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Value at Risk and its estimation

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Definition

Let $S(t)$ denotes the market value of some portfolio. The *Value at Risk* (VaR) of that portfolio at a given time horizon t and confidence level p is the loss in market value over the time horizon t that is exceeded with probability $1 - p$, i.e.

$$\mathsf{P}\{S(t) - S(0) < -\mathrm{VAR}\} = 1 - p.$$

The Derivative Policy Group has proposed a standard that would set a time horizon t of two weeks and a confidence level $p = 0.99$.

Statistically speaking, this value at risk measure is the "**0.01** critical value" of the probability distribution of changes in market value.

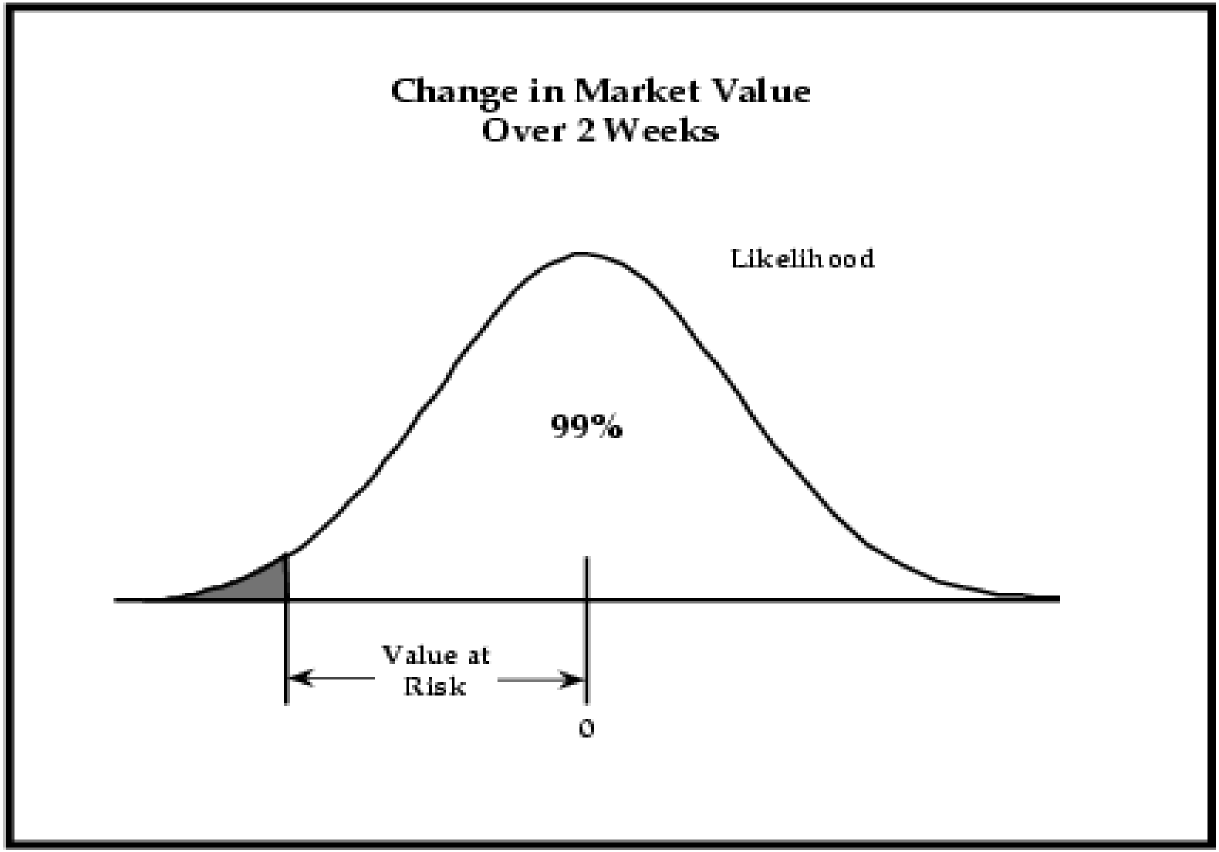

Figure 1: Value at Risk (DPG Standard)

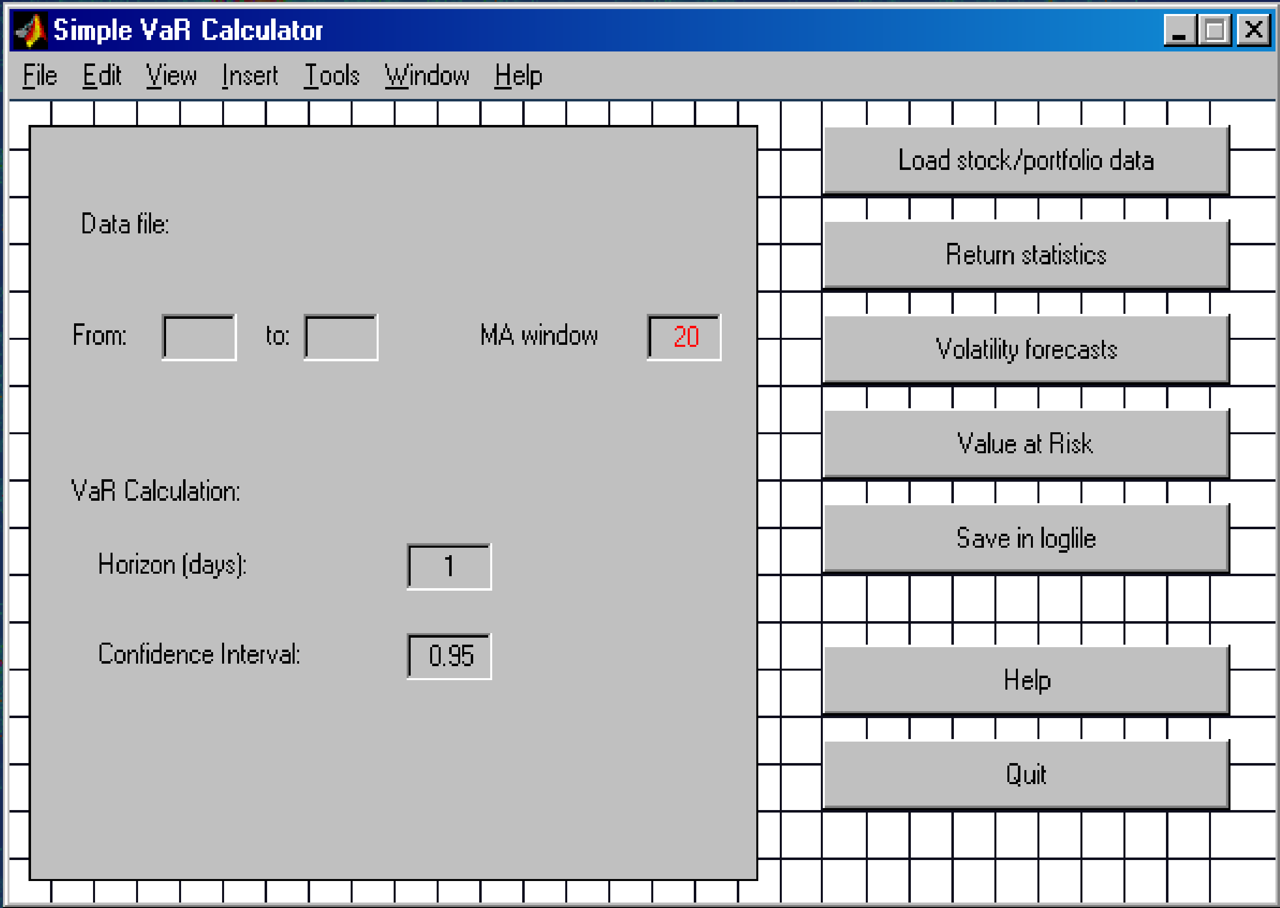

Figure 2: Simple VaR-calculator

Different approaches

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- **Traditional**. Identifying events and causes and linking them statistically using actuarial-based methods.
- **Algorithmics**. To model losses caused by rare events and unexpected failures in systems, people and environmental factors.

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4. Simulate the change in market value of the portfolio, for each scenario of the underlying market returns. Generate a sufficient number of scenarios to estimate VaR with the desired level of accuracy.

Example

There are **418** underlying assets covered by RiskMetrics on July **29**, **1996**. A portfolio of *plain-vanilla* options on these assets is simulated by Monte Carlo:

- Independently, any option is a European call with probability **0.5** and European put with the same probability.
- Independently, the time to expiration is **1** month with probability **0.4**, **3** months with probability **0.3**, **6** months with probability **0.2** and **1** year with probability **0.1**.
- Independently, the ratio of exercise price to forward price is log-normally distributed with mean **1**. Its logarithm has standard deviation **0.1**.

Methods for estimating the VaR

1. **Actual**. Monte Carlo simulation of all underlying asset prices and computation of each option price for each scenario by an exact formula.

2. **Delta**. Monte Carlo simulation of all underlying asset prices and approximation of each option price for each scenario by a *delta-approximation* of its change in value.

3. **Gamma**. Monte Carlo simulation of all underlying asset prices and approximation of each option price for each scenario by the *delta-gamma approximation* $\mathbf{Y(\Delta, \Gamma)}$ of its change in value.

4. **Analytical-Gamma**. The approximation $\boldsymbol{c(p)}\,\sqrt{\mathbf{Var}(\boldsymbol{Y}(\mathbf{\Delta}, \mathbf{\Gamma}))}$, where $\boldsymbol{c(p)}$ is the $\boldsymbol{p}$-critical value of the standard normal density.

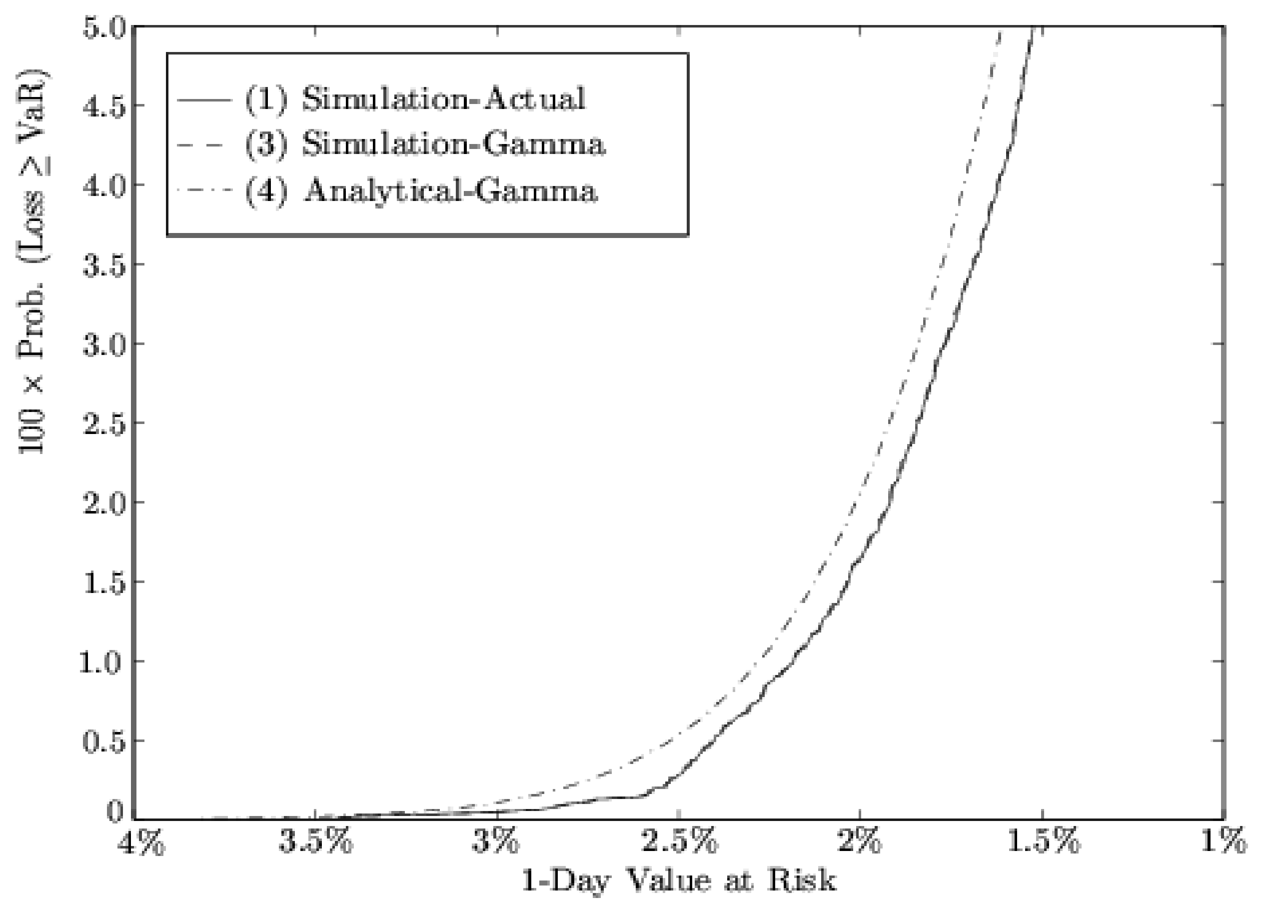


Figure 3: Value at Risk of long option portfolio — plain-vanilla model

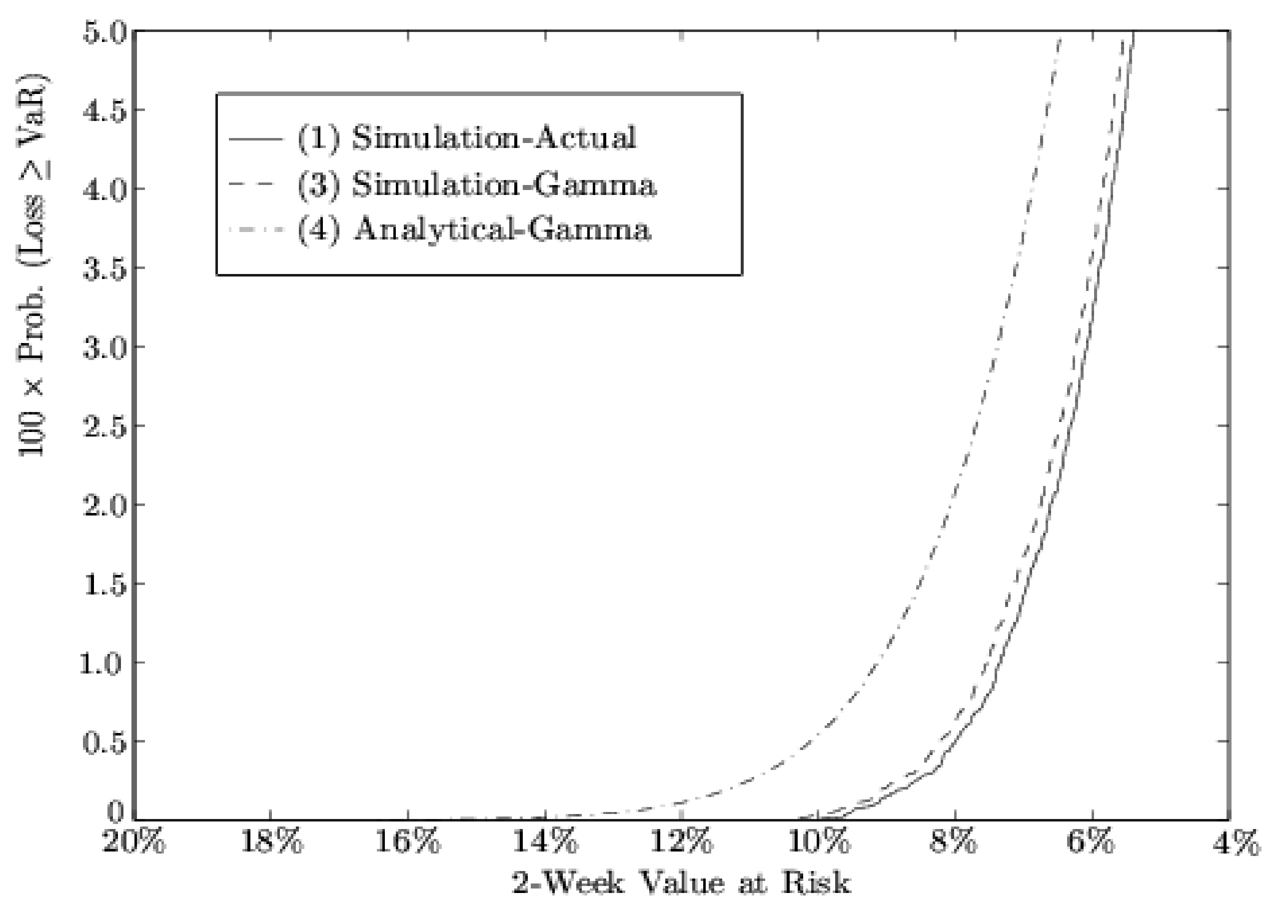


Figure 4: Value at Risk of long option portfolio — plain-vanilla model

Correlated jumps

Consider the so called jump-diffusion model **3**, in that half of the variance of the annual return of each asset is associated with a jump, with an expected arrival rate of **1** jump per year. We have a more dramatic comparison of different methods.

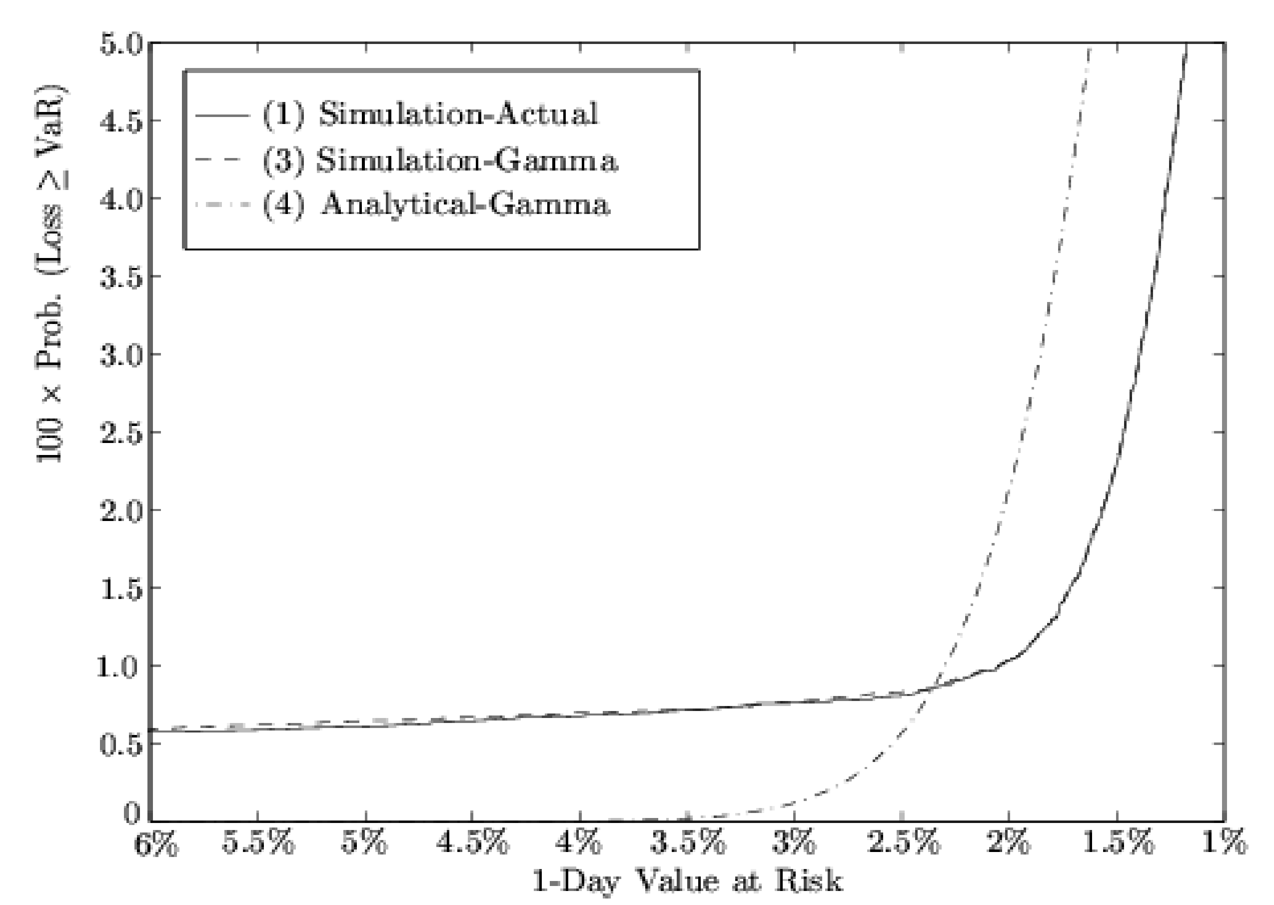


Figure 5: Value at Risk for short option portfolio — jump-diffusion model 3

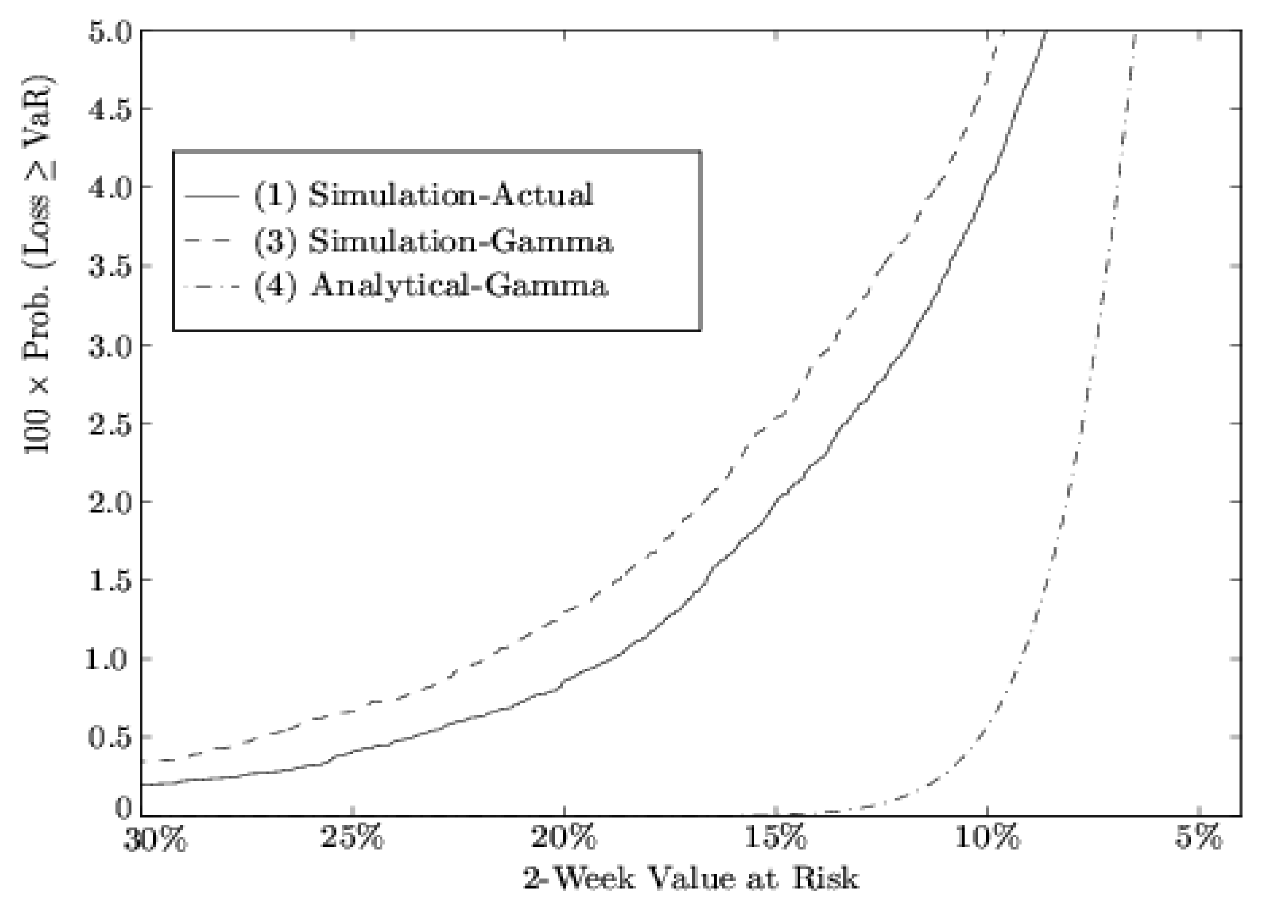


Figure 6: Value at Risk for short option portfolio — jump-diffusion model 3

The basic model of return risk

Let $\boldsymbol{R}_t$ denotes the return of some underlying asset at day $\boldsymbol{t}$. Then

$$R_{t+1} = \mu_t + \sigma_t \varepsilon_{t+1},$$

where

$$\mu_t = \mathsf{E}\{R_{t+1}/\mathfrak{F}_t\},$$
$$\sigma_t^2 = \mathsf{Var}\{R_{t+1}/\mathfrak{F}_t\},$$
$$\mathsf{E}\{\varepsilon_{t+1}/\mathfrak{F}_t\} = 0,$$
$$\mathsf{Var}\{\varepsilon_{t+1}/\mathfrak{F}_t\} = 1.$$

A model is called *plain-vanilla* if $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are constant parameters, and $\boldsymbol{\varepsilon}_t$ are independent standard normal random variables (white noise). Experience shows that in practice distribution of tails is heavy.

Jump-diffusion model

$$R_t = R_0 \exp\{\alpha t + X_t\},$$

$$X_t = \beta W_t + \sum_{k=0}^{N(t)} \nu Z_k,$$

where W_t is a standard Wiener process and $N(t)$ is the number of jumps that occur by time t. This is Poisson process with intensity λ. All jumps νZ_k are independent and normally distributed with mean zero and standard deviation ν.

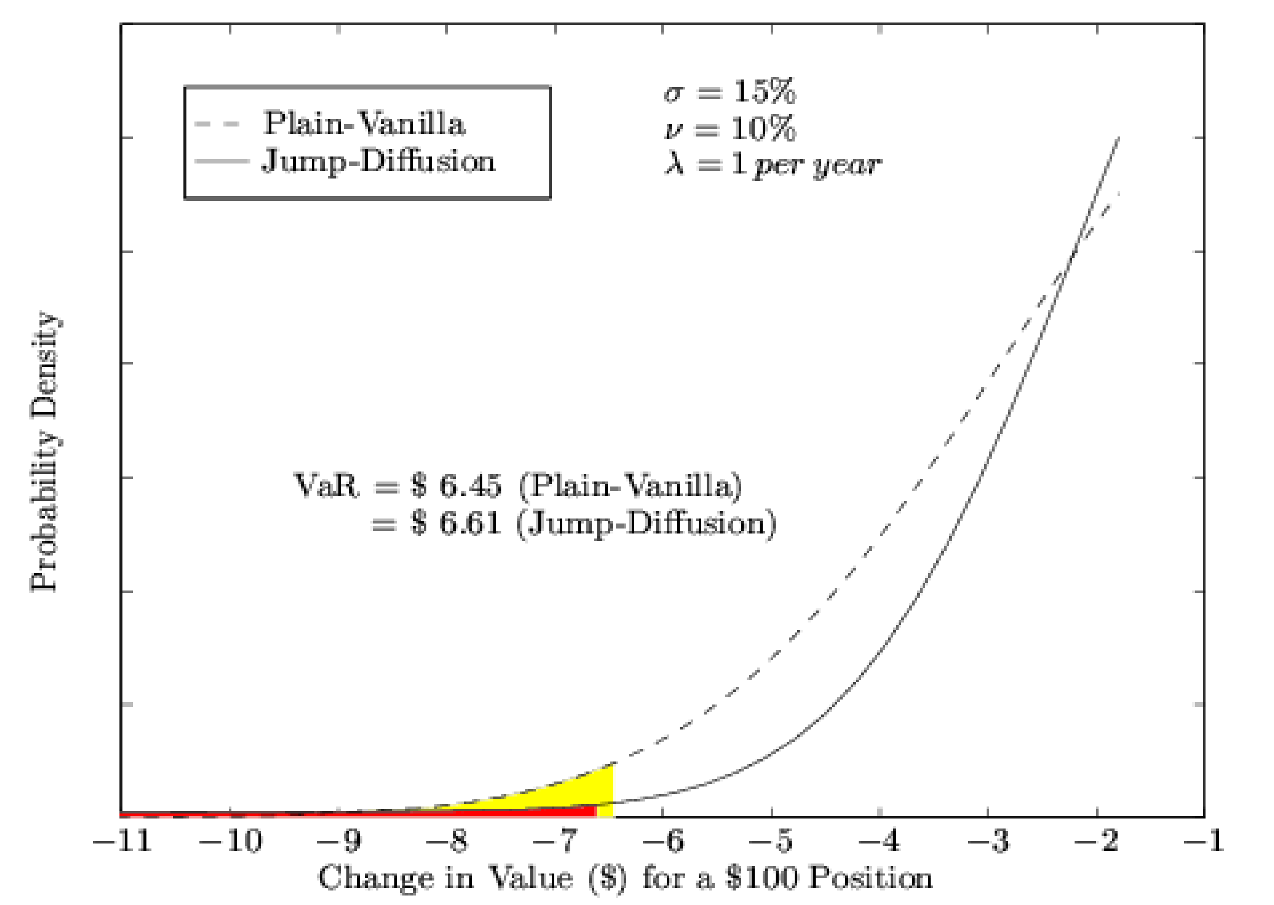


Figure 7: 2-Week 99%-**VAR** for underlying asset, one jump per year

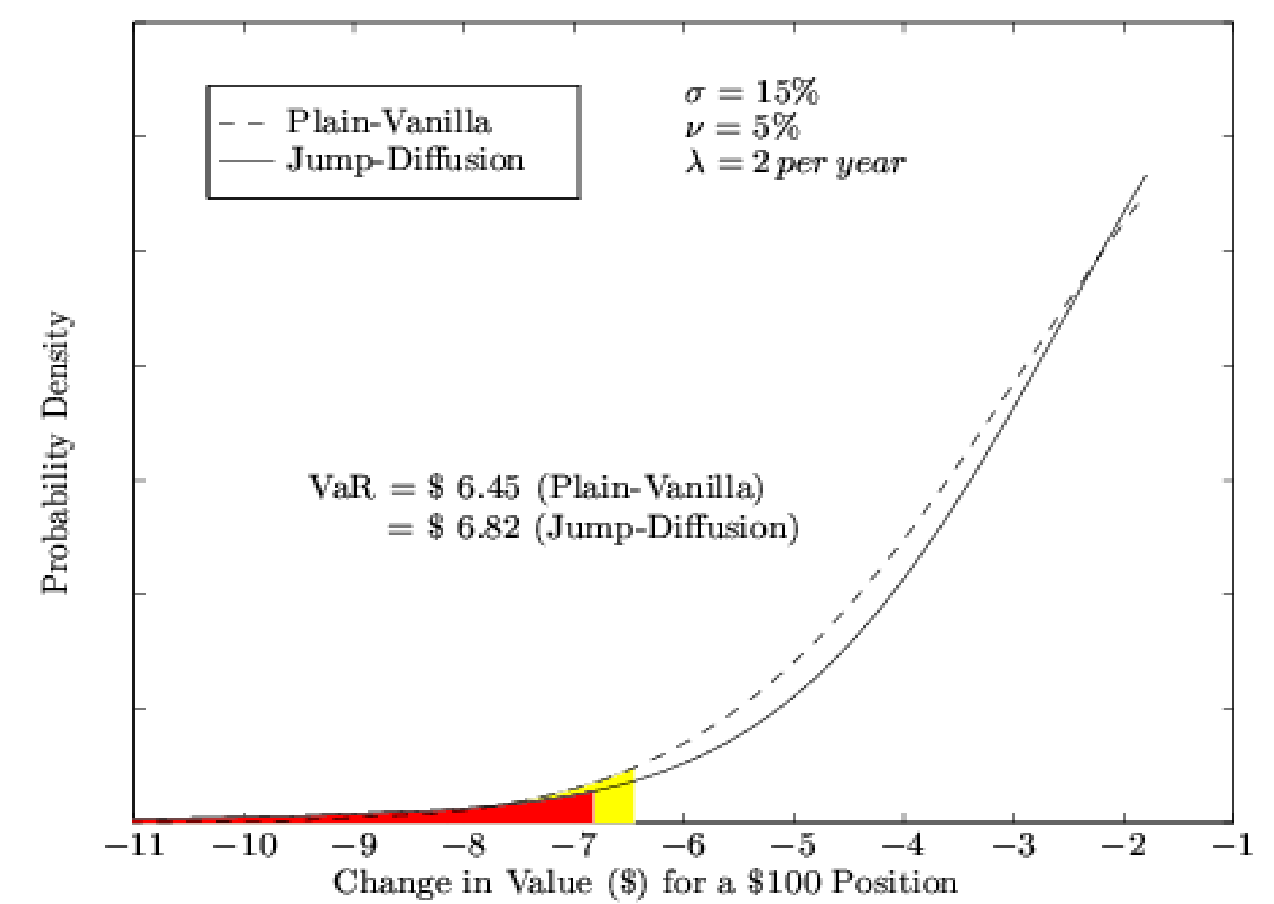


Figure 8: 2-Week 99%-**VAR** for underlying asset, two jumps per year

Stochastic volatility

We will consider only Markovian models of the form

$$\sigma_t = F(\sigma_{t-1}, z_t, t),$$

where z_t is white noise.

In the model of *regime-switching volatility*, volatility behaves according to a finite-state Markov chain. A model on Fig. 9 illustrates *persistence*, i.e., relatively high (low) recent volatility implies a relatively high (low) forecast of volatility in the near future.

The model of *log-auto-regressive volatility* is given by

$$\log \sigma_t^2 = \alpha + \gamma \log \sigma_{t-1}^2 + \kappa z_t,$$

where α, γ and κ are constants. A value of γ near $\mathbf{0}$ implies low persistence, while a value near $\mathbf{1}$ implies high persistence.

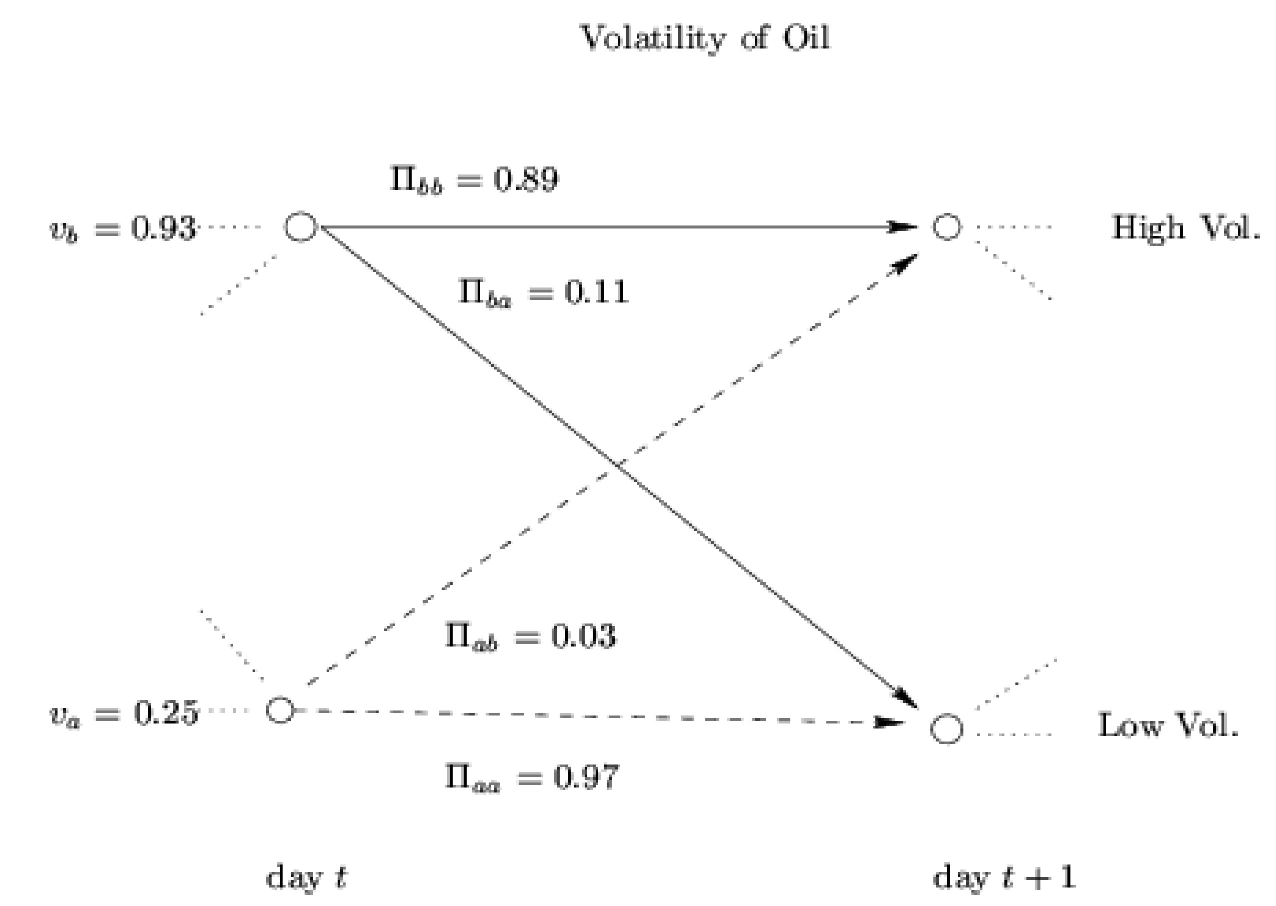


Figure 9: Regime-switching volatility estimates for light crude oil

The GARCH (**G**eneral **A**uto**R**egressive **C**onditional **H**eteroscedasticity) model assumes that

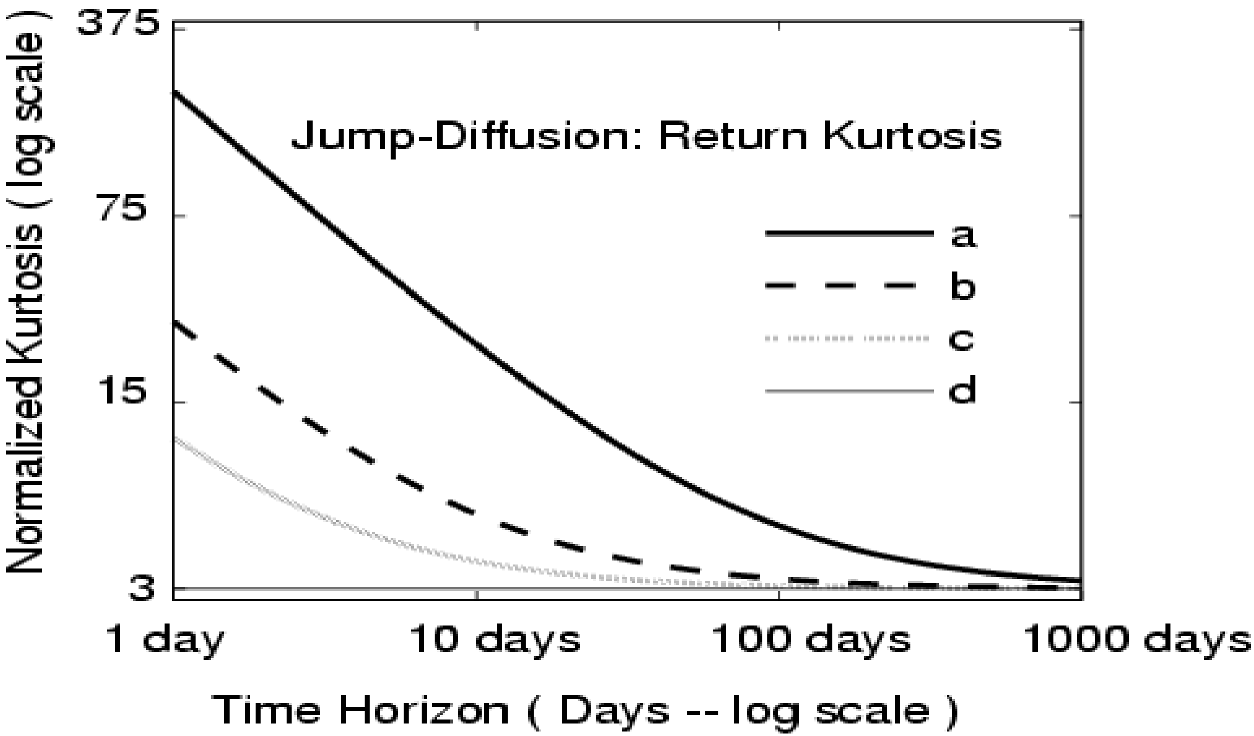
$$\sigma_t^2 = \alpha + \beta(R_t - \mu)^2 + \gamma\sigma_{t-1}^2.$$

For example, estimated GARCH parameters associated with crude oil have maximum likelihood estimates given by

$$\alpha = 0.155, \quad \beta = 0.292, \quad \gamma = 0.724.$$

The cross-market inference can be accounted by the multivariate GARCH model, for example

$$\begin{pmatrix} \sigma_{a,t}^2 \\ \sigma_{ab,t}^2 \\ \sigma_{b,t}^2 \end{pmatrix} = \alpha + \beta \begin{pmatrix} R_{a,t}^2 \\ R_{a,t}R_{b,t} \\ R_{b,t}^2 \end{pmatrix} + \gamma \begin{pmatrix} \sigma_{a,t-1}^2 \\ \sigma_{ab,t-1}^2 \\ \sigma_{b,t-1}^2 \end{pmatrix}.$$

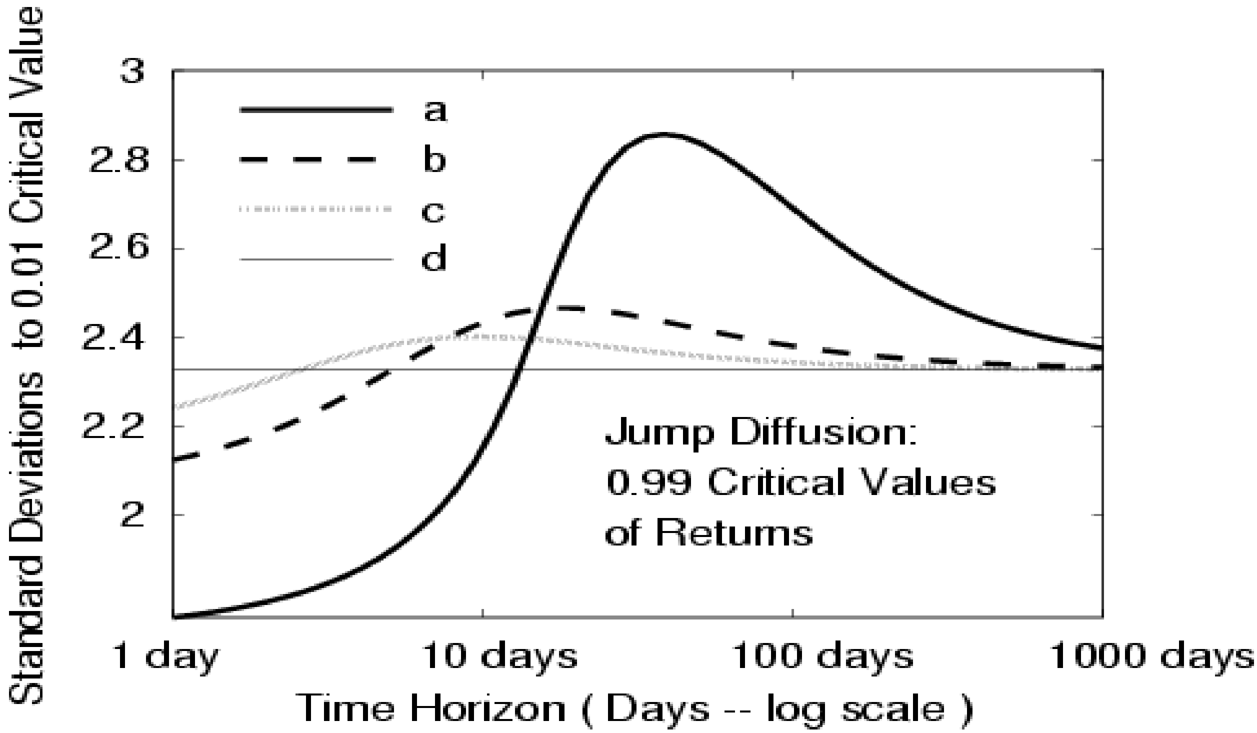


Kurtosis of return is shown for the following cases:

(a) $\sigma = 15\%$, $\lambda = 1.0$, $\nu = 10\%$; (b) $\sigma = 15\%$, $\lambda = 2.0$, $\nu = 5\%$

(c) $\sigma = 15\%$, $\lambda = 3.0$, $\nu = 3.33\%$; (d) plain-vanilla with $\sigma = 15\%$.

Figure 10: Term structure of kurtosis for the jump-diffusion model



99% critical value is shown for the following cases:
(a) $\sigma = 15\%$, $\lambda = 1.0$, $\nu = 10\%$; (b) $\sigma = 15\%$, $\lambda = 2.0$, $\nu = 5\%$
(c) $\sigma = 15\%$, $\lambda = 3.0$, $\nu = 3.33\%$; (d) plain-vanilla with $\sigma = 15\%$.

Figure 11: Term structure of **0.99** critical value of the jump-diffusion model

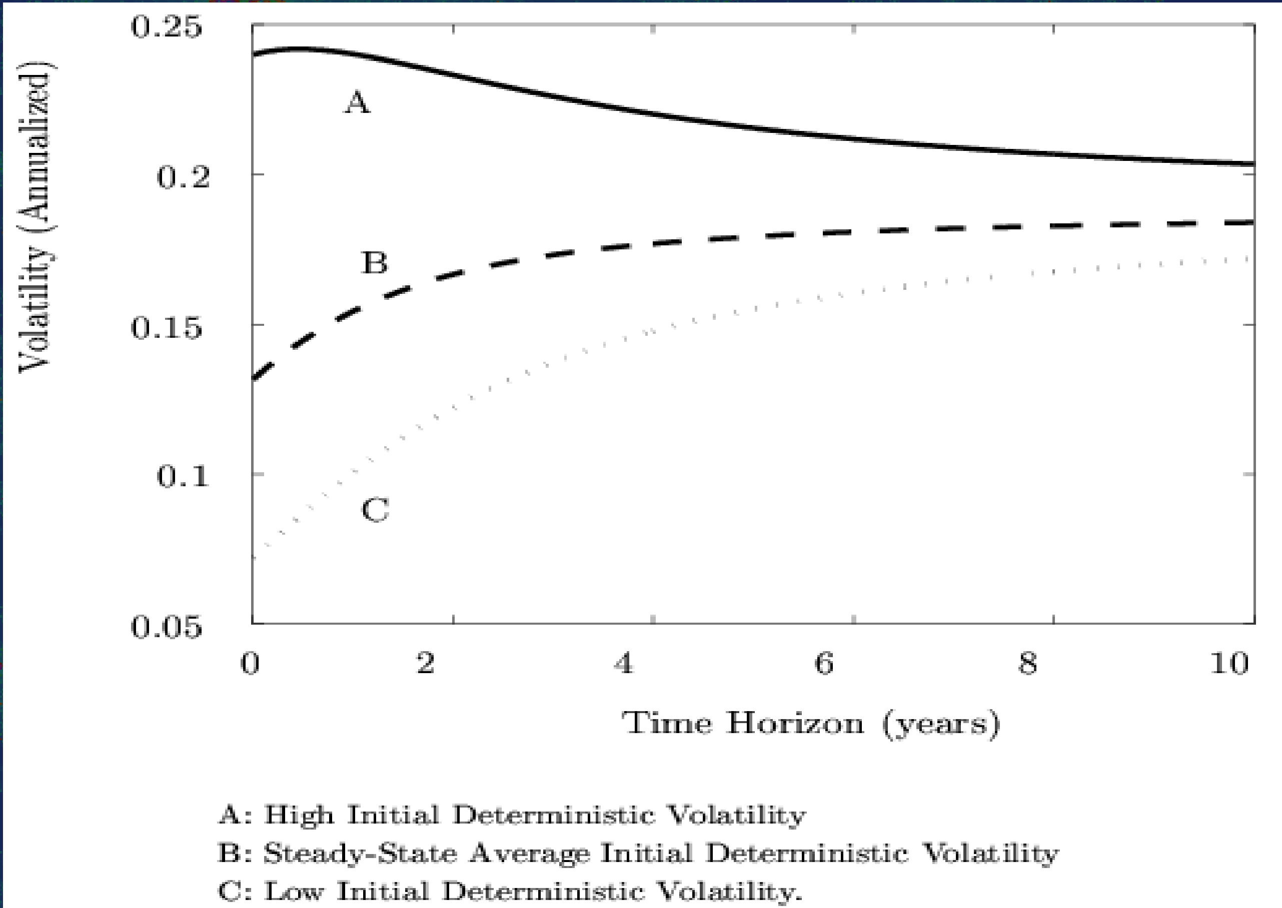


Figure 12: Term structure of volatility (Hang Seng Index — estimated)

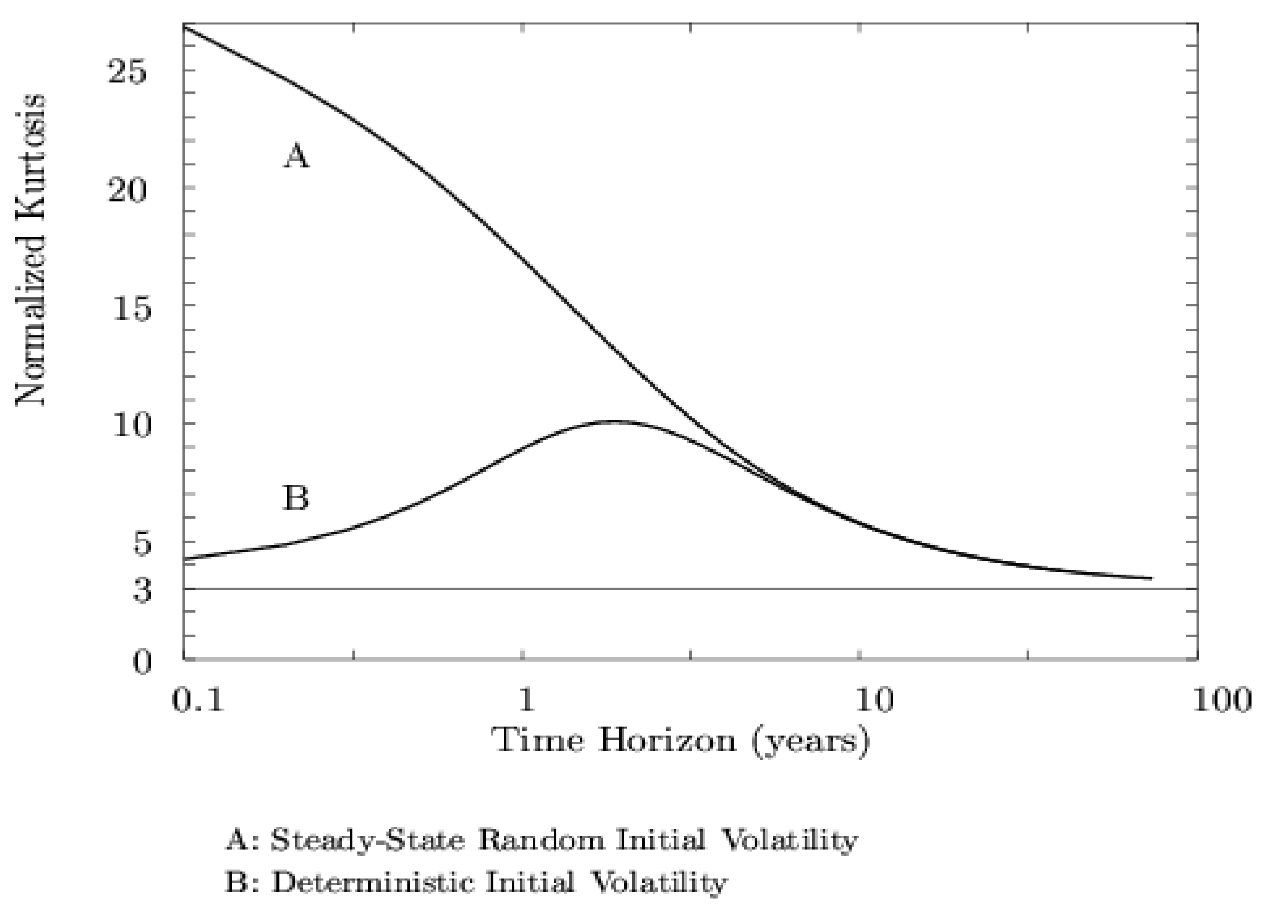


Figure 13: Long-run kurtosis of stochastic volatility model

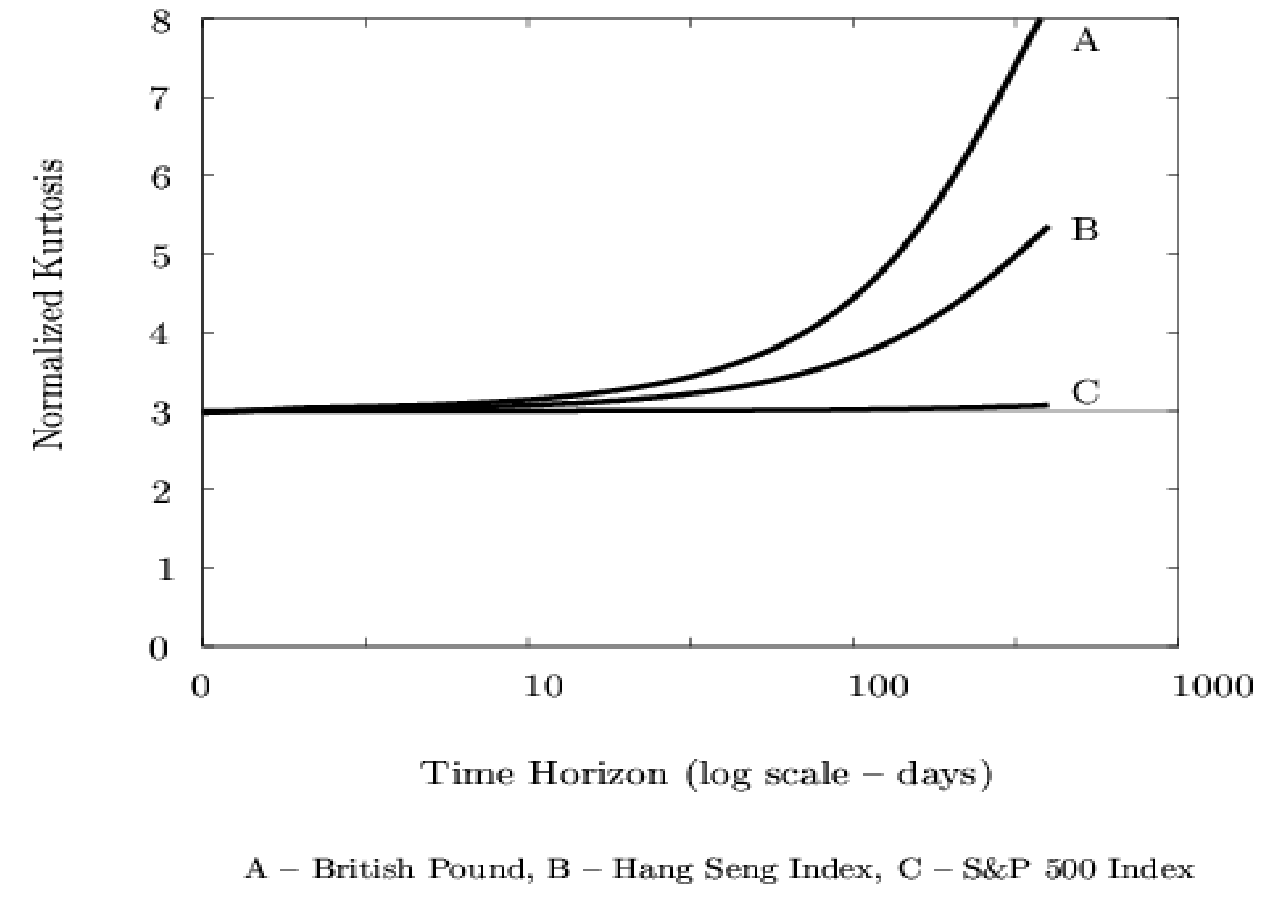


Figure 14: Estimated term structure of kurtosis for stochastic volatility

Estimating current volatility

The *historical volatility* for returns $R_t, r_{t+1}, \ldots, R_T$ is the usual estimate

$$\hat{\sigma}^2_{t,T} = \frac{1}{T - t} \sum_{s=t+1}^{T} (R_s - \hat{\mu}_{t,T}),$$

where

$$\hat{\mu}_{t,T} = \frac{R_{t+1} + \cdots + R_T}{T - t}.$$

The constant volatility model can not be applied to essentially every major market, as shown on Fig. 15.

The *Black–Scholes implied volatility* $\sigma = \sigma^{BS}(C_t, P_t, \tau, K, r)$ is calculated numerically.

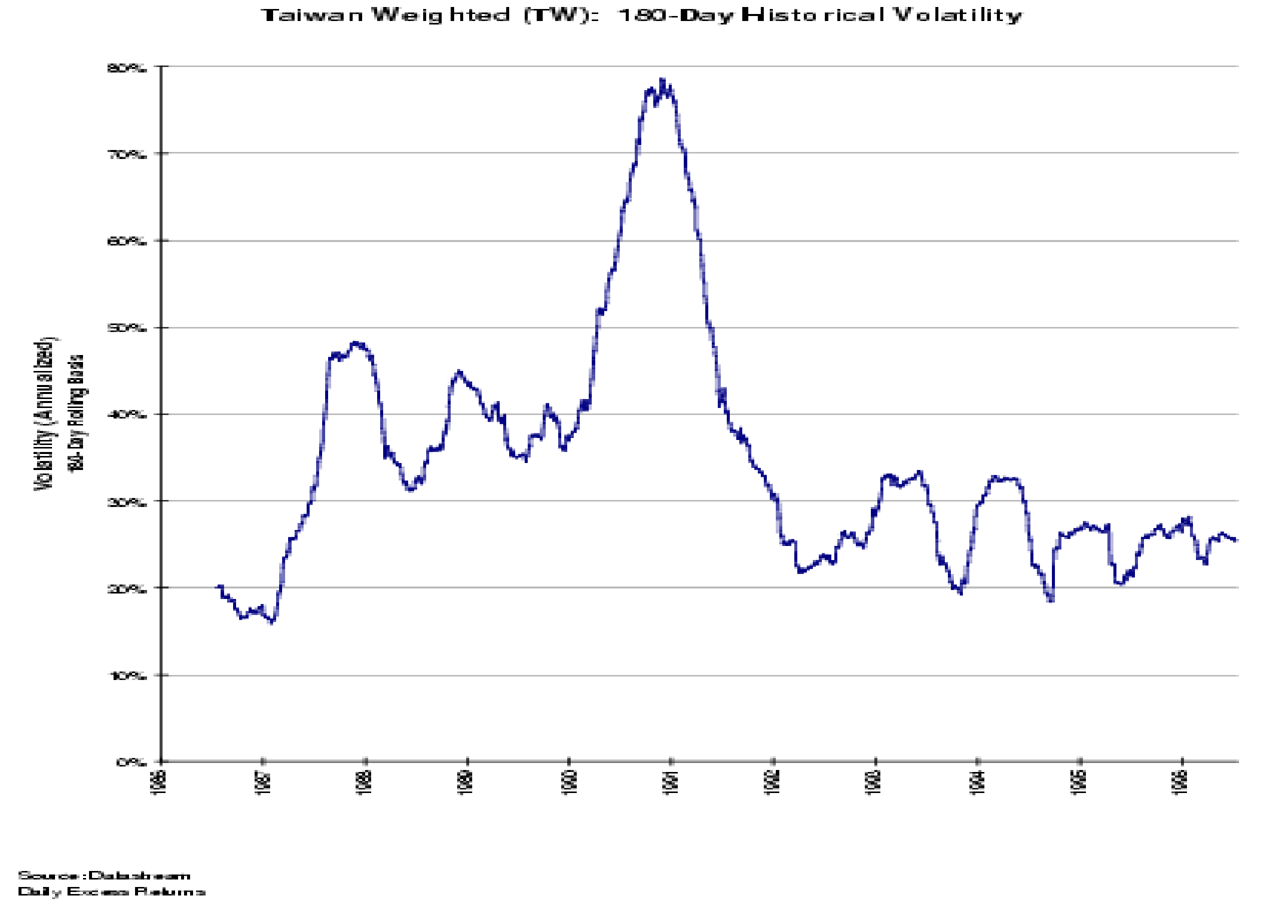


Figure 15: Rolling volatility for Taiwan equity index.

VaR calculations for derivatives

The delta approximation:

$$f(y + x) = f(y) + f'(y)x + o(x).$$

The delta-gamma approximation:

$$f(y + x) = f(y) + f'(y)x + \frac{1}{2}f''(y)x^2 + o(x^2).$$

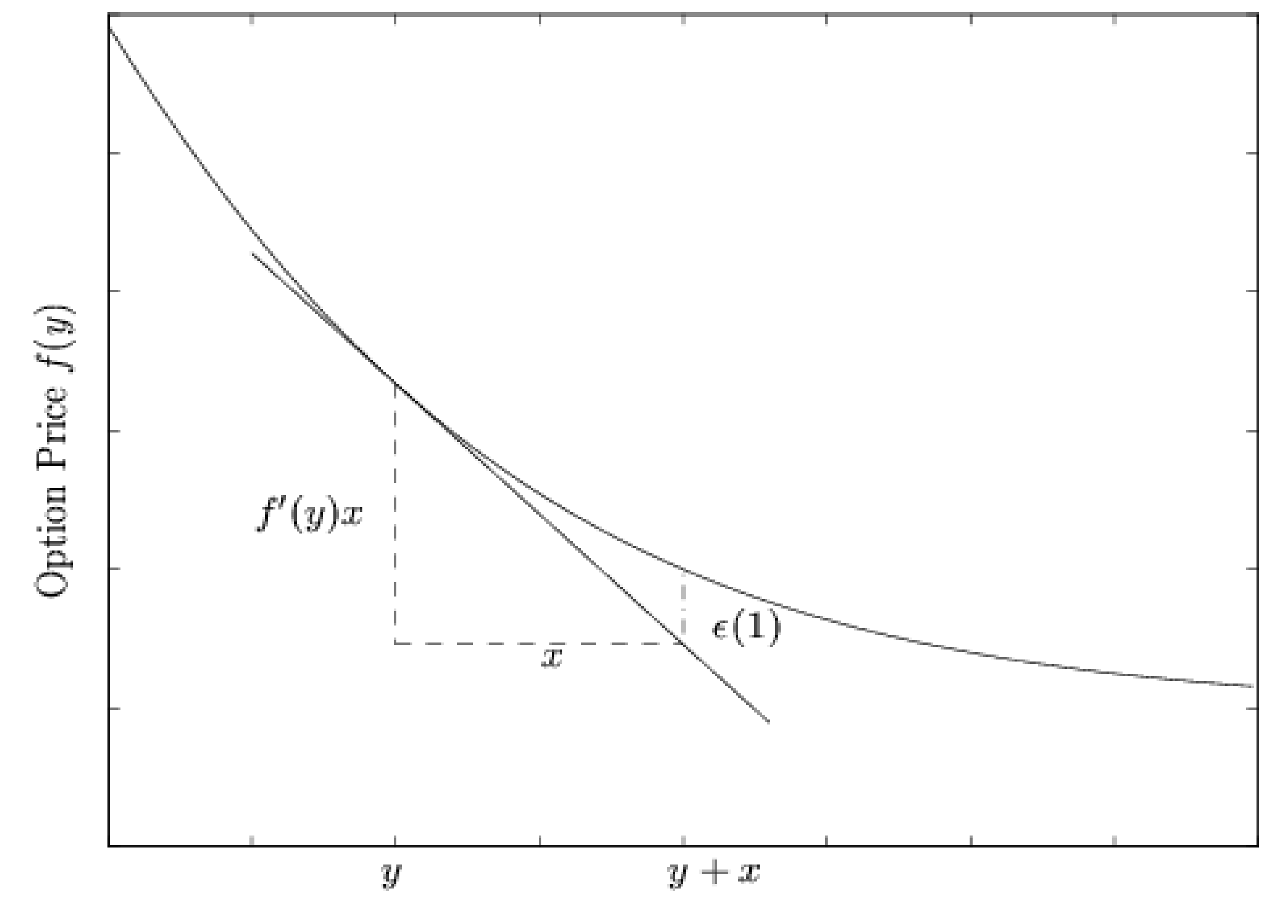


Figure 16: The delta (first-order) approximation

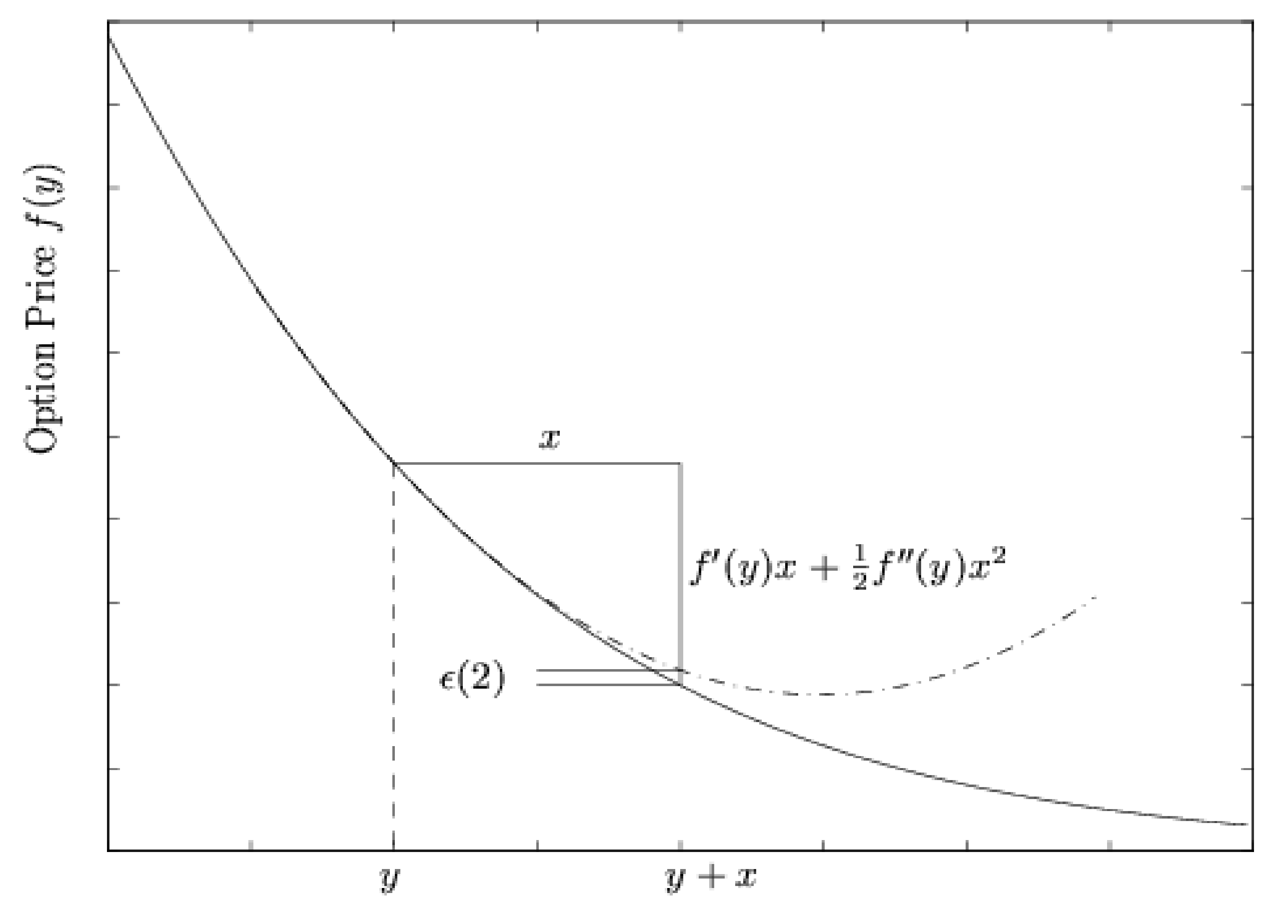


Figure 17: Delta-gamma hedging, second order approximation

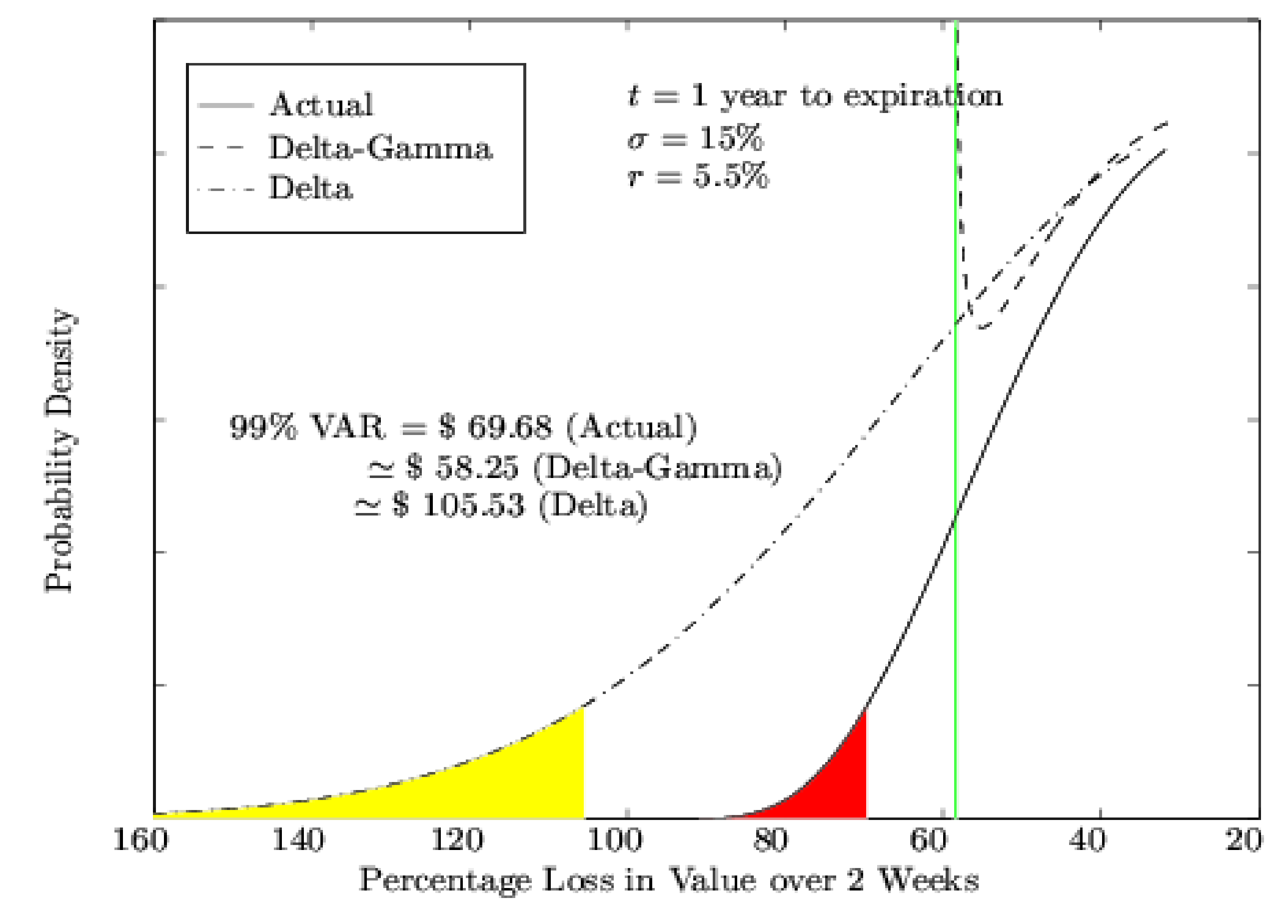


Figure 18: 2-Week loss on **20**% out-of-money put (plain-vanilla returns)

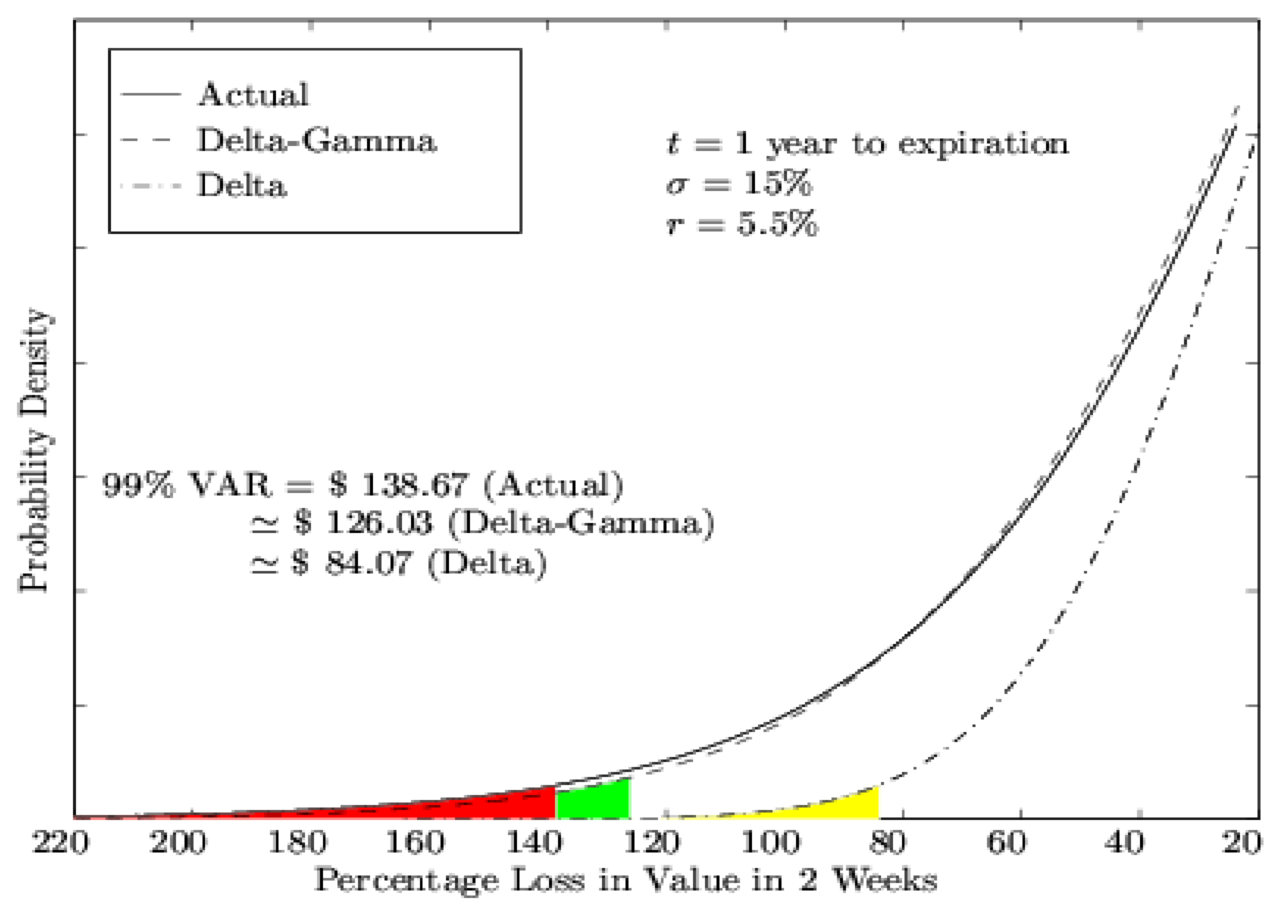


Figure 19: 2-Week loss on short **20**% out-of-money put (plain-vanilla returns)

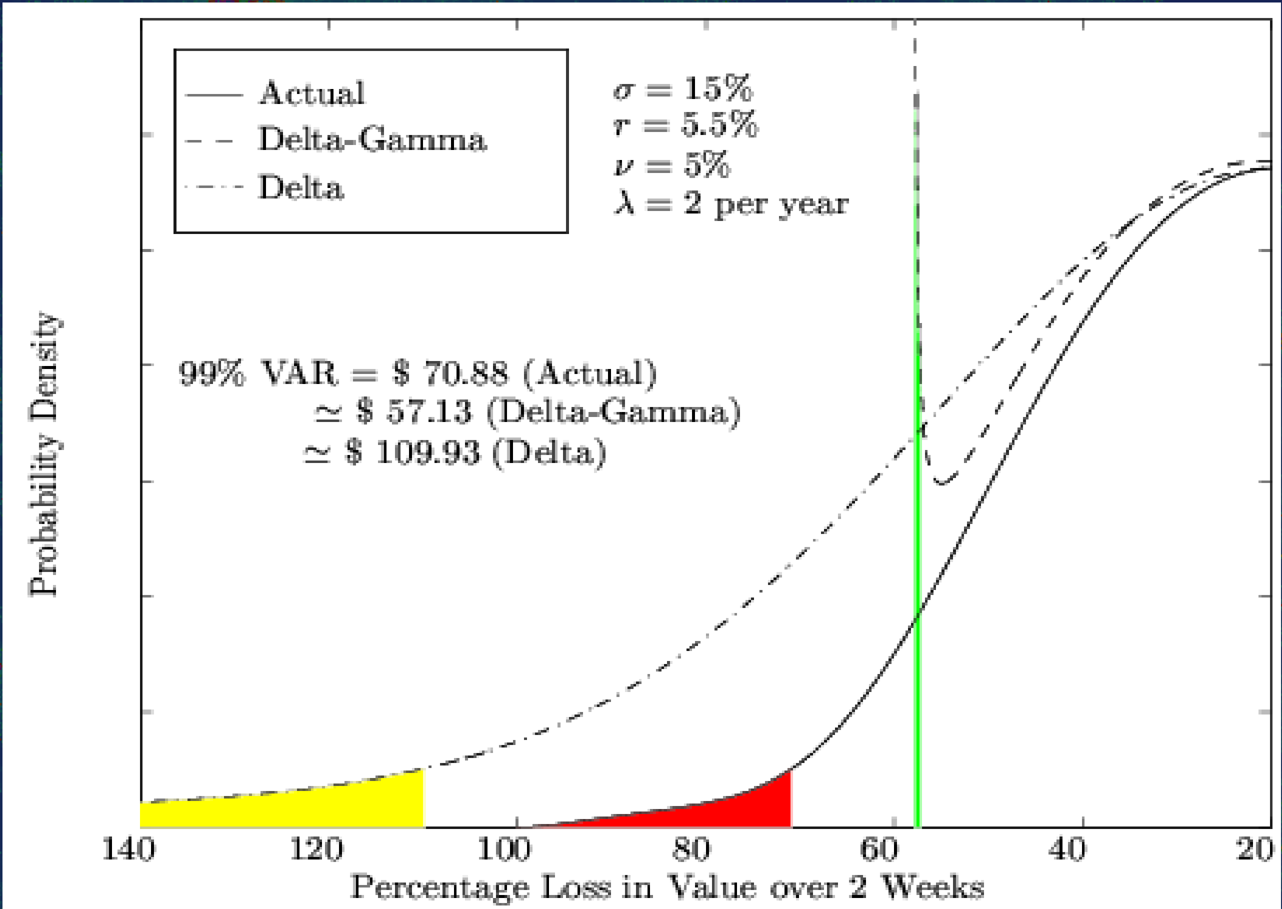


Figure 20: 2-Week loss on short **20%** out-of-money put (jump-diffusion)

Portfolio VaR

Let X_i denotes the difference between the i-th risk factor and its expected value. The total change in value for the entire book has the delta-gamma approximation:

$$Y(\Delta, \Gamma) = \sum_{j=1}^{n} \Delta_j X_j + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \Gamma_{jk} X_j X_k.$$

The portfolio variance is equal to

$$\begin{aligned} \text{Var}(Y(\Delta, \Gamma)) = \sum_{j,k} \Delta_j \Delta_k \,\text{Cov}(X_j, X_k) + \sum_{i,j,k} \Delta_j \Gamma_{jk} \,\text{Cov}(X_i, X_j X_k) \\ + \frac{1}{4} \sum_{i,j,k,l} \Gamma_{ij} \Gamma_{kl} \,\text{Cov}(X_i X_j, X_k X_l). \end{aligned}$$

Back to methods

Simulating fat tailed distributions

Suppose one wants to simulate a random variable X of zero mean and unit variance with a given kurtosis. Let η be the Bernoully random variable:

$$\mathsf{P}\{\eta = 1\} = p, \qquad \mathsf{P}\{\eta = 0\} = 1 - p.$$

Let Z be the standard normal random variable independent on η. Define X as

$$X = \begin{cases} \alpha Z, & \eta = 1, \\ \beta Z, & \eta = 0. \end{cases}$$

Then we have $\mathsf{Var}(X) = p\alpha^2 + (1-p)\beta^2$, $\mathsf{E}(X^4) = 3(p\alpha^4 + (1-p)\beta^4)$. Now we can choose α, β and p.

How many scenarios is enough?

Let $\xi_1, \xi_2, \ldots,$ be an independently and identically distributed sequence of random variables with $\mathbf{E}(\xi_i) = \mu$. Let

$$\hat{\mu}(k) = \frac{\xi_1 + \cdots + \xi_k}{k}.$$

Let $g(\theta)$ denotes the moment-generating function of ξ_i, that is

$$g(\theta) = \mathbf{E}[\exp(\theta\xi_i)].$$

According to the *large deviations theorem*, under mild regularity conditions

$$\mathbf{P}\{\hat{\mu}(k) \geq \delta\} \leq e^{-k\gamma(\theta)},$$

where $\gamma(\theta) = \delta\theta - \log[g(\theta)]$.

In our application we let

$$\xi_i = \begin{cases} 1, & X_i > -\mathrm{VAR}, \\ 0, & X_i \le -\mathrm{VAR}. \end{cases}$$

Let

$$\hat{p}(k) = \frac{\xi_1 + \cdots + \xi_k}{k}$$

be the estimate of p. Maximising $\gamma(\theta)$ with respect to θ, we have

$$\mathsf{P}\{\hat{p}(k) \ge \delta\} \le \exp(-k\Gamma),$$

where

$$\Gamma = \delta \log \delta + (1-\delta)\log(1-\delta) - \delta \log p - (1-\delta)\log(1-p).$$

For example, let $p = 0.95$ and $\delta = 0.975$. Then $\Gamma = 0.008$. For a confidence of $c = 0.99$ we see that

$$k = -\frac{1}{\Gamma}\log(c) = 576.$$

Bootstrapped simulation from historical data

In a stationary statistical environment — no problems.

In the case of significant non-stationarity, we update the historical asset distribution. For example,

$$\hat{R}_i = R_i \frac{\hat{V}}{V},$$

where V is the historical volatility and $\hat{V}$ is a recent volatility estimate, or

$$\hat{R}_t = \hat{C}^{1/2} C^{-1/2} R_t,$$

where R_t denotes the vector of historical returns, C denotes the historical covariance matrix for returns across a group of assets under consideration, $\hat{C}$ denotes the updated estimate.