplets and swaptions.