Relation between Swaption BPV and Delta

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1 Introduction

This short note describes the relation between BPV and delta for the interest rate derivative European swaption. The notation is taken from Björk (2004).

2 Swap

The value of a payer swap is

$$\mathbf{PS_n^N}(t;K) = p_n(t) - p_N(t) - K \sum_{i=n+1}^{N} \alpha_i p_i(t) \tag{1}$$

The forward swap rate $R_n^N(t)$ of the $T_n \times (T_N - T_n)$ swap is the value of K for which $\mathbf{PS_n^N}(t;K) = 0$, i.e.

$$R_n^N(t) = \frac{p_n(t) - p_N(t)}{\sum_{i=n+1}^{N} \alpha_i p_i(t)} \tag{2}$$

where $\alpha_i = T_i - T_{i-1}$. Also, define $S_n^k(t)$ as

$$S_n^k(t) = \sum_{i=n+1}^{k} \alpha_i p(t, T_i) \tag{3}$$

3 Swaption

A swaption is a option to enter a swap. A European swaption with expiration date T_n is an option to enter into a $T_n \times (T_n - T_n)$ swap at time T_n. Thus the pay-off from a swaption is

$$X_n^N = \max\left[\mathbf{PS_n^N}(T_n;K), 0\right] = \max\left[R_n^N(T_n) - K, 0\right] S_n^N(t) \tag{4}$$

where we have used (1).

Thus, when using $S_n^N(t)$ as our numeraire, the swaption is simply a option on R_n^N with strike K. Assuming that $\log R_n^N$ is normally distributed under the pricing measure we arrive at Black's formula. Black's formulae for pricing a $T_N \times (T_N - T_n)$ European payer swaption with strike K is given by

$$\mathbf{PSN}_n^N(t) = S_n^N(t)\left[R_n^N(t)N(d_1) - KN(d_2)\right] \tag{5}$$

where

$$= \frac{1}{\sigma_{n,N}\sqrt{T_n - t}}\left[\ln\left(\frac{R_n^N(t)}{K}\right) + \frac{1}{2}\sigma_{n,N}^2(T_n - t)\right] \tag{6}$$

$$d_2 = d_1 - \sigma_{n,N}\sqrt{T_n - t} \tag{7}$$

3.1 Relation between delta and BPV

In this section we describe the relation between two commonly used key figures - the option delta and the option BPV(Basis point Value), which is the option sensitivity to parallel shifts in the entire yield curve. Below sensitivity w.r.t. the yield curve is denoted by $\partial/\partial r$. We start by differentiate (5) with respect to the entire yield curve, i.e.

$$\begin{aligned}
\frac{\partial \mathbf{PSN_n^N}(t)}{\partial r} &= \frac{\partial S_n^N(t)}{\partial r}\left[R_n^N(t)N(d_1) - KN(d_2)\right] + S_n^N(t)\frac{\partial}{\partial r}\left[R_n^N(t)N(d_1) - KN(d_2)\right] \\
&= \frac{\partial S_n^N(t)}{\partial r}\left[R_n^N(t)N(d_1) - KN(d_2)\right] + S_n^N(t)\frac{\partial}{\partial R_n^N(t)}\frac{\partial R_n^N(t)}{\partial r}\left[R_n^N(t)N(d_1) - KN(d_2)\right] \\
&= \frac{\partial S_n^N(t)}{\partial r}\left[R_n^N(t)N(d_1) - KN(d_2)\right] + \frac{\partial R_n^N(t)}{\partial r}\frac{\partial}{\partial R_n^N(t)}S_n^N(t)\left[R_n^N(t)N(d_1) - KN(d_2)\right] \\
&= \frac{\partial S_n^N(t)}{\partial r}\left[R_n^N(t)N(d_1) - KN(d_2)\right] + \frac{\partial R_n^N(t)}{\partial r}\underbrace{\frac{\partial}{\partial R_n^N(t)}\mathbf{PSN_n^N}(t)}_{\Delta} \\
&= \frac{\partial S_n^N(t)}{\partial r}\left[R_n^N(t)N(d_1) - KN(d_2)\right] + \frac{\partial R_n^N(t)}{\partial r}\Delta \\
&= \frac{\partial S_n^N(t)}{\partial r}\left[\frac{\mathbf{PS_n^N}(t)}{S_n^N(t)}\right]
\end{aligned}$$

which means that

$$\Delta = \frac{\mathrm{BPV} - \frac{\partial}{\partial r}S_n^N(t)\left[\frac{\mathbf{PSN_n^N}(t)}{S_n^N(t)}\right]}{\partial R_n^N(t)/\partial r} \tag{8}$$

Litteratur

Björk, T. (2004). *Arbitrage Theory in Continuous Time* (1 ed.). Oxford University Press.