Appendices

A. Simulating Fat Tailed Distributions

Suppose one wants to simulate a random variable of zero mean and unit variance, but with a given degree of tail fatness (fourth moment). Sticking to the more-or-less "bell-curved" shapes for the probability density of returns (and ignoring skewness), one could adopt the following approach,[65] based on the idea that a random variable has fat tails if it can be expressed as a random mixture of normal random variables of different variances.

First draw a random variable Y whose outcomes are 1 and 0, with respective probabilities p and $1 - p$. Independently, draw a standard normal random variable Z. Let α and β be the standard deviations of the two normals to be "mixed." If the outcome of Y is 1, let $X = \alpha Z$. If the outcome of Y is 0, let $X = \beta Z$. We want to choose α and β so that the variance of X is 1. We have $\mathrm{var}(X) = p\alpha^2 + (1-p)\beta^2 = 1$, so that

$$\beta = \sqrt{\frac{1 - p\alpha^2}{1 - p}}.$$

Now we can choose p and α to achieve a given kurtosis or 0.99 critical value. The kurtosis of X is $E(X^4) = 3(p\alpha^4 + (1-p)\beta^4)$.

B. How Many Scenarios is Enough?

This appendix shows how to compute an answer to the following sort of question, which can be used to decide how many scenarios is "sufficient" for measuring the percentile measure for the loss on a given portfolio of positions.

"Suppose an event occurs with probability p. With k independent trials, what is the likelihood $\pi(k)$ that we estimate p to be δ or larger?"

For the case of risk management, the event of concern is whether losses are no greater than some critical level. The danger to be avoided is over-estimation of the

[65] This approach was suggested by Robert Litterman of Goldman Sachs at a meeting in March 1996 of the Financial Research Initiative at Stanford University.

probability of this event, for it would leave a firm's risk manager with undue confidence regarding the firm's risk. For example, The key "error likelihood" $\pi(k)$ depends of course on the number k of scenarios simulated. As k goes to infinity, the law of large numbers implies that $\pi(k)$ goes to zero. For example, we can show the following. If $p = 0.95$ and $\delta = 0.975$, then regardless of the distribution of the underlying market values of positions in the book, $\pi(k) \leq e^{-0.008k}$. With 1000 scenarios, for instance, the error probability is less than 3.5 parts per 10,000.

We obtain this upper bound on $\pi(k)$ along with the following general result, which allows us to derive error probabilities for other cases of estimated percentile and assumed true percentiles than $p = 0.95$ and $\delta = 0.975$. For example, with $p = 0.95$ and $\delta = 0.975$, the number of scenarios necessary to keep the error probability below 0.01 is approximately 4600.

In order to state the general result, we suppose that $Y_1, Y_2, \ldots,$ is an independently and identically distributed (*i.i.d.*) sequence of random variables, with $E(Y_i) = \mu$. We know that $\widehat{\mu}(k) = (Y_1 + \cdots + Y_k)/k \to \mu$ almost surely, but at what rate? We let g denote the moment-generating function of Y_i, that is,

$$g(\theta) = E[\exp(\theta Y_i)].$$

Large Deviations Theorem. Under mild regularity,[66]

$$P\left[\widehat{\mu}(k) \geq \delta\right] \leq e^{-k\gamma(\theta)},$$

where $\gamma(\theta) = \delta\theta - \log[g(\theta)]$.

We can get an optimal upper bound of this form by maximizing $\gamma(\theta)$ with respect to θ. Under purely technical[67] conditions, the solution θ^* provides an upper bound $\exp[-k\gamma(\theta^*)]$ that, asymptotically with k, cannot be improved.

In our application, we suppose that X is the random variable whose percentiles are of interest. We let Y_i be a "binomial random variable" (that is, a "Bernoulli trial"), with an outcome of 1 if the i-th simulated outcome of X is above the cutoff percentile level, and zero otherwise. The probability that $Y_i = 1$ is some number p, the true quantile score for this cutoff, which is 0.95 in the above example. We let $\widehat{p}(k) = (Y_1 + \cdots + Y_k)/k$, be the estimate of p. We are checking to see how likely it is that our estimate is larger than δ, which is 0.975 in the above example.

[66] For details, see Durrett [1991].

[67] Again, see Durrett [1991].

Optimizing on θ, we have

$$P(\widehat{p}(k) \geq \delta) \sim \exp(-k\Gamma),$$

where

$$\Gamma = \delta \log \delta + (1-\delta)\log(1-\delta) - \delta \log p - (1-\delta)\log(1-p).$$

With $p = 0.95$ and $\delta = 0.975$, $\Gamma = 0.008$. We can now solve the equation $\exp(-\Gamma \times k) = c$ for k. For a confidence of $c = 0.99$, we see that

$$k = -\frac{1}{\Gamma}\log(c) = 576 \text{ simulations.}$$

That is, the probability that $\widehat{p}(k) \geq 0.975$ is roughly $\exp(-576 \times 0.008) = 0.01$. For $\delta = 0.96$, in order to achieve 99-percent confidence, $k = 4600$ simulations are suggested.

C. Multi-Factor Scenarios

Continuing the discussion of scenario analysis begun in Section 5, we could consider the expected change in the Canadian Government forward curve conditioned on a given move in the U.S. forward curve. For illustration, we could suppose that the risk factors associated with the U.S. forward rate curve are unexpected movements of $X_1, \ldots, X_m$ basis points at each of k respective maturities, and that the scenario outcome for the forward curve shift is the vector $x = (x_1, \ldots, x_m)$ of basis point changes at the respective maturities. For example, x could be the forward curve shift vector associated with the first principle component of U.S. forward curve changes.

For some other given risk factor X_k, say the unexpected change in the Canadian 5-year forward rate, we are interested in computing the expected change in X_k given the outcome $X_1 = x_1$, $X_2 = x_2$, $\ldots$, $X_m = x_m$. Assuming joint normality of the rates, we have

$$E(X_k \,|X_1, X_2, X_3, \ldots, X_m) = (X_1, \ldots, X_m)^{\top} A^{-1} q,$$

where $A = \text{cov}(X_1, \ldots, X_m)$ is the $k \times k$ covariance matrix of $X_1, \ldots, X_k$, and q is the vector whose i-th element is $\text{cov}(X_i, X_m)$. Evaluating this conditional expectation at the scenario shift $(X_1, \ldots, X_m) = (x_1, \ldots, x_m)$, we have the desired result. One can do this for each Canadian rate to get the expected response of the Canadian forward curve. An approximation of the re-valuation of Canadian fixed-income products at this shift can be done on a delta basis. (For straight bonds, this is an easy calculation.)

Of course, our focus in this example on Canadian and U.S. forward rates is simply for illustration. We could have used equity returns, foreign exchange rates, or other risk factors. The only necessary information is the covariance matrix for the risk factors, and the scenario of concern.

D. Tail-Fatness of Jump-Diffusion Models

The calculations for tail-fatness of the jump-diffusion model considered in Section 2 are shown below for reference only.

D.1 Critical Values

We consider the return on the asset that undergoes a jump diffusion

$$\begin{aligned} S_t &= S_0 \exp(\alpha t + X_t) \\ X_t &= \beta B_t + \sum_{k=0}^{N(t)} \nu Z_k, \end{aligned} \tag{1}$$

where B is a standard Brownian motion and $N(t)$ is the number of jumps that occur by time t. Each jump νZ_k is normally distributed with mean zero and standard deviation ν. The arrival rate of jumps (Poisson) is λ. Then the total volatility σ is defined by

$$\sigma^2 = \beta^2 + \lambda \nu^2. \tag{2}$$

We are interested in the critical value, at confidence p and time horizon t, that is, $C_{p,t} = \{x : P(X_t \leq x) = p\}$. Using the law of iterated expectations and conditioning on the number of jumps,

$$\begin{aligned} P(X_t \leq x) &= \sum_{k=0}^{\infty} p(k,t) P(X_t \leq x | N(t) = k) \\ &= \sum_{k=0}^{\infty} p(k,t) N(0, \sqrt{\beta^2 t + k\nu^2}, x), \end{aligned} \tag{3}$$

where $p(k,t)$ is the Poisson probability of k arrivals within t units of time, given by

$$p(k,t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \tag{4}$$

and $N(a,s,x)$ is the probability that a random variable that is distributed normally with mean 0 and standard deviation s has an outcome less than x.

D.2 Kurtosis

The normalized kurtosis is defined by

$$K_t = \frac{E(X_t^4)}{[Var(X_t)]^2} = \frac{E(X_t^4)}{\sigma^4 t^2}. \tag{5}$$

After tedious calculation, the numerator can be shown to be

$$\begin{aligned} E(X_t^4) &= E[(\beta B_t + \sum_{k=0}^{N(t)} Z_k)^4] \\ &= E[3(\beta^2 t + N(t)\nu^2)^2] = 3[\beta^4 t^2 + (\lambda + \lambda^2 t)\nu^4 t + 2\nu^2\beta^2\lambda t^2]. \end{aligned} \tag{6}$$

E. Option Portfolio Value-at-Risk

Table 3 is a summary of the moments of the simulated distribution of the "short" option portfolio described in Section 4.5, based on different models for the underlying. Specifically, in Jump-diffusion Model 1, the diffusion part is simulated using the RiskMetrics covariance matrix for July 29, 1996, while each asset jumps independently with poisson arrival of intensity $\lambda = 1$ per year. The standard deviation of jumps size for each asset is taken to be half of the corresponding RiskMetrics standard deviation. So, in this example, the total covariance matrix is not matched to that of RiskMetrics. Jump-diffusion Models 2 and 3, however, are parameterized in such a way that the total covariance matrix of the underlying assets is matched to that of RiskMetrics. Model 2 has one-fifth of its total covariance coming from jumps, and four-fifths from diffusion, while in Model 3 half of the total covariance comes from jumps, and half from diffusion. The moments for the long portfolio are of exactly the same magnitude, except for sign reversals for odd moments (mean and skewness).

Table 4 shows the estimated 1% and 0.4% values at risk (critical values) of the "predominantly short" option portfolio, designated "Portfolio S," over time horizons of one day and two weeks. The portfolio is normalized to an initial market value of -100 dollars. The "predominantly long" option portfolio, designated "Portfolio L," has a total market value of \$100. Of course, the left tail of the distribution of value changes for Portfolio L can be obtained from the right tail of Portfolio S. Table 5 shows the estimated 1% and 0.4% critical values for Portfolio L over one day or two weeks, estimated by the various approximation methods described in Section 4.

Table 3: Moments of the Simulated Distribution

Time Span	Model	Method	mean	s-dev	skewness	kurtosis
Overnight	pl. vanilla	Actual	-100.02	0.97	-0.15	3.06
		Gamma	-100.02	0.98	-0.15	3.07
		Delta	-99.88	0.97	-0.03	3.04
	jump-duffsion (Model 1)	Actual	-100.02	1.02	-0.15	3.72
		Gamma	-100.03	1.04	-0.25	4.73
		Delta	-99.86	1.02	-0.07	4.47
	jump-duffsion (Model 2)	Actual	-100.00	1.10	-8.65	233.85
		Gamma	-100.01	1.17	-11.44	343.38
		Delta	-99.87	0.98	-1.94	59.17
	jump-duffsion (Model 3)	Actual	-100.01	1.18	-7.43	103.24
		Gamma	-100.02	1.26	-8.38	122.42
		Delta	-99.87	0.99	1.02	76.31
2 Weeks	pl. vanilla	Actual	-100.22	3.76	-0.48	3.40
		Gamma	-100.28	3.90	-0.51	3.48
		Delta	-98.18	3.73	-0.06	3.03
	jump-duffsion (Model 1)	Actual	-100.23	3.96	-0.42	3.45
		Gamma	-100.39	4.21	-0.52	3.89
		Delta	-97.90	3.91	-0.04	3.17
	jump-duffsion (Model 2)	Actual	-100.26	4.37	-2.35	18.01
		Gamma	-100.38	4.87	-3.47	34.89
		Delta	-98.22	3.81	-0.10	6.29
	jump-duffsion (Model 3)	Actual	-100.27	4.69	-2.65	16.07
		Gamma	-100.42	5.25	-3.20	21.74
		Delta	-98.26	3.79	-0.17	7.99

F. Sample Statistics for Daily Returns

For reference purposes, we record in the table below some sample statistics for daily returns for the period 1986 to 1996 for a selection of equity indices, foreign currencies, and commodities. The statistics for foreign equity returns are in local currency terms. The raw price data were obtained from Datastream.

Table 4: Critical Values of the "Short Option" Portfolio

Model	Method		Overnight 1%	Overnight 0.4%	2 Weeks 1%	2 Weeks 0.4%
pl. vanilla	Analytical	Delta	2.27	3.27	8.70	12.54
		Gamma	2.28	3.28	9.12	13.15
	Simulation	Actual	2.41	2.80	10.48	12.18
		Gamma	2.42	2.81	10.91	12.77
		Delta	2.18	2.51	7.10	8.42
jump-diffusion (Model 1)	Analytical	Delta	2.19	3.16	8.42	12.14
		Gamma	2.20	3.17	8.81	12.70
	Simulation	Actual	2.51	3.05	10.58	12.44
		Gamma	2.60	3.10	11.63	14.03
		Delta	2.25	2.70	6.97	8.54
jump-diffusion (Model 2)	Analytical	Delta	2.28	3.29	8.74	12.59
		Gamma	2.29	3.30	9.16	13.20
	Simulation	Actual	2.15	2.53	15.57	23.14
		Gamma	2.15	2.54	17.80	26.81
		Delta	1.93	2.24	7.51	11.30
jump-diffusion (Model 3)	Analytical	Delta	2.29	3.30	8.77	12.64
		Gamma	2.30	3.31	9.19	13.25
	Simulation	Actual	2.05	8.83	18.89	25.01
		Gamma	2.05	9.54	21.85	28.57
		Delta	1.63	2.60	9.68	13.55

Shown are the annualized sample standard deviation (volatility), the sample skewness, sample normalized kurtosis, the number of days on which the return was more than 10 sample standard deviations below the mean, the number of days on which the return was more than 5 sample standard deviations below the mean, the number of days on which the return was more than 5 sample standard deviations above the mean, the number of days on which the return was more than 10 sample standard deviations above the mean, the number of standard deviations to the 0.4 percent critical value of the sample distribution, and the number of standard deviations to the 99.6 percent

Table 5: Critical Values of the "Long Option" Portfolio

Model	Method		Overnight 1%	Overnight 0.4%	2 Weeks 1%	2 Weeks 0.4%
pl. vanilla	Analytical	Delta	2.27	3.27	8.70	12.54
		Gamma	2.28	3.28	9.12	13.15
	Simulation	Actual	2.18	2.45	7.34	8.17
		Gamma	2.19	2.45	7.53	8.39
		Delta	2.41	2.64	10.49	11.49
jump-diffusion (Model 1)	Analytical	Delta	2.19	3.16	8.42	12.14
		Gamma	2.20	3.17	8.81	12.70
	Simulation	Actual	2.29	2.61	7.89	8.86
		Gamma	2.30	2.70	8.17	9.29
		Delta	2.56	2.93	11.18	12.60
jump-diffusion (Model 2)	Analytical	Delta	2.28	3.29	8.74	12.59
		Gamma	2.29	3.30	9.16	13.20
	Simulation	Actual	1.99	2.25	7.16	7.92
		Gamma	1.99	2.26	7.37	8.18
		Delta	2.20	2.51	11.02	14.11
jump-diffusion (Model 3)	Analytical	Delta	2.29	3.30	8.77	12.64
		Gamma	2.30	3.31	9.19	13.25
	Simulation	Actual	1.64	1.92	6.50	7.52
		Gamma	1.65	1.93	6.70	7.81
		Delta	1.87	2.37	12.52	16.20

critical value of the sample distribution.

Table 6: Sample Return Statistics for Selected Markets

Daily Return Statistics 1986-1996

Name	Vol. (Annual)	Skew	Kurtosis	< - 10 sd	< - 5 sd	> 5 sd	> 10 sd	0.4%	99.6%
S&P 500	15.9%	-4.8	110.7	1	5	3	0	-3.64	2.76
NASDAQ	15.2%	-5.1	109.7	2	2	2	0	-3.57	2.62
NYSE All Share	14.7%	-5.2	121.4	1	5	2	0	-3.54	2.68
Mexico Bolsa (Pesos)	26.3%	-0.2	7.8	0	3	4	0	-3.59	3.75
Mexico Bolsa (US$)	32.0%	0.0	14.9	0	6	4	0	-3.98	3.44
FTSE 100	15.0%	-1.7	28.6	2	4	3	0	-3.37	2.94
FTSE All Share	13.6%	-1.9	29.1	2	5	4	0	-3.94	2.94
German DAX 30 Perf.	19.7%	-0.9	15.6	1	7	8	0	-4.25	3.37
France DS Mkt.	17.5%	-0.9	13.1	0	7	4	0	-3.72	3.14
France CAC 40	19.4%	-0.6	10.9	0	6	4	0	-3.37	3.17
Sweden Veckans Aff.	18.4%	-0.4	12.1	0	8	9	0	-4.76	3.96
Milan B.C.I.	20.3%	-0.9	13.3	1	5	1	0	-3.58	3.25
Swiss Perf.	15.6%	-2.6	32.0	3	8	3	0	-4.65	3.71
Australia All Ord.	17.1%	-7.8	198.3	1	7	3	0	-3.54	2.34
Nikkei 500	20.4%	-0.4	34.8	2	4	6	1	-3.70	3.50
Hang Seng	26.7%	-6.4	143.6	2	6	1	0	-4.08	2.93
Bangkok S.E.T.	25.0%	-0.6	9.8	0	9	5	0	-4.86	3.59
Taiwan Weighted	36.1%	-0.1	5.0	0	0	1	0	-3.08	3.20
US$: English Pound	11.0%	-0.2	5.9	0	1	1	0	-3.60	3.20
US$: Mexican New Peso	18.9%	-8.1	217.5	4	12	8	1	-5.15	3.20
US$: German Mark (DM)	11.3%	-0.1	5.3	0	0	1	0	-3.40	3.40
US$: French Franc (FF)	10.9%	0.0	5.7	0	0	2	0	-3.50	3.45
US$: Swedish Krone (SK)	10.5%	-0.6	10.9	1	3	4	0	-3.50	3.30
US$: Italian Lira	11.3%	-0.6	8.6	0	4	0	0	-3.60	3.30
US$: Swiss Franc (SF)	12.6%	0.0	4.9	0	0	1	0	-3.20	3.30
US$: Australian Dollar	9.4%	-0.7	8.1	0	4	1	0	-4.30	2.75
US$: Japanese Yen	11.1%	0.4	8.1	0	3	5	0	-3.10	3.68
US$: Hong Kong Dollar	0.8%	-0.6	17.4	0	11	10	0	-5.40	4.30
US$: Thai Baht	2.4%	0.7	33.8	1	9	13	2	-4.80	4.95
US$: Canadian Dollar	4.4%	-0.1	7.2	0	3	6	0	-3.80	3.25
Gold (First Nearby	13.4%	-0.7	11.4	0	4	3	0	-4.24	3.44
Oil (First Nearby)	38.7%	-0.8	21.8	1	7	7	0	-4.14	4.07
Oil (Sixth Nearby)	27.5%	-0.6	15.6	1	3	3	0	-3.82	3.82