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VASIČEK BEYOND THE NORMAL

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A general Ornstein-Uhlenbeck (OU) process is obtained upon replacing the Brownian motion appearing in the defining stochastic differential equation with a general Lévy process. Certain properties of the Brownian ancestor are distribution-free and carry over to the general OU process. Explicit expressions are obtainable for expected values of a number of functionals of interest also in the general case. Special attention is paid here to gamma- and Poisson-driven OU processes. The Brownian, Poisson, and gamma versions of the OU process are compared in various respects; in particular, their aptitude to describe stochastic interest rates is discussed in view of some standard issues in financial and actuarial mathematics: prices of zero-coupon bonds, moments of present values, and probability distributions of present values of perpetuities. The problem of possible negative interest rates finds its resolution in the general setup by taking the driving Lévy process to be nondecreasing.

KEY WORDS: Ornstein-Uhlenbeck process, background driving Lévy process, gamma process, compound Poisson process, stochastic interest rates, positive interest rates, perpetuity

1. INTRODUCTION

1.1. The Vasiček Model

The Brownian-motion-driven Ornstein-Uhlenbeck (BOU) process $(r_t)_{t\geq 0}$ is the solution to the stochastic differential equation

$$dr_t = \alpha(\rho - r_t)\,dt + \sigma\,dW_t, \tag{1.1}$$

where $\alpha > 0$, ρ, and $\sigma > 0$ are constants, and W is a standard Brownian motion. The notation r_t stems from financial mathematics, where this process is a commonly used model, attributed to Vasiček (1977), for the spot rate of interest. The notion is that the interest rate drifts systematically toward its long-term mean ρ at a rate proportional to its current deviation from this mean, and that this systematic behavior is disturbed by short-term erratic fluctuations.

1.2. Beyond Vasiček

The state space of the BOU process is all of the real line. Therefore, the Vasiček model is not apt to describe interest rates that cannot go negative. This (claimed-to-be) weakness

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is remedied by the CIR model (Cox, Ingersoll, and Ross 1985), which replaces σ in (1.1) with $\sigma\sqrt{r_t}$. Either model is inappropriate in situations where the interest rate may be discontinuous. Models accommodating jump behavior of interest rates have been proposed by a number of authors; see Shirakawa (1991) and, for a treatment in the general Heath-Jarrow-Morton framework, see Björk (1995), Björk, Kabanov, and Runggaldier (1997), and Glasserman and Kou (2003).

1.3. Vasiček beyond the Normal: Outline of the Study

The essential features of the Ornstein-Uhlenbeck process are preserved if we replace the Brownian motion with some other process with independent and stationary increments—that is, a Lévy process. General results of this kind are worked out in Section 2 where we find the Laplace transforms for transition probabilities, for the stationary distribution, and for integrals of the general OU process. Section 3 establishes some basic distribution theoretical results for the case when the driving process is compound Poisson. It turns out that the marginal distributions of the compound Poisson OU process and of integrals of this process are themselves essentially compound Poisson. Other candidate driving processes are the negative binomial process, which turns out to be just a special case of the compound Poisson, and the gamma process, which is treated rather casually because it is mathematically less tractable. Sections 4–6 discuss aspects relevant for financial/actuarial applications: moments of present values of payment streams, arbitrage-free zero-coupon bond markets, and the probability distribution of a perpetual annuity. Comparisons with the traditional normal Vasiček model are made throughout. If the driving Lévy process is nondecreasing (as is possible for gamma and Poisson processes), we can confine the state space of the OU process to the positive half line, thereby offering a resolution to the problem with possible negative interest rates in the traditional Vasiček model. Section 7 summarizes and discusses the results of the study and adds some considerations about statistical inference. Brief accounts of the general Itô's formula and Laplace transforms are placed in an Appendix. These are taken as prerequisites throughout.

Eberlein and Raible (1998) investigated a certain class of Heath-Jarrow-Morton-type bond prices driven by Lévy processes and, pursuing ideas of Carverhill (1994), proved that the short rate is Markovian under what they call Vasiček and Ho-Lee volatility structures. They paid particular attention to hyperbolic Lévy motion.

The present paper takes a different approach, starting from a general Lévy-driven Vasiček short rate, which is automatically Markov, discussing its potential role in a wider framework of actuarial and financial problems, and focusing in particular on the compound Poisson version of the model.

2. THE GENERAL ORNSTEIN-UHLENBECK PROCESS

2.1. The Driving Lévy Process

Let $(X_t)_{t\geq 0}$ be a stochastic process defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $(\mathcal{F}_t)_{t\geq 0}$ be its natural filtration. We shall assume that X is square integrable and that its paths are RCLL. Moreover, we assume that it is a Lévy process (see Appendix A.3). Then there exist constants μ and σ such that $\mathbb{E}[X_t] = \mu t$ and $\mathbb{V}\mathrm{ar}[X_t] = \sigma^2 t$. In shorthand,

$$X_t \sim (\mu t, \sigma^2 t). \tag{2.1}$$

This generalizes to

$$\int_0^\infty f_s\, dX_s \sim \left(\mu \int_0^\infty f_s\, ds, \sigma^2 \int_0^\infty f_s^2\, ds\right), \tag{2.2}$$

valid for any deterministic function f such that the integrals exist in the Riemann sense. The proof parallels that of (A.4) in Appendix A.3.

2.2. The Ornstein-Uhlenbeck Process

We define the *general Ornstein-Uhlenbeck* (*OU*) *process* as the solution to the stochastic differential equation

$$dr_t = -\alpha r_t\, dt + dX_t. \tag{2.3}$$

Upon defining the standardized process W by

$$W_t = \frac{X_t - \mu t}{\sigma}, \qquad X_t = \mu t + \sigma W_t,$$

we can recast (2.3) in the form (1.1) with

$$\rho = \frac{\mu}{\alpha}. \tag{2.4}$$

The standardized process shares some essential properties with Brownian motion; it has independent and stationary zero mean increments, hence is a martingale, and $\mathbb{V}\mathrm{ar}[W_t] = t$.

2.3. Transition Probabilities

The OU process is Markov, so we need only to determine its simple transition probabilities—that is, the probability law (written $\mathcal{L}$) of r_u conditional on r_t (written $r_u|_{r_t}$) for $t < u$. We set out by solving the differential equation (2.3), which goes by standard rules of calculus since we are working with an integrating factor that is continuous and of bounded variation. Carrying the first term on the right of (2.3) over to the left, then multiplying with $e^{\alpha t}$ to form the complete differential $d(e^{\alpha t} r_t)$, which is Itô's formula, given as (A.1) in the Appendix, and finally integrating over $(t, u]$, we arrive at

$$r_u = e^{-\alpha(u-t)} r_t + \int_t^u e^{-\alpha(u-s)}\, dX_s. \tag{2.5}$$

Now we mobilize the results in Appendixes A.2 and A.3. The first step is to identify the Laplace transform (LT) of X_t, $\ell_{X_t}(\eta) = \exp\{\phi(\eta)t\}$. Then, applying (A.5) and (A.4) to the last term in (2.5), we get the first part of the following theorem. The second part is a straightforward consequence of (2.2) plus a rearrangement based on (2.4).

THEOREM 2.1. *The transition probabilities of the OU process are given by*

$$r_u|_{r_t} \stackrel{\mathcal{L}}{=} e^{-\alpha(u-t)} r_t + R,$$

where $R = \int_0^{u-t} e^{-\alpha s}\, dX_s$ *has the LT*

$$\ell_R(\eta) = e^{\int_0^{u-t} \phi(\eta e^{-\alpha s})\, ds}.$$

The first two moments are distribution-free and are given by

$$r_u|_{r_t} \sim \left(\rho + e^{-\alpha(u-t)}(r_t - \rho), \sigma^2 \frac{1 - e^{-2\alpha(u-t)}}{2\alpha}\right).$$

2.4. The Integrated Process

For fixed $t < T < \infty$ and an integrable deterministic function q, consider the random variable $\int_t^T q_u r_u \, du$, which plays a role in the actuarial/financial context. From (2.5) we get

$$\int_t^T q_u r_u \, du = \int_t^T q_u e^{-\alpha(u-t)} \, du \, r_t + \int_t^T q_u e^{-\alpha u} \, du \int_t^u e^{\alpha s} \, dX_s.$$

Upon changing the order of integration in the last term on the right (Itô applied to the product of $X_{1,u} = \int_u^T q_s e^{-\alpha s} \, ds$ and $X_{2,u} = \int_t^u e^{\alpha s} \, dX_s$) and introducing

$$Q_t^T = \int_t^T q_u e^{-\alpha(u-t)} \, du, \tag{2.6}$$

we arrive at

$$\int_t^T q_u r_u \, du = Q_t^T r_t + \int_t^T Q_u^T \, dX_u. \tag{2.7}$$

Obviously, under integrability conditions, (2.7) remains valid also in the limit $T = \infty$. Copying the proof of Theorem 2.1, we obtain the following theorem.

THEOREM 2.2. *Let $(r_t)_{t \geq 0}$ be an OU process. For $t < T < \infty$ and q an integrable deterministic function,*

$$\left.\int_t^T q_u r_u \, du\right|_{r_t} \overset{\mathcal{L}}{=} Q_t^T r_t + R,$$

where $R = \int_t^T Q_u^T \, dX_u$ has LT

$$\ell_R(\eta) = e^{\int_t^T \phi(\eta Q_u^T) \, du}.$$

The first two moments are distribution-free and are given by

$$\left.\int_t^T q_u r_u \, du\right|_{r_t} \sim \left(Q_t^T r_t + \mu \int_t^T Q_u^T \, du, \sigma^2 \int_t^T \left(Q_u^T\right)^2 du\right)$$
$$= \left(\int_t^T q_u \, du \, \rho + Q_t^T (r_t - \rho), \sigma^2 \int_t^T \left(Q_u^T\right)^2 du\right).$$

To verify the last equality, deduce from (2.6) that

$$dQ_t^T = \left(\alpha Q_t^T - q_t\right) dt, \quad \int_t^T Q_u^T \, du = \frac{1}{\alpha}\left(\int_t^T q_u \, du - Q_t^T\right), \tag{2.8}$$

and recall (2.4).

2.5. Stationary Distribution

Define the process Z by

$$Z_u = \int_0^u e^{-\alpha s}\, dX_s.$$

The limit Z_∞ is almost surely finite. Since $Z_\infty = e^{-\alpha u}\int_u^\infty e^{-\alpha(s-u)}\, dX_s + Z_u$, we have

$$Z_\infty \overset{\mathcal{L}}{=} e^{-\alpha u} Z' + Z_u, \tag{2.9}$$

with Z' independent of Z_u and $Z' \overset{\mathcal{L}}{=} Z_\infty$. Now, putting $t = 0$ in (2.5) and using (A.5), we obtain

$$r_u \overset{\mathcal{L}}{=} e^{-\alpha u} r_0 + Z_u,$$

and by (2.9) we conclude with the following theorem.

THEOREM 2.3. *The stationary distribution of the OU process is that of the random variable* $Z_\infty = \int_0^\infty e^{-\alpha s}\, dX_s$, *with the LT*

$$\ell_{Z_\infty}(\eta) = e^{\int_0^\infty \phi(\eta e^{-\alpha s})\, ds} = e^{\frac{1}{\alpha}\int_0^\eta \frac{\phi(s)}{s}\, ds}, \tag{2.10}$$

and mean and variance given by

$$Z_\infty \sim \left(\rho, \frac{\sigma^2}{2\alpha}\right).$$

The second expression in (2.10) is obtained from the first upon substituting $\eta e^{-\alpha s}$ in the integral.

The formula (2.10) was obtained also by Barndorff-Nielsen, Jensen, and Sørensen (1998) by a different method.

3. SPECIAL ORNSTEIN-UHLENBECK PROCESSES

3.1. A General Remark

We now study some special driving processes X. To verify that they are well-defined Lévy processes, one has only to check that the LT of X_t is of the form (A.3).

3.2. The Brownian OU Process Revisited

The classical BOU model adds normality to (2.1) to make it $X_t \sim \mathrm{N}(\mu t, \sigma^2 t)$, with LT

$$\ell_{X_t}(\eta) = e^{-\mu t\eta + \frac{1}{2}\sigma^2 t\eta^2}; \quad \text{hence } \phi(\eta) = -\mu\eta + \frac{1}{2}\sigma^2\eta^2. \tag{3.1}$$

Inserting this in the general results above, we rediscover well-known results about the BOU process, which amount to adding the qualifying N to $(\cdot,\cdot)$ in Theorems 2.1, 2.2 (a small piece of news), and 2.3.

3.3. The Poisson-Driven OU Process

Let C be a compound Poisson process defined by

$$C_t = \sum_{i=1}^{N_t} Y_j, \tag{3.2}$$

where N is a homogeneous Poisson process with intensity λ, independent of $Y_1, Y_2, \ldots$, which are i.i.d. replicates of some random variable Y with distribution G, say. Then $C_t \sim \mathrm{CPO}(\lambda t, Y)$, the compound Poisson distribution with frequency parameter λt and compounding variate Y, with LT

$$\ell_{C_t}(\eta) = e^{\lambda t(\ell_Y(\eta)-1)}. \tag{3.3}$$

By (A.4),

$$\ell_{\int_0^\infty f_s\, dC_s}(\eta) = e^{\lambda \int_0^\infty (\ell_Y(\eta f_s)-1)\, ds}. \tag{3.4}$$

In particular, for $0 \leq t < u < \infty$,

$$\ell_{\int_t^u f_s\, dC_s}(\eta) = e^{\lambda(u-t)\left(\int_t^u \ell_Y(\eta f_s)\frac{ds}{u-t}-1\right)}. \tag{3.5}$$

Inspection of (3.5) and comparison with (3.3) yield

$$\int_t^u f_s\, dC_s \sim \mathrm{CPO}(\lambda(u-t), f_S Y), \tag{3.6}$$

where $S \sim \mathrm{U}[t, u]$, the uniform distribution over $[t, u]$, and S and Y are independent.

It is sometimes convenient to adopt the marked point process point of view (see, e.g., Brémaud 1981, or Karr 1991), and write (3.2) as

$$C_t = \int_0^t \int_y dN_s(dy), \quad dC_t = \int_y dN_t(dy), \tag{3.7}$$

where $N_t(A) = \sum_{j=1}^{N_t} 1_A(Y_j)$. Then, for any process $\{H_t(y); t \geq 0, y \in \mathbb{R}\}$, jointly measurable in (ω, t, y), predictable in (ω, t) for each fixed y, and suitably integrable, the process M defined by

$$dM_t = \int_y H_t(y)\{dN_t(dy) - G(dy)\,\lambda\, dt\} \tag{3.8}$$

is a martingale. (Here ω denotes the generic point in the event space Ω.)

The Poisson OU (POU) process is defined by letting X in (2.3) be a compound Poisson process with drift,

$$X_t = \nu t + C_t. \tag{3.9}$$

Then, with the notation adopted for the general case,

$$\mu = \nu + \lambda \mathbb{E}[Y], \quad \sigma^2 = \lambda \mathbb{E}[Y^2]. \tag{3.10}$$

By (3.3),

$$\phi(\eta) = -\nu\eta + \lambda(\ell_Y(\eta) - 1). \tag{3.11}$$

From Theorems 2.1 and 2.2, or directly from (2.5), (2.7), (2.8), (3.6), and (3.9), and from Theorem 2.3 we gather the following results.

THEOREM 3.1. *The POU process possesses the following properties:*
For $t < u < \infty$,

$$r_u|_{r_t=r} \stackrel{\mathcal{L}}{=} \frac{\nu}{\alpha} + e^{-\alpha(u-t)}\left(r - \frac{\nu}{\alpha}\right) + R,$$

where $R = \int_0^{u-t} e^{-\alpha s}\, dC_s \sim \mathrm{CPO}(\lambda(u-t), e^{-\alpha S} Y)$, *with* $S \sim \mathrm{U}[0, u-t]$ *and independent of* Y.

For $t < T < \infty$ *and* q *an integrable deterministic function,*

$$\left.\int_t^T q_u r_u\, du\right|_{r_t=r} \stackrel{\mathcal{L}}{=} \int_t^T q_u\, du \frac{\nu}{\alpha} + Q_t^T\left(r - \frac{\nu}{\alpha}\right) + R,$$

where $R = \int_t^T Q_u^T\, dC_u \sim \mathrm{CPO}(\lambda(T-t), Q_S^T Y)$, *with* $S \sim \mathrm{U}[t, T]$ *and independent of* Y.

The stationary distribution of the POU process is that of $\nu/\alpha + Z_\infty$, *where*

$$Z_\infty = \int_0^\infty e^{-\alpha s}\, dC_s,$$

with LT

$$\ell_{Z_\infty}(\eta) = e^{\lambda \int_0^\infty \left(\ell_Y(\eta e^{-\alpha s}) - 1\right) ds} = e^{\frac{\lambda}{\alpha} \int_0^\eta \frac{\ell_Y(s) - 1}{s}\, ds}.$$

The compound Poisson distribution can be calculated with various techniques. The Panjer recursion is widely referred to in actuarial literature (see, e.g., Panjer and Willmot 1992).

The POU process has paths of bounded variation. It makes jumps distributed as Y at Poisson epochs, and on intervals between jumps it drifts exponentially toward ν/α, as is seen from the first relation in Theorem 3.1. In particular, if $Y > 0$ and $r_0 > \nu/\alpha$, then the process stays strictly above ν/α. Thus, $(\nu/\alpha, \infty)$ is the natural state space of the process in this case. Similarly, if $Y < 0$, then the natural state space is $(-\infty, \nu/\alpha)$.

Every Lévy process can be obtained as the weak limit of a sequence of compound Poisson processes. For the BOU this is seen, for example, by inspection of (3.11): putting $\nu = \mu - \lambda \mathbb{E}[Y]$ and inserting the second-order Taylor expansion of $\ell_Y(\eta)$, we find that

$$\phi(\eta) = -(\mu - \lambda \mathbb{E}[Y])\eta + \lambda\left(-\mathbb{E}[Y]\eta + \frac{1}{2}\mathbb{E}[Y^2]\eta^2 - \frac{1}{6}\mathbb{E}[Y^3]\eta'^3\right),$$

with η' between 0 and η. Canceling terms, letting $\lambda \to \infty$, and rescaling Y such that $\sigma^2 = \lambda \mathbb{E}[Y^2]$ remains fixed, we arrive at the function ϕ in (3.1). By the usual extension argument, the limit result carries over to the process.

3.4. The Gamma OU Process

Let C be a gamma process defined by the stationary and independent increment properties and $C_t \sim \mathrm{GA}(\gamma t, \delta)$, the gamma distribution with density by

$$g_{C_t}(c) = \frac{\delta^{\gamma t}}{\Gamma(\gamma t)} c^{\gamma t - 1} e^{-\delta c},$$

$c > 0$. Here Γ denotes the gamma function, and the shape parameter γ and the inverse scale parameter δ are both strictly positive. One easily calculates

$$\mathbb{E}\left[C_t^p e^{-qC_t}\right] = \frac{\delta^{\gamma t}}{\Gamma(\gamma t)} \frac{\Gamma(\gamma t + p)}{(\delta + q)^{\gamma t + p}}. \tag{3.12}$$

In particular, $\mathbb{E}[C_t] = \gamma t/\delta$, $\mathbb{V}\text{ar}[C_t] = \gamma t/\delta^2$, and

$$\ell_{C_t}(\eta) = e^{-\gamma t \ln(1+\eta/\delta)}. \tag{3.13}$$

The gamma-driven OU process (GOU) is defined by letting X in (2.3) be

$$X_t = \nu t + C_t.$$

Then

$$\mu = \nu + \gamma/\delta, \quad \sigma^2 = \gamma/\delta^2.$$

By (3.13),

$$\phi(\eta) = -\nu\eta - \gamma \ln(1 + \eta/\delta), \tag{3.14}$$

which is to be put into Theorems 2.1–2.3.

The gamma process, being nondecreasing with independent increments, is a pure jump process. Its jump epochs form a dense set on the positive half line, however, so to study the dynamics of functions of this process, we would need a more general version of Itô's formula than the one presented in the Appendix. We shall not pursue such matters because the dynamics equations are not constructive anyway.

To see that the GOU process is BOU in the limit, inspect the second-order Taylor expansion of (3.14),

$$\begin{aligned} \phi(\eta) &= -\nu\eta - \gamma \left(\eta/\delta - \frac{1}{2}(\eta/\delta)^2 + \frac{1}{3}\theta^3\right) \\ &= -(\nu + \gamma/\delta)\eta + \frac{1}{2}(\gamma/\delta^2)\eta^2 - \frac{1}{3}\gamma\theta^3, \end{aligned} \tag{3.15}$$

where $|\theta| \leq |\eta/\delta|^3$. For fixed μ and σ^2, take $\gamma = \delta^2\sigma^2$, $\nu = \mu - \gamma/\delta$, and let $\delta \to \infty$, to obtain that (3.15) tends to the expression for $\phi(\eta)$ of the BOU process given in (3.1).

3.5. The Negative Binomial OU Process

Let B be the compound negative binomial process defined by

$$B_t = \sum_{i=1}^{N_t} Z_j,$$

where N is a negative binomial process, independent of $Z_1, Z_2, \ldots$, which are i.i.d. replicates of some random variable Z. The process N can be constructed by taking C to be the gamma process described in the previous subsection and, conditional on C, letting N have independent increments with

$$N_t - N_s|_C \sim \text{PO}(C_t - C_s), \quad s < t. \tag{3.16}$$

Applying (3.12) to

$$\mathbb{P}[N_t = n] = \mathbb{E}\{\mathbb{P}[N_t = n \mid C]\} = \mathbb{E}\left[\frac{1}{n!}C_t^n e^{-C_t}\right],$$

we find that $N_t \sim \mathrm{NB}(\gamma t, (1+\delta)^{-1})$, the negative binomial distribution given by

$$\mathbb{P}[N_t = n] = \binom{n+\gamma t-1}{n}\left(\frac{\delta}{1+\delta}\right)^{\gamma t}\left(\frac{1}{1+\delta}\right)^{n}, \quad n = 0, 1, \ldots. \tag{3.17}$$

Combining (3.16) and (3.3), we find

$$\mathbb{E}[e^{-\eta B_t}] = \mathbb{E}\{\mathbb{E}[e^{-\eta B_t} \mid C]\} = \mathbb{E}[e^{C_t(\ell_Z(\eta)-1)}],$$

and, by (3.13),

$$\ell_{B_t}(\eta) = e^{-\gamma t \ln\left(1+\frac{1-\ell_Z(\eta)}{\delta}\right)}. \tag{3.18}$$

As the driving process for the compound negative binomial OU process we take $X_t = \nu t + B_t$.

Now, this case needs no separate treatment because the compound negative binomial process is, in fact, a special case of the compound Poisson process in Section 3.3, with intensity

$$\lambda = \gamma \ln\left(1+\frac{1}{\delta}\right)$$

and jump size distribution given by

$$\ell_Y(\eta) = 1 - \frac{\ln\left(1+\frac{1-\ell_Z(\eta)}{\delta}\right)}{\ln\left(1+\frac{1}{\delta}\right)}. \tag{3.19}$$

To see this, start from the obvious relationship

$$\ell_{B_t}(\eta) = \mathbb{P}[N_t = 0] + (1-\mathbb{P}[N_t = 0])\ell_{B_t \mid N_t>0}(\eta).$$

Insert in this relationship the expression (3.18) and, from (3.17), $\mathbb{P}[N_t = 0] = \exp(-\gamma t \ln(1+1/\delta))$, solve with respect to $\ell_{B_t \mid N_t>0}(\eta)$, and verify that the limit of this function as t tends to 0 is the expression in (3.19). The claimed result now follows by inspection of (3.18) and (3.3).

3.6. Positive and Almost Normal Vasiček Interest

Let the driving Lévy process in (2.3) be of the form

$$X_t = \beta t + V_t,$$

where β is constant and V_t is a nondecreasing Lévy process with no deterministic drift term. Thus, V_t is a pure jump process (see Bertoin 1996). The solution r_t to (2.3) must then be bounded from below by the solution $\underline{r}_t$ to the deterministic differential equation

$$dr_t = -\alpha r_t\, dt + \beta\, dt,$$

which is

$$\underline{r}_t = e^{-\alpha t} r_0 + \left(1-e^{-\alpha t}\right)\frac{\beta}{\alpha}.$$

If $r_0 \geq 0$ and $\beta \geq 0$, it follows that $r_t \geq 0$ for all $t > 0$.

Thus we have identified a class of general OU processes that live on the positive half line and meet the requirement that interest rates be positive.

Moreover, recalling the weak convergence results obtained in Sections 3.3 and 3.4, we conclude that we can construct OU processes of compound Poisson and gamma types, that are positive and are arbitrarily close to a classical Vasiček (BOU) process.

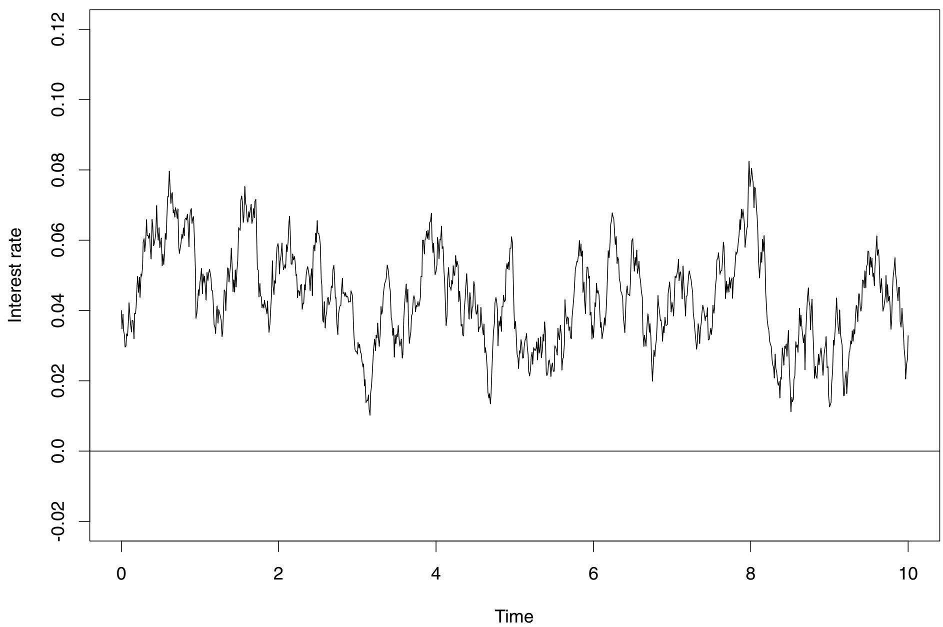

FIGURE 3.1. First simulated path of the BOU process.

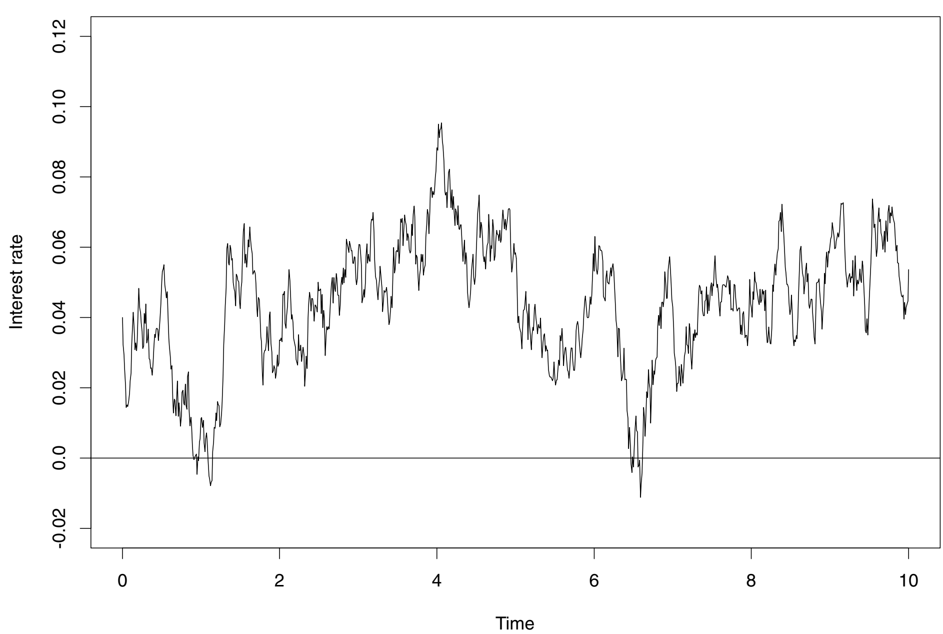

FIGURE 3.2. Second simulated path of the BOU process.

The point is illustrated in Figures 3.1–3.4, which display simulated paths of OU processes, all with the same initial value $r_0 = 0.04$ and the same parameters $\alpha = 5$, $\rho = 0.04$, and $\sigma^2 = 0.002$. Figures 3.1 and 3.2 show two outcomes of the BOU process, one of which turns negative. Figure 3.3 shows an outcome of the POU process with $\alpha = 5$, $\lambda = 20$, and

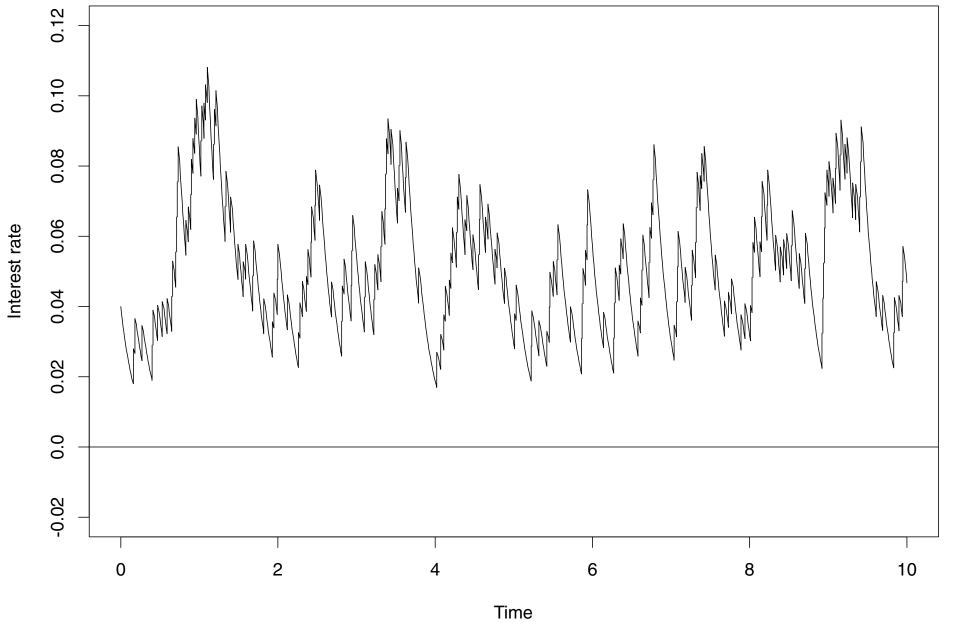

FIGURE 3.3. Simulated path of the POU process.

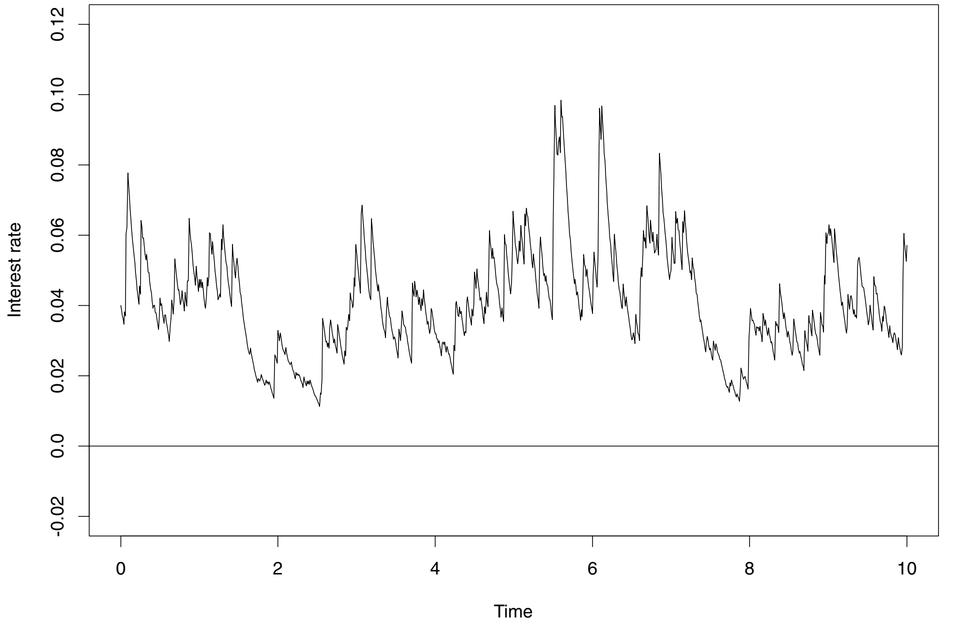

FIGURE 3.4. Simulated path of the GOU process.

$Y = 0.01$. Figure 3.4 shows an outcome of the GOU process with $\alpha = 5$, $\gamma = 20$, and $\delta = 100$. The basic features of the paths are much the same, but we know that the POU and GOU processes specified here could never produce negative values.

Barndorff-Nielsen and Shephard (1999) advocate positive OU processes as models for stochastic volatility.

4. MOMENTS OF PRESENT VALUES

4.1. Motivation

Let $(P_t)_{t\geq 0}$ be an RCLL stochastic process that is independent of the OU process. Interpreting P as a stream of payments and r_t as the interest rate at time t, the present value at time t of the payments in $(t, T]$ is

(4.1) $$\int_t^T e^{-\int_t^s r_u\,du}\,dP_s.$$

In the actuarial context, where solvency of insurers is a major issue, one takes the first three conditional moments (or mean, coefficient of variation, and skewness) of this present value as basic measures of the riskiness of the business (see, e.g., Parker 1994, and Norberg 1995). To remain focused on stochastic interest rates, assume henceforth that P is deterministic and continuous. The kth noncentral moment of the present value in (4.1) is (Fubini and symmetry)

(4.2) $$k!\int\cdots\int_{t<s_1<\cdots<s_k\leq T}\mathbb{E}\left[e^{-\int_t^T\sum_{i=1}^k 1_{(t,s_i]}(u)r_u\,du}\right]dP_{s_1}\cdots dP_{s_k}.$$

Thus, the moments involve expected values of the form

(4.3) $$p_t^T=\mathbb{E}\left[e^{-\int_t^T q_u r_u\,du}\middle|\,r_t\right]=e^{-Q_t^T r_t+\int_t^T\phi\left(Q_u^T\right)du},$$

the latter expression coming from Theorem 2.2. We shall study this function under the BOU and POU scenarios.

4.2. Brownian Case

For the classical Vasiček model in Section 3.2, we get from (3.1) that (4.3) specializes to

(4.4) $$p_t^T=e^{-Q_t^T r_t-\mu\int_t^T Q_u^T\,du+\frac{1}{2}\sigma^2\int_t^T\left(Q_u^T\right)^2du}.$$

Applying Itô's formula (A.1) for continuous processes to (A.4), using (1.1), (2.8), and $d[r,r]_t=\sigma^2\,dt$, we obtain after some easy rearrangements

(4.5) $$dp_t^T=p_t^T\left(q_t r_t-\sigma\,Q_t^T\,dW_t\right).$$

4.3. Poisson Case

For the POU process in Section 3.3, we get from (3.11) that (4.3) specializes to

(4.6) $$p_t^T=e^{-Q_t^T r_t-\nu\int_t^T Q_u^T\,du+\lambda\int_t^T\left(\ell_Y\left(Q_u^T\right)-1\right)du}.$$

Applying Itô's formula (A.1) for bounded variation processes to (4.6), using (2.3) and (2.8), and noting that the continuous part of r_t develops as $dr_t^c=(-\alpha r_t+\nu)\,dt$ and that r_t makes a jump of size y whenever $N_t(dy)$ jumps (by 1), we find after canceling some terms that

(4.7) $$dp_t^T=p_{t-}^T\{q_t r_t-dM_t\},$$

where M is the martingale defined by

$$dM_t = \int_y \left(1 - e^{-Q_t^T y}\right) \{dN_t(dy) - \lambda G(dy)\, dt\}. \tag{4.8}$$

The Poissonian formulas tend to their Brownian counterparts as λ increases when Y is replaced with $Y/\sqrt{\lambda}$. The details are $\lambda \nearrow \infty$, $\sigma \leftarrow \sigma/\sqrt{\lambda}$, and $\nu \leftarrow \mu - \sigma\sqrt{\lambda}$ (which tends to $-\infty$).

5. PRICES OF ZERO-COUPON BONDS

5.1. General Description of the Market

Let $(r_t)_{t\geq 0}$ be an OU process. Consider a financial market consisting of a bank account with price process $B_t = \exp(\int_0^t r_s\, ds)$ and a zero-coupon bond, maturing at time T, with price process given by (4.3) with $q \equiv 1$:

$$p_t^T = e^{-Q_t^T r_t + \int_t^T \phi(Q_u^T)\, du}, \quad t \leq T, \tag{5.1}$$

where

$$Q_t^T = \frac{1 - e^{-\alpha(T-t)}}{\alpha}. \tag{5.2}$$

5.2. Brownian Case

From (4.4) and (4.5) we obtain the well-known results

$$p_t^T = e^{-(T-t)\left(\rho - \frac{\sigma^2}{2\alpha^2}\right) + \frac{1-e^{-\alpha(T-t)}}{\alpha}(\rho - r_t) + \frac{\sigma^2}{4\alpha^3}\left(1 - \left(2 - e^{-\alpha(T-t)}\right)^2\right)}, \tag{5.3}$$

$$dp_t^T = p_t^T \left(r_t - \sigma \frac{1 - e^{-\alpha(T-t)}}{\alpha}\, dW_t\right). \tag{5.4}$$

Referring to standard arbitrage-pricing theory (see, e.g., Duffie 1996), we note that a market with price processes given by (5.3) for zero-coupon bonds with varying maturities T (in some finite time interval) is arbitrage-free since all discounted prices $B_t^{-1} p_t^T$ are martingales (a consequence of (5.4) and $dB_t^{-1} = -B_t^{-1} r_t\, dt$). Any replacement of ρ by $\tilde{\rho}$, say, in (5.3) and (5.4) would produce an alternative set of arbitrage-free prices. This is seen upon rewriting (1.1) as

$$dr_t = \alpha(\tilde{\rho} - r_t)\, dt + \sigma\, d\tilde{W}_t, \tag{5.5}$$

with $\tilde{W}_t = W_t + (\alpha/\sigma)(\rho - \tilde{\rho})\, t$, and transforming $\mathbb{P}$ to an equivalent measure $\tilde{\mathbb{P}}$ under which $\tilde{W}$ is a standard Brownian motion.

The market is complete since there is one source of randomness.

5.3. Poisson Case

In the POU case the bond prices are given by (4.6)–(4.8) with Q_t^T defined by (5.2). In particular,

$$dp_t^T = p_{t-}^T (r_t\, dt - dM_t), \tag{5.6}$$

with M defined by (4.8).

Again, a market with zero-coupon bonds of varying maturities T is arbitrage-free because all discounted prices are martingales. Any replacement of λ by $\tilde{\lambda} > 0$ and G by an equivalent measure $\tilde{G}$ (with the same support as G) would produce an alternative set of arbitrage-free prices. This follows by change to the equivalent measure $\tilde{\mathbb{P}}$ under which N is a compound Poisson process with intensity $\tilde{\lambda}$ and Y has distribution $\tilde{G}$ (see, e.g., Lando 1995).

The path properties of the basic process remain unaltered under a change to an equivalent measure. In the Poisson case the path properties are simply that N is a pure jump process with isolated jumps in the range of Y. The path properties of the POU process are preserved under the equivalent measure since λ and G do not appear in the dynamics given by (2.3) and (3.9). Only probabilities and expected values are affected by a change of measure. In particular, under the equivalent measure the mean μ in (3.10) changes to $\tilde{\mu} = \nu + \tilde{\lambda}\tilde{\mathbb{E}}[Y]$ and the long-term mean changes to $\tilde{\rho} = \tilde{\mu}/\alpha$. (Similar remarks apply for the BOU case. The dynamics equation (5.5) is the same as (1.1), of course, but it serves to visualize how the long-term mean is shifted under the change of measure.)

The "number of random sources" is the cardinality of the support of the distribution G. Therefore, a market consisting of a finite number of assets can be complete only if G has finite support.

For this case the structure of bond prices has been investigated by Björk (1995). In particular, (5.6) specializes to

$$dp_t^T = p_{t-}^T \left(r_t\, dt - \left\{ 1 - e^{-\sigma \frac{1-e^{-\alpha(T-t)}}{\alpha}} \right\} (dN_t - \lambda\, dt) \right),$$

which should be compared with (5.4).

6. PERPETUITIES

6.1. Definition and General Properties

A *perpetuity* is a perpetual annuity contributing the total amount s in any finite time interval $(0, s]$. Taking our stand at time t, the present value of the future payments is

$$V_t = \int_t^\infty e^{-\int_t^s r_u\, du}\, ds.$$

The probability distribution of V_0 is an issue in actuarial science; see Dufresne (1990), Geman and Yor (1993), Paulsen (1993), and Norberg (1999) for the case where $\int_0^t r_s\, ds$ is replaced with a Brownian motion with drift, and see Delbaen (1993) for the CIR model.

Assume that $(r_t)_{t\geq 0}$ is an OU process. Then V_t is a finite-valued random variable whose conditional distribution, given $r_t = r$, is independent of t:

$$P(r, v) = \mathbb{P}[V_t \leq \vartheta \mid r_t = r].$$

We want to determine this distribution.

Introduce the martingale M^* defined by

$$M_t^* = \mathbb{P}[V_0 \leq \vartheta \mid \mathcal{F}_t]. \tag{6.1}$$

Upon decomposing V_0 as

$$V_0 = \int_0^t e^{-\int_0^s r_u\, du}\, ds + e^{-\int_0^t r_u\, du}\, V_t,$$

and defining

$$U_t = e^{\int_0^t r_u\,du}\left(\vartheta - \int_0^t e^{-\int_0^s r_u\,du}\,ds\right),$$

the event $[V_0 \le \vartheta]$ can be recast as $[V_t \le U_t]$. Thus, by the Markov property of the OU process, the martingale in (6.1) is nothing but

$$M_t^* = P(r_t, U_t). \tag{6.2}$$

The process U_t is a continuous and bounded variation process with dynamics

$$dU_t = (r_t U_t - 1)\,dt. \tag{6.3}$$

The path properties of the process r_t depend on the driving Lévy process, so to proceed from here we must distinguish between different cases.

6.2. Brownian Case

Applying Itô's formula to (6.2), using (6.3) and (1.1) with W being standard Brownian motion, we find

$$\begin{aligned} dM_t^* &= \frac{\partial}{\partial r}P(r_t, U_t)(\alpha(\rho - r_t)\,dt + \sigma\,dW_t) + \frac{\partial}{\partial\vartheta}P(r_t, U_t)(r_t U_t - 1)\,dt \\ &\quad + \frac{1}{2}\frac{\partial^2}{\partial r^2}P(r_t, U_t)\sigma^2 dt. \end{aligned}$$

Since M^* is a martingale, the drift term must vanish almost surely, and we obtain the differential equation

$$\frac{\partial}{\partial r}P(r,\vartheta)\alpha(\rho - r) + \frac{\partial}{\partial v}P(r,\vartheta)(r\vartheta - 1) + \frac{1}{2}\frac{\partial^2}{\partial^2 r}P(r,\vartheta)\sigma^2 = 0,$$

which is to be solved subject to

$$P(r, 0) = 0, \qquad P(r, \infty) = 1. \tag{6.4}$$

6.3. Poisson Case

Applying Itô's formula for bounded variation processes to (6.2), using (6.3) and (2.3) with X_t given by (3.7) and (3.9), we find

$$\begin{aligned} dM_t^* &= \frac{\partial}{\partial r}P(r_t, U_t)(\vartheta - \alpha r_t)\,dt + \frac{\partial}{\partial v}P(r_t, U_t)(r_t U_t - 1)\,dt \\ &\quad + \int_y \{P(r_{t-} + y, U_{t-}) - P(r_{t-}, U_{t-})\}\,dN_t(dy). \end{aligned}$$

By use of (3.8), and arguing again that the drift term on the right must vanish, we obtain the integral-differential equation

$$\begin{aligned} &\frac{\partial}{\partial r}P(r,\vartheta)(\nu - \alpha r) + \frac{\partial}{\partial\vartheta}P(r,\vartheta)(r\vartheta - 1) \\ &\quad + \int_y \{P(r + y, \vartheta) - P(r,\vartheta)\}\,G(dy)\lambda = 0, \end{aligned}$$

which is to be solved subject to (6.4).

7. SUMMARY: COMPARING THE MODELS

7.1. State Spaces and Path Properties

With a view to the discussions in Sections 1 and 3.6, let us look at our OU models as candidate descriptions of stochastic interest rates.

Presumably, those who advocate that r_t should be positive have in mind interest rates that are subject to institutional regulation—for example, the interest rate stipulated by a national bank or the nominal interest rates on loans and deposits in a commercial bank. Actuaries may feel comfortable with the normal Vasiček model because to them the interest rate is usually the rate of return on the investment portfolio of the insurance company, which admittedly may be negative. Anyway, the Poisson and gamma versions of the Vasiček model can be specified so as to be positive almost surely.

Other approaches to modeling of positive interest rates, not of the OU type, are presented in Flesaker and Hughston (1996) and Rogers (1997); see also the references therein.

7.2. Computational Aspects

Conditional probabilities and expected values that do not possess explicit expressions must be computed numerically as solutions to differential equations. Sections 6.2 and 6.3 exhibit archetypal examples in this respect: in the Brownian scenario one has to deal with second-order partial differential equations, whereas in the Poisson scenario one has to deal with first-order partial integral-differential equations.

Computation goes by discretization. In principle, first- and second-order derivatives are replaced with first- and second-order differences, and integrals appearing in integral-differential equations are replaced with sums.

Sometimes ad hoc methods may simplify the problem. For instance, consider the problem of computing the moments in Section 4. The multiple integral in (4.2) is numerically cumbersome, so one would instead construct differential equations along the lines of Section 6; see Norberg (1995) for an example involving time-continuous Markov chains. However, for the first-order moment one could alternatively do the following: The expected value of (4.1) is $\int_t^T p_t^s \, dP_s$, where p_t^s is the bond price given by (5.1). When interest is of the OU type, this price is easily shown to be of the form $\exp(-\int_t^s r_u^* \, du)$, where r_u^* depends on t and r_t. One can, therefore, calculate the expected present value at a given time t under the simple assumption of deterministic interest rates.

A comprehensive study of numerical solutions to Lévy-driven stochastic differential equations, with ample and up-to-date references, is presented by Protter and Talay (1997).

7.3. Statistics

Let us first take the theoretical point of view that statistical inference amounts to determining estimators and tests that are optimal in some well-defined mathematical sense within a given model. The only constraint is that the statistics used must involve only a finite number of observations (in a finite world every practical calculation must use a finite algorithm).

This constraint raises problems in the diffusion world, where the paths of the basic processes are of unbounded variation and thus cannot be completely observed. One has to assume that the process is observed only at some finite set of times. For the BOU process one can then perform a standard maximum likelihood estimation of the parameters since the increments of the process have a joint normal distribution and thus an explicit likelihood function. Optimality can never be attained because a refinement of a given grid of observational points would always improve the estimation. Estimation problems in diffusion models are broadly discussed by Sørensen (1997).

In the Poisson world things may be simpler because only a finite number of jumps takes place in any finite time interval. It is then conceivable that the times and the sizes of the jumps are observed, which means that also the driving process is completely observed. If this is the case, the intensity λ is estimated consistently by N_t/t, and the jump size distribution is estimated consistently from the i.i.d. replicates $Y_1, \ldots, Y_{N_t}$ (condition on the ancillary statistic N_t, which tends to infinity almost surely as t increases). The POU process is piecewise deterministic, and we realize that α and ν can be determined without any error already at the time of the second jump or from, for example, three observations of the process in an interval between two jumps. If, on the other hand, the POU process could be observed only at certain prefixed times, then we would be worse off than in the BOU case because the likelihood is not a closed expression (it is a product of infinite weighted sums of convolutions).

Now, statistics is not just a mathematical matter. When it comes to empirical studies, one must confront the model with data and test its appropriateness for the situation at hand. Suppose we have data for the development of some interest rate and have in mind an OU type of model. The POU variant would be rejected immediately for much the same reasons that were given in its favor above; the path properties of a factual interest rate would never comply with the piecewise deterministic nature of the POU process. Presumably, the BOU process would appear to be resistant against the facts of life, for some time at least, beause its paths can have virtually any shape (no jumps though), but also this model would be rejected in the end, as would any model in any empirical context, of course. By way of conclusion, and referring to Figures 3.1–3.4, the models should be judged by their broad features and their ability to approximate rather than duplicate reality.

7.4. Combined Driving Processes

One can let the driving process X be a sum of independent processes of the prototypes investigated above. Results for such combined driving processes are easily compiled from the results on the pure cases. For instance, in Theorems 2.1–2.3 one should just let the LT be the product of the LTs of the individual terms.

APPENDIX

A.1. Itô's Formula

The following version of Itô's formula, compiled from Protter (1990), is valid for any semimartingale $X = (X_1, \ldots, X_m)$ with paths that are RCLL (right-continuous with left-limits) and have at most a finite number of jumps in any finite interval, and any scalar field f such that the relevant derivatives exist and are continuous:

$$df(X_t) = \sum_i \frac{\partial f}{\partial x_i}(X_t)\, dX^c_{i,t} + \frac{1}{2}\sum_{i,j}\frac{\partial^2 f}{\partial x_i \partial x_j}(X_t) d[X_i, X_j]^c_t + f(X_t) - f(X_{t-}). \tag{A.1}$$

Here X_i^c denotes the continuous part of X_i, and $[X_i, X_j]$ is the optional covariance process of X_i and X_j. If X is continuous, then the last two terms on the right of (A.1) vanish. If either X_i or X_j is of bounded variation, then $d[X_i, X_j]_t^c$ appearing in the second term on the right of (A.1) vanishes. If $dX_{i,t} = f_t\, dW_t$, with f deterministic (predictable is enough) and W a standard Brownian motion, then $d[X_i, X_i]_t = f_t^2\, dt$.

A.2. The Laplace Transform

The *Laplace transform (LT)* of a random variable X is

$$\ell_X(\eta) = \mathbb{E}\left[e^{-\eta X}\right],$$

considered as a function of $\eta \in \mathbb{R}$ for which the expected value exists.

Let $(X_t)_{t\geq 0}$ be an RCLL stochastic process with X_0 fixed. For a deterministic function f such that $\int_0^\infty f_s\, dX_s$ is a well-defined random variable, we have

$$\ell_{\int_0^\infty f_s\, dX_s}(\eta) = \mathbb{E}\left[e^{-\eta \int_0^\infty f_s\, dX_s}\right]. \tag{A.2}$$

Considered as a functional (function of f), (A.2) determines the probability law of the process $(X_t)_{t\geq 0}$ because for any piecewise constant function f it specializes to the joint LT of increments of the process.

For a general account of the Laplace functional for point processes, see Daley and Vere-Jones (1988) or Karr (1991).

A.3. Lévy Processes

A Lévy process is a stochastic process $(X_t)_{t\geq 0}$ that is continuous in probability and has stationary and independent increments and $X_0 = 0$. In terms of the LT these properties can be expressed as

$$\ell_{X_{s+t}-X_s}(\eta) \equiv \ell_{X_t}(\eta), \quad \ell_{X_{s+t}}(\eta) \equiv \ell_{X_s}(\eta)\ell_{X_{s+t}-X_s}(\eta),$$

for s and t nonnegative. Combining the two, we find that $\ell_{X_t}(\eta)$, considered as a function of t, satisfies the functional equation

$$\ell_{X_{s+t}}(\eta) \equiv \ell_{X_s}(\eta)\ell_{X_t}(\eta),$$

which characterizes the exponential function. We conclude that there must exist a function ϕ such that

$$\ell_{X_t}(\eta) = e^{\phi(\eta)t}. \tag{A.3}$$

Conversely, if a function of the form $e^{\phi(\eta)t}$ is an LT for each $t > 0$, then there exists a Lévy processes as described above (Kolmogorov's consistency condition will be automatically satisfied).

The result (A.3) extends readily to

$$\ell_{\int_0^\infty f_s\, dX_s}(\eta) = e^{\int_0^\infty \phi(\eta f_s)\, ds}, \tag{A.4}$$

valid for any deterministic, real-valued function f that is bounded and Riemann integrable (hence square integrable). The proof is standard: For f an indicator function of a finite interval, (A.4) reduces to (A.3) by the stationary increments property. The result extends to piecewise constant functions f by the independent increments property. Finally, it extends to limits of piecewise constant functions by almost sure and dominated convergence.

In particular, for $t < u < \infty$, the identity $\int_t^u \phi(\eta f_{u-s})\, ds = \int_0^{u-t} \phi(\eta f_s)\, ds$ implies that

$$\int_t^u f_{u-s}\, dX_s \overset{\mathcal{L}}{=} \int_0^{u-t} f_s\, dX_s. \tag{A.5}$$

For general accounts on Lévy processes, see Protter (1990) and Bertoin (1996).

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