Multiple-Curve Valuation with One-Factor Hull-White Model

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Abstract

This paper shows the one-factor Hull-White model can be used in the modern derivative multiple-curve valuation framework. For vanilla instruments such as caps and European swaptions, closed-form pricing formulae are derived.

1 Introduction

After the credit crunch in 2007, it was realized that discounting and rate curves needed to be altered, this led to the multiple-curve valuation framework. For example, F Mercurio [7] extended the LIBOR market model to the multiple-curve valuation framework, N. Moreni and A. Pallaicini [8] extended HJM models, and C Kenyon [4] extended short-rate models. More examples can be found in [6], [1], [3], [5] and [9]. The classical one-factor Hull-White model is an interest rate term structure model that is still popular in the market today because it is simple, tractable and easy to calibrate. In the multiple-curve valuation framework, rates with different tenors are associated with different discount factor curves. In general, for a single rate derivative such as an interest rate swap, one need two curves, one for discounting and one for the rate, hence a two-factor model is needed (see [4] for details on such a two-factor Hull-White model). For such a model, two sets of Hull-White parameters and a correlation need to be calibrated to market quotes. But sometimes, quotes are not available. For example, if we want use an OIS curve for discounting, quotes for overnight rate caps or swaptions are not available on the market. Hence we need to guess some parameters before we perform calibration. The valuation would then not be consistent among participates, and a bad guess would probably lead to a bad valuation. Though the one-factor Hull-White model seems not to fit in the multiple-curve valuation framework, in this paper we show how the one-factor Hull-White model can be used in the multiple-curve valuation framework under some reasonable conditions. The model parameters can be calibrated using the close-form pricing formulae derived in a similar way as in the single-curve valuation framework. These calibrated model parameters can also serve as better guesses for the two-factor Hull-white model. These formulae generalize some results of M. Henrard [3].

2 Multiple Curve Valuation Framework

A positive function $D(t)$ in time t is called a discount factor curve if $D(0) = 1$. A discount factor curve is usually decreasing and can define an implied rate of tenor τ at future time T as follows

$$L(T, T+\tau) = \frac{1}{\tau}(\frac{D(T)}{D(T+\tau)} - 1). \tag{1}$$

We call this rate an implied rate to distinguish from the old forward LIBOR rate.
If $L(t, T, T+\tau) = E^{T+\tau}[R(T)]$ for some rate of tenor τ, where $E^{T+\tau}$ is the $(T+\tau)$-forward measure, then the curve $D(t)$ is also called a rate curve for the rate $R(T)$.

Before the credit crunch, a discount factor curve is always a rate curve for a LIBOR rate of any tenor. In the multiple curve valuation framework, a 3-month LIBOR curve is not a rate curve for the 6-month LIBOR. In general, we need two or more curves to price a single-currency instrument: one for discounting, others are for rates that are related to the instrument. For example, to price an interest rate swap where a payer makes a series of fixed semi-annual payments in USD and receives a series of semi-annual payments in USD that depend on the future level of 3-month LIBOR rates, we need two discount factor curves, one, denoted by $D_d(t)$, for discounting and the other, denoted by $D_L(t)$, for the 6-month LIBOR rate. Assume the notional is 1, then the price of the swap is

$$\sum_{i=1}^{n} \alpha_i E^{T_i}[L_{6m}(T_{i-1})] D_d(T_i) - K \sum_{i=1}^{n} \alpha_i D_d(T_i) = \sum_{i=1}^{n} \alpha_i L(T_{i-1}, T_i) D_d(T_i) - K \sum_{i=1}^{n} \alpha_i D_d(T_i) \tag{2}$$

These curves can be bootstrapped from market quotes. For example, a discounting curve can be build from quotes on OIS instruments by an ordinary bootstrapping procedure, then the discount factor curve for the 3-month LIBOR can be build from 3-month LIBOR swaps by the bootstrapping procedure again using the above pricing formula.

3 The One-Factor Hull-White Model

In this section we describe the one-factor Hull-White model following [2]. The Hull-White model is a short-rate model that is described by the following equation

$$dr(t) = [\theta(t) - ar(t)]dt + \sigma dw(t), \tag{3}$$

where a is a constant called the mean reversion parameter, σ is a constant volatility (which can be generalized to a deterministic function in t) and $w(t)$ is a Brownian motion. $\theta(t)$ is defined by a market discount factor curve $D(t)$. Let $f(t)$ be the instantaneous forward rate at time 0 for the maturity t, i.e.

$$f(t) = -\partial ln D(t)/\partial t. \tag{4}$$

Then we have

$$\theta(t) = \partial f(t)/\partial t + af(t) + \frac{\sigma^2}{2a}(1 - e^{-2at}). \tag{5}$$

A solution to the equation 3 is

$$r(t) = r(s)e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-(t-s)} + \int_s^t e^{-a(t-u)}\sigma(u)dw(u) \tag{6}$$

where

$$\alpha(t) = f(t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2. \tag{7}$$

Under the Hull-White model, the price at time t of a zero coupon bond paying off 1 at time T is

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}, \tag{8}$$

where

$$B(t,T) = \frac{1}{a}[1 - e^{-a(T-t)}], \tag{9}$$

and

$$A(t,T) = \frac{D(T)}{D(t)} exp\{B(t,T)f(t) - \frac{\sigma^2}{4a}(1 - e^{-2at})B(t,T)^2\} \tag{10}$$

Let $x(t)$ be the short-rate in the equation 3 by setting $\theta(t) = 0$, $x(t)$ then satisfies the following equation under the T-forward measure

$$dx(t) = [-B(t,T)\sigma^2 - ax(t)]dt + \sigma dw^T(t), \tag{11}$$

and

$$x(t) = x(s)e^{-a(t-s)} - M^T(s,t) + \int_s^t \sigma e^{-a(t-u)} dw^T(u), \tag{12}$$

where

$$M^T(s,t) = x(s)e^{-a(t-s)} - M^T(s,t) + \int_s^t \sigma e^{-a(t-u)} dw^T(u), \tag{13}$$

Hence, the price of a European call-option with strike X and maturity T on a zero coupon bond $P(T,S)$ is

$$ZBC(t,T,S,X) = P(t,S)\phi(h) - XP(t,T)\phi(h-\sigma_p), \tag{14}$$

where

$$\sigma_p = \sigma\sqrt{\frac{1-e^{2a(T-t)}}{2a}}B(T,T+\tau) \tag{15}$$

and

$$h = \frac{1}{\sigma_p} ln \frac{P_d(t,T+\tau)}{P_d(t,T)} + \sigma_p/2 \tag{16}$$

and the swaption price at time $t < T$ is then given by

$$PS(t,T,N,X) = N\sum_{i=1}^{n} c_i ZBP(t,T,T_i,x_i). \tag{17}$$

where N is the notional, T_i is the i-th coupon payment date, $c_i = c\tau_i$ for $i < n$ and $c_n = 1 + c\tau_n$ where c is the fixed coupon rate and

$$x_i = A(T,T_i)e^{-B(T,T_i)r^*}. \tag{18}$$

where r^* is the value of the spot rate at time T for which

$$\sum_{i=1}^{n} c_i A(T,T_i)e^{-B(T,T_i)r^*} x_i = 1. \tag{19}$$

4 One-factor Hull-White Model With Two Curves

Suppose there are two curves, one for discounting and denoted by $D_d(t)$ and the other for a rate denoted by $D_r(t)$. For example, $D_d(t)$ is an OIS discount factor curve and $D_r(t)$ the 3-month LIBOR rate curve. Then the implied rate of $D_r(t)$ is the expected 3-month LIBOR rate.
Based on the discount factor curve $D_d(t)$, we can define a short rate as follows

$$dr_d(t) = [\theta_d(t) - ar_d(t)]dt + \sigma dw(t), \tag{20}$$

This gives a zero coupon bond pricing formula:

$$P_d(t,T) = A(t,T)e^{-B(t,T)r(t)}, \tag{21}$$

where

$$B(t,T) = \frac{1}{a}[1 - e^{-a(T-t)}], \tag{22}$$

and

$$A(t,T) = \frac{D_d(T)}{D_d(t)} exp\{B(t,T)f(t) - \frac{\sigma^2}{4a}(1-e^{-2at})B(t,T)^2\} \tag{23}$$

$P_d(t,T)$ can defines a term structure of the implied rate of $D_d(t)$. In particular, $P_d(0,T) = D_d(T) = D_r(T)D_d(T)/D_r(T)$. Let $P_r(0,T) = D_r(T)$ and $D_s(T) = \frac{D_r(T)}{D_d(T)}$, then $P_r(0,T) = P_d(0,T)D_s(T)$. Since $D_s(0) = 1$, $D_s(t)$ is also a discount factor curve that represnts the difference between $D_r(t)$ and $D_d(r)$ or the "spread" between two implied rates. In general, the difference between these two curves is small (ignorable before the credit crunch). If we assume the difference is deterministic and not changed as time goes then we can define a term structure of the implied rate of $D_r(t)$ as follows:

$$P_r(t,T) = P_d(t,T)D_s(T)/D_s(t), \tag{24}$$

Remark 1 *$D_s(t)$ can be regarded as a survival probability curve. In other words, if $P_d(t,T)$ is regarded as a default-free zero coupon bond price, then $P_r(t,T)$ is a defaultable zero coupon bond price with survival probability $D_s(T)/D_s(t)$ and zero recovery assuming default has not occured prior t.*

Proposition 1 *For any rate tenor τ, let $L_d(T,T+\tau) = \frac{1}{\tau}(\frac{1}{P_d(T,T+\tau)} - 1)$ and $L_r(T,T+\tau) = \frac{1}{\tau}(\frac{1}{P_r(T,T+\tau)} - 1)$, then*

$$L_r(T,T+\tau) = bL_d(T,T+\tau) + \frac{1}{\tau}(b-1), \tag{25}$$

where $b = \frac{D_s(T)}{D_s(T+\tau)}$

Proof. Let $b = D_s(T)/D_s(T+\tau)$, then by Equation 24 we have

$$\begin{aligned}
L_r(T,T+\tau) &= \frac{1}{\tau}(\frac{1}{P_r(T,T+\tau)} - 1) \\
&= \frac{1}{\tau}(\frac{b}{P_d(T,T+\tau)} - 1) \\
&= \frac{b}{\tau}(\frac{1}{P_d(T,T+\tau)} - \frac{1}{b}) \\
&= \frac{b}{\tau}(\frac{1}{P_d(T,T+\tau)} - 1 + 1 - \frac{1}{b}) \\
&= bL_d(T,T+\tau) + \frac{1}{\tau}(b-1)
\end{aligned}$$

The implied rate $L_d(T,T+\tau)$ from the discounting curve may not represent any existing rate, but it is a martingale under the forward measure. Hence, it enable us to calculate the implied rate of the rate curve $D_r(t)$ under the same measure.

First of all, we show that the price of a standard FRA on the rate $L_r(T,T+\tau)$ with notional N and strike K can be easily derived in terms of an FRA on the rate $L_d(T,T+\tau)$ as follows:

$$\begin{aligned}
\mathbf{FRA}(N, L_r(T,T+\tau), K) &= D_d(T)E^{T+\tau}[P_d(T,T+\tau)N\tau(L_r(T,T+\tau) - K)] \\
&= D_d(T)E^{T+\tau}[P_d(T,T+\tau)N\tau(bL_d(T,T+\tau) + \frac{1}{\tau}(b-1) - K)]
\end{aligned}$$

$$
\begin{aligned}
&= D_d(T)E^{T+\tau}[P_d(T,T+\tau)bN\tau(L_d(T,T+\tau)+\frac{1}{\tau}(1-1/b)-K/b)] \\
&= D_d(T)E^{T+\tau}[P_d(T,T+\tau)bN\tau(L_d(T,T+\tau)-(K/b+\frac{1}{\tau}(1/b-1)))] \\
&= \mathbf{FRA}(N', L_d(T,T+\tau), K')
\end{aligned}
$$

where $N' = bN$ and $K' = K/b + \frac{1}{\tau}(1/b-1)$. This means that the standard FRA is equivalent to an FRA on $L_d(T,T+\tau)$ with notional N' and strike K'.

A caplet that expires at time T has a payoff

$$P_d(T,T+\tau)N\tau(L_r(T,T+\tau)-K)^+, \tag{26}$$

where N is the notional and K the strike.
An similar argument as above shows that the caplet is equivalent to a caplet on $L_d(T,T+\tau)$ with a strike of $K/b+\frac{1}{\tau}(1/b-1)$ and a notional of bN. Since the later caplet can be valued in the single-curve valuation framework as an option on a zero coupon bond, see [2], hence the price of the caplet is

$$
\begin{aligned}
\mathbf{Caplet}(N, L_r(T,T+\tau), K) &= bN(1+K/b+\frac{1}{\tau}(1/b-1)\tau)ZBP(T,T+\tau,K') \\
&= N(1+K\tau)ZBP(T,T+\tau,K')
\end{aligned}
$$

where $K' = \frac{b}{1+\tau K}$ and ZBP is the price of European put option on the zero coupon bond with respect to D_d. Comparing with the single-curve pricing formula, the difference is just a scale factor of b on the strike. Therefore we have

Proposition 2 *The price of a caplet with respect to $L_r(T,T+\tau)$ is*

$$N[bP_d(t,T)\phi(-h+\sigma_p)-(1+K\tau)P_d(t,S)\phi(-h)] \tag{27}$$

where $\sigma_p = \sigma\sqrt{\frac{1-e^{2a(T-t)}}{2a}}B(T,T+\tau)$ *and* $h = \frac{1}{\sigma_p}ln\frac{P_d(t,T+\tau)}{P_d(t,T)}+\sigma_p/2$

To price a European swaption, in the single curve case, we find the value of a spot rate which satisfies

$$\sum_{i=1}^{n} c_i A(T,T_i)e^{-B(T,T_i)r^*} = 1. \tag{28}$$

In the two-curve case, by the Proposition 1, the ith floating payment amount is

$$
\begin{aligned}
N\tau_i L_r(T_{i-1},T_i) &= N\tau_i(b_i L_d(T_{i-1},T_i)+\frac{1}{\tau_i}(b_i-1)) \\
&= N\tau_i\bar{b}L_d(T_{i-1},T_i)+N(b_i-1)+N\tau_i(b_i-\bar{b})L_d(T_{i-1},T_i)
\end{aligned}
$$

where $\bar{b}$ is the average of all b_i's. The last term $N\tau_i(b_i-\bar{b}_i)L_d(T_{i-1},T_i)$ is small and negligible because $1-b_i$ is small. But it is better to approximate it with $N\tau_i(b_i-\bar{b}_i)L_d(0,\tau_i)$. Then the value of the floating leg with notional of 1 is approximately

$$\textbf{Floating Leg} = \sum_{i=1}^{n}\tau_i L_r(T_{i-1},T_i)P_d(t,T_i)+P_d(t,T_n)$$

$$\approx \sum_{i=1}^{n} P_d(t,T_i)[\tau_i \bar{b} L_d(T_{i-1},T_i) + (b_i - 1) + \tau_i(b_i - \bar{b})L_d(0,\tau_i)] + P_d(t,T_n)$$

$$= \bar{b} + (1-\bar{b})P_d(t,T_n) + \sum_{i=1}^{n} P_d(t,T_i)[(b_i - 1) + \tau_i(b_i - \bar{b})L_d(0,\tau_i)].$$

Remark 2 *If the spread is flat, i.e.* $b_i = \bar{b}$ *for each* i*, then the above approximation is exact.*

For simplicity, we assume that the fixed leg has the same frequency of the floating leg. Then the price of a swap is

$$\sum_{i=1}^{n}(c_i - g_i)P_d(t,T_i) - \bar{b}, \tag{29}$$

where $g_i = (b_i - 1) + \tau_i(b_i - \bar{b}_i)L_d(0,\tau_i)$ for $i < n$ and $g_n = (b_n - \bar{b}) + \tau_n(b_n - \bar{b})L_d(0,\tau_n)$.

In general, g_i is very small, so we can assume $c_i > g_i$. Then a similar argument to the single-curve case can be used to show the following

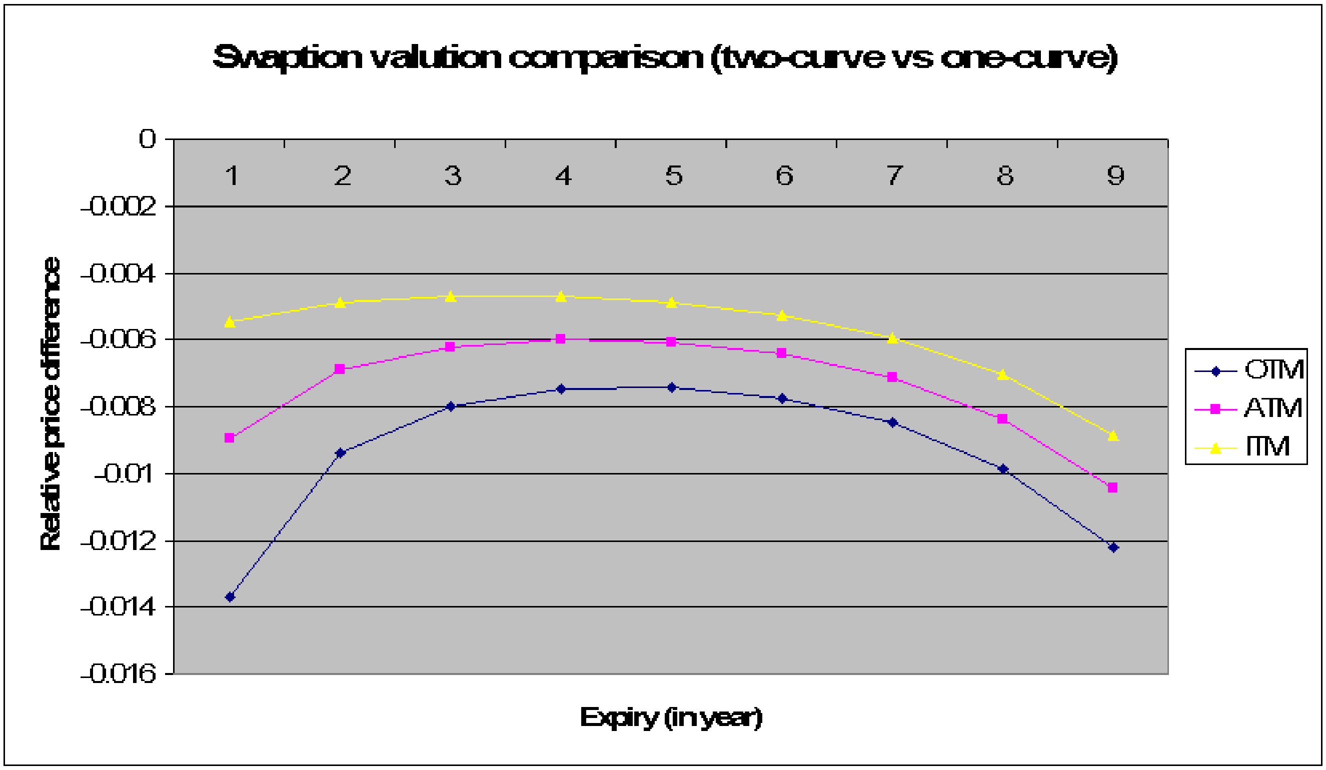
Proposition 3 *The price of a receiver European swaption is*

$$N\sum_{i=1}^{n}(c_i - g_i)ZBC(T,T_i,X_i), \tag{30}$$

where $X_i = A_d(T,T_i)e^{-B(T,T_i)r^*}$ *and* $r*$ *is a spot rate at time* T *such that*

$$\sum_{i=1}^{n}(c_i - g_i)A_d(T,T_i)e^{-B(T,T_i)r^*} = \bar{b}. \tag{31}$$

To compare the two-curve pricing method with the one-curve one, we used nine receiver swaptions with three different status (in the money, at the money and out the money). The underlying swap will mature in ten years, these nine swaptions are options to enter the swap at different times (one-year, two-year, ..., nine-year). The second discount factor curve
is obtained from the first one by adding 10 basis points. The comparison result is shown in the folloing graph.



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