Swaption skews and convexity adjustments

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Abstract

We test both the SABR model [4] and the shifted-lognormal mixture model [2] as far as the joint calibration to swaption smiles and CMS swap spreads is concerned. Such a joint calibration is essential to consistently recover implied volatilities for non-quoted strikes and CMS adjustments for any expiry-tenor pair.

1 Introduction

Derivatives with payoffs depending on one or several swap rates have become increasingly popular in the interest rate market. Typical examples are the CMS spreads and CMS spread options, which pay the difference between long and short maturities swap rates (floored at zero in the option case).

To be correctly priced, such derivatives need a model that incorporate as much information as possible on both swaption volatilities and CMS swaps that are quoted by the market. In fact, it is only the joint calibration to swaption smiles and CMS swap spreads

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that allows one to consistently recover the distribution of swap rates (under the associated swap measure) together with the related CMS convexity adjustments.

To quantify a CMS convexity adjustment one typically employs a well-known market formula, which is derived from efficient approximations and the use of Black's [1] model with an at-the-money implied volatility, see Hagan [4]. The presence, however, of away-from-the-money quotes, at least for the most liquid maturities and tenors, renders necessary a correction of the classical adjustment, which has to account for the information contained in the quoted smile. Such a correction comes from static replication arguments and is based on the calculation of the integral of payer swaption prices over strikes from zero to infinity. Therefore, if swaption quotes were available for every possible strike, a CMS convexity adjustment would be model independent and fully determined by the market swaption smile.

However, swaption implied volatilities are only quoted by the market up to some maximum strike, so that volatility modelling is required to infer consistent CMS adjustments, along with a robust calibration procedure that includes market quotes for CMS adjustments in the given data set.

In this article, we show some examples of calibration of the SABR model to both swaption volatilities and CMS adjustments. We also note that typical redundancy issues related to the SABR parameters can be removed whenever market quotes of CMS adjustments are considered. Results are then compared with those obtained in case of the uncertain-parameter model of [3] and [2]. We conclude with two appendices, where we detail the calculations that lead to the adjustment formula we use in this paper. In particular, we show that one can use two different replication arguments, which turn out to be equivalent as far as CMS convexity adjustments are concerned.

2   The classical convexity adjustment

Let us fix a maturity T_a and a set of times $\mathcal{T}_{a,b} := \{T_{a+1}, \ldots, T_b\}$, with associated year fractions all equal to $\tau > 0$. The forward swap rate at time t for payments in $\mathcal{T}$ is defined by

$$S_{a,b}(t) = \frac{P(t, T_a) - P(t, T_b)}{\tau \sum_{j=a+1}^{b} P(t, T_j)},$$

where $P(t, T)$ denotes the time-t discount factor for maturity T.

Denoting respectively by $Q^{T_a+\delta}$ and $Q^{a,b}$ the $(T_a + \delta)$-forward measure and the forward swap measure associated to $S_{a,b}$, and by $E^{T_a+\delta}$ and $E^{a,b}$ the related expectations, the

convexity adjustment for the swap rate $S_{a,b}(T_a)$ can be approximated, as in [4] or [6], by

$$\mathbf{CA}(S_{a,b}; \delta) := E^{T_a+\delta}(S_{a,b}(T_a)) - S_{a,b}(0) \approx S_{a,b}(0)\, \theta(\delta) \left(\frac{E^{a,b}\big(S_{a,b}^2(T_a)\big)}{S_{a,b}^2(0)} - 1 \right), \quad (1)$$

where

$$\theta(\delta) := 1 - \frac{\tau S_{a,b}(0)}{1 + \tau S_{a,b}(0)} \left(\frac{\delta}{\tau} + \frac{b-a}{(1 + \tau S_{a,b}(0))^{b-a} - 1} \right)$$

and δ is the accrual period of the swap rate[1]. Expression (1) depends on the distributional assumption on $S_{a,b}$. For instance, the classical Black-like dynamics for the swap rate under $Q^{a,b}$, see [1],

$$dS_{a,b}(t) = \sigma_{a,b}^{\text{ATM}} S_{a,b}(t)\, dZ^{a,b}(t), \quad (2)$$

where $\sigma_{a,b}^{\text{ATM}}$ is the at-the-money implied volatility for $S_{a,b}$ and $Z^{a,b}$ is a $Q^{a,b}$-standard Brownian motion, implies that

$$E^{a,b}\big(S_{a,b}^2(T_a)\big) = S_{a,b}^2(0)\, e^{(\sigma_{a,b}^{\text{ATM}})^2\, T_a},$$

which leads to the classical convexity adjustment

$$\mathbf{CA}^{\text{Black}}(S_{a,b}; \delta) \approx S_{a,b}(0)\, \theta(\delta) \left(e^{(\sigma_{a,b}^{\text{ATM}})^2\, T_a} - 1 \right). \quad (3)$$

This is a "flat-smile" quantity, since model (2) leads to flat implied volatilities. In fact, a single volatility input is required for the calculation of (3).

In presence of a market smile, however, the adjustment is necessarily more involved, if we aim to incorporate consistently the information coming from the quoted implied volatilities. A procedure to derive a smile-consistent adjustment is illustrated in the following.

3   Smile-consistent convexity adjustment

The market quotes swaption volatilities for different strikes, at least for the most liquid maturities and tenors, so that the classical assumption of lognormal dynamics needs to be modified to properly account for the quoted smile. To this end, one can calibrate a suitable extension of (2) to market data and then value accordingly the expectation in (1). This is the approach we follow in this article, with specific application to the SABR model [5] and to an uncertain-parameter model [3], [2].

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[1]We consider only the common case of a swap rate fixing at the beginning of the accrual period and paying at its end. We also set the payment frequency of the swap fix-leg to one payment per year. The extension to the general case is, anyway, rather straightforward.

One may also wonder whether swaption volatilities contain all the information that is necessary for a consistent calculation of CMS convexity adjustments, thus rendering superfluous the introduction of alternative swap-rate dynamics. In fact, the second moment of $S_{a,b}(T_a)$ can be replicated exactly as follows:

$$E^{a,b}\big(S_{a,b}^2(T_a)\big) = 2 \int_0^\infty E^{a,b}\big((S_{a,b}(T_a) - K)^+\big) \, dK \,, \tag{4}$$

where, by standard no-arbitrage pricing theory, the integrand $E^{a,b}\big((S_{a,b}(T_a) - K)^+\big)$ is the price (divided by the annuity term) of the payer swaption with strike K and written on $S_{a,b}$. Denoting by $\sigma_{a,b}^{\mathrm{M}}(K)$ the market implied volatility for strike K, and assuming that $\sigma_{a,b}^{\mathrm{M}}(K)$ is known for every K, the expectation in the LHS of (4) can be expressed in terms of market observables as

$$E^{a,b}\big(S_{a,b}^2(T_a)\big) = 2 \int_0^\infty \mathrm{Bl}(K, S_{a,b}(0), v_{a,b}^{\mathrm{M}}(K)) \, dK \tag{5}$$

where

$$\mathrm{Bl}(K, S, v) := S\Phi\left(\frac{\ln(S/K) + v^2/2}{v}\right) - K\Phi\left(\frac{\ln(S/K) - v^2/2}{v}\right),$$
$$v_{a,b}^{\mathrm{M}}(K) := \sigma_{a,b}^{\mathrm{M}}(K)\sqrt{T_a} \,,$$

and Φ denotes the standard normal cumulative distribution function. Therefore, if implied volatilities were available in the market for every possible strike, even arbitrarily large ones,[2] CMS convexity adjustments would be completely determined by the related swaption smile thanks to (1) and (5).

However, volatility quotes in the market are provided only up to some strike $\bar{K}$, so that re-writing (5) as

$$E^{a,b}\big(S_{a,b}^2(T_a)\big) = 2 \int_0^{\bar{K}} \mathrm{Bl}(K, S_{a,b}(0), v_{a,b}^{\mathrm{M}}(K)) \, dK + 2 \int_{\bar{K}}^\infty E^{a,b}\big((S_{a,b}(T_a) - K)^+\big) \, dK, \tag{6}$$

only the value of first integral can be inferred from market swaption data.[3] The value of the second integral is not negligible in general, and it can have a strong impact in the calculation of the second moment of $S_{a,b}(T_a)$.

The previous considerations lead to the conclusion that volatility modelling is required for a consistent derivation of CMS convexity corrections. To this end, two different approaches are possible. The first one is based on specifying a static parametric form for the

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[2]This happens, for instance, in case implied volatilities are given by some explicit functional form.
[3]We assume that a continuum of quotes is obtained by suitably interpolating the market ones.

whole smile curve so as to explicitly integrate (5). A second and more reliable approach is to consider a dynamical model for the swap rate in order to infer from it the volatility smile surface to use for the above integration. As we will see in our tests below, the SABR model [5] combines both approaches, whereas the uncertain-parameter model [3], [2] is a clear application of the latter.

4 The SABR model

Hagan *et al.* [5] propose a stochastic volatility model for the evolution of the forward price of an asset under the asset's canonical measure. In this model, which is commonly referred to by the acronym SABR, the forward-asset dynamics is of constant-elasticity-of-variance (CEV) type with a stochastic volatility that follows a driftless geometric Brownian motion, possibly instantaneously correlated with the forward price itself. This model is a market popular choice for swaption smile analysis.

4.1 Model definition

The SABR model assumes that $S_{a,b}(t)$ evolves under the associated forward swap measure $Q^{a,b}$ according to

$$
\begin{aligned}
dS_{a,b}(t) &= V(t)S_{a,b}(t)^{\beta}\,dZ^{a,b}(t),\\
dV(t) &= \epsilon V(t)\,dW^{a,b}(t),\\
V(0) &= \alpha,
\end{aligned}
\tag{7}
$$

where $Z^{a,b}$ and $W^{a,b}$ are $Q^{a,b}$-standard Brownian motions with

$$
dZ^{a,b}(t)\,dW^{a,b}(t) = \rho\,dt,
$$

and where $\beta \in [0,1]$, ϵ and α are positive constants and $\rho \in [-1,1]$.

Using singular perturbation techniques, Hagan *et al.* [5] derive the following approximation for the implied volatility $\sigma^{\mathrm{imp}}(K, S_{a,b}(0))$ of the swaption with maturity T_a, payments in $\mathcal{T}$ and strike K:

$$
\begin{aligned}
\sigma^{\mathrm{imp}}(K, S_{a,b}(0)) \approx &\frac{\alpha}{(S_{a,b}(0)K)^{\frac{1-\beta}{2}}\left[1 + \frac{(1-\beta)^2}{24}\ln^2\left(\frac{S_{a,b}(0)}{K}\right) + \frac{(1-\beta)^4}{1920}\ln^4\left(\frac{S_{a,b}(0)}{K}\right)\right]}\frac{z}{x(z)}\\
&\cdot\left\{1 + \left[\frac{(1-\beta)^2\alpha^2}{24(S_{a,b}(0)K)^{1-\beta}} + \frac{\rho\beta\epsilon\alpha}{4(S_{a,b}(0)K)^{\frac{1-\beta}{2}}} + \epsilon^2\frac{2-3\rho^2}{24}\right]T_a\right\},
\end{aligned}
\tag{8}
$$

where

$$z := \frac{\epsilon}{\alpha}(S_{a,b}(0)K)^{\frac{1-\beta}{2}} \ln\left(\frac{S_{a,b}(0)}{K}\right)$$

and

$$x(z) := \ln\left\{\frac{\sqrt{1 - 2\rho z + z^2} + z - \rho}{1 - \rho}\right\}.$$

4.2 Convexity adjustment

Formula (8) provides us with an (efficient) approximation for the SABR implied volatility for each strike K. It is market practice, however, to consider (8) as exact and to use it as a functional form mapping strikes into implied volatilities. Under this assumption, we can calculate, at least numerically, the CMS convexity adjustment implied by dynamics (7), by integrating the RHS of (5). We obtain

$$\mathbf{CA}^{\mathrm{SABR}}(S_{a,b}; \delta) = S_{a,b}(0)\,\theta(\delta)\left(\frac{2}{S_{a,b}^2(0)}\int_0^\infty \mathrm{Bl}\big(K, S_{a,b}(0), v^{\mathrm{imp}}(K, S_{a,b}(0))\big)\,dK - 1\right), \quad (9)$$

where

$$v^{\mathrm{imp}}(K, S_{a,b}(0)) := \sigma^{\mathrm{imp}}(K, S_{a,b}(0))\sqrt{T_a}.$$

The values of the SABR parameters, needed to calculate (9), are obtained through calibration of (8) to the corresponding swaption smile/skew. The parameter $\beta \in [0, 1]$ can be fixed either to a historical value or according to heuristic considerations. Some popular choices are $\beta = 0$ (normal model), $\beta = \frac{1}{2}$ (square-root model) and $\beta = 1$ (lognormal model).

Remark 4.1. *The behavior of the SABR implied volatility for strikes tending to infinity must be studied carefully so as to ensure that the integral in (9) is finite. Following Lee [8], it is enough to check that the implied volatility growth is bounded for large strikes by $\sqrt{\ln K}$. This behavior is satisfied by the SABR model with the prominent exception of the case $\beta = 1$, where in the limit for large strikes we have:*

$$\lim_{K \to +\infty}\frac{\sigma^{\mathrm{imp}}(K, S_{a,b}(0))}{\sqrt{\ln K}} = \lim_{K \to +\infty}\frac{\epsilon\sqrt{\ln K}}{\ln\left(\frac{\epsilon}{2\alpha}\ln K\right)} = +\infty.$$

We will see below a numerical example showing that, ceteris paribus, SABR convexity adjustments indeed diverge for β approaching one.

5 The uncertain-parameter model (UPM)

Gatarek [3] and Brigo *et al.* [2] propose UPMs for the evolution of some given assets under given reference measures. The uncertainty in the model parameters is described by a random vector and is introduced to capture, in an extremely simple way, stylized facts coming from the option market. As immediate consequence of the model assumptions, the assets marginal densities are mixtures of shifted-lognormal densities, which directly leads to closed-form formulas for European-style option prices.

UPMs can also be used for the evolution of swap rates under the associated forward swap measures. This is described in the following.

5.1 Model definition

The UPM for the swap rate assumes that $S_{a,b}(t)$ evolves under the associated forward swap measure $Q^{a,b}$ according to a displaced geometric Brownian motion with uncertain parameters, namely

$$dS_{a,b}(t) = \sigma^I_{a,b}(S_{a,b}(t) + \alpha^I_{a,b}) \, dZ^{a,b}(t), \tag{10}$$

where I is a discrete random variable, independent of the Brownian motion $Z_{a,b}$, taking values in the set $\{1,\ldots,m\}$ with probabilities $\lambda^{a,b}_i := Q^{a,b}(I = i) > 0$, and where $\sigma^i_{a,b}$ and $\alpha^i_{a,b}$ are positive constants for each i.

The random value of I, and hence of the pair $(\sigma^I_{a,b}, \alpha^I_{a,b})$, is drawn at an infinitesimal instant after time 0, reflecting the initial uncertainty on which scenario will occur in the (near) future.

The UPM (10) is characterized by marginal densities that are mixtures of shifted log-normal densities. Option prices are thus given, in closed form, by mixtures of modified Black's prices. Precisely, the price at time zero of a European payer swaption with maturity T_a and strike K, with underlying swap paying on times $\mathcal{T}$, is equal to

$$\mathbf{PS}(0; a, b, K) = \sum_{h=a+1}^{b} \tau P(0, T_h) \sum_{i=1}^{m} \lambda^{a,b}_i \operatorname{Bl}(K + \alpha^i_{a,b}; S_{a,b}(0) + \alpha^i_{a,b}; \sigma^i_{a,b}\sqrt{T_a}). \tag{11}$$

5.2 Convexity adjustment

The model tractability can also be exploited to calculate explicitly the CMS convexity correction implied by the above UPM dynamics:

$$
\begin{aligned}
E^{a,b}\big(S_{a,b}^2(T_a)\big) &= E^{a,b}\big(E^{a,b}\big(S_{a,b}^2(T_a)\big|I\big)\big) \\
&= \sum_{i=1}^m \lambda_i^{a,b}\, E^{a,b}\big(S_{a,b}^2(T_a)\big|I=i\big) \\
&= S_{a,b}^2(0)\left[1+\sum_{i=1}^m \lambda_i^{a,b}\left(\frac{S_{a,b}(0)+\alpha_{a,b}^i}{S_{a,b}(0)}\right)^2\left(e^{(\sigma_{a,b}^i)^2 T_a}-1\right)\right],
\end{aligned}
\tag{12}
$$

where the last calculation is carried out by noting that the explicit solution to (10) is

$$
S_{a,b}(T_a) = (S_{a,b}(0)+\alpha_{a,b}^I)e^{-\frac{1}{2}(\sigma_{a,b}^I)^2 T_a+\sigma_{a,b}^I Z^{a,b}(T_a)} - \alpha_{a,b}^I,
$$

and hence that the swap rate, conditional on I, is distributed as a shifted lognormal random variable.

The CMS convexity correction for the UPM (10) is then obtained by substituting the expectation (12) into the general formula (1), leading to

$$
\mathbf{CA}^{\mathrm{UPM}}(S_{a,b};\delta) = S_{a,b}(0)\,\theta(\delta)\sum_{i=1}^m \lambda_i^{a,b}\left(\frac{S_{a,b}(0)+\alpha_{a,b}^i}{S_{a,b}(0)}\right)^2\left(e^{(\sigma_{a,b}^i)^2 T_a}-1\right)
\tag{13}
$$

6 Model calibration

Swaption volatility models can be calibrated by taking into account market information on both the related implied volatilities, quoted for different (but finite) strikes, and derivatives, such as CMS swaps, which depend on swap rate convexity adjustments. This is the approach we follow in this article, providing examples of calibration of the SABR model and the UPM to a set of market data that includes both swaption volatilities and CMS-swap spreads.

The reason why we resort to such a joint calibration is because implied volatilities by themselves do not allow to uniquely identify the four parameters of the SABR model. In fact, several are the combinations of parameters β and ρ that produce (almost) equivalent fittings to the finite set of market volatilities available for given maturity and tenor. The β parameter, therefore, can be fixed almost arbitrarily when calibrating the model to the

Date	Rate	Date	Rate	Date	Rate	Date	Rate
29-Sep-05	2.12%	19-Dec-07	2.48%	30-Sep-19	3.50%	30-Sep-31	3.79%
03-Oct-05	2.12%	30-Sep-08	2.56%	30-Sep-20	3.54%	30-Sep-32	3.80%
07-Oct-05	2.15%	30-Sep-09	2.67%	30-Sep-21	3.58%	30-Sep-33	3.80%
31-Oct-05	2.15%	30-Sep-10	2.77%	30-Sep-22	3.62%	29-Sep-34	3.80%
30-Nov-05	2.16%	30-Sep-11	2.87%	29-Sep-23	3.65%	28-Sep-35	3.80%
21-Mar-06	2.19%	28-Sep-12	2.97%	30-Sep-24	3.68%	29-Sep-45	3.81%
15-Jun-06	2.23%	30-Sep-13	3.07%	30-Sep-25	3.71%	30-Sep-55	3.80%
21-Sep-06	2.28%	30-Sep-14	3.16%	30-Sep-26	3.73%		
20-Dec-06	2.32%	30-Sep-15	3.24%	30-Sep-27	3.75%		
20-Mar-07	2.36%	30-Sep-16	3.31%	29-Sep-28	3.76%		
21-Jun-07	2.40%	29-Sep-17	3.38%	28-Sep-29	3.78%		
20-Sep-07	2.44%	28-Sep-18	3.44%	30-Sep-30	3.78%		

Table 1: EUR zero-coupon continuously-compounded spot rates (ACT/365).

quoted smile, and an "implied" value for it can only be inferred as soon as we include suitable non "plain vanilla" instruments in our data set.

As far as the UPM is concerned, instead, we do not observe the same problem of parameter determination, since the market swaption smile is usually well accommodated by a unique choice of UPM parameters. The resulting CMS adjustments, however, are likely to be underestimated, so that also in this case a more robust calibration is to be achieved by fitting CMS swap spreads, too.

Our examples of calibration are based on Euro data as of 28 September 2005, which we report in Tables 1, 2, 3 and 4. We list swaption volatilities for different strikes and CMS-swap spreads for different CMS-rate tenors.

Our calibrations are performed by minimizing the square percentage difference between model quantities (prices or volatilities and CMS adjustments or spreads) and the corresponding market ones. Since typical market bid-ask spread for CMS-swap quotes can be rather large, up to ten or fifteen basis points for long expiries and tenors, each error is weighted in inverse proportion to the bid-ask spread.

The calibration procedures we will follow in the SABR and UPM cases are different and specifically designed to take into account the different features of the two models.

Expiry	Tenor	Strike							
		-200	-100	-50	-25	25	50	100	200
1y	10y	11.51%	3.24%	1.03%	0.37%	-0.22%	-0.22%	0.21%	2.13%
5y	10y	7.80%	2.63%	1.02%	0.44%	-0.33%	-0.53%	-0.63%	-0.17%
10y	10y	6.39%	2.25%	0.91%	0.40%	-0.31%	-0.52%	-0.71%	-0.47%
20y	10y	5.86%	2.07%	0.85%	0.37%	-0.30%	-0.51%	-0.73%	-0.62%
30y	10y	5.44%	1.92%	0.79%	0.35%	-0.29%	-0.52%	-0.79%	-0.85%
1y	20y	9.45%	2.74%	1.17%	0.46%	-0.24%	-0.25%	0.15%	1.62%
5y	20y	7.43%	2.56%	1.00%	0.43%	-0.32%	-0.51%	-0.60%	-0.10%
10y	20y	6.59%	2.34%	0.94%	0.41%	-0.32%	-0.54%	-0.72%	-0.43%
20y	20y	6.11%	2.19%	0.90%	0.40%	-0.32%	-0.55%	-0.77%	-0.61%
30y	20y	5.46%	1.92%	0.79%	0.35%	-0.29%	-0.50%	-0.72%	-0.69%
1y	30y	9.17%	2.67%	1.19%	0.47%	-0.25%	-0.27%	0.13%	1.58%
5y	30y	7.45%	2.58%	1.01%	0.44%	-0.33%	-0.52%	-0.61%	-0.13%
10y	30y	6.73%	2.38%	0.96%	0.42%	-0.33%	-0.53%	-0.68%	-0.35%
20y	30y	6.20%	2.22%	0.91%	0.40%	-0.32%	-0.54%	-0.74%	-0.55%
30y	30y	5.39%	1.90%	0.78%	0.35%	-0.28%	-0.50%	-0.72%	-0.68%

Table 2: Market volatility smiles for the selected expiry-tenor pairs. Strikes are expressed as absolute differences in basis points *w.r.t* the at-the-money values.

Expiry	Tenor		
	10y	20y	30y
1y	17.60%	15.30%	14.60%
5y	16.00%	14.80%	14.30%
10y	14.40%	13.60%	13.10%
20y	13.10%	12.10%	11.90%
30y	12.90%	12.30%	12.30%

Table 3: Market at-the-money volatilities.

| | **Tenor** | | |
Maturity	10y	20y	30y
5y	94.1	124.1	130.3
10y	82.0	104.8	110.6
15y	72.5	91.3	98.3
20y	66.7	84.2	92.9
30y	64.6	85.2	97.9

Table 4: Market CMS swap spreads in basis points.

6.1 Market swaption smiles

Swaption volatilities are quoted by the market for different strikes K as a difference $\Delta\sigma^{\mathrm{M}}_{a,b}$ with respect to the at-the-money level

$$\Delta\sigma^{\mathrm{M}}_{a,b}(\Delta K) := \sigma^{\mathrm{M}}_{a,b}(K^{\mathrm{ATM}} + \Delta K) - \sigma^{\mathrm{ATM}}_{a,b}$$

usually for $\Delta K \in \{\pm 200, \pm 100, \pm 50, \pm 25, 0\}$, where the ΔK values are expressed in basis points. The set of market pairs (a, b) is denoted by $\mathcal{S}$.

We stress that smile quotes are not provided for all the swaption tenors and expiries, for which at-the-money volatilities are available. Interpolation schemes are then to be employed to complete the missing quotes.

6.2 CMS swap spreads

CMS swaps are interest rate swaps whose fixed leg is replaced by a sequence of CMS rates paid every three months, while the floating leg is a sequence of three-month Libor rates plus a spread, here referred to as CMS swap spread, which is received with the same frequency.

The market quotes the spread $X_{n,c}$ which sets to zero the no-arbitrage value of a CMS swap starting today with payment dates T'_i, with $i = 1, \ldots, n$, and paying the c-year swap rate $S'_{i,c}$ set in T'_{i-1}, with $T'_0 = 0$. This definition leads to an explicit relationship for the spread in term of related CMS convexity adjustments:

$$X_{n,c} = \frac{\sum_{i=1}^{n} \left(S'_{i,c}(0) + \mathbf{CA}(S'_{i,c}; \delta) \right) P(0, T'_i)}{\sum_{i=1}^{n} P(0, T'_i)} - \frac{1 - P(0, T'_n)}{\delta \sum_{i=1}^{n} P(0, T'_i)}, \tag{14}$$

where all the accrual periods are set equal to δ. The set of market pairs (n, c) is denoted by $\mathcal{X}$.

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Note that equation (14) can also be used in a reverse way to infer convexity adjustments $\mathbf{CA}(S'_{n,c}; \delta)$, with $n = 1, \ldots, N$, from CMS spreads $X_{n,c}$ by iteratively solving a set of N equations (14). The market, however, quotes the spread only for few CMS swap maturities and tenors (usually 5, 10, 15, 20 and 30 years). Since the payment frequency is four times per year, these quotes contain too little information to bootstrap directly all convexity adjustments from CMS swap spreads. This further motivates a joint calibration with swaption smile data.

6.3   The SABR calibration procedure

We assume that all relevant swap rates evolve according to dynamics (7). Precisely, we assume that implied volatilities are given by the functional form (8) and that each swap rate is associated with different parameters α, ϵ and ρ. The parameter β is instead assumed to be equal across different maturities and tenors.[4]

The joint calibration to swaption smiles and CMS convexity adjustments is carried out through the following two-stage procedure:

1. Set the β parameter at an initial guess.

2. For each expiry-tenor pair $(a, b) \in \mathcal{S}$, *i.e.* for each underlying swap rate:

   – Select an initial set of SABR parameters $\alpha^0 = \alpha^0_{a,b}$, $\epsilon^0 = \epsilon^0_{a,b}$ and $\rho^0 = \rho^0_{a,b}$, with the β parameter previously fixed.

   – Calibrate the associated parameters $\alpha = \alpha_{a,b}$, $\epsilon = \epsilon_{a,b}$ and $\rho = \rho_{a,b}$ to the corresponding swaption volatility smile, obtaining $\alpha_{a,b}(\beta)$, $\epsilon_{a,b}(\beta)$ and $\rho_{a,b}(\beta)$.

   – Calculate[5] the CMS adjustment (9) with parameters β, $\alpha_{a,b}(\beta)$, $\epsilon_{a,b}(\beta)$ and $\rho_{a,b}(\beta)$.

3. For each market pair $(n, c) \in \mathcal{X}$, with $c = b - a$, calculate the corresponding CMS swap spread (14) using the adjustments for swap rates $S'_{i,c}$, $i = 1, \ldots, n$, where $\mathbf{CA}(S'_{i,c}; \delta)$ has been calculated in the previous step[6]. Denote the obtained spread by $X_{n,c}(\beta)$.

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[4]We are not assuming a swap model in a strict sense, but simply that swaption volatilities are given in terms of the SABR functional form.

[5]The CMS convexity adjustment (9) is calculated by means of the QUADPACK subroutine package for the numerical computation of one-dimensional integrals, see Piessens *et al.* [7].

[6]If a convexity adjustment is needed for a swap rate whose volatility is not quoted by the market, the value is obtained by (cubic spline) interpolation.

| | **β** | | | | | | |
Maturity	0.2	0.3	0.4	0.5	0.6	0.7	0.8
5	93.4	94.0	94.0	94.0	94.1	94.1	94.9
10	80.6	81.3	81.5	81.8	82.2	83.0	85.3
15	70.4	71.6	72.1	72.9	74.3	78.5	129.8
20	63.0	65.8	66.6	68.1	71.2	82.1	306.1
30	56.2	62.0	63.7	66.6	73.5	104.2	1206.4

Table 5: CMS swap spreads in basis points for ten-year CMS swaps with different maturities under SABR volatilities calibrated to the swaption smile for different choices of β.

4. Iterate over β, repeating steps 2 to 4, until the CMS swap spreads $X_{n,c}(\beta)$, for all market pairs $(n, c) \in \mathcal{X}$, are as close as possible to the corresponding market quotes.

The inner optimizations on $\alpha_{a,b}$, $\epsilon_{a,b}$ and $\rho_{a,b}$, for each market pair (a, b), and the outer optimization on β are all performed by using standard and consolidated minimization algorithms.

Remark 6.1. *Though the parameter β can assume any value between zero and one, in practice we have to bound it to achieve a successful calibration. In fact, as already noticed in Remark 4.1, values of β approaching one lead to divergent values for convexity adjustments. As a numerical confirmation, we show in Table 5 the CMS swap spreads $X_{n,10}(\beta)$ for a ten-year underlying swap rate and for different maturities T'_n, after calibration, with fixed β, to whole swaption smile as of 28 September 2005. We notice that the value of the spread can increase up to a factor of twelve according to the choice made for β within the conservative range of $[0.2, 0.8]$ used in the example.*

6.4 The UPM calibration procedure

We now assume that swap rates evolve according to the UPM (10). Precisely, we assume that swaption prices are given by formula (11), where we set $m = 2$ for all swap rates, and where the parameters $\lambda^{a,b}$, $\alpha_{a,b}$ and $\sigma_{a,b}$ are different for different expiry-tenor pairs (a, b).[7] CMS convexity adjustments are explicitly given by equation (13).

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[7]Also in this case, we are not dealing with a proper swap model, but simply with a suitable pricing function that can be justified in terms of single swap-rate dynamics.

Contrary to the previous case, we here follow a one-step calibration procedure, which has proven to be rather fast and yield robust results:[8]

1. Bootstrap from the CMS swap spreads quoted by market the convexity adjustments of each swap rate $S_{a,b}$, $(a,b) \in \mathcal{S}$, by choosing a suitable functional form.[9]

2. Select an initial set of parameters for the UPM for each expiry-tenor pair $(a,b) \in \mathcal{S}$.

3. For each (a,b), calibrate the parameters to the related swaption smile and the convexity adjustment calculated at step 1.

4. Check that the CMS swap spreads predicted by the model are in accordance with the quoted values, so as to ensure that the functional form used at step 1 is reliable. Otherwise, restart from the first step choosing a different functional form.

In our example, we used a Nelson-Siegel function and obtained that the CMS swap spread implied by a calibration to swaption smiles and the bootstrapped CMS adjustments are quite close to corresponding market ones, see also Table 6 below.

6.5 Calibration results

Calibration results for the SABR model and the and UPM are described in tables 6 and 7 and represented in figures 1 and 2. As we can see, both models accommodate market data in a satisfactory way, with the SABR model that usually performs the UPM. Calibration errors could be lowered by increasing the number of mixtures in the UPM and by considering different β parameters for different swap rates. This, however, must be done at the cost of increasing the computation time.

7 Conclusions

We have considered the SABR and uncertain parameter models for the evolution of swap rates under their associated measures and reviewed the pricing of swaptions. We have then derived explicit formulas for the CMS convexity adjustments implied by both models, noting that such adjustments contain supplementary information with respect to that of the quoted swaption smile.

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[8]The same procedure, when applied to the SABR calibration, is less efficient and robust due to the previously mentioned problems on the determination of β and the fact that the calculation of CMS adjustments is much more time consuming than in the UPM case.

[9]In our example, we interpolated along expiries the adjustments for swap rates with the same tenor.

UPM Maturity	Tenor 10y	20y	30y	SABR Maturity	Tenor 10y	20y	30y
5y	0.8	1.4	2.4	5y	0.1	0.2	0.9
10y	1.4	2.4	4.3	10y	0.2	0.9	2.6
15y	1.5	2.6	4.8	15y	0.4	1.0	3.3
20y	1.1	2.0	4.3	20y	1.4	0.4	2.7
30y	1.2	2.1	3.7	30y	2.1	0.2	1.5

Table 6: Absolute differences in basis points between market CMS swap spreads and those induced by the UPM and SABR model, respectively. All calibration errors are within typical bid-ask spreads.
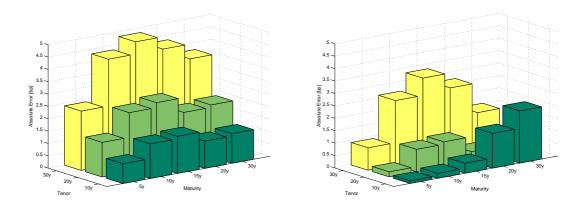

Figure 1: Absolute differences in basis points between market CMS swap spreads and those induced by the UPM (left side) and SABR model (righth side).

UPM Strike	Swaption 5x10	5x20	5x30	SABR Strike	Swaption 5x10	5x20	5x30
-200	8.5	11.5	21.0	-200	2.1	2.6	2.3
-100	1.4	1.3	3.0	-100	1.2	1.8	1.5
-50	0.0	1.0	2.0	-50	0.9	1.1	0.8
-25	1.0	0.6	0.3	-25	1.0	0.7	1.1
0	0.7	0.5	0.5	0	1.0	0.7	0.8
25	1.9	1.7	1.4	25	1.2	0.8	1.5
50	1.2	0.9	1.2	50	1.5	1.4	1.1
100	0.9	0.7	0.7	100	1.4	1.9	1.1
200	1.2	0.9	0.9	200	1.7	2.0	1.5

UPM Strike	Swaption 10x10	10x20	10x30	SABR Strike	Swaption 10x10	10x20	10x30
-200	7.2	13.7	20.0	-200	1.5	1.9	1.5
-100	5.9	10.0	12.0	-100	0.7	1.2	0.5
-50	0.6	1.7	3.5	-50	1.1	0.8	1.1
-25	2.6	1.6	1.7	-25	0.7	0.4	0.8
0	3.4	3.7	4.0	0	0.5	0.8	0.7
25	3.9	5.6	5.3	25	0.7	0.4	1.0
50	3.4	3.6	4.9	50	1.1	1.0	1.1
100	0.2	0.4	0.5	100	1.2	1.7	0.8
200	1.1	0.9	1.0	200	1.2	1.5	1.1

UPM Strike	Swaption 20x10	20x20	20x30	SABR Strike	Swaption 20x10	20x20	20x30
-200	7.1	15.8	16.7	-200	1.9	2.4	2.7
-100	6.9	12.5	13.3	-100	1.1	1.2	1.7
-50	1.1	5.1	4.9	-50	1.7	1.4	1.7
-25	0.8	0.5	0.4	-25	0.3	0.9	0.6
0	2.1	1.7	2.1	0	0.5	0.6	0.5
25	2.8	3.6	4.4	25	0.8	0.4	0.8
50	2.4	3.6	3.9	50	1.1	1.6	1.4
100	0.1	0.4	0.1	100	1.4	1.8	1.6
200	0.4	0.5	0.4	200	1.4	1.7	1.7

Table 7: Absolute differences in basis points between market implied volatilities and those induced by the UPM and the SABR model, respectively. Strikes are expressed as absolute differences in basis points *w.r.t* the at-the-money values.
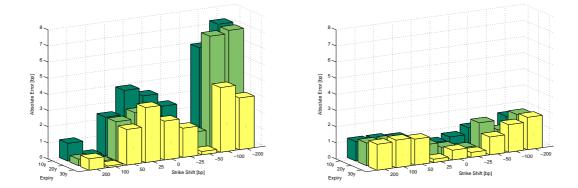
Figure 2: Absolute differences in basis points between market implied volatilities and those induced by the UPM (left side) and the SABR model (righth side) for swaption on ten year swap rate.

We have finally provided an example of calibration of both models to market data, which avoids employing heuristic considerations to fix model parameters, such as the SABR β parameter. The considered data set comprises the swaption smiles and CMS swap spreads quoted by the market.

Both the SABR and uncertain-parameter models can well interpret market data. In fact, they impose themselves as effective pricing tools for CMS derivatives such as CMS spreads and CMS spread options, which are very sensitive to the swaption smile, without resorting to a fully consistent market model.

References

[1] Black, F. (1976). The pricing of commodity contracts. *Journal of Financial Economics* 3, 167-179.

[2] Brigo, D., Mercurio, F., and Rapisarda, F. (2004) Smile at the uncertainty. *Risk* 17(5), 97-101.

[3] Gatarek, D. (2003) LIBOR market model with stochastic volatility. Deloitte&Touche. Available online at:
http://papers.ssrn.com/sol3/papers.cfm?abstract_id=359001

[4] Hagan, P.S. (2003) Convexity Conundrums: Pricing CMS Swaps, Caps, and Floors. *Wilmott magazine*, March, 38-44

[5] Hagan, P.S., Kumar, D., Lesniewski, A.S., Woodward, D.E. (2002) Managing Smile Risk. *Wilmott magazine*, September, 84-108.

[6] Pelsser, A. (2003) Mathematical foundation of convexity correction. *Quantitative Finance* 3, 59-65

[7] Piessens, R., de Doncker-Kapenga, E., Uberhuber, C.W., and Kahaner, D.K. (1983) QUADPACK: a subroutine package for automatic integration. Springer Verlag.

[8] Lee, R.W. (2004) The moment formula for implied volatility at extreme strikes. *Mathematical Finance* 14(3), 469-480.

Appendix A: CMS option pricing with cash-settled swaptions

We denote by $G_{a,b}(S)$ the annuity term in the cash-settled swaptions associated to $S_{a,b}$,

$$G_{a,b}(S) := \sum_{j=1}^{b-a} \frac{\tau}{(1+\tau S)^j} = \begin{cases} \frac{1}{S}\left[1 - \frac{1}{(1+\tau S)^{b-a}}\right] & S > 0 \\ \tau(b-a) & S = 0 \end{cases}$$

and set $f(S) := 1/G_{a,b}(S)$. Standard replication arguments imply

$$f(S)(S-K)^+ = f(K)(S-K)^+ + \int_K^{+\infty} [f''(x)(x-K) + 2f'(x)](S-x)^+\, dx, \qquad (15)$$

or equivalently,

$$(S-K)^+ = f(K)(S-K)^+ G_{a,b}(S) + \int_K^{+\infty} [f''(x)(x-K) + 2f'(x)](S-x)^+ G_{a,b}(S)\, dx, \quad (16)$$

so that, taking expectation on both sides,

$$E^{T_a}[(S_{a,b}(T_a) - K)^+] = f(K)E^{T_a}[(S_{a,b}(T_a) - K)^+ G_{a,b}(S_{a,b}(T_a))]$$
$$+ \int_K^{+\infty} [f''(x)(x-K) + 2f'(x)]E^{T_a}[(S_{a,b}(T_a) - x)^+ G_{a,b}(S_{a,b}(T_a))]\, dx.$$

Since $(S_{a,b}(T_a) - K)^+ G_{a,b}(S_{a,b}(T_a))$ is the payoff of a cash-settled swaption whose (forward) price

$$E^{T_a}[((S_{a,b}(T_a) - K)^+ G_{a,b}(S_{a,b}(T_a)))]$$

is, by market practice,[10] equal to

$$c_{a,b}(K) G_{a,b}(S_{a,b}(0)),$$

where

$$c_{a,b}(x) := \mathrm{Bl}(x, S_{a,b}(0), v_{a,b}^{\mathrm{M}}(x)),$$

we finally have:

$$E^{T_a}[(S_{a,b}(T_a) - K)^+] = f(K) c_{a,b}(K) G_{a,b}(S_{a,b}(0))$$
$$+ \int_K^{+\infty} [f''(x)(x - K) + 2 f'(x)] c_{a,b}(x) G_{a,b}(S_{a,b}(0)) dx.$$

In the standard case of a payoff occurring at time $T = T_a + \delta$, the calculation of

$$E^T[(S_{a,b}(T_a) - K)^+]$$

is carried out by resorting to the approximation

$$\frac{P(t, T_a)}{P(t, T_a + \delta)} \approx (1 + \tau S_{a,b}(t))^{\delta/\tau},$$

leading to

$$E^T[(S_{a,b}(T_a) - K)^+] \approx (1 + \tau S_{a,b}(0))^{\delta/\tau} E^{T_a}\left[(S_{a,b}(T_a) - K)^+ \frac{1}{(1 + \tau S_{a,b}(T_a))^{\delta/\tau}}\right].$$

Setting

$$\bar{f}(S) := \frac{f(S)}{(1 + \tau S)^{\delta/\tau}},$$

and applying (15) to function $\bar{f}$, we finally have, remembering the market price of cash-settled swaptions,

$$E^T[(S_{a,b}(T_a) - K)^+] \approx (1 + \tau S_{a,b}(0))^{\delta/\tau}\left[\bar{f}(K) c_{a,b}(K) G_{a,b}(S_{a,b}(0))\right.$$
$$\left. + \int_K^{+\infty} [\bar{f}''(x)(x - K) + 2 \bar{f}'(x)] c_{a,b}(x) G_{a,b}(S_{a,b}(0)) dx\right]. \tag{17}$$

---

[10]We assume that the implied volatilities for cash-settled and physically-settled swaptions coincide.

The convexity adjustment $E^T[S_{a,b}(T_a)] - S_{a,b}(0)$ is then obtained by setting $K = 0$ in (17) and noting that

$$\bar{f}(0) = f(0) = \frac{1}{\tau(b-a)}.$$

Appendix B: CMS option pricing with physically-settled swaptions

A CMS option price can also be calculated by moving to the forward swap measure $Q^{a,b}$

$$E^T[(S_{a,b}(T_a) - K)^+] = \frac{\sum_{j=a+1}^{b} P(0, T_j)}{P(0, T)} E^{a,b}\left(\frac{(S_{a,b}(T_a) - K)^+ P(T_a, T)}{\sum_{j=a+1}^{b} P(T_a, T_j)}\right).$$

Using Hagan's (2003) approximation

$$\frac{\sum_{j=a+1}^{b} \tau P(t, T_j)}{P(t, T)} \approx (1 + \tau S_{a,b}(t))^{\delta/\tau} \sum_{j=1}^{b-a} \frac{\tau}{(1 + \tau S_{a,b}(t))^j} = \frac{1}{\bar{f}(S_{a,b}(t))},$$

we obtain

$$E^T[(S_{a,b}(T_a) - K)^+] \approx \frac{1}{\bar{f}(S_{a,b}(0))} E^{a,b}\big[\bar{f}(S_{a,b}(T_a))(S_{a,b}(T_a) - K)^+\big].$$

Applying again (15) to function $\bar{f}$ and taking $Q^{a,b}$-expectation on both sides, we have

$$E^{a,b}\big[\bar{f}(S_{a,b}(T_a))(S_{a,b}(T_a) - K)^+\big]$$
$$= \bar{f}(K)E^{a,b}\big[(S_{a,b}(T_a) - K)^+\big] + \int_K^{+\infty} [\bar{f}''(x)(x - K) + 2\bar{f}'(x)]E^{a,b}\big[(S_{a,b}(T_a) - x)^+\big]\, dx.$$

Hence, we can finally write

$$E^T[(S_{a,b}(T_a) - K)^+] \approx \frac{1}{\bar{f}(S_{a,b}(0))}\left[\bar{f}(K)c_{a,b}(K) + \int_K^{+\infty} [\bar{f}''(x)(x - K) + 2\bar{f}'(x)]c_{a,b}(x)\, dx\right],$$
$$(18)$$

which coincides with formula (17), derived in the cash-settled case, since

$$(1 + \tau S_{a,b}(0))^{\delta/\tau} G_{a,b}(S_{a,b}(0)) = \frac{1}{\bar{f}(S_{a,b}(0))}.$$

Remark 7.1. *The convexity adjustment (1) can be obtained as a particular case of (18) by setting $K = 0$, approximating linearly the function $\bar{f}$ around $S_{a,b}(0)$,*

$$\bar{f}(S) \approx \bar{f}(S_{a,b}(0)) + \bar{f}'(S_{a,b}(0))[S - S_{a,b}(0)],$$

and noting, after some algebra, that

$$\theta(\delta) = \frac{\bar{f}'(S_{a,b}(0))}{\bar{f}(S_{a,b}(0))} S_{a,b}(0).$$