Portfolio Insurance Strategies : OBPI versus CPPI

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Abstract

The purpose of this article is to analyze and compare two standard portfolio insurance methods: OBPI and CPPI. Various criteria are used such as comparison of payoffs at maturity, stochastic or "quantile" dominance of their returns...Dynamic hedging properties are also examined, in particular classical delta hedging.

1 Introduction

Portfolio insurance is designed to give the investor the ability to limit downside risk while allowing some participation in upside markets. Such methods allow investors to recover, at maturity, a given percentage of their initial capital, in particular in falling markets. There exist various portfolio insurance models, among them the *Option Based Portfolio Insurance* (OBPI) and the *Constant Proportion Portfolio Insurance* (CPPI).

The OBPI, introduced by Leland and Rubinstein (1976), consists of a portfolio invested in a risky asset S (usually a financial index such as the S&P) covered by a listed put written on it. Whatever the value of S at the terminal date T, the portfolio value will be always greater than the strike K of the put. At first glance, the goal of the OBPI method is to guarantee a fixed amount only at the terminal date. In fact, as recalled and analyzed in this paper, the OBPI method allows one to get a portfolio insurance at any time. Nevertheless, the European put with suitable strike and maturity may be not available on the market. Hence it must be synthesized by a dynamic replicating portfolio invested in a riskfree asset (for instance, T-bills) and in the risky asset.

The CPPI was introduced by Perold (1986) (see also Perold and Sharpe (1988)) for fixed-income instruments and Black and Jones (1987) for equity instruments. This method uses a simplified strategy to allocate assets dynamically over time. The investor starts by setting a floor equal to the lowest acceptable value of the

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portfolio. Then, she computes the cushion as the excess of the portfolio value over the floor and determines the amount allocated to the risky asset by multiplying the cushion by a predetermined multiple. Both the floor and the multiple are functions of the investor's risk tolerance and are exogenous to the model. The total amount allocated to the risky asset is known as the exposure. The remaining funds are invested in the reserve asset, usually T-bills.

The higher the multiple, the more the investor will participate in a sustained increase in stock prices. Nevertheless, the higher the multiple, the faster the portfolio will approach the floor when there is a sustained decrease in stock prices. As the cushion approaches zero, exposure approaches zero too. In continuous time, this keeps portfolio value from falling below the floor. Portfolio value will fall below the floor only when there is a very sharp drop in the market before the investor has a chance to trade.

Bookstaber and Langsam (2000) analyze properties of both these portfolio insurance models. They focus on path dependence, showing that only option-replicating strategies provide path independence. They deal also with the problem of the time horizon and in particular time-invariant or perpetual strategies (further studied in Black and Perold (1992)).

Black and Rouhani (1989) compare CPPI with OBPI when the put option has to be synthesized. They compare the two payoffs and examine the role of both expected and actual volatilities. They show that "OBPI performs better if the market increases moderately. CPPI does better if the market drops or increases by a small or large amount".

The present paper extends their analysis to various criteria, introducing systematically the probability distributions of the two portfolio values. It is also focussed on the dynamics of both methods. In the first section, basic properties are recalled. Payoffs at maturity are compared by means of stochastic dominance, first four moments and some of their quantiles. In particular, the role of the insured amount is emphasized. In the second section, it is proved that the OBPI method is a generalized CPPI where the multiple is allowed to vary. The properties of this varying multiple are spelled out. It also focuses on the dynamics of both methods. Hedging properties involved by these two strategies are studied, when the option has to be synthesized. The "greeks" of the OBPI and the CPPI are derived. Their features show the different nature of the dynamic properties of the two strategies[1].

2 Comparison between standard OBPI and CPPI at maturity

To compare these two strategies, the natural first step is to examine their performances at maturity. The analysis of payoff functions gives a first insight. However, this comparison must take account of probabilities of for example bullish or bearish markets. This leads us to develop methods based on the first four moments. Methods based on quantiles can also be introduced and are developed in what follows.

[1] All the proofs are gathered in the Appendix.

2.1 Definition of the two strategies

The portfolio manager is assumed to invest in two basic assets : a money market account, denoted by B, and a portfolio of traded assets such as a composite index, denoted by S. The period of time considered is $[0, T]$. The strategies are self-financing.

The value of the riskless asset B evolves according to : $dB_t = B_t r dt$, where r is the deterministic interest rate.

The dynamics of the market value of the risky asset S are given by the classic diffusion process :

$$dS_t = S_t \left[\mu dt + \sigma dW_t\right]$$

where W_t is a standard Brownian motion.

The OBPI method consists basically of purchasing q shares of the asset S and q shares of European put options on S with maturity T and exercise price K. To simplify the presentation, we shall assume that q is normalized and set equal to one[2].

Thus, the portfolio value V^{OBPI} is given at the terminal date by :

$$V_T^{OBPI} = S_T + (K - S_T)^+$$

which is also : $V_T^{OBPI} = K + (S_T - K)^+$, due to the Put/Call parity. This relation shows that the insured amount at maturity is the exercise price, K.

The value V_t^{OBPI} of this portfolio at any time t in the period $[0, T]$ is :

$$V_t^{OBPI} = S_t + P(t, S_t, K) = K.e^{-r(T-t)} + C(t, S_t, K)$$

where $P(t, S_t, K)$ and $C(t, S_t, K)$ are the Black-Scholes values of the European put and call.

Note that, for all dates t before T, the portfolio value is always above the deterministic level $Ke^{-r(T-t)}$.

The amount insured at the final date is often expressed as a percentage p of the initial investment V_0 (with $p \leq e^{rT}$). Since, here, this amount is equal to the strike K itself, it is required that K is an increasing function of the percentage p, determined from the relation [3] :

$$pV_0(K) = p(K.e^{-rT} + C(0, S_0, K)) = K.$$

The CPPI method consists of managing a dynamic portfolio so that its value is above a floor F at any time t. The value of the floor gives the dynamical insured amount. It is assumed to evolve according to :

$$dF_t = F_t r dt$$

Obviously, the initial floor F_0 is less than the initial portfolio value V_0^{CPPI}. The difference $V_0^{CPPI} - F_0$ is called the cushion, denoted by C_0. Its value C_t at any time t in $[0, T]$ is given by :

$$C_t = V_t^{CPPI} - F_t$$

[2] Note that, if the initial investment value V_0 is fixed, then the number of shares q is a decreasing function of the strike K. Nevertheless, by the homogeneity property of portfolio values with respect to q, we can normalize q to 1 without loss of generality.

[3] This relation can also take account of the smile effect.

Denote by e_t the exposure, which is the total amount invested in the risky asset. The standard CPPI method consists of letting $e_t = mC_t$ where m is a constant called the multiple. The interesting case is when $m > 1$, that is, when the payoff function is convex.

The value of this portfolio V_t^{CPPI} at any time t in the period $[0, T]$ is[4] :

$$V_t^{CPPI}(m, S_t) = F_0.e^{rt} + \alpha_t.S_t^m$$
$$\text{where} \quad \alpha_t = \left(\frac{C_0}{S_0^m}\right)\exp[\beta t] \quad \text{and} \quad \beta = \left(r - m\left(r - \frac{1}{2}\sigma^2\right) - m^2\frac{\sigma^2}{2}\right)$$

Thus, the CPPI method is parametrized by F_0 and m. The OBPI has just one parameter, the strike K of the put. In order to compare the two methods, first the initial amounts V_0^{OBPI} and V_0^{CPPI} are assumed to be equal, secondly the two strategies are supposed to provide the same guarantee K at maturity. Hence, $F_T = K$ and then $F_0 = Ke^{-rT}$. Moreover, the initial value C_0 of the cushion is equal to the call price $C(0, S_0, K)$. Note that these two conditions do not impose any constraint on the multiple, m. In what follows, this leads us to consider CPPI strategies for various values of the multiple m.[5]

2.2 Comparison of the payoff functions

Is it possible that the payoff function of one of these two strategies lies above the other for all S_T values ? Since $V_0^{OBPI} = V_0^{CPPI}$, the absence of arbitrage implies the following result.

Proposition 1 *Neither of the two payoffs is greater than the other for all terminal values of the risky asset. The two payoff functions intersect one another.*

Figure 1 below illustrates what happens for a numerical example with typical values for the financial markets (parameters : μ, σ, r) : $S_0 = 100$, $\mu = 10\%$, $\sigma = 20\%$, $T = 1$, $K = S_0 = 100$, $r = 5\%$. Note that as m increases, the payoff function of the CPPI becomes more convex.

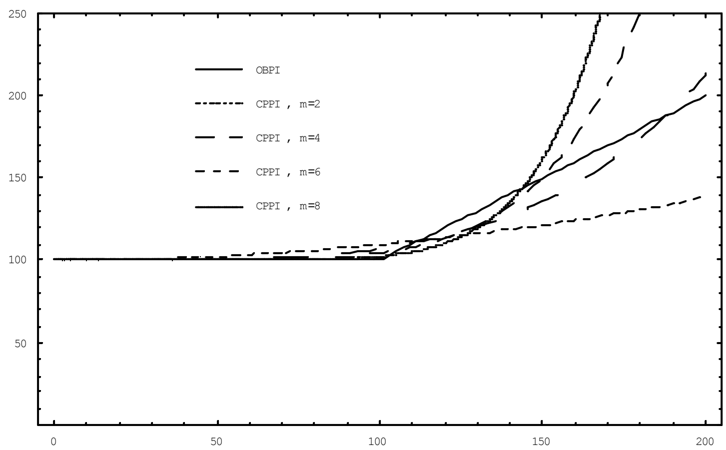

Figure 1 : CPPI and OBPI Payoffs as functions of S

[4] Details about this formula are provided in the Appendix.

[5] Note that the multiple must not be too high as shown for example in Prigent (2001) or in Bertrand and Prigent (2002).

We can check in this example that the two curves intersect one another for the different values of m considered ($m = 2$, $m = 4$, $m = 6$ and $m = 8$).

CPPI performs better for large fluctuations of the market while OBPI performs better in moderate bullish markets.

2.3 Comparison with the stochastic dominance criterion

To take account of the risky dimension of the terminal payoff functions for the two methods, first-order stochastic dominance is introduced.

Recall that a random variable X stochastically dominates a random variable Y at the first order ($X \succ Y$) if and only if the cumulative distribution function of X, denoted by F_X, is always below the cumulative distribution function F_Y of Y.

Proposition 2 *Neither of the two strategies stochastically dominates the other at first order.*

2.4 Comparison of the expectation, variance, skewness and kurtosis

When dealing with options, the mean-variance approach is not always justified since payoffs are not linear. Thus, we examine simultaneously the first four moments and the semi-variance of the rates of portfolio returns R_T^{OBPI} and R_T^{CPPI}.

Proposition 3 *For any parametrization of the financial markets, there exists at least one value for m such that the OBPI strategy dominates (is dominated), in a mean-variance (mean-semivariance) sense, (by) the CPPI one.*

The following example gives an illustration with the previous values of the parameters. The multiple m, solution of $E[R_T^{OBPI}] = E[R_T^{CPPI}]$, is equal to 5.77647. Table 1 contains the first four moments and the semi-volatility for the OBPI with an at-the-money call and for the corresponding CPPI with this particular value of the multiple[6].

Table 1
comparison of the first four moments

	OBPI	CPPI
expectation	8.61176 %	8.61176 %
volatility	16.8625 %	23.2395 %
semi volatility	9.1676%	7.7666%
relative skewness	1.49114	9.70126
relative kurtosis	5.4576	357.73

The OBPI dominates the CPPI in a mean-variance sense but is dominated by the CPPI if semi volatility is considered (as confirmed below by the relative skewness). Nevertheless, the CPPI has a higher positive relative skewness than the OBPI. Hence with respect to this criterion, CPPI should be preferred to OBPI. However, CPPI relative kurtosis is much higher than OBPI one. This feature is explained by the dominance of the CPPI payoff for small and high values of the risky asset S, as shown in figure 1. Note that, here, owing to the insurance feature, kurtosis arises mainly in the right tail of the distribution.

[6] The same qualitative result holds for calls in- and out-of-the-money.

2.5 Comparison of "quantiles"

In the present situation, where the distributions to be compared are strongly asymmetric, the study of the moments is not sufficient. The whole distribution has to be considered.

Figure 1 has showed comparison of payoff functions. The preceding analysis has to be extended by weighting each payoff by its probability of occurrence. Figure 2 below loosely illustrates the situation, where both payoff functions and risky asset density are depicted.

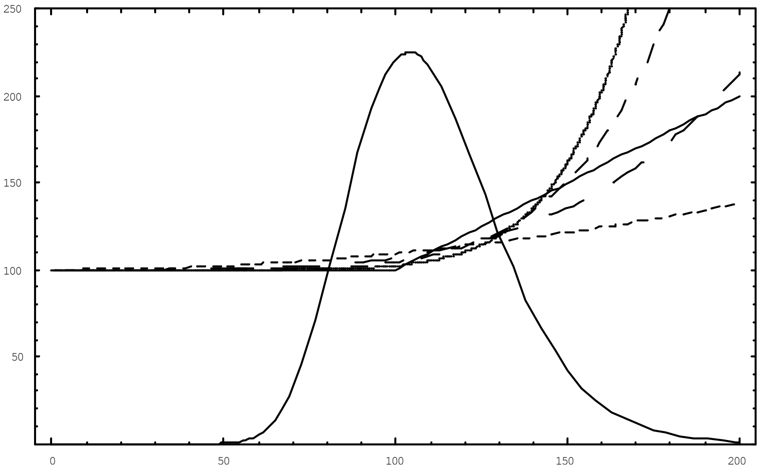

Figure 2 : CPPI and OBPI payoffs and probability of S.

The study of the distribution of the quotient of the CPPI value to the OBPI one allows a closer inspection of the effect of probabilities.

The plot of the cumulative distribution function of $\frac{V_T^{OBPI}}{V_T^{CPPI}}$ for different values[7] of K is :

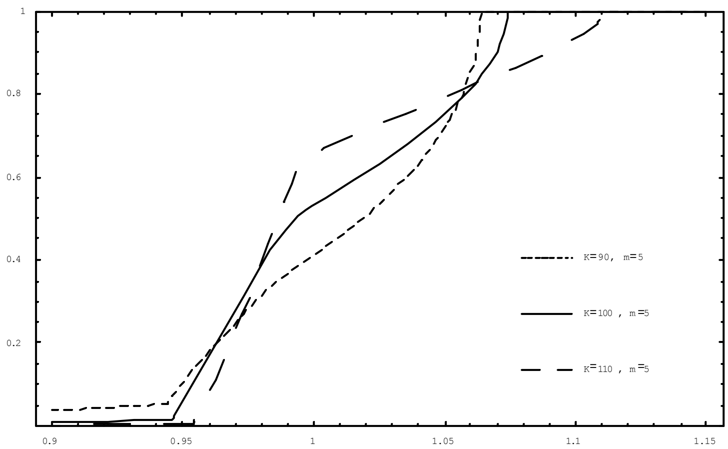

Figure 3 : Cumulative distribution of $\frac{V_T^{OBPI}}{V_T^{CPPI}}$.

This figure shows in particular that:

[7] Note that smile effect can also be taken into account, if a "Black-Scholes world" is no longer assumed.

- For the at-the-money call ($K = 100$), the probability that the CPPI portfolio value is higher than the OBPI one is approximately 0.5, meaning that neither of the strategies "dominates" the other.

- This is no longer true for $K = 90$ where the probability that the CPPI portfolio value is above the OBPI one is about 0.4. For $K = 110$, this probability takes the value 0.7. This arises because the probability of exercising the call decreases with the strike. Recall that the strike K is an increasing function of the insured percentage p of the initial investment. Thus, as p rises, the CPPI method seems to be more desirable than the OBPI method[8].

  Notice that, for in- and out-of-the-money calls, extreme values of the quotient are more likely to appear : on the one hand, the CPPI portfolio value can be at least equal to 106% of the OBPI portfolio value with probability 5% (respectively 0%) when $K = 90$ (respectively $K = 110$). On the other hand, the CPPI portfolio value can be at most equal to 94% of the OBPI portfolio value with probability 0% (respectively 18%) when $K = 90$ (respectively $K = 110$).

  The same qualitative results are obtained for other usual values of the multiple (m between 2 and 8).

3 The dynamic behavior of OBPI and CPPI

In many situations, the use of traded options is not possible[9]. For example, the portfolio to be insured may be a diversified fund for which no single option is available. The insurance period may also not coincide with the maturity of a listed option. Thus, for all these reasons, the OBPI put has often to be synthesized.

In this framework, both CPPI and OBPI induce dynamic management of the insured portfolio. As proved in what follows, the OBPI method is a generalized CPPI. Portfolio rebalancing implies hedging risk and transaction costs. Hence, hedging properties of both methods are to be analyzed, in particular the behavior of the quantity to invest on the risky asset at any time during the management period (the "delta" of the option).

3.1 OBPI as a generalized CPPI

For the CPPI method, the multiple is the key parameter that fixes the amount invested in the risky asset at any time. It also plays the role of a weight between performance and risk. Knowing the importance of the multiple, does there exist such an "implicit" parameter for the OBPI ?

Proposition 4 *The OBPI method is equivalent to the CPPI method in which the multiple is allowed to vary and is given by* $m^{OBPI}(t, S_t) = \frac{S_t N(d_1(t,S_t))}{C(t,S_t,K)}$.

[8] For $K = 90$, $p = 87,97\%$, for $K = 100$, $p = 94,72\%$ and for $K = 110$, $p = 99,39\%$

[9] As for OTC options, they have several drawbacks since they introduce a default risk, they are not liquid and their prices are often less competitive.

Thus, the OBPI multiple is a function of the risky asset value[10] S. It is equal to the amount invested on the risky asset to replicate the call option divided by the OBPI cushion which is the call value. It is a decreasing function of the risky asset value S as illustrated by the following figure.

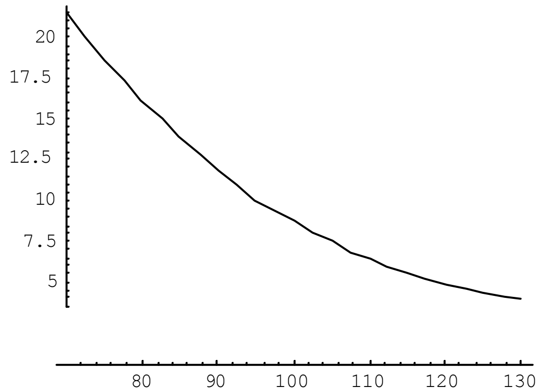

Figure 4 : OBPI multiple as a function of S at $t = 0.5$.

The OBPI multiple takes higher values than the standard CPPI multiple, except when the associated call is in-the-money. In particular, in a rising market, the OBPI method prevents the portfolio being over-invested in the risky asset, as the multiple is low.

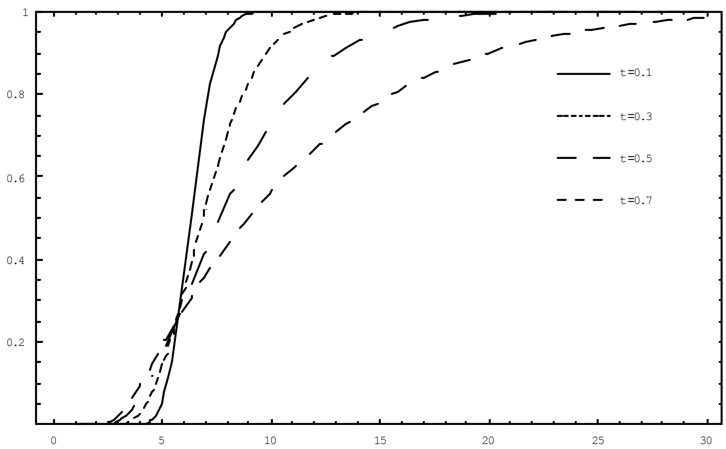

Figure 5 : OBPI multiple cumulative distribution.

Figure 5 shows the evolution of the OBPI multiple cumulative distribution with time[11]. As time increases, the probability of obtaining high values of the multiple increases. This essentially comes from the rise in the variance with time.

We now study the dynamic properties of the two strategies and in particular their "greeks".

[10] Such more general multiples have been introduced and studied in Prigent (2001).

[11] Near the maturity of the call, the multiple is undetermined when the call is not exercisable since the cushion and the exposure are nil. Otherwise, the multiple converges to $\frac{S_T}{S_T - K}$.

3.2 The Delta

The delta of the OBPI is obviously the delta of the call. For the CPPI, it is given by :

$$\Delta^{CPPI} = \frac{\partial V_t^{CPPI}}{\partial S_t} = \alpha m S_t^{m-1}.$$

The following figure shows the evolution of the delta as a function of the risky asset value S_t.

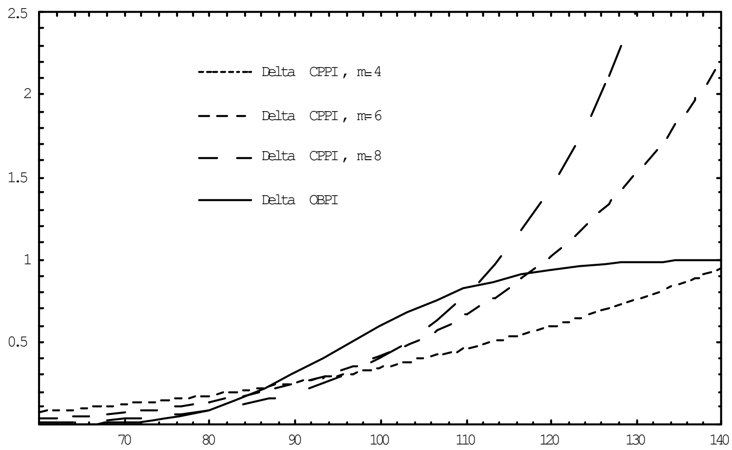

Figure 6 : CPPI and OBPI delta as functions of S.

It can be observed in the previous figure that the behavior of the delta of the two strategies are different. For the CPPI, not surprisingly, the delta becomes more convex with m and the delta can be greater than one.

For a large range of the values of the risky asset, the delta of the OBPI is greater than that of the CPPI. Moreover, this happens for the most likely values of the underlying asset (i.e. around the money). In order to be more precise, the probability that the delta of the OBPI is greater than that of the CPPI has to be calculated for various market parametrizations. It can be observed that, in probability, CPPI is significantly less sensitive to the risky asset than OBPI as shown in the following tables. Notice that this finding has important practical implications.

Table 2

Probability $P[\Delta^{OBPI} > \Delta^{CPPI}]$ for different m and t

m	$t = 0.1$	$t = 0.3$	$t = 0.5$	$t = 0.7$	$t = 0.9$
3	1	$0,974$	$0,91$	$0,834$	$0,745$
4	1	$0,966$	$0,888$	$0,798$	$0,699$
5	$0,98$	$0,903$	$0,816$	$0,724$	$0,622$
6	$0,703$	$0,753$	$0,73$	$0,667$	$0,57$
7	$0,371$	$0,619$	$0,675$	$0,645$	$0,553$
8	$0,231$	$0,545$	$0,654$	$0,652$	$0,56$
9	$0,185$	$0,514$	$0,652$	$0,673$	$0,583$
10	$0,172$	$0,508$	$0,658$	$0,7$	$0,614$

Table 3
Probability $P[\Delta^{OBPI} > \Delta^{CPPI}]$ for different m and σ

m	$\sigma = 5\%$	$\sigma = 10\%$	$\sigma = 15\%$	$\sigma = 20\%$	$\sigma = 25\%$
3	1,000	0,991	0,970	0,945	0,921
4	1,000	0,987	0,961	0,930	0,876
5	1,000	0,983	0,946	0,860	0,759
6	0,999	0,978	0,884	0,748	0,672
7	0,999	0,949	0,782	0,661	0,636
8	0,999	0,881	0,685	0,616	0,630
9	0,992	0,788	0,616	0,599	0,640
10	0,96	0,69	0,58	0,60	0,66

Table 4
Probability $P[\Delta^{OBPI} > \Delta^{CPPI}]$ for different m and μ

m	$\mu = 5\%$	$\mu = 10\%$	$\mu = 15\%$	$\mu = 20\%$	$\mu = 25\%$
3	0,925	0,945	0,960	0,972	0,981
4	0,910	0,930	0,943	0,951	0,953
5	0,861	0,860	0,850	0,830	0,801
6	0,774	0,748	0,712	0,667	0,616
7	0,706	0,661	0,610	0,554	0,495
8	0,670	0,616	0,558	0,497	0,436
9	0,657	0,599	0,537	0,475	0,413
10	0,657	0,598	0,536	0,473	0,411

The previous features are made clearer by examining the distribution of the ratio $\frac{\Delta^{OBPI}}{\Delta^{CPPI}}$. Figure 7 shows that the probability that the CPPI delta is smaller than the OBPI one is a decreasing function of the strike K (or equivalently, of the insured amount). Note that, for small values of K, the range of possible values of the ratio $\frac{\Delta_t^{OBPI}}{\Delta_t^{CPPI}}$ spreads out.

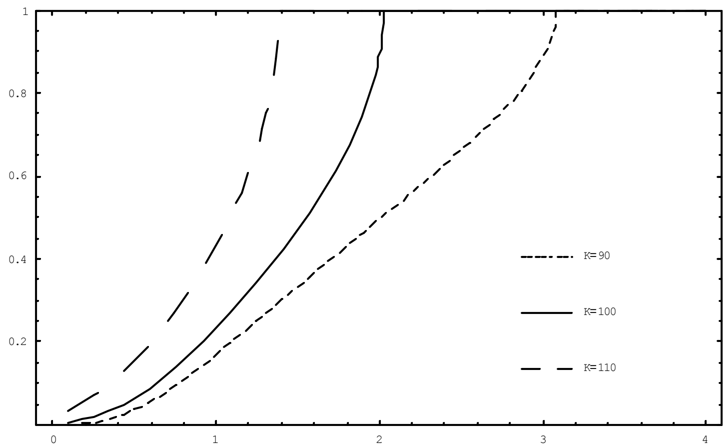

Figure 7 : Cumulative distribution of $\frac{\Delta_t^{OBPI}}{\Delta_t^{CPPI}}$ at $t = 0.5$.

Figure 8 below shows the evolution of the delta with time. Whatever the level of S compared to the level of the insured level at maturity, K, the delta of the CPPI is decreasing with time. For the OBPI, the evolution of the delta depends obviously on the moneyness of the option.

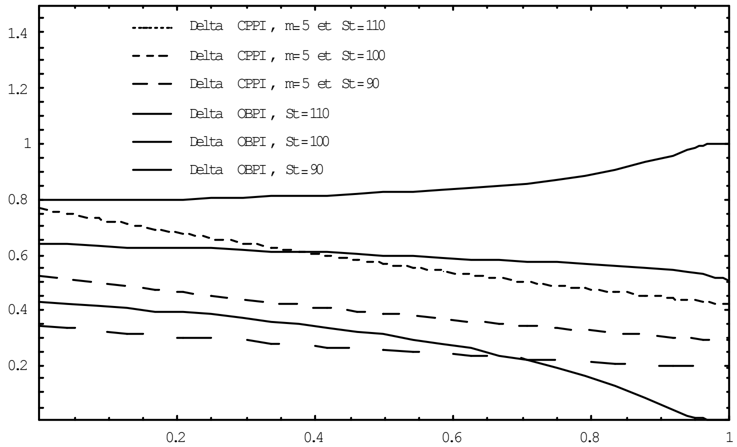

Figure 8 : CPPI and OBPI delta as functions of time.

More surprisingly, it can be shown that the delta of the CPPI is decreasing with the actual volatility (since $m > 1$). The same feature arises when examining the vega of the CPPI since they depend on the same way on this actual volatility (see later). For the OBPI, the result depends on the moneyness of the option.

3.3 The Gamma

The gamma of the CPPI is equal to :

$$\Gamma^{CPPI} = \frac{\partial \Delta_t^{CPPI}}{\partial S_t} = \alpha m(m-1) S_t^{m-2}.$$

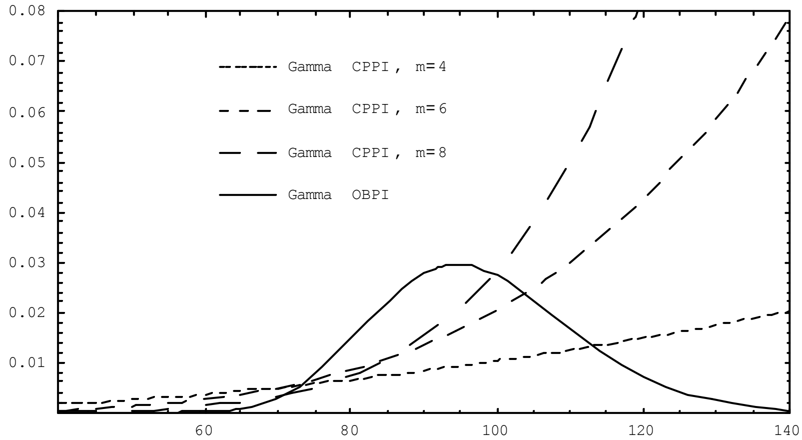

Figure 9 : CPPI and OBPI gamma as functions of S at $t = 0.5$ and for $K = 100$.

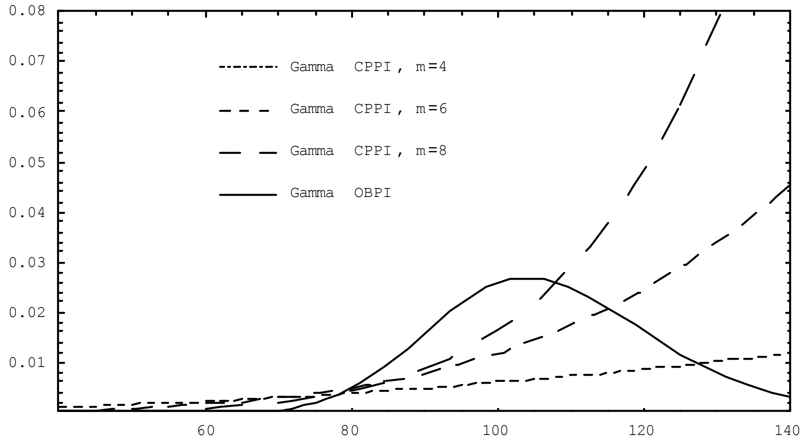

Figure 10 : CPPI and OBPI gamma as functions of S at $t = 0.5$ and for $K = 110$.

For the CPPI, it is always for high values of S that the gamma is important. Nevertheless, for usual values of m, the CPPI gamma is smaller than the OBPI one for a large range of values of S. This is particularly true for $K = 110$. This fact is important as the magnitude of transaction costs are directly linked to the gamma. Again, the CPPI method seems to be better suited when the insured percentage, p, of the initial investment is high.

Moreover, the gamma of the CPPI is monotonically decreasing with time, although it does not reach zero at maturity. Recall that, for a call, the gamma will go to zero as the expiration date approaches if the call is in-the-money or out-of-the-money, but will become very large if it is exactly at-the-money.

3.4 The Vega

The vega of the CPPI is defined as[12] :

$$
\begin{aligned}
&vega^{CPPI} = \frac{\partial V_t^{CPPI}}{\partial \sigma} \\
&= C(0, S_0, K) \left(\frac{S_t}{S_0}\right)^m ((m - m^2)\sigma t) \exp[\beta t] \\
&= ((m - m^2)\sigma t)\, V_t^{CPPI}
\end{aligned}
$$

Thus, the sensitivity of the CPPI value with respect to the actual volatility is negative as $m > 1$.

[12] In the following calculation, we do not take into account the effect of the volatility on $C(0, S_0, K)$ because the call enters in the CPPI formula only to insure the compatibility, at time 0, with the OBPI. Furthermore, $C(0, S_0, K)$ depends only on the expected volatility and not on the actual one.

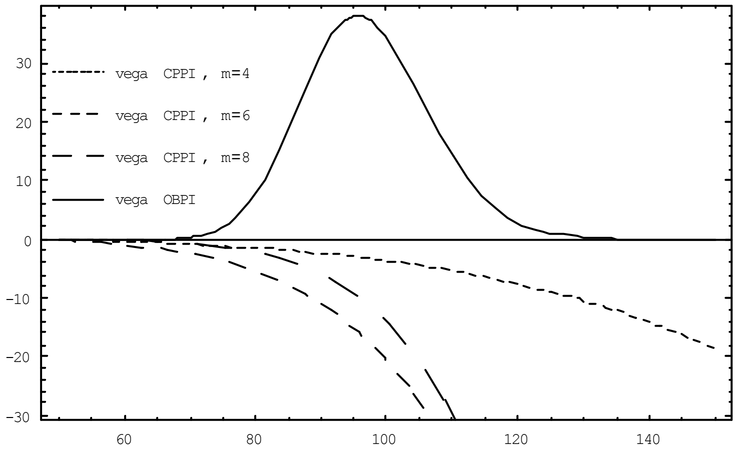

Figure 11 : CPPI and OBPI Vega as functions of S at $t = 0.5$.

As noted in Black and Rouhani (1989), as actual volatility increases, CPPI payoffs decline. Furthermore, the higher the multiple, the more they decrease.

4 Conclusion

We have examined the two main portfolio insurance methods : OBPI and CPPI. In the first part, we have shown that comparison with usual criteria such as first order stochastic dominance and various moments of their rates of return does not allow one to discriminate clearly between the two strategies. This comes from the non linearity of their payoff functions. Nevertheless, the study of the whole distribution of their returns has allowed us to shed light on the effect of the insured amount at maturity. As it increases, the CPPI strategy seems more relevant than the OBPI one. This arises mainly because the OBPI call has less chance to be exercised.

We then analyzed the dynamic properties of these two methods, showing in particular how the OBPI method can be considered as a generalized CPPI method. The difference in their sensitivity to the risky asset fluctuations has been put forward, in particular when examining their gamma according to the insured percentage of the initial investment.

5 Appendix

5.1 Calculation of the CPPI value

The value at t of the CPPI portfolio is given by $dV_t = (V_t - e_t)\frac{dB_t}{B_t} + e_t\frac{dS_t}{S_t}$

Recall that $V_t = C_t + F_t$, $e_t = mC_t$ and $dF_t = rdt$. Thus, the cushion value C must satisfy :

$$\begin{aligned} dC_t &= d(V_t - F_t) \\ &= (V_t - e_t)\frac{dB_t}{B_t} + (e_t)\frac{dS_t}{S_t} - dF_t \\ &= (C_t + F_t - mC_t)\frac{dB_t}{B_t} + (mC_t)\frac{dS_t}{S_t} - dF_t \\ &= (C_t - mC_t)\frac{dB_t}{B_t} + (mC_t)\frac{dS_t}{S_t} \\ &= C_t[(m(\mu - r) + r)dt + m\sigma dW_t] \end{aligned}$$

Thus : $C_t = C_0 \exp[((m(\mu - r) + r - \frac{m^2\sigma^2}{2})t + m\sigma W_t)]$

By using the relation : $S_t = S_0 \exp\left[\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right)t\right]$, it can be deduced that :

$$W_t = \frac{1}{\sigma}\,\square\, \ln\left(\frac{S_t}{S_0}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)t\Big]$$

Substituting this expression for W_t into the expression for C_t leads to :

$$\begin{aligned} C_t(m, S_t) &= C_0\left(\tfrac{S_t}{S_0}\right)^m \exp\left[\left(r - m\left(r - \tfrac{1}{2}\sigma^2\right) - m^2\tfrac{\sigma^2}{2}\right)t\right] \\ &= \alpha_t . S_t^m \\ \text{where} \quad \alpha_t &= \left(\tfrac{C_0}{S_0^m}\right)\exp[\beta t] \quad \text{and} \quad \beta = \left(r - m\left(r - \tfrac{1}{2}\sigma^2\right) - m^2\tfrac{\sigma^2}{2}\right) \end{aligned}$$

The portfolio value is then obtained :

$$V_t^{CPPI}(m, S_t) = F_0 . e^{rt} + \alpha_t . S_t^m$$

5.2 First order stochastic dominance

Lemma 5 *Consider two functions $f : A \to B$ and $g : A \to B$, where f or g is increasing and the inverse of f and g exists. Then :*

$$f^{-1} \,\square\, g^{-1} \Longleftrightarrow g \,\square\, f.$$

Proof : the following equivalent relations are deduced:

$$\begin{aligned} f^{-1} \,\square\, g^{-1} \Longleftrightarrow \quad & \forall x \in A,\ f^{-1}[f(x)] \,\square\, g^{-1}[f(x)] \\ & \forall x \in A,\ x \,\square\, g^{-1}[f(x)] \qquad \blacksquare \\ & \forall x \in A,\ g(x) \,\square\, f(x) \ \ (g \text{ increasing}) \end{aligned}$$

Proof of Proposition 2 : We proceed by contradiction. Consider first the following functions

$$\begin{aligned} f &: [S^*, +\infty[\ \to [S^* - K, +\infty[\ \text{ such that } f(x) = \alpha . x^m, \\ g &: [S^*, +\infty[\ \to [S^* - K, +\infty[\ \text{ such that } g(x) = x - K, \end{aligned}$$

where f and g are the two payoff functions restricted to a particular domain.

S^* is the smallest solution of the equation $x - K = \alpha . x^m$ (the other solution will be noted as S^{**}, see the figure of the payoff functions).

- Suppose now that : $V_T^{CPPI} \succ V_T^{OBPI}$. Then :

  $$\forall z \geq 0, P\left[(S_T - K)^+ \,\square\, z\right] \geq P\left[\alpha . S_T^m \,\square\, z\right],$$

  so in particular, $\forall z \geq S^* - K$.

But, for $z \geq S^* - K$, we get :

$$\begin{cases} P\left[(S_T - K)^+ \,\square\, z\right] = P\left[S_T \,\square\, S^*\right] + P\left[S^* \,\square\, S_T \,\square\, z + K\right], \\ P\left[\alpha . S_T^m \,\square\, z\right] = P\left[S_T \,\square\, S^*\right] + P\left[S_T \,\square\, S^* \textit{ and } \alpha . S_T^m \,\square\, z\right]. \end{cases}$$

Thus, for $z \geq S^* - K$, the relation $V_T^{CPPI} \succ V_T^{OBPI}$ can be stated as :

$$\begin{aligned} & P\left[S^* \,\square\, S_T \text{ and } g(S_T) \,\square\, z\right] \geq P\left[S^* \,\square\, S_T \text{ and } f(S_T) \,\square\, z\right] \\ \Longleftrightarrow \quad & P\left[S^* \,\square\, S_T \,\square\, g^{-1}(z)\right] \geq P\left[S^* \,\square\, S_T \,\square\, f^{-1}(z)\right]. \end{aligned}$$

Hence, $\forall z \geq S^{*}-K$, $g^{-1}(z) \geq f^{-1}(z)$.

By the previous lemma, it can be shown that : $\forall x \geq S^{*}$, $g(x) \square f(x)$.

This result yields a contradiction with the definition of the two payoff functions : for any $x \in \left]S^{*}, S^{**}\right[$, $g(x) > f(x)$. Thus, the CPPI can never stochastically dominate the OBPI strategy at first order.

- The converse can also be proved in the following way :

$$
\begin{aligned}
V_T^{OBPI} \succ V_T^{CPPI} &\iff \forall z>0,\ P\left[S_T \square \left(\tfrac{z}{\alpha}\right)^{\frac{1}{m}}\right] \geq P\left[S_T \square z+K\right] \\
&\iff \forall z>0,\ \left(\tfrac{z}{\alpha}\right)^{\frac{1}{m}} \geq z+K.
\end{aligned}
$$

For any finite values of the parameters $(K,\ \alpha,\ m)$, the previous inequality is never satisfied at $z=0$. Hence, the OBPI can never stochastically dominate at the first-order the CPPI strategy. ■

5.3 OBPI as a generalized CPPI

Proof : Recall that :

$$V_T^{OBPI}=S_T+(K-S_T)^{+}=K+(S_T-K)^{+}.$$

Thus :

$$V_t^{OBPI}=Ke^{-r(T-t)}+C(t, S_t, K),$$

where $Ke^{-r(T-t)}=F_t$ is the time t value of the floor and $C(t, S_t, K)$ is the cushion at time t. By definition, the cushion is defined as $\frac{e_t}{m_t}$. Here, the cushion is simply the call and the exposure is the total amount invested in the risky asset, equal to $S_t N\left(d_1\left(t, S_t\right)\right)$. Finally, the desired result for the multiple is obtained :

$$m_t^{OBPI}=\frac{S_t N(d_1(t,S_t))}{C(t,S_t,K)} \blacksquare$$

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