

HULL-WHITE ONE FACTOR MODEL: RESULTS AND IMPLEMENTATION

QUANTITATIVE RESEARCH



ABSTRACT. Details regarding the implementation of the Hull-White one factor model are provided. The details concern the model description and parameters, the vanilla instruments pricing and the Monte Carlo implementation.

1. INTRODUCTION

This document provides a brief description of the Hull-White / extended Vasicek model (Hull and White [1990]) and possible implementations. A general overview of the model can be found in Brigo and Mercurio [2006]. When a specific volatility function is required, a piecewise constant volatility and constant mean reversion is used. The document provides the results necessary for the implementation, the theoretical developments are not provided. The reader is referred to the original papers and books mentioned in the text for the theoretical justifications. The notations of the multi-curves framework are the one of Henrard [2010a].

2. IBOR AND SWAPS

Let $P^D(t, u)$ be the discounting curve. Given the forward curve $P^j(t, u)$, the Ibor fixing in t_0 for the period $[t_1, t_2]$ is

$$(1) \quad I_{t_0}^j(t_1, t_2) = \frac{1}{\delta} \left(\frac{P^j(t_0, t_1)}{P^j(t_0, t_2)} - 1 \right)$$

The forward Ibor rate is defined in t by

$$(2) \quad F_t^j(t_0, t_1) = \frac{1}{\delta} \left(\frac{P^j(t, t_0)}{P^j(t, t_1)} - 1 \right)$$

where δ is the accrual fraction between t_0 and t_1 in the day count fraction associated to the Ibor rate.

An swap (IRS) exchanges a set of n fixed cash-flows against floating (Ibor) rates paid in their natural day count convention on their natural periods. An IRS is described by a set of fixed coupons or cash flows c_i at dates \tilde{t}_i ($1 \leq i \leq \tilde{n}$). For those flows, the discounting is used. It also contains a set of floating coupons over the periods $[t_{i-1}, t_i]$ with $t_i = t_{i-1} + j$ ($1 \leq i \leq n$)¹.

The value of a (fixed rate) receiver IRS is

$$(3) \quad \sum_{i=1}^{\tilde{n}} c_i P^D(t, \tilde{t}_i) - \sum_{i=1}^n P^D(t, t_i) \left(\frac{P^j(t, t_{i-1})}{P^j(t, t_i)} - 1 \right).$$

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Version 1.1.

¹In practice, due to weekends and holidays, the periods used for the fixings can be slightly different from the payment dates. We will not make that distinction here.

Let $\tilde{\delta}_i$ be the accrual fraction associated to the fixed coupons paid in \tilde{t}_i . The level or present value of a basis point of the swap is given by

$$\text{PVBP}_t = \sum_{i=1}^n \tilde{\delta}_i P^D(t, \tilde{t}_i).$$

The (forward) swap rate is given by

$$S_t = \frac{\sum_{i=1}^n P^D(t, t_i) \left(\frac{P^j(t, t_{i-1})}{P^j(t, t_i)} - 1 \right)}{\text{PVBP}_t}.$$

We define

$$(4) \quad \beta_t^j(u, u+j) = \frac{P^j(t, u)}{P^j(t, u+j)} \frac{P^D(t, u+j)}{P^D(t, u)}.$$

With that definition, a floating coupon price is

$$P^D(t, t_i) \left(\frac{P^j(t, t_{i-1})}{P^j(t, t_i)} - 1 \right) = \beta_t^j(t_{i-1}, t_i) P^D(t, t_{i-1}) - P^D(t, t_i).$$

Throughout this note we assume the constant spread hypothesis **S0** of [Henrard \[2010a\]](#).

The swap is often represented by its cash-flow equivalent $(t_i, d_i)_{i=0, \dots, n}$. The date t_0 is the settlement date and $(t_i)_{i=1, \dots, m}$ the payment dates (fixed and floating legs). The cash flow amounts are $c_0 = -\beta^j(t_0, t_1)$, d_i ($i = 1, \dots, n-1$) the coupons including the adjustments for floating coupons and $d_n = 1 + c_{\tilde{n}}$ the final coupon plus 1 for the notional.

Note that the swap and Ibor rates can be computed using only the re-based discount factors in any numeraire as both involve only ratios.

3. MODEL

A term structure model describes the behavior of $P^D(t, u)$, the price in t of the zero-coupon bond paying 1 in u ($0 \leq t, u \leq T$). When the discount curve $P^D(t, \cdot)$ is absolutely continuous, which is something that is always the case in practice as the curve is constructed by some kind of interpolation, there exists $f(t, u)$ such that

$$(5) \quad P^D(t, u) = \exp \left(- \int_t^u f(t, s) ds \right).$$

The short rate associated to the curve is $(r_t)_{0 \leq t \leq T}$ with $r_t = f(t, t)$. The cash-account numeraire is

$$N_t = \exp \left(\int_0^t r_s ds \right).$$

3.1. Short rate model. A good reference for the description of the short rate approach is [\[Brigo and Mercurio, 2006, Section 3.3\]](#). The book description refers to the case of the constant volatility model. The stochastic (one factor) equation for the short rate is, in the cash-account numeraire,

$$(6) \quad dr_t = (\theta(t) - ar_t) dt + \eta(t) dW_t.$$

When the numeraire is changed from the cash-account to $P^D(t, u)$, the equation is

$$(7) \quad dr_t = (\theta(t) - ar_t - \eta(t)\nu(t, u)) dt + \eta(t) dW_t$$

where ν is defined below.

3.2. Heath-Jarrow-Morton. The idea of [Heath et al. \[1992\]](#) was to model f with a stochastic differential equation

$$df(t, u) = \mu(t, u)dt + \sigma(t, u)dW_t$$

for some suitable μ and σ and deducing the behavior of P^D from there. To ensure the arbitrage-free property of the model, a relationship between the drift and the volatility is required. The model technical details can be found in the original paper or in the chapter *Dynamical term structure model* of [Hunt and Kennedy \[2004\]](#).

To simplify the writing, the notation

$$\nu(t, u) = \int_t^u \sigma(t, s)ds$$

is used.

The equations of the model in the cash-account numeraire measure associated to N_t are

$$df(t, u) = \sigma(t, u)\nu(t, u)dt + \sigma(t, u)dW_t.$$

The following *separability hypothesis* will be used:

H: The function σ is deterministic and satisfies $\sigma(s, t) = g(s)h(t)$ for some positive functions g and h .

The [Hull and White \[1990\]](#) volatility model satisfies the condition (H) with $\nu(s, t) = (1 - \exp(-a(t-s)))\eta(s)/a$ and $\sigma(s, t) = \eta(s)\exp(-a(t-s))$.

The model is analyzed with a piecewise constant volatility. By this we mean that there exists $0 = \tau_0 < \tau_1 < \dots < \tau_n = +\infty$ such that $\eta(s) = \eta_i$ for $\tau_{i-1} \leq s \leq \tau_i$.

4. PRELIMINARY RESULTS

The forward volatility of a re-based zero-coupon bond is the positive number defined by

$$\alpha(\theta_0, \theta_1, u, v)^2 = \int_{\theta_0}^{\theta_1} (\nu(s, v) - \nu(s, u))^2 ds.$$

The expiry dates are between some of the dates defining the piecewise constant function. The dates are denoted $\tau_p \leq \theta_0 < \theta_1 \leq \tau_q$. To shorten the notation an intermediary notation is used: $r_p = \theta_0 < r_l = \tau_l < r_q = \theta_1$. With those notations, one has

$$\alpha(\theta_0, \theta_1, u, v) = (\exp(-au) - \exp(-av)) \sqrt{\frac{1}{2a^3} \sum_{l=p}^{q-1} \eta_l^2 (\exp(2ar_{l+1}) - \exp(2ar_l))}.$$

Lemma 1. *Let $0 \leq t \leq s \leq u, v$. In a HJM one factor model, the price of the zero coupon bond can be written in the $P^D(\cdot, u)$ numeraire as*

$$(8) \quad \frac{P^D(s, v)}{P^D(s, u)} = \frac{P^D(t, v)}{P^D(t, u)} \exp\left(-\alpha(t, s, u, v)X_{s,t} - \frac{1}{2}\alpha^2(t, s, u, v)\right)$$

for a standard normally distributed random variable $X_{s,t}$ independent of \mathcal{F}_t .

Note that, thanks to the separability hypothesis, the variable $X_{s,t}$ is the same for all maturities v .

To speed-up Monte-Carlo simulations on several periods, it is useful to decompose the volatility into a maturity dependent part and an expiry part. Let $g(s) = \eta(s)\exp(as)$ and $h(t) = \exp(at)$. We define $H(u) = \int_0^u h(t)dt = \exp(au)/a$. The expiry dependent part is

$$\gamma(\theta_0, \theta_1) = \int_{\theta_0}^{\theta_1} g^2(s)ds = \frac{1}{2a} \sum_{l=p}^{q-1} \eta_l^2 (\exp(2ar_{l+1}) - \exp(2ar_l)).$$

The volatility is $\alpha(\theta_0, \theta_1, u, v) = \sqrt{\gamma(\theta_0, \theta_1)(H(v) - H(u))}$.

Let $s_0 = 0 < s_1 \cdots < s_i < \cdots$ and $Z_i = \int_{s_{i-1}}^{s_i} g(s) dW_s$. The stochastic variables can be written as

$$\alpha(0, s_i, u, v) X_{0, s_i} = (H(v) - H(u)) Y_i$$

where $Y_i = \sum_{j=1}^i Z_j$. The random variables Y_i are such that $Y_i = Y_{i-1} + Z_i$ with Z_i independent normally distributed with variance $\gamma(\tau_{i-1}, \tau_i)$.

5. CAP/FLOOR

The Cap and Floors are a set of caplet/floorlet. Each caplet is a zero-coupon bond on its period. The price of zero-coupon option is described in [Brigo and Mercurio, 2006, Section 3.3.2]. It can also be deduced from the coupon bond formula with $n = 1$. In this case the exercise boundary κ is explicit. In this section the price is described in the multi-curves framework under the constant spread hypothesis **S0**.

The expiry date is denoted θ . The start and end date of the rate are t_0 and t_1 and the fixing accrual factor is δ_I . The strike is denoted K , the payment date is t_p and the payment accrual factor is δ_p . For standard caplet, the payment date is equal to the end of fixing period ($t_p = t_1$). For non-standard caplet (in arrear, short/long tenor) the dates can be significantly different.

The volatility used is $\alpha_i = \alpha(0, \theta, t_p, t_i)$ ($i = 1, 2$).

Theorem 1 (Explicit cap/floor formula in Hull-White). *In the extended Vasicek model, the price of a cap with strike K is given at time 0 by*

$$\frac{\delta_p}{\delta_I} P^D(0, t_p) \left(\frac{P^j(0, t_0)}{P^j(0, t_1)} N(-\kappa - \alpha_0) - (1 + \delta_I K) N(-\kappa - \alpha_1) \right)$$

where κ is given by

$$\kappa = \frac{1}{\alpha_1 - \alpha_0} \left(\ln \left(\frac{(1 + \delta_I K) P^j(0, t_1)}{P^j(0, t_0)} \right) - \frac{1}{2} (\alpha_1^2 - \alpha_0^2) \right).$$

The price of a floor is given by

$$\frac{\delta_p}{\delta_I} P^D(0, t_p) \left((1 + \delta_I K) N(\kappa + \alpha_1) - \frac{P^j(0, t_0)}{P^j(0, t_1)} N(\kappa + \alpha_0) \right).$$

Note that book formulas can be slightly different as they usually do not consider the difference between expiry date (θ) and start date (t_0) and the difference between payment date t_p and end of fixing period t_1 .

6. EUROPEAN SWAPTIONS

6.1. Jamshidian trick (or decomposition). The standard pricing formula for physical delivery swaption in the model uses the Jamshidian decomposition proposed in Jamshidian [1989]. The details are also available in [Brigo and Mercurio, 2006, Section 3.11.1]. As this is not the most efficient implementation, we don't detail it here.

6.2. Explicit formula: physical delivery swaption. The formula was initially proposed in Henrard [2003]. It is adapted here for the multi-curve framework under the deterministic spread hypothesis **S0**.

Theorem 2 (Exact swaption price in Hull-White model). *Suppose we work in the HJM one-factor model with a separable volatility term satisfying (H) and in the multi-curves framework with hypothesis **S0**. Let $\theta \leq t_0 < \cdots < t_n$, $c_0 < 0$ and $c_i \geq 0$ ($1 \leq i \leq n$). The price of an European receiver swaption (with physical delivery), with expiry θ on a swap with cash-flows representation (t_i, d_i) is given at time t by the \mathcal{F}_t -measurable random variable*

$$\sum_{i=0}^n d_i P^D(t, t_i) N(\kappa + \alpha_i)$$

where κ is the \mathcal{F}_t -measurable random variable defined as the (unique) solution of

$$(9) \quad \sum_{i=0}^n d_i P^D(t, t_i) \exp\left(-\frac{1}{2}\alpha_i^2 - \alpha_i \kappa\right) = 0$$

and

$$\alpha_i = \alpha(t, \theta, \theta, t_i).$$

The price of the payer swaption is

$$-\sum_{i=0}^n d_i P^D(t, t_i) N(-\kappa - \alpha_i)$$

Note that the original proof required that $d_i > 0$ ($1 \leq i \leq n$) while here we have coupon equivalent like $1 - \beta^j$ with usually $\beta^j > 1$; some of the coupon equivalent are potentially slightly negative.

6.3. Approximated formula: physical delivery swaption. The results of this section are from [Henrard \[2009\]](#).

The forward values of the zero-coupon bonds and swap without the initial notional are given in t by

$$P_t^i = \frac{P^D(t, t_i)}{P^D(t, t_0)} \quad \text{and} \quad B_t = \frac{\sum_{i=1}^n d_i P^D(t, t_i)}{P^D(t, t_0)}.$$

Those value are the value re-based by the numeraire $P(\cdot, t_0)$

Let $\nu^i(t) = (\nu(t, t_i) - \nu(t, t_0))$ ($0 \leq i \leq n$). In the martingale probability associated to the numeraire $P^D(\cdot, t_0)$, the re-based prices are martingale that satisfy the equations

$$dP_t^i = -P_t^i \nu^i(t) \cdot dW_t^0.$$

The zero-coupon bond prices are exactly log-normal as the volatility $\nu^i(t)$ is deterministic.

The re-based swap value satisfy

$$dB_t = -\sum_{i=1}^n c_i P_t^i \nu^i(t) \cdot dW_t^0.$$

Using the notation $\alpha_t^i = c_i P_t^i / B_t$ and $\sigma(t) = \sum \alpha_t^i \nu^i(t)$, the equation becomes

$$dB_t = -B_t \sigma(t) \cdot dW_t^0.$$

This is formally a log-normal equation but the σ coefficient is state dependent.

A *strike* value of the different parameters is selected. The swap price is *at-the-money* at expiry when

$$\sum_{i=0}^n c_i P_\theta^i = 0$$

The discounting value of the zero-coupon bond can be approximated (initial freeze) by

$$P_\theta^i = \frac{P^D(0, t_i)}{P^D(0, t_0)} \exp\left(-\tau_i X_i - \frac{1}{2}\tau_i^2\right)$$

with

$$\tau_i^2 = \int_0^\theta (\nu_0^i(t))^2 ds = \alpha^2(0, \theta, t_i, t_0)$$

and the X_i standard normally distributed random variables. By choosing (arbitrarily) to have all the stochastic variables X_i equal at the strike the (one dimensional) equation to solve is

$$\sum_{i=0}^n c_i P_0^i \exp\left(-\tau_i \bar{x} - \frac{1}{2}\tau_i^2\right) = 0.$$

Obtaining the solution to the above equation requires to solve a one dimensional equation equivalent to the one solved in the swaption price (see above). For numerical reasons one may prefers not

to have to solve this type of equation. The above equation can be replaced by its first order approximation

$$\sum_{i=0}^n c_i P_0^i \left(1 - \tau_i \bar{x} - \frac{1}{2} \tau_i^2 \right),$$

the solution of which is explicit:

$$\bar{x} = \frac{\sum_{i=0}^n c_i P_0^i - \frac{1}{2} \sum_{i=0}^n c_i P_0^i \tau_i^2}{\sum_{i=0}^n c_i P_0^i \tau_i}.$$

The zero-coupon prices, in the exponential case and the approximated first order case, are given by

$$P_K^i = P_0^i \exp \left(-\tau_i \bar{x} - \frac{1}{2} \tau_i^2 \right), \quad \text{respectively} \quad P_K^i = P_0^i \left(1 - \tau_i \bar{x} - \frac{1}{2} \tau_i^2 \right).$$

The rates and bond prices are

$$\prod_{j=0}^{i-1} (1 + \delta_j L_K^j) = (P_K^i)^{-1} \quad \text{and} \quad B_K = \sum_{i=1}^n c_i P_K^i = K.$$

By defining

$$\alpha_K^i = \frac{c_i P_K^i}{B_K}$$

the swaption can be priced with a (option strike dependent) approximated volatility

$$(10) \quad \bar{\sigma}_K(t) = \frac{1}{2} \left(\sum_{i=1}^n (\alpha_0^i + \alpha_K^i) \nu^i(t) \right).$$

Note that in the multi-factor model, the volatility $\bar{\sigma}_K(t)$ is a vector, as ν_0 and ν_K are.

Theorem 3 (Approximated swaption price in Hull-White model). *In the extended Vasicek model, the price, with initial freeze and corrector approximation, of a receiver swaption is given at time 0 by*

$$R_0 = P^D(0, t_0) (B_0 N(\kappa_K + \bar{\sigma}_K) - K N(\kappa_K))$$

where

$$\kappa_K = \frac{1}{\bar{\sigma}_K} \left(\ln(B_0/K) - \frac{1}{2} \bar{\sigma}_K^2 \right)$$

and

$$\bar{\sigma}_K^2 = \int_0^\theta (\bar{\sigma}_K(t))^2 dt.$$

The price of a payer swaption is

$$P_0 = P^D(0, t_0) (K N(-\kappa_K) - B_0 N(-\kappa_K - \bar{\sigma}_K))$$

6.4. Approximated formula: cash-settled swaption. An efficient approximated formula for cash-settled swaptions is proposed in [Henrard, 2010b, Appendix A].

7. INTEREST RATE FUTURES

A general pricing formula for eurodollar futures in the Gaussian HJM model was proposed in Henrard [2005]. The formula extended a previous result proposed in Kirikos and Novak [1997]. The extension to the multi-curve framework was proposed in Henrard [2010a] and is reproduced here.

The futures are liquid only for the three month Ibor up to two or three years. To a lesser extent some one month futures are available on the shorter part of the curve.

The future fixing date is denoted t_0 . The fixing is on the Libor rate between $t_1 = \text{Spot}(t_0)$ and $t_2 = t_1 + j$. The accrual factor for the period $[t_1, t_2]$ is δ , the fixing is linked to the yield curve by

$$1 + \delta L_{t_0}^j = \frac{P^j(t_0, t_1)}{P^j(t_0, t_2)}.$$

The futures price is Φ_t^j . On the fixing date, the relation between the price and the rate is

$$\Phi_{t_0}^j = 1 - L_{t_0}^j.$$

The futures margining is done on the futures price (multiplied by the notional and divided by 4).

Theorem 4. *Let $0 \leq t \leq t_0 \leq t_1 \leq t_2$. In the HJM one-factor model on the discount curve under the hypotheses **D**, **L** and **SI**, the price of the futures fixing on t_0 for the period $[t_1, t_2]$ with accrual factor δ is given by*

$$(11) \quad \begin{aligned} \Phi_t^j &= 1 - \frac{1}{\delta} \left(\frac{P^j(t, t_1)}{P^j(t, t_2)} \gamma(t) - 1 \right) \\ &= 1 - \gamma(t) F_t^j + \frac{1}{\delta} (1 - \gamma(t)) \end{aligned}$$

where

$$\gamma(t) = \exp \left(\int_t^{t_0} \nu(s, t_2) (\nu(s, t_2) - \nu(s, t_1)) ds \right).$$

Proof. Using the generic pricing future price process theorem [Hunt and Kennedy, 2004, Theorem 12.6],

$$\Phi_t^j = E_{\mathbb{N}} \left[1 - L_{t_0}^j \middle| \mathcal{F}_t \right].$$

In $L_{t_0}^j$, the only non-constant part is the ratio of j -discount factors which is, up to $\beta_{t_0}^j$, the ratio of D -discount factors. Using [Henrard, 2005, Lemma 1] twice, we obtain, with W_s the Brownian motion associated to the N -numeraire,

$$\begin{aligned} \frac{P^D(t_0, t_1)}{P^D(t_0, t_2)} &= \frac{P^D(t, t_1)}{P^D(t, t_2)} \exp \left(-\frac{1}{2} \int_t^{t_0} \nu^2(s, t_1) - \nu^2(s, t_2) ds \right. \\ &\quad \left. + \int_t^{t_0} \nu(s, t_1) - \nu(s, t_2) dW_s \right). \end{aligned}$$

Only the second integral contains a stochastic part. This integral is normally distributed with variance $\int_t^{t_0} (\nu(s, t_1) - \nu(s, t_2))^2 ds$. So the expected discount factors ratio value is reduced to

$$\begin{aligned} \frac{P^D(t, t_1)}{P^D(t, t_2)} &\exp \left(-\frac{1}{2} \int_t^{t_0} \nu^2(s, t_1) - \nu^2(s, t_2) ds \right. \\ &\quad \left. + \int_t^{t_0} (\nu(s, t_1) - \nu(s, t_2))^2 ds \right) \end{aligned}$$

The coefficient $\beta_{t_0}^j$ is independent of the D -discount factors ratio and a martingale, hence we have the announced result. \square

8. OTHER INSTRUMENTS

8.1. Bermudan swaptions. The pricing of Bermudan swaptions in the one-factor Gaussian HJM model using an iterate numerical integration procedure is presented in Henrard [2008b].

8.2. CMS and CMS cap/floor. The pricing of CMS and CMS cap/floor in the one-factor Gaussian HJM model using approximations is presented in Henrard [2008a].

9. MONTE CARLO

Suppose we work in the $P^D(., u)$ numeraire. The numeraire choice should be adapted for each product (see below).

Let $(s_i)_{i=0, \dots, n_F}$ with $s_0 = 0$ be the fixing and expiry dates used in the pricing. The fixing dates can be related to Ibor or swap (CMS) rates. The dates used in s_i for payments and to compute the fixings are denoted $(t_{i,j})_{i=1, \dots, n_F; j=0, \dots, n_R(i)}$. Some examples are given below. The payment dates are $t_{i,j}$ with $0 \leq j \leq n_P(i)$ and the dates used for the fixing computation are $t_{i,j}$ with $n_P(i) + 1 \leq j \leq n_R(i)$.

The re-based discount factor initial values are denoted

$$P_{0,i,j}^F = \frac{P^D(0, t_{i,j})}{P^D(0, u)}.$$

The re-based discount factors used in the pricing are, for $0 \leq i \leq n_F$, $i \leq j \leq n_F$ and $0 \leq k \leq n_R(i)$

$$P_{i,j,k}^F = \frac{P^D(s_i, t_{j,k})}{P^D(s_i, u)}.$$

At each fixing dates s_i , amounts to be paid in $t_{i,j}$ ($0 \leq j \leq n_P(i)$) are

$$C_{i,j} = F \left((P_{l,l,k}^F)_{0 \leq l \leq i; n_P(l)+1 \leq k \leq n_R(l)} \right).$$

The amounts will be made explicit for each specific product (see below).

In general, for each i , there is a limited number of payment dates (one or two). Also the amount often depend directly only on $P_{i,i,k}^F$; the dependence on the previous rates is indirect through $C_{i-1, \cdot}$.

In the Monte Carlo approach, the price on one path of the above option is computed as

$$P^D(0, u) \sum_{i=0, \dots, n} \sum_{0 \leq j \leq n_P(i)} P_{i,i,j}^F C_{i,j}.$$

9.1. HJM (jump between fixing dates). The values $P_{i,i,j}^F$ are simulated from their initial values $P_{0,i,j}^F$ and repeated HJM evolution (8). For the evolution, one needs to compute the quantities $\alpha(s_{l-1}, s_l, u, t_{i,j})$. Those quantities need to be computed only once and not for each path. The one step evolution is

$$P_{l,i,k}^F = P_{l-1,i,k}^F \exp \left(-\frac{1}{2} \alpha^2(s_{l-1}, s_l, u, t_{i,k}) - \alpha(s_{l-1}, s_l, u, t_{i,k}) X_l \right).$$

For one path, the $(X_l)_{1 \leq l \leq n_F}$ are independent standard normally distributed draws. Even if only the final value $P_{i,i,k}^F$ is used, it is important to estimate $P_{l,i,k}^F$ from X_l to have a coherence in the paths. The final value can be computed as

$$P_{i,i,k}^F = P_{0,i,k}^F \exp \left(-\frac{1}{2} \sum_{l=1}^i \alpha^2(s_{l-1}, s_l, u, t_{i,k}) - \sum_{l=1}^i \alpha(s_{l-1}, s_l, u, t_{i,k}) X_l \right).$$

This last expression is faster to implement than the recursive one above (less exponential and multiplication to compute).

The simulation is done with *long jumps*, i.e. the steps are from one fixing date to the next, without extra time discretization.

9.2. HJM (one very long jump to each fixing date). The idea is that each discounting factor $P_{i,i,k}^F$ is computed directly from its initial value and one random variable Y_i by

$$P_{i,i,k}^F = P_{0,i,j}^F \exp \left(H_{i,j} Y_i - \frac{1}{2} H_{i,j}^2 \gamma(0, s_i) \right)$$

with $H_{i,j} = H(t_{i,j}) - H(u)$ and $Y_i = Y_{i-1} + Z_i$ as described in Section 4.

The variables Y_i are not independent; they can be viewed as equal on some time interval with one of them continuing after. The covariance between Y_i and Y_j (with $i \leq j$) is the variance of Y_i and is

$$\sum_{k=1}^i \int_{s_{k-1}}^{s_k} g^2(s) ds = \int_0^{s_i} g^2(s) ds = \gamma(0, s_i).$$

As the variables are not independent anymore, we simulate independent variables and multiply them by the Cholesky decomposition of the covariance matrix.

9.3. European swaptions. For the European swaption, there is only one fixing date, the expiry date: $s_1 = \theta$. Let $(d_i, t_i)_{0 \leq i \leq n}$ be the cash flow equivalent of the underlying swap as described in Section 2. Let $t_{1,j} = t_{1,n+1+j} = t_j$ ($0 \leq j \leq n$), $n_R(1) = 2n$ and $n_P(1) = n$. With that notation, the payoff of the European swaption is

$$\left(\sum_{j=1}^{n_P(1)} d_j P_{1,1,j}^F \right)^+,$$

or in the Monte-Carlo notations,

$$C_{1,j} = d_j \mathbb{1} \left(\sum_{j=1}^{n_P(1)} d_j P_{1,1,j}^F > 0 \right).$$

As the cash flows are regularly distributed over the period (almost -1 in t_0 and $1 + c_n$ in t_n), the numeraire choice has a very small impact. We suggest to use $u = t_0$.

9.4. CMS and CMS cap/floor. The description is done for only one payment. There is only one fixing date s_1 and one payment date $t_{1,0}$. Let $(d_i, t_i)_{0 \leq i \leq n}$ be the cash flow equivalent of the underlying swap as described in Section 2. Let $t_{1,j+1} = t_j$ ($0 \leq j \leq n$), $n_R(1) = n + 1$ and $n_P(1) = 1$. For a cap, $\omega = 1$ and for a floor $\omega = -1$. With that notation, the payoff of the CMS cap/floor is in $t_{1,0}$,

$$C_{1,0} = (\omega(S_{s_1} - K))^+.$$

There is only one cashflow, paid in $t_{1,0}$; we suggest to use $u = t_{1,0}$.

9.5. Ratchet (on Ibor). The ratchet is described by a set of fixing dates s_i ($1 \leq i \leq n_F$). At each fixing date a Ibor rate is recorded. To each fixing date a payment date $t_{i,0}$ is associated and the fixing rate depends on two dates $t_{i,j}$ ($1 \leq j \leq 2$). The Ibor rate I_{s_i} is given by Equation 1.

To each coupon are associated three coefficients: a memory α_i^M , a multiplicative coefficient β_i^M and an additive term γ_i^M . The amount paid is the coupon multiplied by the accrual fraction and the notional.

For a floating first coupon, the amount is

$$C_{1,0} = \beta_1^M I_{s_1} + \gamma_1^M$$

and for a fixed first coupon c the amount is $C_{1,0} = c$. From the second coupon, each coupon is associated an upper barrier (cap) with memory α_i^C , multiplicative coefficient β_i^C and an additive term γ_i^C and a down barrier (floor) with memory α_i^F , multiplicative coefficient β_i^F , and additive term γ_i^F . The coupon, paid in $t_{i,0}$, is given by

$$C_{i,0} = \max(\alpha_i^F C_{i-1,0} + \beta_i^F I_{s_i} + \gamma_i^F, \min(\alpha_i^C C_{i-1,0} + \beta_i^C I_{s_i} + \gamma_i^C, \alpha_i^M C_{i-1,0} + \beta_i^M I_{s_i} + \gamma_i^M)).$$

10. IMPLEMENTATION

The one-factor Hull-White model parameters with constant mean reversion and piecewise constant volatility are available from the object `HullWhiteOneFactorPiecewiseConstantParameters`. The different functions related to the model (α , γ , H , etc.) are available in the class `HullWhiteOneFactorPiecewiseConstantInterestRateModel`.

The pricing of Ibor cap/floor is available in `CapFloorIborHullWhiteMethod`.

The pricing of interest rate futures is available in `InterestRateFutureHullWhiteMethod`.

The pricing of European physical delivery swaptions with the exact formula is available in `SwaptionPhysicalFixedIborHullWhiteMethod` and the approximated formula in `SwaptionPhysicalFixedIborHullWhiteApproximationMethod`. The pricing is also available with a numerical integration in `SwaptionPhysicalFixedIborHullWhiteNumericalIntegrationMethod`. This last method is more for testing and analysis than for production; the exact formula is faster and more precise.

The pricing of European cash-settled swaptions by approximation is available in `SwaptionCashFixedIborHullWhiteApproximationMethod`. The numerical integration approach is available in `SwaptionCashFixedIborHullWhiteNumericalIntegrationMethod`.

The pricing of Bermuda swaptions is available in `SwaptionBermudaFixedIborHullWhiteNumericalIntegrationMethod`.

The pricing of CMS coupons is available in `CouponCMSHullWhiteApproximationMethod`; the numerical integration approach is available in `CouponCMSHullWhiteNumericalIntegrationMethod`. The pricing of CMS cap/floor by approximation is available in `CapFloorCMSHullWhiteApproximationMethod`; the numerical integration approach is available in `CapFloorCMSHullWhiteNumericalIntegrationMethod`.

The pricing with Monte Carlo is available in `HullWhiteMonteCarloMethod` and `MonteCarloDiscountFactorCalculator`.

REFERENCES

- D. Brigo and F. Mercurio. *Interest Rate Models, Theory and Practice*. Springer Finance. Springer, second edition, 2006. 1, 2, 4
- D. Heath, R. Jarrow, and A. Morton. Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation. *Econometrica*, 60(1):77–105, January 1992. 3
- M. Henrard. Explicit bond option and swaption formula in Heath-Jarrow-Morton one-factor model. *International Journal of Theoretical and Applied Finance*, 6(1):57–72, February 2003. 4
- M. Henrard. Eurodollar futures and options: Convexity adjustment in HJM one-factor model. Working paper 682343, SSRN, March 2005. Available at ssrn.com/abstract=682343. 6, 7
- M. Henrard. CMS swaps and caps in one-factor Gaussian models. Working Paper 985551, SSRN, February 2008a. URL ssrn.com/abstract=985551. Available at ssrn.com/abstract=985551. 7
- M. Henrard. Bermudan swaptions in Gaussian HJM one-factor model: Analytical and numerical approaches. Working Paper 1287982, SSRN, 2008b. URL ssrn.com/abstract=1287982. Available at ssrn.com/abstract=1287982. 7
- M. Henrard. Efficient swaptions price in Hull-White one factor model. Quantitative finance, arXiv, 2009. URL arxiv.org/abs/0901.1776. Available at arxiv.org/abs/0901.1776. 5
- M. Henrard. The irony in the derivatives discounting - Part II: the crisis. *Wilmott Journal*, 2(6): 301–316, December 2010a. 1, 2, 6
- M. Henrard. Cash-settled swaptions: How wrong are we? Technical report, OpenGamma, 2010b. URL ssrn.com/abstract=1703846. Available at ssrn.com/abstract=1703846. 6
- J. Hull and A. White. Pricing interest rate derivatives securities. *The Review of Financial Studies*, 3:573–592, 1990. 1, 3
- P. J. Hunt and J. E. Kennedy. *Financial Derivatives in Theory and Practice*. Wiley series in probability and statistics. Wiley, second edition, 2004. ISBN 0-470-86359-5. 3, 7

- F. Jamshidian. An exact bond option formula. *The journal of Finance*, XLIV(1):205–209, march 1989. 4
- G. Kirikos and D. Novak. Convexity conundrums. *Risk*, pages 60–61, March 1997. 6

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