

# An alternative derivation of the Vasicek model <sup>1</sup>

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## **Abstract**

This note gives a brief description of how the Vasicek model can be derived from the potential approach described by Rogers (1997), Rutkowski (1997) and Flesaker & Hughston (1996). In particular, a relaxation of the supermartingale assumption required in these papers for positive interest allows us to derive the Vasicek (1977) model.

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We follow the formulation of Rutkowski (1997).

Suppose that  $A_t = \exp(\sigma X(t) - \mu t)$  where  $X(t)$  is an Ornstein-Uhlenbeck process under the forward measure  $P_\infty$ : that is,  $dX(t) = -\alpha X(t)dt + d\hat{Z}(t)$  where  $\hat{Z}(t)$  is a Brownian motion under  $P_\infty$ . Hence  $X(T)|\mathcal{F}_t \sim N(\exp(-\alpha(T-t)), (1 - \exp(-2\alpha(T-t)))/2\alpha)$ . Following Rutkowski (1997) (but relaxing the conditions that  $A_t$  be a supermartingale) we define

$$\begin{aligned}
P(t, T) &= \frac{E_{P_\infty}[A_T | \mathcal{F}_t]}{A_t} \\
&= \exp \left[ -\mu(T-t) + \frac{\sigma^2}{4\alpha} (1 - e^{-2\alpha(T-t)}) - \sigma (1 - e^{-\alpha(T-t)}) X(t) \right] \\
\Rightarrow f(t, T) &= -\frac{\partial}{\partial T} \log P(t, T) \\
&= \mu + \alpha\sigma e^{-\alpha(T-t)} X(t) - \frac{\sigma^2}{2} e^{-2\alpha(T-t)} \\
\Rightarrow r(t) &= \mu + \alpha\sigma X(t) - \frac{\sigma^2}{2}
\end{aligned}$$

With a few lines of algebra we can also see that

$$\begin{aligned}
dP(t, T) &= P(t, T) \left[ r(t)dt + \frac{\sigma^2}{2} (1 - e^{-2\alpha(T-t)}) dt - \sigma (1 - e^{-\alpha(T-t)}) d\hat{Z}(t) \right. \\
&\quad \left. + \frac{\sigma^2}{2} (1 - e^{-\alpha(T-t)})^2 dt \right] \\
&= P(t, T) \left[ r(t)dt - \sigma (1 - e^{-\alpha(T-t)}) d\tilde{Z}(t) \right]
\end{aligned}$$

$$\text{where } d\tilde{Z}(t) = d\hat{Z}(t) - \sigma dt$$

With an appropriate change from the measure  $P_\infty$  to what we will call the risk-neutral measure  $Q$ ,  $\tilde{Z}(t)$  is a Brownian motion under  $Q$ .

Since we have  $r(t) = \mu + \alpha\sigma X(t) - \frac{\sigma^2}{2}$  we thus find that

$$\begin{aligned}
dr(t) &= -\alpha(r(t) - \tilde{\mu})dt + \tilde{\sigma}d\tilde{Z}(t) \\
\text{where } \tilde{\mu} &= \mu + \frac{1}{2}\sigma^2 \\
\tilde{\sigma} &= \alpha\sigma
\end{aligned}$$

$$\text{and } dP(t, T) = P(t, T) [r(t)dt + S(t, T)d\tilde{Z}(t)]$$

$$\text{where } S(t, T) = -\frac{\tilde{\sigma}}{\alpha} (1 - e^{-\alpha(T-t)})$$

## References

- Flesaker, B., and Hughston, L.P. (1996) "Positive interest," *Risk* 9(1), 46-49.
- Langetieg, T. (1980) "A multivariate model of the term structure," *Journal of Finance* 35, 71-97.
- Rogers, L.C.G. (1997) "The potential approach to the term-structure of interest rates and foreign exchange rates," *Mathematical Finance* 7, 157-164.
- Rutkowski, M. (1997) "A note on the Flesaker & Hughston model of the term structure of interest rates," *Applied Mathematical Finance* 4, 151-163.
- Vasicek, O. (1977) "An equilibrium characterisation of the term structure," *Journal of Financial Economics* 5, 177-188.