Portfolio Credit Risk

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The Goodrich-Rabobank swap: 1983

- **B.F. Goodrich (BBB-rated)**: 5.5 million (11% fixed) once a year, LIBOR + 0.5% (Semiannual)
- **Rabobank (AAA-rated)**: 5.5 million (11% annual) once a year, (LIBOR − y)% Semiannual
- **Morgan Guarantee Trust**: Swap

U.S. Savings Banks

Belgian dentists
Review of basic concepts
Cash flow valuation

Fundamental principle: time is money

Present value of cash flows is given by the formula

\[ \text{Value} = \sum p_i e^{-r_i t_i} \]

\( p_i \) is a payment at time \( t_i \)

\( r_i \) continuously compounded interest rate for \( t_i \)

This assumes payments will occur with probability 1 (no default risk).
Credit premium

The discounted value of cash flows, when there is probability of default, is given by

$$Value = \sum p_i e^{-r_it_i} q_i$$

$q_i$ denotes the probability that the counter-party is solvent at time $t_i$.

The larger the default risk ($q$ small), the smaller its value.

The higher the credit risk ($q$ small), the higher the payments, to preserve the same present value.
The credit spread

Since $q_i \leq 1$, we can write

$$q_i = e^{-h_i t_i}, \quad h_i = -\frac{1}{t_i} \ln(q_i)$$

the loan is now valued as

$$\text{Value} = \sum p_i e^{-(r_i + h_i) t_i}$$

Default-prone interest rate rate increases.
First model: two credit states

What is the credit spread?
Assume only 2 possible credit states: solvency and default

Assume the probability of solvency in a fixed period (one year, for example), conditional on solvency at the beginning of the period, is given by a fixed amount: $q$

According to this model, we have:

$$q_i = q^{t_i}$$

which gives rise to a constant credit spread:

$$h_i = h = -\ln(q)$$
The general Markov model

In other words, when the default process follows a Markov chain,

<table>
<thead>
<tr>
<th></th>
<th>Solvency</th>
<th>Default</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solvency</td>
<td>$q$</td>
<td>$1 - q$</td>
</tr>
<tr>
<td>Default</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

the credit spread is constant, and equals

$$h_i = h = -\ln(q)$$
Goodrich-Morgan swap

The fixed rate loan
The leg to consider for Credit Risk is the one between JPMorgan and BF Goodrich.

Cashflows of the leg (in million USD):
- 0.125 upfront
- 5.5 per yr, during 8 years

Assume:
- constant spread $h = 180$ bpi
- 2 state transition probabilities matrix
G-RB CreditMetrics: expected cashflows

- Since
  
  \[ \text{Expected[cashflows]} = \left( \text{\$cashflows} \right) \times \text{Prob\{non\_default\}} \]

- Then
  
  \[ \text{E[cashflows]} = 0.125 + \text{Sum}(5.5 \times \text{P\{nondefault @ each year\}}) \]

- But at the same time
  
  \[ \text{E[cashflow]} = 0.125 \left( \sum_{i=1}^{8} r_{ht} \exp(-i) \right) \]
G-RB CreditMetrics: probability of default

- Under our assumptions:
  \[ P \{\text{non-default}\} = \exp(-h) \]
  \[ = \exp(-.018) \]
  \[ = .98216 \]
  
  - constant for each year

- The 2 state matrix:

\[
\begin{array}{ll}
\text{BBB} & \text{D} \\
\hline
\text{BBB} & .9822 & .0178 \\
\text{D} & 0 & 1 \\
\end{array}
\]
G-RB CreditMetrics: compute cashflows

- **Inputs**
  - \( P\{\text{default of BBB corp.}\} = 1.8\%; \)
    \[
    1 - \exp(0.018) = 0.9822
    \]
  - The gvmnt zero curve for August 1983 was
    \[ r = (0.08850, 0.09297, 0.09656, 0.0987855, 0.10550, 0.104355, 0.11770, 0.118676) \]
    for years (1,2,3,4,5,6,7,8)
G-RB CreditMetrics: cashflows (cont)

- $E[cashflows] = \sum_{i=1}^{8} 25(5.5) \exp(())(0.982)^i$
  $= 23.0527$

- Risk-less Cashflows $= \sum_{i=1}^{8} 25(5.5) \exp(()) r_i^f$
  $= 24.67581$
Therefore

\[ E[\text{loss}] = 1 - \left( \frac{E[\text{cashflows}]}{\text{Non-Risk Cashflow}} \right) \]

= .065776

i.e. the proportional expected loss is around 6.58% of USD 24.67581 million

Or roughly

\[ E[\text{loss}] = 1.623 \text{ (USD million)} \]
Non-constant spreads

A default/no-default model (such as CreditRisk+) leads to constant spreads, unless probabilities vary with time.

In order to fit non-constant spreads, and be able to fit the model to market observations, one needs to assume either:

- Time-varying default probabilities
- Multi-rating systems (such as credit-metrics)
Markov Processes

Transition Probabilities
Constant in time

\[ p_{11}, p_{12}, p_{23}, p_{56} \]

\[ t=0 \quad \ldots \quad t=1 \quad \ldots \quad t=2 \]
### Transition probabilities

Conditional probabilities, which give rise to a matrix with \( n \) credit states,

\[
\begin{array}{cccccc}
  p_{11} & p_{12} & \cdots & \cdots & \cdots & p_{1n} \\
p_{21} & p_{22} & & & & \\
p_{31} & & & & & \\
& & & & & \\
& & & & & \\
p_{n1} & \cdots & \cdots & \cdots & p_{nn} \\
\end{array}
\]

\( P_{ij} = \text{cond prob of changing from state } i \text{ to state } j \)
There are corporations whose business is to rate the credit quality of corporations, governments, and also of specific debt issues. The main ones are:

- Moody’s Investors Service,
- Standard & Poor’s,
- Fitch IBCA,
- Duff and Phelps Credit Rating Co

**S&P’s**

- **AAA**: Highest Quality; Capacity to pay interest and repay principal is extremely strong
- **AA**: High Quality
- **A**: Strong payment capacity
- **BBB**: Adequate payment capacity
- **BB**: Likely to fulfill obligations; ongoing uncertainty
- **B**: High risk obligations
- **CCC**: Current vulnerability to default
- **D**: In bankruptcy or default, or other marked shortcoming
## Standard and Poor’s Markov model

<table>
<thead>
<tr>
<th></th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>0.9081</td>
<td>0.0833</td>
<td>0.0068</td>
<td>0.0006</td>
<td>0.0012</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>AA</td>
<td>0.0070</td>
<td>0.9065</td>
<td>0.0779</td>
<td>0.0064</td>
<td>0.0006</td>
<td>0.0014</td>
<td>0.0002</td>
<td>0.0000</td>
</tr>
<tr>
<td>A</td>
<td>0.0009</td>
<td>0.0227</td>
<td>0.9105</td>
<td>0.0552</td>
<td>0.0074</td>
<td>0.0026</td>
<td>0.0001</td>
<td>0.0006</td>
</tr>
<tr>
<td>BBB</td>
<td>0.0002</td>
<td>0.0033</td>
<td>0.0595</td>
<td>0.8693</td>
<td>0.0530</td>
<td>0.0117</td>
<td>0.0012</td>
<td>0.0018</td>
</tr>
<tr>
<td>BB</td>
<td>0.0003</td>
<td>0.0014</td>
<td>0.0067</td>
<td>0.0773</td>
<td>0.8053</td>
<td>0.0884</td>
<td>0.0100</td>
<td>0.0106</td>
</tr>
<tr>
<td>B</td>
<td>0.0000</td>
<td>0.0011</td>
<td>0.0024</td>
<td>0.0043</td>
<td>0.0648</td>
<td>0.8346</td>
<td>0.0407</td>
<td>0.0520</td>
</tr>
<tr>
<td>CCC</td>
<td>0.0022</td>
<td>0.0000</td>
<td>0.0022</td>
<td>0.0130</td>
<td>0.0238</td>
<td>0.1124</td>
<td>0.6486</td>
<td>0.1979</td>
</tr>
<tr>
<td>D</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>
Long term transition probabilities

Transition probability between state \( i \) and state \( j \), in two time steps, is given by

\[
p_{ij}^{(2)} = \sum_k p_{ik} \cdot p_{kj}
\]

In other word, if we denote by \( A \) the one-step conditional probability matrix, the two-step transition probability matrix is given by

\[ A^2 \]
Transition probabilities in general

If $A$ denotes the transition probability matrix at one step (one year, for example), the transition probability after $n$ steps (30 is specially meaningful for credit risk) is given by

$$A^n$$

For the same reason, the quarterly transition probability matrix should be given by

$$A^{1/4}$$

This gives rise to a number of important practical issues.
Credit Loss
Credit Exposure

It is the maximum loss that a portfolio can experience at any time in the future, taken with a certain confidence level.

Evolution of the mark-to-market of a 20-month swap

Exposure (99%)

Exposure (95%)
Recovery Rate – Loss Given Default

When default occurs, a portion of the value of the portfolio can usually be recovered. Because of this, a recovery rate is always considered when evaluating credit losses. It represents the percentage value which we expect to recover, given default.

Loss-given-default is the percentage we expect to lose when default occurs:

\[ R = 1 - LGD \]
Each counterparty has a certain probability of defaulting on their obligations. Some models include a random variable which indicates whether the counterparty is solvent or not. Other models use a random variable which measures the credit quality of the counterparty. For the moment, we will denote by $b$ the random variable which is 1 when the counterparty defaults, and 0 when it does not. The modeling of how it changes from 0 to 1 will be dealt with later.
For an instrument or portfolio with *only one* counterparty, we define:

\[
\text{Credit Loss} = b \times \text{Credit Exposure} \times \text{LGD}
\]

- **Random variable:**
  - Depends on the credit quality of the counterparty

- **Number:**
  - Depends on the market risk of the instrument or portfolio

- **Number:**
  - Usually, this number is a universal constant (55%), but more refined models relate it to the market and the counterparty
Measuring the distribution of credit losses (2)

For a portfolio with several counter-parties, we define:

\[
\text{Credit Loss} = \sum (b_i \times \text{Credit Exposure}_i \times \text{LGD}_i)
\]

Random variable:
Normally different for different counterparties

Number:
Normally different for different portfolios, same for the same portfolios

Number:
Usually, this number is a universal constant (55%), but more refined models relate it to the market and the counterparty
The traditional approach to measuring credit risk is to consider only the net replacement value:

\[ NRV = \sum_i (\text{Credit Exposures}) \]

This is a rough statistic, which measures the amount that would be lost if all counter-parties default at the same time, and at the time when all portfolios are worth most, and with no recovery rate.
Credit loss distribution

The credit loss distribution is often very complex. As with Markowitz theory, we try to summarize its statistics with two numbers: its expected value, and its standard deviation.

In this context, this gives us two values:
- The expected loss
- The unexpected loss
Credit VaR / Worst Credit Loss

Worst Credit Loss represents the credit loss which will not be exceeded with some level of confidence, over a certain time horizon.

A 95%-WCL of $5M on a certain portfolio means that the probability of losing more than $5M in that particular portfolio is exactly 5%.

CVaR represents the credit loss which will not be exceeded in excess of the expected credit loss, with some level of confidence over a certain time horizon:

A daily CVaR of $5M on a certain portfolio, with 95% means that the probability of losing more than the expected loss plus $5M in one day in that particular portfolio is exactly 5%.
Using credit risk measurements in trading

- Marginal contribution to risk
  When considering a new instrument to be traded as part of a certain book, one needs to take into account the impact of the new deal in the credit risk profile at the time the deal is considered. An increase of risk exposure should lead to a higher premium or to a deal not being authorized. A decrease in risk exposure could lead to a more competitive price for the deal.

- Remuneration of capital
  Imagine a deal with an Expected Loss of $1M, and an unexpected loss of $5M. The bank may impose a credit reserve equal to $5M, to make up for potential losses due to default; this capital which is immobilized will require remuneration; because of this, the price of any credit-prone contract should equal

  \[
  \text{Price} = \text{Expected Loss} + \text{(portion)} \text{ Unexpected loss}
  \]
When two counterparties enter into multiple contracts, the cashflows over all the contracts can be, by agreement, merged into one cashflow. This practice, called netting, is equivalent to assuming that when a party defaults on one contract it defaults in all the contracts simultaneously.

Netting may affect the credit-risk premium of particular contracts. Assuming that the default probability of a party is independent from the size of exposures it accumulates with a particular counter-party, the expected loss over several contracts is always less or equal than the sum of the expected losses of each contract. The same result holds for the variance of the losses (i.e. the variance of losses in the cumulative portfolio of contracts is less or equal to the sum of the variances of the individual contracts). Equality is achieved when contracts are either identical or the underlying processes are independent.
Expected Credit Loss: General framework

In the general framework, the expected credit loss is given by

$$ECL = \text{Expectation}\left[b \times CE \times LGD \right] = \int \left[b \times CE \times LGD \right] f (b, CE, LDG) \, db \, dCE \, dLGD$$

- Joint probability density for all three random variables:
  - default status (b)
  - Credit Exposure
  - Loss given default
Expected Credit Loss: Special case

Because calculating the joint probability distribution of all relevant variables is hard, most often one assumes that their distributions are independent. In that case, the ECL formula simplifies to:

\[
ECL = E[b] \times E[CE] \times E[LDG]
\]
Example

Consider a commercial mortgage, with a shopping mall as collateral. Assume the exposure of the deal is $100M, an expected probability of default of 20% (std of 10%), and an expected recovery of 50% (std of 10%).

Calculate the expected loss in two ways:

1. Assuming independence of recovery and default (call it *x*).
2. Assuming a –50% correlation between the default probability and the recovery rate (call it *y*).

What is your best guess as to the numbers *x* and *y*.

1. *x*=$10M, *y*=$10M.
2. *x*=$10M, *y*=$20M.
3. *x*=$10M, *y*=$5M.
4. *x*=$10M, *y*=$10.5M.
Example

Consider a commercial mortgage, with a shopping mall as collateral. Assume the exposure of the deal is $100M, an expected probability of default of 20% (std of 10%), and an expected recovery of 50% (std of 10%).

Calculate the expected loss in two ways:

- Assuming independence of recovery and default (call it $x$)
- Assuming a –50% correlation between the default probability and the recovery rate (call it $y$).

What is your best guess as to the numbers $x$ and $y$.

1. $x =$10M, $y =$10M.
2. $x =$10M, $y =$20M.
3. $x =$10M, $y =$5M. **Cannot be: $x$ has to be smaller than $y**
4. $x =$10M, $y =$10.5M.
Tree-based model

\[
\begin{aligned}
p^{++} - p^{+-} - p^{-+} + p^{--} &= -0.5 \\
p^{++} + p^{+-} &= 0.5 \\
p^{-+} + p^{--} &= 0.5 \\
p^{++} + p^{--} &= 0.5
\end{aligned}
\]
Correlating default and recovery

Assume two equally likely future credit states, given by default probabilities of 30% and 10%.
Assume two equally likely future recovery rates, given by 60% and 40%.
With a –50% correlation between them, the expected loss is

\[
EL = 100M \times (0.375 \times 0.6 \times 0.3 + 0.375 \times 0.4 \times 0.1 + 0.125 \times 0.4 \times 0.3 + 0.125 \times 0.6 \times 0.1)
\]
\[
= 100M \times (0.0825 + 0.0225)
\]
\[
= 10.5M
\]
Consider the swap between Goodrich and MGT. Assume a total exposure averaging $10M (50% std), a default rate averaging 10% (3% std), fixed recovery (50%).

Calculate the expected loss in two ways

- Assuming independence of exposure and default (call it $x$)
- Assuming a –50% correlation between the default probability and the exposure (call it $y$).

What is your best guess as to the numbers $x$ and $y$.

1. $x=$500,000, $y=$460,000.
2. $x=$500,000, $y=$1M.
3. $x=$500,000, $y=$500,000.
4. $x=$500,000, $y=$250,000.
Consider the swap between Goodrich and MGT. Assume a total exposure averaging $10M (50% std), a default rate averaging 10% (3% std), fixed recovery (50%).

Calculate the expected loss in two says

- Assuming independence of exposure and default (call it \( x \))
- Assuming a –50% correlation between the default probability and the exposure (call it \( y \)).

What is your best guess as to the numbers \( x \) and \( y \).

1. \( x = \$500,000, y = \$450,000 \).
2. \( x = \$500,000, y = \$1M \).
3. \( x = \$500,000, y = \$500,000 \).
4. \( x = \$500,000, y = \$250,000 \).

Cannot be: \( x \) has to be larger than \( y \).
Correlating default and exposure

Assume two equally likely future credit states, given by default probabilities of 13% and 7%.
Assume two equally likely exposures, given by $15M and $5M.
With a −50% correlation between them, the expected loss is

\[
EL = 0.5 \times (0.125 \times 15M \times 0.13 + 0.125 \times 5M \times 0.07 + 0.375 \times 15M \times 0.07 + 0.375 \times 5M \times 0.13)
\]
\[
= 0.5 \times ($0.24M + $0.04 + $0.40M + $0.24M)
\]
\[
= $460,000
\]
Example 23-2: FRM Exam 1998 Question 39

“Calculate the 1 yr expected loss of a $100M portfolio comprising 10 B-rated issuers. Assume that the 1-year probability of default of each issuer is 6% and the recovery rate for each issuer in the event of default is 40%.”
“Calculate the 1 yr expected loss of a $100M portfolio comprising 10 B-rated issuers. Assume that the 1-year probability of default of each issuer is 6% and the recovery rate for each issuer in the event of default is 40%.”

\[0.06 \times 100M \times 0.6 = 3.6M\]
Variation of example 23-2.

“Calculate the 1 yr unexpected loss of a $100M portfolio comprising 10 B-rated issuers. Assume that the 1-year probability of default of each issuer is 6% and the recovery rate for each issuer in the event of default is 40%.

Assume, also, that the correlation between the issuers is

1. 100% (i.e., they are all the same issuer)
2. 50% (they are in the same sector)
3. 0% (they are independent, perhaps because they are in different sectors)"
Solution

1. The loss distribution is a random variable with two states: default (loss of $60M, after recovery), and no default (loss of 0). The expectation is $3.6M. The variance is

\[ 0.06 \times (60M-3.6M)^2 + 0.94 \times (0-3.6M)^2 = 200(M^2) \]

The unexpected loss is therefore

\[ \sqrt{200} = 14M. \]
2. The loss distribution is a sum of 10 random variables $X_i$, each with two states: default (loss of $6M, after recovery), and no default (loss of 0). The expectation of each of them is $0.36M$. The variance of each is (as before) 2. The variance of their sum is

\[
\text{Var} = \sum_{i,j} \mathbb{E}[X_i X_j] \mu_i \mu_j \\
= \sum_i \sigma_i^2 + \sum_{i \neq j} \sigma_{i,j} \\
= \sum_i \sigma_i^2 + \sum_{i \neq j} 0.5 \sigma_i \sigma_j \\
= 110
\]
3. The loss distribution is a sum of 10 random variable, each with two states: default (loss of $6M, after recovery), and no default (loss of 0). The expectation of each of them is $0.36M. The standard deviation of each is (as before) $1.4M.

The standard deviation of their sum is

\[ \sqrt{10} \times 1.4M = 5M \]

Note: the number of defaults is given by a Poisson distribution. This will be of relevance later when we study the CreditRisk+ methodology.
“Which loan is more risky? Assume that the obligors are rated the same, are from the same industry, and have more or less the same sized idiosyncratic risk: A loan of
1. $1M with 50% recovery rate.
2. $1M with no collateral.
3. $4M with a 40% recovery rate.
4. $4M with a 60% recovery rate.”
“Which loan is more risky? Assume that the obligors are rated the same, are from the same industry, and have more or less the same sized idiosyncratic risk: A loan of

1. $1M with 50% recovery rate.
2. $1M with no collateral.
3. $4M with a 40% recovery rate.
4. $4M with a 60% recovery rate.”

The expected exposures times expected LGD are:

1. $500,000
2. $1M
3. $2.4M. Riskiest.
4. $1.6M

“Which of the following conditions results in a higher probability of default?
1. The maturity of the transaction is longer
2. The counterparty is more creditworthy
3. The price of the bond, or underlying security in the case of a derivative, is less volatile.
4. Both 1 and 2.”

“Which of the following conditions results in a higher probability of default?

1. The maturity of the transaction is longer
2. The counterparty is more creditworthy
3. The price of the bond, or underlying security in the case of a derivative, is less volatile.
4. Both 1 and 2.”

Answer

1. True
2. False, it should be “less”, nor “more”
3. The volatility affects (perhaps) the value of the portfolio, and hence exposure, but not the probability of default (*)
Expected and unexpected losses must take into account, not just a static picture of the exposure to one cash flow, but the variation over time of the exposures, default probabilities, and express all that in today’s currency.

This is done as follows: the PV ECL is given by

\[ PV - ECL = \sum_t E[CL_t] \times PV_t \]
Expected loss: an approximation

Rewrite:

\[ PV - ECL = \sum_t E[CL_t] \times PV_t \]

\[ = \sum_t p_t \times E[CE_t] \times (1 - f) \times PV_t \]

\[ \equiv Ave_t \{p_t\} \times Ave_t \{ECE_t\} \times (1 - f) \times \sum_t PV_t \]

Each number changes with time

Each is replaced by an amount independent of time: their average

Note: in the book, the term \((1-f)\) is assumed to be independent of time. In some situations, such as commercial mortgages, this will underestimate the credit risk.
A swap
The portfolio

- 5 year swap
- BBB counterparty
- $100M notional
- 6% annual interest rate (discount factor)
- 45% recovery rate
- Annual periods
### TABLE 23-1 Computation of Expected Credit Loss for a Swap

<table>
<thead>
<tr>
<th>Year</th>
<th>P(default) (%)</th>
<th>Expos. ECE&lt;sub&gt;t&lt;/sub&gt;</th>
<th>LGD (1 - f)</th>
<th>Discount PV&lt;sub&gt;t&lt;/sub&gt;</th>
<th>Total (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.22 0.220 0.220</td>
<td>$1.660</td>
<td>0.55</td>
<td>0.9434</td>
<td>$0.1895</td>
</tr>
<tr>
<td>2</td>
<td>0.54 0.321 0.320</td>
<td>$1.497</td>
<td>0.55</td>
<td>0.8900</td>
<td>$0.2345</td>
</tr>
<tr>
<td>3</td>
<td>0.88 0.342 0.340</td>
<td>$1.069</td>
<td>0.55</td>
<td>0.8396</td>
<td>$0.1678</td>
</tr>
<tr>
<td>4</td>
<td>1.55 0.676 0.670</td>
<td>$0.554</td>
<td>0.55</td>
<td>0.7921</td>
<td>$0.1617</td>
</tr>
<tr>
<td>5</td>
<td>2.28 0.741 0.730</td>
<td>$0.000</td>
<td>0.55</td>
<td>0.7473</td>
<td>$0.0000</td>
</tr>
<tr>
<td>Total</td>
<td>2.280</td>
<td></td>
<td></td>
<td>4.2124</td>
<td>$0.7535</td>
</tr>
<tr>
<td>Average</td>
<td>0.456</td>
<td>$0.956</td>
<td>0.55</td>
<td>4.2124</td>
<td>$1.01</td>
</tr>
</tbody>
</table>
## TABLE 23-2 Computation of Expected Credit Loss for a Bond

<table>
<thead>
<tr>
<th>Year t</th>
<th>P(default) (%)</th>
<th>Expos. ECE_t</th>
<th>LGD (1 − f)</th>
<th>Discount PV_t</th>
<th>Total (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.22</td>
<td>$100.0</td>
<td>0.55</td>
<td>0.9434</td>
<td>$11.415</td>
</tr>
<tr>
<td>2</td>
<td>0.54</td>
<td>$100.0</td>
<td>0.55</td>
<td>0.8900</td>
<td>$15.664</td>
</tr>
<tr>
<td>3</td>
<td>0.88</td>
<td>$100.0</td>
<td>0.55</td>
<td>0.8396</td>
<td>$15.701</td>
</tr>
<tr>
<td>4</td>
<td>1.55</td>
<td>$100.0</td>
<td>0.55</td>
<td>0.7921</td>
<td>$29.189</td>
</tr>
<tr>
<td>5</td>
<td>2.28</td>
<td>$100.0</td>
<td>0.55</td>
<td>0.7473</td>
<td>$30.002</td>
</tr>
<tr>
<td>Total</td>
<td>2.280</td>
<td>$100.0</td>
<td>0.55</td>
<td>4.2124</td>
<td>$101.97</td>
</tr>
<tr>
<td>Average</td>
<td>0.456</td>
<td>$100.0</td>
<td>0.55</td>
<td>4.2124</td>
<td>$105.65</td>
</tr>
</tbody>
</table>
A worked-out example
A simple bond

Consider a bond issued from a default-prone party, paying two $5 coupons after the end of the second and fourth years. We assume that throughout the duration of the bond the interest rates are at 0%. (This assumption simplifies discounting.)

The default-prone party has a yearly default probability of 7% and when it defaults no money can be recovered (recovery rate = 1 – severity = 0). We assume that the default-free party maintains a risk-capital to cover for the standard deviation of losses that is adjusted annually and that it demands a certain return on this risk-capital.
Figure 1.3.1. Survival and default probabilities. D = default, ND = not default. The nodes are one year apart.
Expected loss calculation

There are two, equivalent in this case, ways to compute the expected loss. Since the value of the contract is always non-negative to the default-free party, we do not need to discard any future events (as already explained this is not a limitation, as every contract can be decomposed into contracts that have always non-negative or non-positive value).

One way to compute the expected loss is to compute the expected cashflows. The default probabilities are shown in figure 1.3.1, where the nodes are one year apart. There are two cashflows of $5 each, and the expected cashflow is

\[ EC = 5p_{nd \in (0,2]} + 5p_{nd \in (0,4]} = 8.065 \]

where \( p_{nd \in (0,2]} \) is the probability that the default-prone party does not default in the time interval between years 0 and 2. The expected loss is

\[ EL = 10 - EC = 1.935 \]
An equivalent way...

The second way is to calculate loss based on the yearly exposures

\[
\begin{align*}
\text{Exposure(year 1\textsuperscript{−})} &= \$10 \\
\text{Exposure(year 2\textsuperscript{−})} &= \$10 \\
\text{Exposure(year 3\textsuperscript{−})} &= \$5 \\
\text{Exposure(year 4\textsuperscript{−})} &= \$5
\end{align*}
\]

where no correction due to discounting was included, since interest rates are flat at 0\% and Exposure(year 1\textsuperscript{−}), the value of the contract just before year 1.
The expected losses are

\[ EL = \text{Exposure(year 1\(^-\))} p_{d \in (0,1]} + \text{Exposure(year 2\(^-\))} p_{d \in (1,2]} + \text{Exposure(year 3\(^-\))} p_{d \in (2,3]} + \text{Exposure(year 4\(^-\))} p_{d \in (3,4]} \]

\[ = 10 \times 0.07 + 10 \times 0.0651 + 5 \times 0.0605 + 5 \times 0.0563 \]

\[ = $1.935 \]

where \( p_{d \in (2,3]} \) is the probability that the default-prone party defaults between the years 2 and 3.
The unexpected loss

The variance of the losses is

\[ \text{varL}_{(0,1)} = \text{Exposure}^2(y_{1^-}p_{d \in (0,1)} - (\text{Exposure}(y_{1^-})p_{d \in [0,1]})^2 \]

\[ = (\text{EL}(1))^2 \left( \frac{1}{p_{d \in [0,1]}} - 1 \right) = 6.51 \]

\[ \text{varL}_{(1,2)} = \text{Exposure}^2(y_{2^-}p_{d \in [1,2)} - (\text{Exposure}(y_{2^-})p_{d \in [1,2]})^2 \]

\[ = (\text{EL}(2))^2 \left( \frac{1}{p_{d \in [1,2]}} - 1 \right) = 6.08 \]

\[ \text{varL}_{(2,3)} = (\text{EL}(3))^2 \left( \frac{1}{p_{d \in [2,3]}} - 1 \right) = 1.42 \]

\[ \text{varL}_{(3,4)} = (\text{EL}(4))^2 \left( \frac{1}{p_{d \in [3,4]}} - 1 \right) = 1.33 \]
Credit reserve.

If, for example, a risk-capital of two standard deviations is required, the default-free party anticipates to use risk-capital equal to $5.10 at year 0, $4.93 at year 1, $2.38 at year 2 and $2.31 at year 3. A yearly return of 10% on such capital leads to an additional surcharge of $1.47.

**REMARK** Notice that a high enough return rate would lead to the possibility of arbitrage (in this case arbitrage corresponds to an initial credit-risk premium of more than $10).
Credit VaR
Credit VaR

It is the unexpected credit loss, at some confidence level, over a certain time horizon.

If we denote by \( f(x) \) the distribution of credit losses over the prescribed time horizon (typically one year), and we denote by \( c \) the confidence level (i.e. 95%), then the Worst-Credit-Loss (WCL) is defined to be

\[
\int_{WCL}^{\infty} f(x) \, dx = 1 - c
\]

Credit VaR = (Worst-Credit-Loss) – (Expected Credit Loss)

Leads to Reserve capital
A risk analyst is trying to estimate the Credit VaR for a risky bond. The Credit VaR is defined as the maximum unexpected loss at a confidence level of 99.9% over a one month horizon. Assume that the bond is valued at $1M one month forward, and the one year cumulative default probability is 2% for this bond, what is your estimate of the Credit VaR for this bond assuming no recovery?

1. $20,000
2. 1,682
3. 998,318
4. 0
Example 23-5: FRM exam 1998

A risk analyst is trying to estimate the Credit VaR for a risky bond. The Credit VaR is defined as the maximum unexpected loss at a confidence level of 99.9% over a one month horizon. Assume that the bond is valued at $1M one month forward, and the one year cumulative default probability is 2% for this bond, what is your estimate of the Credit VaR for this bond assuming no recovery?

1. $20,000
2. 1,682
3. **998,318**
4. 0

- If $d$ is the monthly probability of default, $(1-d)^{12} = (0.98)$, so $d = 0.00168$.
- ECL = $1,682$
- WCL(0.999) = WCL(1-0.00168) = $1,000,000$.
- CVaR = $1,000,000 - 1,682 = 998,318$. 
A risk analyst is trying to estimate the Credit VaR for a portfolio of two risky bonds. The Credit VaR is defined as the maximum unexpected loss at a confidence level of 99.9% over a one month horizon. Assume that both bonds are valued at $500,000 one month forward, and the one year cumulative default probability is 2% for each of these bonds. What is your best estimate of the Credit VaR for this portfolio assuming no default correlation and no recovery?

1. $841
2. $1,682
3. $10,000
4. $249,159
Example 23-6: FRM exam 1998

A risk analyst is trying to estimate the Credit VaR for a portfolio of two risky bonds, each worth $250K. The Credit VaR is defined as the maximum unexpected loss at a confidence level of 99.9% over a one month horizon. Assume the one year cumulative default probability is 2% for each of these bonds. What is your best estimate of the Credit VaR for this portfolio assuming no default correlation and no recovery?

1. $841
2. $1,682
3. $10,000
4. $249,159

- If $d$ is the monthly probability of default, $(1-d)^{12} = 0.98$, so $d = 0.00168$.
- ECL = $840
- WCL(0.999) = WCL(1-0.00168) = $250,000.
- CVaR = $250,000 - $840 = $249,159.
As before, the monthly default probability is $d=0.00168$

<table>
<thead>
<tr>
<th>Default</th>
<th>Probability</th>
<th>Loss</th>
<th>pxL</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 bonds</td>
<td>$d^2=0.00000282$</td>
<td>$500,000$</td>
<td>$1.4$</td>
</tr>
<tr>
<td>1 bond</td>
<td>$2d(1-d)=0.00336$</td>
<td>$250,000$</td>
<td>$839.7$</td>
</tr>
<tr>
<td>0 bonds</td>
<td>$(1-d)^2=0.9966$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

The 99.9 loss quantile is about $\$500,000$

Weighted worst-case loss (WCL) = $\$250,000$

Expected loss (EL) = $\$839.70$

Conditional value at risk (CVaR) = $\$500,000 - $839.70 = 249,159$
Exercise

Credit VaR
Consider a stock $S$ valued at $1$ today, which after one period can be worth $S_T$: $2$ or $0.50$. Consider also a convertible bond $B$, which after one period will be worth $\max(1, S_T)$. Assume the stock can default ($p=0.05$), after which event $S_T=0$ (no recovery). Determine which is the following three portfolios has lower 95%-Credit-VaR:

1. $B$
2. $B-S$
3. $B+S$
Goodrich

Calculating credit exposure
Credit VaR

- Credit Exposure
  - How much one can lose due to counterparty default
  
  \[
  \max( \text{Swap Value}_t, 0 )
  \]
Credit VaR

- 99% Credit VaR
  - Sort losses and take the 99'th percentile
Expected Shortfall

- Expected Loss given 99% VaR
  - Take the average of the exposure greater than 99% percentile.
Simulation

- Monte Carlo simulation
- 10,000 simulations
- Simulate
  - Interest Rates
  - Credit Spreads
Interest Rates

- Black-Karasinski Model

\[ d \ln r = a \left( \frac{\theta}{\alpha} - \ln r \right) dt + \sigma_r dW \]

<table>
<thead>
<tr>
<th>Tenor</th>
<th>Init. IR</th>
<th>Mean</th>
<th>Vol</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5yrs</td>
<td>8.18%</td>
<td>7.99%</td>
<td>5.98%</td>
</tr>
<tr>
<td>10yrs</td>
<td>10.56%</td>
<td>8.93%</td>
<td>5.64%</td>
</tr>
</tbody>
</table>

Est. from Bonds
Spreads

➤ Vasicek Model

\[ ds = a \left( \frac{\theta}{\alpha} - s \right) dt + \sigma_s dW \]

<table>
<thead>
<tr>
<th>Tenor</th>
<th>Init. IR</th>
<th>Mean</th>
<th>Vol</th>
</tr>
</thead>
<tbody>
<tr>
<td>5yrs</td>
<td>2.4%</td>
<td>2.546%</td>
<td>0.535%</td>
</tr>
</tbody>
</table>
Algorithm

- IR-Spread
  - Choleski Decomposition
  - Sample from Normal distribution
  - Interest Rate-Spread Corr

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9458</td>
<td>0.53</td>
</tr>
<tr>
<td>0.9458</td>
<td>1</td>
<td>0.53</td>
</tr>
<tr>
<td>0.53</td>
<td>0.53</td>
<td>1</td>
</tr>
</tbody>
</table>

Corr of spread to 5yr IR
Est. from New Car Sales and Bond rates 71-83

Corr of 6mo and 10yr rate
Est. from Bond Data
Algorithm

- Iterate the Black-Karasinski
- Calculate the Value of the Swap as the difference of the values of Non-Defaultable Fixed and Floating Bonds
- After 10,000 calculate the credit VaR and the expected shortfall
Simulation: Credit Exposure

Credit Exposure

Exposure

$14,000
$12,000
$10,000
$8,000
$6,000
$4,000
$2,000
$0,000

Time/Month

Series1
Simulation: Expected Shortfall

Expected Shortfall

Shortfall

$0.00000

$2.00000

$4.00000

$6.00000

$8.00000

$10.00000

$12.00000

$14.00000

$16.00000

$18.00000

Time/Month

1 7 13 19 25 31 37 43 49 55 61 67 73 79 85 91

Series1
Credit models
Portfolio Credit Risk Models

- CreditMetrics
  - JPMorgan
- CreditRisk+
  - Credit Suisse
- KMV
  - KMV
- CreditPortfolioView
  - McKinsey
### Defining characteristics

<table>
<thead>
<tr>
<th>Risk Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Default-mode models only take into account default as a credit event.</td>
</tr>
<tr>
<td>MtM models consider changes in market values and credit ratings as they affect the value of the instruments.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Bottom-up vs. Top-down</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top-down models ignore details of each individual transaction and focus on the impact of each instrument on a large list of risk sources. Appropriate for retail portfolios.</td>
</tr>
<tr>
<td>Bottom-up models focus on the risk profile of each instrument. Appropriate for corporate or sovereign portfolios.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Conditional vs. unconditional models</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conditional models correlate default probabilities to macroeconomic factors (i.e., default frequencies increase during a recession).</td>
</tr>
<tr>
<td>Unconditional models focus on the counter-party; changes in macroeconomic factors can affect the counter-party default parameters.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Structural/reduced form models</th>
</tr>
</thead>
<tbody>
<tr>
<td>Structural models explain correlations by the joint movement of assets.</td>
</tr>
<tr>
<td>Reduced form models link default statistics with factor models to industrial and country specific variables.</td>
</tr>
</tbody>
</table>
CreditMetrics

- Credit risk is driven by movements in bond ratings.
- Analyses the effect of movements in risk factors to the exposure of each instrument in the portfolio (instrument exposure sensitivity).
- Credit events are “rating downgrades”, obtained through a matrix of migration probabilities.
- Each instrument is valued using the credit spread for each rating class.
- Recovery rates are obtained from historical similarities.
- Correlations between defaults are inferred from equity prices, assigning each obligor to a combination of 152 indices (factor decomposition).
- All this information is used to simulate future credit losses.
- It does not integrate market and credit risk.
Simulation of one asset: a bond

FIGURE 23-3 Building the Distribution of Bond Values

<table>
<thead>
<tr>
<th>Probability</th>
<th>Value</th>
<th>Sum</th>
<th>(Vi-m)^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>0.02%</td>
<td>$109.37</td>
<td>0.02</td>
</tr>
<tr>
<td>AAA</td>
<td>0.33%</td>
<td>$109.19</td>
<td>0.36</td>
</tr>
<tr>
<td>AAA</td>
<td>5.95%</td>
<td>$108.66</td>
<td>6.47</td>
</tr>
<tr>
<td>BBB</td>
<td>86.93%</td>
<td>$107.55</td>
<td>93.49</td>
</tr>
<tr>
<td>BB</td>
<td>5.30%</td>
<td>$102.02</td>
<td>5.41</td>
</tr>
<tr>
<td>B</td>
<td>1.17%</td>
<td>$98.10</td>
<td>1.15</td>
</tr>
<tr>
<td>CCC</td>
<td>0.12%</td>
<td>$83.64</td>
<td>0.10</td>
</tr>
<tr>
<td>Default</td>
<td>0.18%</td>
<td>$51.13</td>
<td>0.09</td>
</tr>
</tbody>
</table>

Sum= 100.00%
Mean= $107.09
SD= $2.99

Source: CreditMetrics

Stats for MtM
Simulation of one asset: a bond

FIGURE 23-3 Building the Distribution of Bond Values

<table>
<thead>
<tr>
<th>Probability</th>
<th>Value</th>
<th>Σ pi Vi</th>
<th>Σ pi (Vi-m)^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>0.02%</td>
<td>$109.37</td>
<td>0.02</td>
</tr>
<tr>
<td>AAA</td>
<td>0.33%</td>
<td>$109.19</td>
<td>0.36</td>
</tr>
<tr>
<td>AAA</td>
<td>5.95%</td>
<td>$108.66</td>
<td>6.47</td>
</tr>
<tr>
<td>BBB</td>
<td>86.93%</td>
<td>$107.55</td>
<td>93.49</td>
</tr>
<tr>
<td>BB</td>
<td>5.30%</td>
<td>$102.02</td>
<td>5.41</td>
</tr>
<tr>
<td>B</td>
<td>1.17%</td>
<td>$98.10</td>
<td>1.15</td>
</tr>
<tr>
<td>CCC</td>
<td>0.12%</td>
<td>$83.64</td>
<td>0.10</td>
</tr>
<tr>
<td>Default</td>
<td>0.18%</td>
<td>$51.13</td>
<td>0.09</td>
</tr>
</tbody>
</table>

99% CVaR: [$11, $24]
2σ: $6

Source: CreditMetrics

Stats for MtM:

- Mean: $107.09
- SD: $2.99
Correlations in CreditMetrics

Two counter-parties:

\[ r_1 = 0.90 r_{US,Ch} + k_1 \varepsilon_1, \]

\[ r_2 = 0.74 r_{GE,In} + 0.15 r_{GE,Ba} + k_2 \varepsilon_2 \]

\[ \rho_{def}(r_1, r_2) = 0.90 \times 0.74 \rho(r_{US,Ch}, r_{GE,In}) + 0.90 \times 0.15 \rho(r_{US,Ch}, r_{GE,Ba}) = 0.11 \]

152 country indices, 28 country indices, 19 worldwide indices, including:
- US Chemical Industry index
- German Insurance Index
- German Banking index
Simulation of more than one asset

Consider a portfolio consisting of $m$ counterparties, and a total of $n$ possible credit states.

We need to simulate a total of $n^m$ states; their multivariate distribution is given by their marginal distributions (as before) and the correlations given by the regression model.

To obtain accurate results, since many of these states have low probabilities, large simulations are often needed.

It does not integrate market and credit risk: losses are assumed to be due to credit events alone: for example,

- Swaps’ exposures are taken to be their expected exposures.
- Bonds are valued using today’s forward curve and current credit spreads for generated future credit ratings.
Exercise

Pricing the Goodrich swap using the Credit Metrics framework
The full swap

If we consider the full swap, we need to consider the default process $b$ and the interest rate process $r$.

The random variable that describes losses is given by

$$\text{Loss} = 50 \sum_{8 \text{ years}} (11 - \text{libor}_t)_+ e^{-r_t} b_t$$

If we assume the credit process and the market process are independent, we get

$$\text{ECL} = 50 \sum_{8 \text{ years}} \left[ \mathbb{E}(11 - \text{libor}_t)_+ \right] e^{-r_t} \mathbb{E}b_t$$

This will overestimate the risk in the case that the default process and the market process are negatively correlated.
The MonteCarlo approach

Correlation on market variables drive correlations of default events:

\[ \rho(\text{Libor}, \text{GR}) = -0.47 \]

Then,

\[ \rho(\text{Libor}_t, b_t) = -0.47 \]

and

\[ \text{ECL} = 50 \sum_{8 \text{ years}} \left[ E(11 - \text{libor}_t) \right] e^{-r_t} E b_t \]

is calculated with Monte-Carlo techniques.
The CreditMetrics Approach

Assume a 1 year time horizon, and that we wish to calculate the loss statistics for that time horizon. Assume credit ratings with transition probabilities from BBB given by

<table>
<thead>
<tr>
<th>Rating</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>0.02%</td>
</tr>
<tr>
<td>AA</td>
<td>0.33%</td>
</tr>
<tr>
<td>A</td>
<td>5.95%</td>
</tr>
</tbody>
</table>

... and spreads given by

<table>
<thead>
<tr>
<th>Rating</th>
<th>Spread</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>25</td>
</tr>
<tr>
<td>AA</td>
<td>40</td>
</tr>
<tr>
<td>A</td>
<td>100</td>
</tr>
<tr>
<td>BBB</td>
<td>180</td>
</tr>
<tr>
<td>BB</td>
<td>250</td>
</tr>
<tr>
<td>B</td>
<td>320</td>
</tr>
<tr>
<td>CCC</td>
<td>500</td>
</tr>
<tr>
<td>Default</td>
<td></td>
</tr>
</tbody>
</table>
The loss statistics (1 year forward)

The loss statistics can be summarized as follows

<table>
<thead>
<tr>
<th>Credit event</th>
<th>Mtm Change in $K</th>
<th>Spread (bpi)</th>
<th>prob default</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>155</td>
<td>25</td>
<td>0.02</td>
<td>31</td>
</tr>
<tr>
<td>AA</td>
<td>140</td>
<td>40</td>
<td>0.33</td>
<td>462</td>
</tr>
<tr>
<td>A</td>
<td>80</td>
<td>100</td>
<td>2.95</td>
<td>2360</td>
</tr>
<tr>
<td>BBB</td>
<td>0</td>
<td>180</td>
<td>86.93</td>
<td>0</td>
</tr>
<tr>
<td>BB</td>
<td>-70</td>
<td>250</td>
<td>5.3</td>
<td>-3710</td>
</tr>
<tr>
<td>B</td>
<td>-140</td>
<td>320</td>
<td>1.17</td>
<td>-1638</td>
</tr>
<tr>
<td>CCC</td>
<td>-320</td>
<td>500</td>
<td>0.12</td>
<td>-384</td>
</tr>
<tr>
<td>Default</td>
<td>-10000</td>
<td></td>
<td>0.18</td>
<td>-18000</td>
</tr>
</tbody>
</table>

Average: -2609.88
Std: 6457.41
Loss stats over the life of the asset

Expected exposures, and exposure quantiles (in the case of this swap) will generally decrease over the life of the asset. They are pure market variables, which can be calculated with monte carlo methods.

Probability of default, and the probability of other credit downgrades, increase over the live of the asset. They are calculated, either with transition probability matrices, or with default probability estimations (Merton’s model, for instance)

Discount factors will also decrease with time, and are given by the discount curve.

\[ PV - ECL = \sum_t \mathbb{E}[CL_t] \times PV_t \]
Pricing the deal

Assume the ECL=$50,000, and UCL=$200,000.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>$K</th>
</tr>
</thead>
<tbody>
<tr>
<td>GR swap</td>
<td>bps</td>
<td></td>
</tr>
<tr>
<td>Capital at Risk (UL, or CVAR)</td>
<td></td>
<td>200</td>
</tr>
<tr>
<td>Cost of capital is (15-8=7%)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Required net income (8 years)</td>
<td></td>
<td>112</td>
</tr>
<tr>
<td>Tax (40%)</td>
<td></td>
<td>75</td>
</tr>
<tr>
<td>Pretax net income</td>
<td></td>
<td>187</td>
</tr>
<tr>
<td>Operating costs</td>
<td></td>
<td>100</td>
</tr>
<tr>
<td>Credit Provision (ECL)</td>
<td></td>
<td>50</td>
</tr>
<tr>
<td>Hedging costs</td>
<td>0.50</td>
<td></td>
</tr>
<tr>
<td>Required revenue</td>
<td>0.50</td>
<td>327</td>
</tr>
</tbody>
</table>
CreditRisk+

- Uses only two states of the world: default/no-default.
- But allows the default probability to vary with time. As we saw in the review, if one considers only default/no-default states, the default probability must change with time to allow the credit spread to vary (otherwise, the spread is constant, and does not fit observed spreads).
- Defaults are Poisson draws with the specified varying default probabilities.
- Allows for correlations using a sector approach, much like CreditMetrics. However, it divides counter-parties into homogeneous sectors within which obligors share the same systematic risk factors.
- Severity is modeled as a function of the asset; assets are divided into severity bands.
- It is an analytic approach, providing quick solutions for the distribution of credit losses.
- No uncertainty over market exposures.
If we have a number of counter-parties $A$, each with a probability of default given by a fixed $P_A$, which can all be different, then the individual probability generating function is given by

$$F_A(z) = \sum_{n}[\text{Prob } n \text{ defaults}]z^n$$

$$= 1 - p_A + p_Az$$

If defaults are independent of each other, the generating function of all counter-parties is
CreditRisk+: introductory remarks

\[ F(z) = \sum_n \left[ \text{Prob } n \text{ defaults} \right] z^n \]

\[ = \prod_A F_Z(z) \]

\[ = \prod_A \left[ 1 - p_A (1 - z) \right] \]

\[ \approx \exp \left\{ \sum_A p_A (1 - z) \right\} \]

\[ = \exp \{ 1 - \mu z \} \]

\[ = \sum_n \frac{e^{-\mu} \mu^n}{n!} z^n \]

Poisson distribution: std of defaults is \( \sqrt{\mu} \).
# The case for stochastic default rates

Statistics from 1970 to 1996

<table>
<thead>
<tr>
<th>Rating</th>
<th>Average default probability (%)</th>
<th>Standard deviation (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aaa</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Aa</td>
<td>0.03</td>
<td>0.01</td>
</tr>
<tr>
<td>A</td>
<td>0.1</td>
<td>0</td>
</tr>
<tr>
<td>Baa</td>
<td>0.12</td>
<td>0.3</td>
</tr>
<tr>
<td>Ba</td>
<td>1.36</td>
<td>1.3</td>
</tr>
<tr>
<td>B</td>
<td>7.27</td>
<td>5</td>
</tr>
</tbody>
</table>
CreditRisk+: implementation

Each obligor is attached to an economic sector. The average default rate of sector $k$ is given by a number $x_k$, which is assumed to follow a Gamma distribution with parameters $\alpha_k$ and $\beta_k$. This yields a probability of default for each obligor in a sector which has a mean $\mu_k$ and standard deviation $\sigma_k$:

$$\alpha = \frac{\mu^2}{\sigma^2}, \quad \beta = \frac{\sigma^2}{\mu}$$

density function

$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} e^{-x/\beta} x^{\alpha-1}$$

and generating function given by

$$F(z) = \int_0^\infty e^{x(z-1)} f(x) \, dx$$

$$= \left( \frac{1-p}{1-pz} \right)^x, \quad p = \frac{\beta}{1+\beta}$$
According to this, the probability of $n$ defaults is given by the $n$'th term in the power series expansion of the generating function

$$F(z) = \sum_{n=1}^{\infty} \binom{n + \alpha - 1}{n} p^n (1 - p)^\alpha z^n$$

Probability of $n$ defaults
For the entire portfolio, with exposures to several market indices \( k \), the generating function is given by

\[
F(z) = \prod F_k(z) = \prod \left( \frac{1-p_k}{1-p_k z} \right)^{k}
\]
CreditRisk+: loss distribution

It goes from the distribution of default events to the loss distribution, by introducing a unit loss concept, with their associated distributions and generating functions, as follows:

It breaks up the exposure of the portfolio into $m$ exposure bands, each band with an exposure of $v$ units. The individual bands are assumed to be independent, and have a generating function equal to

$$G_j(z) = \sum_n \left[ \text{Prob } n \times v_j \text{ units loss} \right] z^{nv_j}$$

$$= \sum_n \frac{e^{-\mu_j} \mu_j^n}{n!} z^{nv_j}$$

$$= \exp \left\{ \mu_j + \mu_j z^{v_j} \right\}$$

$$= \exp \left\{ \text{Poisson} \left(P_j(z) \right) \right\}$$
For the entire portfolio, with exposures to several market indices $k$, and all exposure bands represented by levels $j$, we have a final explicit expression given by

$$G(z) = \prod_k F_k \left[ \sum_j P_{k,j}(z) \right]$$
KMV and the Merton Model
The Merton Model

Merton (1974) introduced the view that equity value is a call option on the value of the assets of the firm, with a strike price equal to the firm’s debt. In particular, the stock price embodies a forecast of the firm’s default probabilities, in the same way that an option embodies an implied forecast of the option being exercised.
A simple setting

In its simplest situation, assume
- The firm’s value equal to $V$.
- The firm issued a zero-coupon bond due in one time unit equal to $K$.

If, at the end of the time period, the firm’s value is higher than $K$, the bondholders get their bond payment, and the remainder value of spread amongst the shareholders.

If the value of the firm is less than $K$, the bondholders get the value of the firm $V$, and equity value is 0. The firm would then be in default.
Equity values and option prices

In our simple example before, stock value at expiration is

$$S_T = \text{Max}(V_T - K, 0)$$

Since the firm’s value equals equity plus bonds, we have that the value of the bond is

$$B_T = V_T - \text{Max}(V_T - K, 0)$$

$$= \text{Min}(V_T, K)$$
Similarly, the bond value can be expressed as

\[ B_T = K - \text{Max}(K - V_T, 0) \]

In other words, a long position in a risky bond is equivalent to a long position in a risk-free bond plus a short put option.

The shot put option is really a credit derivative, same as the risky bond.

This shows that corporate debt has a payoff similar to a short option position, which explains the left skewness in credit losses.

It also shows that equity is equivalent to an option on the value’s assets; due to the limited liability of the firm, investors can lose no more than their original investment.
Pricing equity

We assume the firm’s value follows a geometric brownian motion process

\[ dV = \mu V dt + \sigma V dz \]

If we assume no transaction costs (including bankruptcy costs)

\[ V = B + S \]

Since stock price is the value of the option on the firm’s assets, we can price it with the Black-Scholes methodology, obtaining

\[ S = VN(d_1) - Ke^{-r\tau} N(d_2) \]

with

\[ d_1 = \frac{-\ln(Ke^{-r\tau} / V)}{\sigma \sqrt{\tau}} + \frac{\sigma \sqrt{\tau}}{2}, \quad d_2 = d_1 - \sigma \sqrt{\tau} \]

Leverage: Debt/Value ratio
Asset volatility
In practice, only equity volatility is observed, not asset volatility, which we must derive as follows:

The hedge ratio

$$dS = \frac{\partial S}{\partial V} dV$$

yields a relationship between the stochastic differential equations for $S$ and $V$, from where we get

$$\sigma_s S = \sigma_v V \frac{\partial S}{\partial V}$$

and

$$\sigma_v = \sigma_s \frac{S \partial V}{V \partial S}$$
Pricing debt

The value of the bond is given by

\[ B = V - S \]

or

\[ B = Ke^{-r\tau} N(d_2) + V\left[1 - N(d_1)\right] \]

This is the same as

\[ \frac{B}{Ke^{-r\tau}} = N(d_2) + \left(\frac{V}{Ke^{-r\tau}}\right)N(-d_1) \]

*Probability of exercising the call, or probability that the bond will not default*
The expected credit loss is the value of the risk-free bond minus the risky bond:

$$ECL = Ke^{-r\tau} - Ke^{-r\tau} N(d_2) - V \left[1 - N(d_1)\right]$$

$$= Ke^{-r\tau} N(-d_2) - VN(-d_1)$$

This is the same as

$$ECL = N(-d_2) \left[ Ke^{-r\tau} - V \frac{N(-d_1)}{N(-d_2)} \right]$$

ECL = prob x Exposure x Loss-Given-Default
Advantages

- Relies on equity prices, not bond prices: more companies have stock prices than bond issues.
- Correlations among equity prices can generate correlations among default probabilities, which would be otherwise impossible to measure.
- It generates movements in EDP that can lead to credit ratings.
Disadvantages

- Cannot be used for counterparties without traded stock (governments, for example)
- Relies on a static model for the firm’s capital and risk structure:
  - the debt level is assumed to be constant over the time horizon.
  - The extension to the case where debt matures are different points in time is not obvious.
- The firm could take on operations that will increase stock price but also its volatility, which may lead to increased credit spread; this is in apparent contradiction with its basic premise, which is that higher equity prices should be reflected in lower credit spreads.
KMV

- KMV was a firm founded by Kealhofer, McQuown and Vasicek, (sold recently to Moody’s), which was a vendor of default frequencies for 29,000 companies in 40 different countries. Much of what they do is unknown.
- Their method is based on Merton’s model: the value of equity is viewed as a call option on the value of the firm’s assets.
- Basic model inputs are:
  - Value of the liabilities (calculated as liabilities (<1 year) plus one half of long term debt)
  - Stock value
  - Volatility
  - Assets
Basic terms

\[ \text{Distance to Default} = \frac{\text{Market Value of Assets} - \text{Default Point}}{\text{Market Value of Assets} + \text{Asset Volatility}} \]
Distance to Default \( t \) = \[ \ln \left( \frac{V_A}{L_t} \right) - \frac{\sigma_A^2}{2} t \]

An approximation

\[ \approx \frac{\ln \left( \frac{V_A}{L_t} \right)}{\sigma_A} \]

\[ \approx \frac{\frac{V_A}{L_t} - 1}{\sigma_A} \]
Example

Consider a firm with:
- $100M Assets
- $80M liabilities
- volatility of $10M (annualized)

Distance from default is calculated as

\[
\frac{A-K}{\sigma} = 2
\]

Default probability is then 0.023 (using a gaussian)
Exercise
The Merton Model

Consider a firm with total asset worth $100, and asset volatility equal to 20%.
The risk free rate is 10% with continuous compounding.
Time horizon is 1 year.
Leverage is 90% (i.e., debt-to-equity ratio 900%)

Find:

➢ The value of the credit spread.
➢ The risk neutral probability of default
➢ Calculate the PV of the expected loss.
Credit Spread

A leverage of 0.9 implies that
\[ Ke^{-0.1} / V = 0.9 \]
which says that K=99.46.

Using Black-Scholes, we get that the call option is worth \( S = \$13.59 \).

The bond price is then
\[ B = V - S = \$100 - \$13.59 = \$86.41 \]
for a yield of
\[ \ln(K / B) = \ln(99.46 / 86.41) = 14.07\% \]
or a credit spread of 4.07\%.
# Option Calculation

## Underlying Data

<table>
<thead>
<tr>
<th>Underlying Type:</th>
<th>Time</th>
<th>Dividend</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Stock Price:</th>
<th>Volatility (% per year):</th>
<th>Risk-Free Rate (% per year):</th>
</tr>
</thead>
<tbody>
<tr>
<td>100.00</td>
<td>20.00%</td>
<td>10.00%</td>
</tr>
</tbody>
</table>

- **Calculate**

## Graph Results

<table>
<thead>
<tr>
<th>Vertical Axis:</th>
<th>Horizontal Axis:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theta</td>
<td>Asset price</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Minimum X value:</th>
<th>Maximum X value:</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>120</td>
</tr>
</tbody>
</table>

- **Draw Graph**

## Option Data

<table>
<thead>
<tr>
<th>Option Type:</th>
<th>Simplicity Volatility</th>
<th>Time to Exercise:</th>
<th>Exercise Price:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytic:</td>
<td></td>
<td>1.0000</td>
<td>99.46</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Price:</th>
<th>Delta (per %):</th>
<th>Gamma (per %):</th>
<th>Vega (per %):</th>
<th>Theta (per day):</th>
<th>Rho (per %):</th>
</tr>
</thead>
<tbody>
<tr>
<td>13.5923574</td>
<td>0.73463442</td>
<td>0.01636677</td>
<td>0.3273533</td>
<td>-0.0253837</td>
<td>0.69877085</td>
</tr>
</tbody>
</table>

- **Graph**

![Graph showing the relationship between asset price and Theta](image)
The risk neutral probability of default

Given by

\[ N(d_2) = 0.6653, \quad EDF = 1 - N(d_2) = 33.47\% \]
Expected loss

It is given by

\[
ECL = N(-d_2) \left[ Ke^{-rt} - V \frac{N(-d_1)}{N(-d_2)} \right] \\
= 0.3347 \times \left[ \$90 - \$100 \times \frac{0.2653}{0.3347} \right] \\
= $3.96
\]
Additional considerations

Variations on the same problem:
- If debt-to-equity ratio is 233%, the spread is 0.36%
- If debt-to-equity ratio is 100%, the spread is about 0.

In other words, the model fails to reproduce realistic, observed credit spreads.
Exercise

Calibrating the asset volatility
The Goodrich Corporation

From company’s financials
- Debt/equity ratio: 2.27
- Shares out: 117,540,000.
- Expected dividend: $0.20/share.

From NYSE, ticker symbol GR
- Stock volatility: 49.59%
- Real rate of return (3 years): 0.06%
- Share price: $17.76 (May 2003)

From interest rate market
- Annual risk free rate: 3.17%

\[
S = 117,540,000 \times 17.76 = 2,087B.
\]
\[
V = S + B = 3.27S = 6.826B
\]
Current debt = $4.759B
Future debt (Strike price)
\[
K = 4.759 e^{0.0317} = 4.912
\]
Dividend =
\[
0.20 \times 117,540,000 = 23,508,000.
\]
Bootstrapping asset volatility

\[ \sigma_V = \sigma_S \frac{S \partial V}{V \partial S} \]
Bootstrapping asset volatility (iterative process)

\[ \sigma_{\text{vol}} = \sigma_S \frac{S \delta V}{V \delta S} \]
\[ = 0.4959 \times 2.087 \times 0.9936 / 6.83 \]
\[ = 14.77\% \]
McKinsey’s Credit Portfolio View

- Introduced in 1997.
- Considers only default/no-default states, but probabilities are time dependent, given by a number $p_t$.
- It is calculated as follows: given macroeconomic variables $x_k$, it uses a multifactor model (Wilson 1997)

$$y_t = \alpha + \sum \beta_k x_k$$

to assign a debtor a country, industry and rating segment. It assigns a probability of default given by

$$p_t = \frac{1}{1 + \exp y_t}$$

- The models uses this set up to simulate the loss distribution.
- The model is convenient to model default probabilities in macroeconomic contexts, but it is inefficient for corporate portfolios.
## Comparative study

### 1 year horizon, 99% confidence

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Assuming 0 correlations</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CM</td>
<td>777</td>
<td>2093</td>
<td>1989</td>
</tr>
<tr>
<td>CR+</td>
<td>789</td>
<td>2020</td>
<td>2074</td>
</tr>
<tr>
<td>Basel</td>
<td>5304</td>
<td>5304</td>
<td>5304</td>
</tr>
<tr>
<td><strong>Assuming correlations</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CM</td>
<td>2264</td>
<td>2941</td>
<td>11436</td>
</tr>
<tr>
<td>CR+</td>
<td>1638</td>
<td>2574</td>
<td>10000</td>
</tr>
<tr>
<td>Basel</td>
<td>5304</td>
<td>5304</td>
<td>5304</td>
</tr>
</tbody>
</table>

- **Three models:** CM, CR+, Basel (8% deposit of loan notional)
- **Three portfolios,** $66.3B total exposure each
  - A: high credit quality, diversified (500 names)
  - B: High credit, concentrated (100 names)
  - C: Low credit, diversified (500)

Models are fairly consistent

Higher discrepancy between models

Correlations increase credit risk

Higher discrepancy between models
Which of the following is used to estimate the probability of default for a firm in the KMV model?

I. Historical probability of default based on the credit rating of the firm (KMV have a method to assign a rating to the firm if unrated)

II. Stock price volatility

III. The book value of the firm’s equity

IV. The market value of the firm’s equity

V. The book value of the firm’s debt

VI. The market value of the firm’s debt

a) I only
b) II, IV and V
c) II, III, VI
d) VI only
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J.P. Morgan’s CreditMetrics uses which of the following to estimate default correlations?

a) CreditMetrics does not estimate default correlations; it assumes zero correlations between defaults
b) Correlations of equity returns
c) Correlations between changes in corporate bond spreads to treasury
d) Historical correlation of corporate bond defaults
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J.P. Morgan’s CreditMetrics uses which of the following to estimate default correlations

a) Bond spreads to treasury
b) History of loan defaults
c) Assumes zero correlations and simulates defaults
d) None of the above
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a) Bond spreads to treasury
b) History of loan defaults
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The KMV credit risk model generates an estimated default frequency (EDF) based on the distance between the current value of the assets and the book value of the liabilities. Suppose that the current value of a firm’s assets and the book value of its liabilities are $500M and $300M, respectively. Assume that the standard deviation of returns on the assets is $100M, and that the returns of the assets are normally distributed. Assuming a standard Merton Model, what is the approximate default frequency (EDF) for this firm?

1. 0.010
2. 0.015
3. 0.020
4. 0.030
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Distance from default is calculated as

\[
\frac{A - K}{\sigma} = 2
\]

Default probability is then 0.023 (using a gaussian)
Which one of the following statements regarding credit risk models is MOST correct?

1. The CreditRisk+ model decomposes all the instruments by their exposure and assesses the effect of movements in risk factors on the distribution of potential exposure.

2. The CreditMetrics model provides a quick analytical solution to the distribution of credit losses with minimal data input.

3. The KMV model requires the historical probability of default based on the credit rating of the firm.

4. The CreditPortfolioView (McKinsey) model conditions the default rate on the state of the economy.
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CR+ assumes fixed exposures
Example 23-11: FRM Exam 2000

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KMV uses the current stock price

requires the historical probability of default