THE BEHAVIOR OF STOCK-MARKET PRICES*

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I. Introduction

For many years the following question has been a source of continuing controversy in both academic and business circles: To what extent can the past history of a common stock's price be used to make meaningful predictions concerning the future price of the stock? Answers to this question have been provided on the one hand by the various chartist theories and on the other hand by the theory of random walks.

Although there are many different chartist theories, they all make the same basic assumption. That is, they all assume that the past behavior of a security's price is rich in information concerning its future behavior. History repeats itself in that "patterns" of past price behavior will tend to recur in the future. Thus, if through careful analysis of price charts one develops an understanding of these "patterns," this can be used to predict the future behavior of prices and in this way increase expected gains.[1]

By contrast the theory of random walks says that the future path of the price level of a security is no more predictable than the path of a series of cumulated random numbers. In statistical terms the theory says that successive price changes are independent, identically distributed random variables. Most simply this implies that the series of price changes has no memory, that is, the past cannot be used to predict the future in any meaningful way.

The purpose of this paper will be to discuss first in more detail the theory underlying the random-walk model and then to test the model's empirical validity. The main conclusion will be that the data seem to present consistent and strong support for the model. This implies, of course, that chart reading, though perhaps an interesting pastime, is of no real value to the stock market investor. This is an extreme statement and the chart reader is certainly free to take exception. We suggest, however, that since the empirical evidence produced by this and other studies in support of the random-walk model is now so voluminous, the counterarguments of the chart reader will be completely lacking in force if they are not equally well supported by empirical work.

* This study has profited from the criticisms, suggestions, and technical assistance of many different people. In particular I wish to express my gratitude to Professors William Alberts, Lawrence Fisher, Robert Graves, James Lorie, Merton Miller, Harry Roberts, and Lester Telser, all of the Graduate School of Business, University of Chicago. I wish especially to thank Professors Miller and Roberts for providing not only continuous intellectual stimulation but also painstaking care in reading the various preliminary drafts.

Many of the ideas in this paper arose out of the work of Benoit Mandelbrot of the IBM Watson Research Center. I have profited not only from the written work of Dr. Mandelbrot but also from many invaluable discussion sessions.

Work on this paper was supported in part by funds from a grant by the Ford Foundation to the Graduate School of Business of the University of Chicago, and in part by funds granted to the Center for Research in Security Prices of the School by the National Science Foundation. Extensive computer time was provided by the 7094 Computation Center of the University of Chicago.

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[1] The Dow Theory, of course, is the best known example of a chartist theory.

II. Theory of Random Walks in Stock Prices

The theory of random walks in stock prices actually involves two separate hypotheses: (1) successive price changes are independent, and (2) the price changes conform to some probability distribution. We shall now examine each of these hypotheses in detail.

A. INDEPENDENCE

1. MEANING OF INDEPENDENCE

In statistical terms independence means that the probability distribution for the price change during time period t is independent of the *sequence* of price changes during previous time periods. That is, knowledge of the *sequence* of price changes leading up to time period t is of no help in assessing the probability distribution for the price change during time period t.[2]

Now in fact we can probably never hope to find a time series that is characterized by *perfect* independence. Thus, strictly speaking, the random walk theory cannot be a completely accurate description of reality. For practical purposes, however, we may be willing to accept the independence assumption of the model as long as the dependence in the series of successive price changes is not above some "minimum acceptable" level.

What constitutes a "minimum acceptable" level of dependence depends, of course, on the particular problem that one is trying to solve. For example, someone who is doing statistical work in the stock market may wish to decide whether dependence in the series of *successive* price changes is sufficient to account for some particular property of the *distribution* of price changes. If the actual dependence in the series is not sufficient to account for the property in question, the statistician may be justified in accepting the independence hypothesis as an adequate description of reality.

By contrast the stock market trader has a much more practical criterion for judging what constitutes important dependence in successive price changes. For his purposes the random walk model is valid as long as knowledge of the past behavior of the series of price changes cannot be used to increase expected gains. More specifically, the independence assumption is an adequate description of reality as long as the actual degree of dependence in the series of price changes is not sufficient to allow the past history of the series to be used to predict the future in a way which makes expected profits greater than they would be under a naïve buy-and-hold model.

Dependence that is important from the trader's point of view need not be important from a statistical point of view, and conversely dependence which is important for statistical purposes need not be important for investment purposes. For example, we may know that on alternate days the price of a security always increases by ϵ and then decreases by ϵ. From a statistical point of view knowledge of this dependence would be important information since it tells us quite a bit about the shape of the distribution of price changes. For trading purposes, however, as long as ϵ is very small, this perfect, negative, statistical dependence is unimportant. Any profits the trader

[2] More precisely, independence means that

$$Pr(x_t = x \mid x_{t-1}, x_{t-2}, \ldots) = Pr(x_t = x),$$

where the term on the right of the equality sign is the unconditional probability that the price change during time t will take the value x, whereas the term on the left is the conditional probability that the price change will take the value x, conditional on the knowledge that previous price changes took the values x_{t-1}, x_{t-2}, etc.

may hope to make from it would be washed away in transactions costs.

In Section V of this paper we shall be concerned with testing independence from the point of view of both the statistician and the trader. At this point, however, the next logical step in the development of a theory of random walks in stock prices is to consider market situations and mechanisms that are consistent with independence in successive price changes. The procedure will be to consider first the simplest situations and then to successively introduce complications.

2. MARKET SITUATIONS CONSISTENT WITH INDEPENDENCE

Independence of successive price changes for a given security may simply reflect a price mechanism which is totally unrelated to real-world economic and political events. That is, stock prices may be just the accumulation of many bits of randomly generated noise, where by noise in this case we mean psychological and other factors peculiar to different individuals which determine the types of "bets" they are willing to place on different companies.

Even random walk theorists, however, would find such a view of the market unappealing. Although some people may be primarily motivated by whim, there are many individuals and institutions that seem to base their actions in the market on an evaluation (usually extremely painstaking) of economic and political circumstances. That is, there are many private investors and institutions who believe that individual securities have "intrinsic values" which depend on economic and political factors that affect individual companies.

The existence of intrinsic values for individual securities is not inconsistent with the random-walk hypothesis. In order to justify this statement, however, it will be necessary now to discuss more fully the process of price determination in an intrinsic-value–random-walk market.

Assume that at any point in time there exists, at least implicitly, an intrinsic value for each security. The intrinsic value of a given security depends on the earnings prospects of the company which in turn are related to economic and political factors some of which are peculiar to this company and some of which affect other companies as well.[3]

We stress, however, that actual market prices need not correspond to intrinsic values. In a world of uncertainty intrinsic values are not known exactly. Thus there can always be disagreement among individuals, and in this way actual prices and intrinsic values can differ. Henceforth uncertainty or disagreement concerning intrinsic values will come under the general heading of "noise" in the market.

In addition, intrinsic values can themselves change across time as a result of either new information or trend. New information may concern such things as the success of a current research and development project, a change in management, a tariff imposed on the industry's product by a foreign country, an increase in industrial production or any other *actual or anticipated* change in a factor which is likely to affect the company's prospects.

[3] We can think of intrinsic values in either of two ways. First, perhaps they just represent market conventions for evaluating the worth of a security by relating it to various factors which affect the earnings of a company. On the other hand, intrinsic values may actually represent equilibrium prices in the economist's sense, i.e., prices that evolve from some dynamic general equilibrium model. For our purposes it is irrelevant which point of view one takes.

On the other hand, an anticipated long-term trend in the intrinsic value of a given security can arise in the following way.[4] Suppose we have two unlevered companies which are identical in all respects except dividend policy. That is, both companies have the same current and anticipated investment opportunities, but they finance these opportunities in different ways. In particular, one company pays out all of its current earnings as dividends and finances new investment by issuing new common shares. The other company, however, finances new investment out of current earnings and pays dividends only when there is money left over. Since shares in the two companies are subject to the same degree of risk, we would expect their expected rates of returns to be the same. This will be the case, however, only if the shares of the company with the lower dividend payout have a higher expected rate of price increase than do the shares of the high-payout company. In this case the trend in the price level is just part of the expected return to equity. Such a trend is not inconsistent with the random-walk hypothesis.[5]

The simplest rationale for the independence assumption of the random walk model was proposed first, in a rather vague fashion, by Bachelier [6] and then much later but more explicitly by Osborne [42]. The argument runs as follows: If successive bits of new information arise independently across time, and if noise or uncertainty concerning intrinsic values does not tend to follow any consistent pattern, then successive price changes in a common stock will be independent.

As with many other simple models, however, the assumptions upon which the Bachelier-Osborne model is built are rather extreme. There is no strong reason to expect that each individual's estimates of intrinsic values will be independent of the estimates made by others (i.e., noise may be generated in a dependent fashion). For example, certain individuals or institutions may be opinion leaders in the market. That is, their actions may induce people to change their opinions concerning the prospects of a given company. In addition there is no strong reason to expect successive bits of new information to be generated independently across time. For example, good news may tend to be followed more often by good news than by bad news, and bad news may tend to be followed more often by bad news than by good news. Thus there may be dependence in either the noise generating process or in the process generating new information, and these may in turn lead to dependence in successive price changes.

Even in a situation where there are dependencies in either the information or the noise generating process, however, it is still possible that there are offsetting mechanisms in the market which tend to produce independence in price changes for individual common stocks. For example, let us assume that there are many sophisticated traders in the stock market and that sophistication can take two forms: (1) some traders may be much better at predicting the appearance of new information and estimating its effects on intrinsic values than others, while (2) some may be much better at doing statistical analyses of price behavior. Thus these two types of sophisticated traders can be roughly thought of as superior intrinsic-value analysts

[4] A trend in the price level, of course, corresponds to a non-zero mean in the distribution of price changes.

[5] A lengthy and rigorous justification for these statements is given by Miller and Modigliani [40].

and superior chart readers. We further assume that, although there are sometimes discrepancies between actual prices and intrinsic values, sophisticated traders in general feel that actual prices usually tend to move toward intrinsic values.

Suppose now that the noise generating process in the stock market is dependent. More specifically assume that when one person comes into the market who thinks the current price of a security is above or below its intrinsic value, he tends to attract other people of like feelings and he causes some others to change their opinions unjustifiably. In itself this type of dependence in the noise generating process would tend to produce "bubbles" in the price series, that is, periods of time during which the accumulation of the same type of noise causes the price level to run well above or below the intrinsic value.

If there are many sophisticated traders in the market, however, they may cause these "bubbles" to burst before they have a chance to really get under way. For example, if there are many sophisticated traders who are extremely good at estimating intrinsic values, they will be able to recognize situations where the price of a common stock is beginning to run up above its intrinsic value. Since they expect the price to move eventually back toward its intrinsic value, they have an incentive to sell this security or to sell it short. If there are enough of these sophisticated traders, they may tend to prevent these "bubbles" from ever occurring. Thus their actions will neutralize the dependence in the noise-generating process, and successive price changes will be independent.

In fact, of course, in a world of uncertainty even sophisticated traders cannot always estimate intrinsic values exactly. The effectiveness of their activities in erasing dependencies in the series of price changes can, however, be reinforced by another neutralizing mechanism. As long as there are important dependencies in the series of successive price changes, opportunities for trading profits are available to any astute chartist. For example, once they understand the nature of the dependencies in the series of successive price changes, sophisticated chartists will be able to identify statistically situations where the price is beginning to run up above the intrinsic value. Since they expect that the price will eventually move back toward its intrinsic value, they will sell. Even though they are vague about intrinsic values, as long as they have sufficient resources their actions will tend to erase dependencies and to make actual prices closer to intrinsic values.

Over time the intrinsic value of a common stock will change as a result of new information, that is, actual or anticipated changes in any variable that affects the prospects of the company. If there are dependencies in the process generating new information, this in itself will tend to create dependence in successive price changes of the security. If there are many sophisticated traders in the market, however, they should eventually learn that it is profitable for them to attempt to interpret both the price effects of current new information and of the future information implied by the dependence in the information generating process. In this way the actions of these traders will tend to make price changes independent.[6]

Moreover, successive price changes may be independent even if there is usually consistent vagueness or uncertainty

[6] In essence dependence in the information generating process is itself relevant information which the astute trader should consider.

surrounding new information. For example, if uncertainty concerning the importance of new information consistently causes the market to underestimate the effects of new information on intrinsic values, astute traders should eventually learn that it is profitable to take this into account when new information appears in the future. That is, by examining the history of prices subsequent to the influx of new information it will become clear that profits can be made simply by buying (or selling short if the information is pessimistic) after new information comes into the market since on the average actual prices do not initially move all the way to their new intrinsic values. If many traders attempt to capitalize on this opportunity, their activities will tend to erase any consistent lags in the adjustment of actual prices to changes in intrinsic values.

The above discussion implies, of course, that, if there are many astute traders in the market, on the average the full effects of new information on intrinsic values will be reflected nearly instantaneously in actual prices. In fact, however, because there is vagueness or uncertainty surrounding new information, "instantaneous adjustment" really has two implications. First, actual prices will initially overadjust to the new intrinsic values as often as they will underadjust. Second, the lag in the complete adjustment of actual prices to successive new intrinsic values will itself be an independent random variable, sometimes preceding the new information which is the basis of the change (i.e., when the information is anticipated by the market before it actually appears) and sometimes following. It is clear that in this case successive price changes in individual securities will be independent random variables.

In sum, this discussion is sufficient to show that the stock market *may* conform to the independence assumption of the random walk model even though the processes generating noise and new information are themselves dependent. We turn now to a brief discussion of some of the implications of independence.

3. IMPLICATIONS OF INDEPENDENCE

In the previous section we saw that one of the forces which helps to produce independence of successive price changes may be the existence of sophisticated traders, where sophistication may mean either (1) that the trader has a special talent in detecting dependencies in series of prices changes for individual securities, or (2) that the trader has a special talent for predicting the appearance of new information and evaluating its effects on intrinsic values. The first kind of trader corresponds to a superior chart reader, while the second corresponds to a superior intrinsic value analyst.

Now although the activities of the chart reader may help to produce independence of successive price changes, once independence is established chart reading is no longer a profitable activity. In a series of independent price changes, the past history of the series cannot be used to increase expected profits.

Such dogmatic statements cannot be applied to superior intrinsic-value analysis, however. In a dynamic economy there will always be new information which causes intrinsic values to change over time. As a result, people who can consistently predict the appearance of *new* information *and* evaluate its effects on intrinsic values will usually make larger profits than can people who do not have this talent. The fact that the activities of these superior analysts help to make successive price changes independent does *not* imply that their expected profits cannot be greater than those of the investor who follows some naïve buy-and-hold policy.

It must be emphasized, however, that the comparative advantage of the superior analyst over his less talented competitors lies in his ability to predict consistently the appearance of *new* information and evaluate its impact on intrinsic values. If there are enough superior analysts, their existence will be sufficient to insure that actual market prices are, on the basis of all *available* information, best estimates of intrinsic values. In this way, of course, the superior analysts make intrinsic value analysis a useless tool for both the average analyst and the average investor.

This discussion gives rise to three obvious question: (1) How many superior analysts are necessary to insure independence? (2) Who are the "superior" analysts? and (3) What is a rational investment policy for an average investor faced with a random-walk stock market?

It is impossible to give a firm answer to the first question, since the effectiveness of the superior analysts probably depends more on the extent of their resources than on their number. Perhaps a single, well-informed and well-endowed specialist in each security is sufficient.

It is, of course, also very difficult to identify *ex ante* those people that qualify as superior analysts. *Ex post*, however, there is a simple criterion. A superior analyst is one whose gains over many periods of time are *consistently* greater than those of the market. Consistently is the crucial word here, since for any given short period of time, even if there are no superior analysts, in a world of random walks some people will do much better than the market and some will do much worse.

Unfortunately, by this criterion this author does not qualify as a superior analyst. There is some consolation, however, since, as we shall see later, other more market-tested institutions do not seem to qualify either.

Finally, let us now briefly formulate a rational investment policy for the average investor in a situation where stock prices follow random walks and at every point in time actual prices represent good estimates of intrinsic values. In such a situation the primary concern of the average investor should be *portfolio analysis*. This is really three separate problems. First, the investor must decide what sort of tradeoff between risk and expected return he is willing to accept. Then he must attempt to classify securities according to riskiness, and finally he must also determine how securities from different risk classes combine to form portfolios with various combinations of risk and return.[7]

In essence in a random-walk market the *security analysis* problem of the average investor is greatly simplified. If actual prices at any point in time are good estimates of intrinsic values, he need not be concerned with whether individual securities are over- or under-priced. If he decides that his portfolio requires an additional security from a given risk class, he can choose that security randomly from within the class. On the average any security so chosen will have about the same effect on the expected return and riskiness of his portfolio.

B. THE DISTRIBUTION OF PRICE CHANGES

1. INTRODUCTION

The theory of random walks in stock prices is based on two hypotheses: (1) successive price changes in an indi-

[7] For a more complete formulation of the portfolio analysis problem see Markowitz [39].

vidual security are independent, and (2) the price changes conform to some probability distribution. Of the two hypotheses independence is the most important. Either successive price changes are independent (or at least for all practical purposes independent) or they are not; and if they are not, the theory is not valid. All the hypothesis concerning the distribution says, however, is that the price changes conform to *some* probability distribution. In the general theory of random walks the form or shape of the distribution need not be specified. Thus any distribution is consistent with the theory as long as it correctly characterizes the process generating the price changes.[8]

From the point of view of the investor, however, specification of the shape of the distribution of price changes is extremely helpful. In general, the form of the distribution is a major factor in determining the riskiness of investment in common stocks. For example, although two different possible distributions for the price changes may have the same mean or expected price change, the probability of very large changes may be much greater for one than for the other.

The form of the distribution of price changes is also important from an academic point of view since it provides descriptive information concerning the nature of the process generating price changes. For example, if very large price changes occur quite frequently, it may be safe to infer that the economic structure that is the source of the price changes is itself subject to frequent and sudden shifts over time. That is, if the distribution of price changes has a high degree of dispersion, it is probably safe to infer that, to a large extent, this is due to the variability in the process generating new information.

Finally, the form of the distribution of price changes is important information to anyone who wishes to do empirical work in this area. The power of a statistical tool is usually closely related to the type of data to which it is applied. In fact we shall see in subsequent sections that for some probability distributions important concepts like the mean and variance are not meaningful.

2. THE BACHELIER-OSBORNE MODEL

The first complete development of a theory of random walks in security prices is due to Bachelier [6], whose original work first appeared around the turn of the century. Unfortunately his work did not receive much attention from economists, and in fact his model was independently derived by Osborne [42] over fifty years later. The Bachelier-Osborne model begins by assuming that price changes from transaction to transaction in an individual security are independent, identically distributed random variables. It further assumes that transactions are fairly uniformly spread across time, and that the distribution of price changes from transaction to transaction has finite variance. If the number of transactions per day, week, or month is very large, then price changes across these differencing intervals will be sums of many independent variables. Under these conditions the central-limit theorem leads us to expect that the daily,

[8] Of course, the theory does imply that the parameters of the distribution should be stationary or fixed. As long as independence holds, however, stationarity can be interpreted loosely. For example, if independence holds in a strict fashion, then for the purposes of the investor the random walk model is a valid approximation to reality even though the parameters of the probability distribution of the price changes may be non-stationary.

For statistical purposes stationarity implies simply that the parameters of the distribution should be fixed at least for the time period covered by the data.

weekly, and monthly price changes will each have normal or Gaussian distributions. Moreover, the variances of the distributions will be proportional to the respective time intervals. For example, if σ^2 is the variance of the distribution of the daily changes, then the variance for the distribution of the weekly changes should be approximately $5\sigma^2$.

Although Osborne attempted to give an empirical justification for his theory, most of his data were cross-sectional and could not provide an adequate test. Moore and Kendall, however, have provided empirical evidence in support of the Gaussian hypothesis. Moore [41, pp. 116–23] graphed the weekly first differences of log price of eight NYSE common stocks on normal probability paper. Although the extreme sections of his graphs seem to have too many large price changes, Moore still felt the evidence was strong enough to support the hypothesis of approximate normality.

Similarly Kendall [26] observed that weekly price changes in British common stocks seem to be approximately normally distributed. Like Moore, however, he finds that most of the distributions of price changes are leptokurtic; that is, there are too many values near the mean and too many out in the extreme tails. In one of his series some of the extreme observations were so large that he felt compelled to drop them from his subsequent statistical tests.

3. MANDELBROT AND THE GENERALIZED CENTRAL-LIMIT THEOREM

The Gaussian hypothesis was not seriously questioned until recently when the work of Benoit Mandelbrot first began to appear.[9] Mandelbrot's main assertion is that, in the past, academic research has too readily neglected the implications of the leptokurtosis usually observed in empirical distributions of price changes.

The presence, in general, of leptokurtosis in the empirical distributions seems indisputable. In addition to the results of Kendall [26] and Moore [41] cited above, Alexander [1] has noted that Osborne's cross-sectional data do not really support the normality hypothesis; there are too many changes greater than $\pm$ 10 per cent. Cootner [10] has developed a whole theory in order to explain the long tails of the empirical distributions. Finally, Mandelbrot [37, Fig. 1] cites other examples to document empirical leptokurtosis.

The classic approach to this problem has been to assume that the extreme values are generated by a different mechanism than the majority of the observations. Consequently one tries a posteriori to find "causal" explanations for the large observations and thus to rationalize their exclusion from any tests carried out on the body of the data.[10] Unlike the statistician, however, the investor cannot ignore the possibility of large price changes before committing his funds, and once he has made his decision to invest, he must consider their effects on his wealth.

Mandelbrot feels that if the outliers are numerous, excluding them takes away much of the significance from any tests carried out on the remainder of the data. This exclusion process is all the more subject to criticism since probability distributions are available which accurately represent the large observations

[9] His main work in this area is [37]. References to his other works are found through this report and in the bibliography.

[10] When extreme values are excluded from the sample, the procedure is often called "trimming." Another technique which involves reducing the size of extreme observations rather than excluding them is called "Winsorization." For a discussion see J. W. Tukey [45].

as well as the main body of the data. The distributions referred to are members of a special class which Mandelbrot has labeled stable Paretian. The mathematical properties of these distributions are discussed in detail in the appendix to this paper. At this point we shall merely introduce some of their more important descriptive properties.

Parameters of stable Paretian distributions.—Stable Paretian distributions have four parameters: (1) a location parameter which we shall call δ, (2) a scale parameter henceforth called γ, (3) an index of skewness, β, and (4) a measure of the height of the extreme tail areas of the distribution which we shall call the characteristic exponent α.[11]

When the characteristic exponent α is greater than 1, the location parameter δ is the expectation or mean of the distribution. The scale parameter γ can be any positive real number, but β, the index of skewness, can only take values in the interval $-1 \leq \beta \leq 1$. When $\beta = 0$ the distribution is symmetric. When $\beta > 0$ the distribution is skewed right (i.e., has a long tail to the right), and the degree of right skewness is larger the larger the value of β. Similarly, when $\beta < 0$ the distribution is skewed left, and the degree of left skewness is larger the smaller the value of β.

The characteristic exponent α of a stable Paretian distribution determines the height of, or total probability contained in, the extreme tails of the distribution, and can take any value in the interval $0 < \alpha \leq 2$. When $\alpha = 2$, the relevant stable Paretian distribution is the normal or Gaussian distribution. When α is in the interval $0 < \alpha < 2$, the extreme tails of the stable Paretian distributions are higher than those of the normal distribution, and the total probability in the extreme tails is larger the smaller the value of α. The most important consequence of this is that the variance exists (i.e., is finite) only in the extreme case $\alpha = 2$. The mean, however, exists as long as $\alpha > 1$.[12]

Mandelbrot's hypothesis states that for distributions of price changes in speculative series, α is in the interval $1 < \alpha < 2$, so that the distributions have means but their variances are infinite. The Gaussian hypothesis, on the other hand, states that α is exactly equal to 2. Thus both hypotheses assume that the distribution is stable Paretian. The disagreement between them concerns the value of the characteristic exponent α.

Properties of stable Paretian distributions.—Two important properties of stable Paretian distributions are (1) stability or invariance under addition, and (2) the fact that these distributions are the only possible limiting distributions for sums of independent, identically distributed, random variables.

By definition, a stable Paretian distribution is any distribution that is stable or invariant under addition. That is, the distribution of sums of independent, identically distributed, stable Paretian variables is itself stable Paretian and, except for origin and scale, has the same form as the distribution of the individual summands. Most simply, stability means that the values of the parameters α and β remain constant under addition.[13]

The property of stability is responsible

[11] The derivation of most of the important properties of stable Paretian distributions is due to P. Levy [29]. A rigorous and compact mathematical treatment of the theory can be found in B. V. Gnedenko and A. N. Kolmogorov [17]. A more comprehensive mathematical treatment can be found in Mandelbrot [37].

[12] For a proof of these statements see Gnedenko and Kolmogorov [17], pp. 179–83.

[13] A more rigorous definition of stability is given in the appendix.

for much of the appeal of stable Paretian distributions as descriptions of empirical distributions of price changes. The price change of a stock for any time interval can be regarded as the sum of the changes from transaction to transaction during the interval. If transactions are fairly uniformly spread over time and if the changes between transactions are independent, identically distributed, stable Paretian variables, then daily, weekly, and monthly changes will follow stable Paretian distributions of exactly the same form, except for origin and scale. For example, if the distribution of daily changes is stable Paretian with location parameter δ and scale paremeter γ, the distribution of weekly (or five-day) changes will also be stable Paretian with location parameter 5δ and scale parameter 5γ. It would be very convenient if the form of the distribution of price changes were independent of the differencing interval for which the changes were computed.

It can be shown that stability or invariance under addition leads to a most important corollary property of stable Paretian distributions; they are the only possible limiting distributions for sums of independent, identically distributed, random variables.[14] It is well known that if such variables have finite variance, the limiting distribution for their sum will be the normal distribution. If the basic variables have infinite variance, however, and if their sums follow a limiting distribution, the limiting distribution must be stable Paretian with $0 < \alpha < 2$.

In light of this discussion we see that Mandelbrot's hypothesis can actually be viewed as a generalization of the central-limit theorem arguments of Bachelier and Osborne to the case where the underlying distributions of price changes from transaction to transaction are allowed to have infinite variances. In this sense, then, Mandelbrot's version of the theory of random walks can be regarded as a broadening rather than a contradiction of the earlier Bachelier-Osborne model.

Conclusion.—Mandelbrot's hypothesis that the distribution of price changes is stable Paretian with characteristic exponent $\alpha < 2$ has far reaching implications. For example, if the variances of distributions of price changes behave as if they are infinite, many common statistical tools which are based on the assumption of a finite variance either will not work or may give very misleading answers. Getting along without these familiar tools is not going to be easy, and before parting with them we must be sure that such a drastic step is really necessary. At the moment, the most impressive single piece of evidence is a direct test of the infinite variance hypothesis for the case of cotton prices. Mandelbrot [37, Fig. 2 and pp. 404–7] computed the sample second moments of the first differences of the logs of cotton prices for increasing sample sizes of from 1 to 1,300 observations. He found that the sample moment does not settle down to any limiting value but rather continues to vary in absolutely erratic fashion, precisely as would be expected under his hypothesis.[15]

As for the special but important case

[14] For a proof see Gnedenko and Kolmogorov [17], pp. 162–63.

[15] The second moment of a random variable x is just $E(x^2)$. The variance is just the second moment minus the square of the mean. Since the mean is assumed to be a constant, tests of the sample second moment are also tests of the sample variance.

In an earlier privately circulated version of [37] Mandelbrot tested his hypothesis on various other series of speculative prices. Although the results in general tended to support his hypothesis, they were neither as extensive nor as conclusive as the tests on cotton prices.

of common-stock prices, no published evidence for or against Mandelbrot's theory has yet been presented. One of our main goals here will be to attempt to test Mandelbrot's hypothesis for the case of stock prices.

C. THINGS TO COME

Except for the concluding section, the remainder of this paper will be concerned with reporting the results of extensive tests of the random walk model of stock price behavior. Sections III and IV will examine evidence on the shape of the distribution of price changes. Section III will be concerned with common statistical tools such as frequency distributions and normal probability graphs, while Section IV will develop more direct tests of Mandelbrot's hypothesis that the characteristic exponent α for these distributions is less than 2. Section V of the paper tests the independence assumption of the random-walk model. Finally, Section VI will contain a summary of previous results, and a discussion of the implications of these results from various points of view.

III. A First Look at the Empirical Distributions

A. INTRODUCTION

In this section a few simple techniques will be used to examine distributions of daily stock-price changes for individual securities. If Mandelbrot's hypothesis that the distributions are stable Paretian with characteristic exponents less than 2 is correct, the most important feature of the distributions should be the length of their tails. That is, the extreme tail areas should contain more relative frequency than would be expected if the distributions were normal. In this section no attempt will be made to decide whether the actual departures from normality are sufficient to reject the Gaussian hypothesis. The only goal will be to see if the departures are usually in the direction predicted by the Mandelbrot hypothesis.

B. THE DATA

The data that will be used throughout this paper consist of daily prices for each of the thirty stocks of the Dow-Jones Industrial Average.[16] The time periods vary from stock to stock but usually run from about the end of 1957 to September 26, 1962. The final date is the same for all stocks, but the initial date varies from January, 1956 to April, 1958. Thus there are thirty samples with about 1,200–1,700 observations per sample.

The actual tests are not performed on the daily prices themselves but on the first differences of their natural logarithms. The variable of interest is

$$u_{t+1} = \log_e p_{t+1} - \log_e p_t , \qquad (1)$$

where p_{t+1} is the price of the security at the end of day $t + 1$, and p_t is the price at the end of day t.

There are three main reasons for using changes in log price rather than simple price changes. First, the change in log price is the yield, with continuous compounding, from holding the security for that day.[17] Second, Moore [41, pp. 13–15] has shown that the variability of simple price changes for a given stock is an increasing function of the price level of the stock. His work indicates that taking

[16] The data were very generously supplied by Professor Harry B. Ernst of Tufts University.

[17] The proof of this statement goes as follows:

$$\frac{p_{t+1}}{p_t} = \exp\left(\log_e \frac{p_{t+1}}{p_t}\right).$$

$$p_{t+1} = p_t \exp\left(\log_e \frac{p_{t+1}}{p_t}\right)$$
$$= p_t \exp(\log_e p_{t+1} - \log_e p_t).$$

logarithms seems to neutralize most of this price level effect. Third, for changes less than ± 15 per cent the change in log price is very close to the percentage price change, and for many purposes it is convenient to look at the data in terms of percentage price changes.[18]

In working with daily changes in log price, two special situations must be noted. They are stock splits and ex-dividend days. Stock splits are handled as follows: if a stock splits two for one on day t, its actual closing price on day t is doubled, and the difference between the logarithm of this doubled price and the logarithm of the closing price for day $t - 1$ is the first difference for day t. The first difference for day $t + 1$ is the difference between the logarithm of the closing price on day $t + 1$ and the logarithm of the actual closing price on day t, the day of the split. These adjustments reflect the fact that the process of splitting a stock involves no change either in the asset value of the firm or in the wealth of the individual shareholder.

On ex-dividend days, however, other things equal, the value of an individual share should fall by about the amount of the dividend. To adjust for this the first difference between an ex-dividend day and the preceding day is computed as

$$u_{t+1} = \log_e (p_{t+1} + d) - \log_e p_t ,$$

where d is the dividend per share.[19]

One final note concerning the data is in order. The Dow-Jones Industrials are not a random sample of stocks from the New York Stock Exchange. The component companies are among the largest and most important in their fields. If the behavior of these blue-chips stocks differs consistently from that of other stocks in the market, the empirical results to be presented below will be strictly applicable only to the shares of large important companies.

One must admit, however, that the sample of stocks is conservative from the point of view of the Mandelbrot hypothesis, since blue chips are probably more stable than other securities. There is reason to expect that if such a sample conforms well to the Mandelbrot hypothesis, a random sample would fit even better.

C. FREQUENCY DISTRIBUTIONS

One very simple way of analyzing the distribution of changes in log price is to construct frequency distributions for the individual stocks. That is, for each stock the empirical proportions of price changes within given standard deviations of the mean change can be computed and compared with what would be expected if the distributions were exactly normal. This is done in Tables 1 and 2. In Table 1 the proportions of observations within 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 4.0, and 5.0 standard deviations of the mean change, as well as the proportion greater than 5 standard deviations from the mean, are computed for each stock. In the first line of the body of the table the proportions for the unit normal distribution are given.

Table 2 gives a comparison of the unit normal and the empirical distributions.

[18] Since, for our purposes, the variable of interest will *always* be the change in log price, the reader should note that henceforth when the words "price change" appear in the text, we are actually referring to the change in log price.

[19] I recognize that because of tax effects and other considerations, the value of a share may not be expected to fall by the full amount of the dividend. Because of uncertainty concerning what the correct adjustment should be, the price changes on ex-dividend days were discarded in an earlier version of the paper. Since the results reported in the earlier version differ very little from those to be presented below, it seems that adding back the full amount of the dividend produces no important distortions in the empirical results.

Each entry in this table was computed by taking the corresponding entry in Table 1 and subtracting from it the entry for the unit normal distribution in Table 1. For example, the entry in column (1) Table 2 for Allied Chemical was found by subtracting the entry in column (1) Table 1 for the unit normal, 0.3830, from the entry in column (1) Table 1 for Allied Chemical, 0.4595.

A positive number in Table 2 should be interpreted as an excess of relative frequency in the empirical distribution over what would be expected for the given interval if the distribution were normal. For example, the entry in column (1) opposite Allied Chemical implies that the empirical distribution contains about 7.6 per cent more of the total frequency within one-half standard deviation of the mean than would be expected if the distribution were normal. The number in column (9) implies that in the empirical distribution about 0.16 per cent more of the total frequency is greater than five standard deviations from the mean than would be expected under the normal or Gaussian hypothesis.

Similarly, a negative number in the table should be interpreted as a deficiency of relative frequency within the

TABLE 1

FREQUENCY DISTRIBUTIONS

Stocks	Intervals								
	0.5 S (1)	1.0 S (2)	1.5 S (3)	2.0 S (4)	2.5 S (5)	3.0 S (6)	4.0 S (7)	5.0 S (8)	>5.0 S (9)
Unit normal	0.3830	0.6826	0.8664	0.9545	0.9876	0.9973	0.999938	0.9999994	0.0000006
Allied Chemical	.4595	.7449	.8782	.9550	.9755	.9869	0.996729	0.9983647	.0016353
Alcoa	.4378	.7260	.8706	.9420	.9765	.9941	1.000000	1.0000000	.0000000
American Can	.4938	.7695	.8983	.9491	.9672	.9844	0.995078	0.9975390	.0024610
A.T.&T.	.5824	.8162	.9237	.9582	.9795	.9860	0.992617	0.9950779	.0049221
American Tobacco	.5394	.7818	.8893	.9462	.9704	.9844	0.994544	0.9968823	.0031177
Anaconda	.4300	.7075	.8785	.9522	.9757	.9933	0.999162	1.0000000	.0000000
Bethlehem Steel	.4792	.7350	.8850	.9483	.9750	.9875	0.996667	0.9991667	.0008333
Chrysler	.4350	.7264	.8794	.9486	.9781	.9905	0.997636	0.9994090	.0005910
Du Pont	.4336	.7257	.8825	.9469	.9775	.9936	0.997586	0.9991955	.0008045
Eastman Kodak	.4410	.7472	.8780	.9467	.9733	.9895	0.998384	0.9983845	.0016155
General Electric	.4631	.7460	.8771	.9427	.9775	.9870	0.997047	0.9994093	.0005907
General Foods	.4489	.7493	.8871	.9467	.9751	.9844	0.997869	0.9992898	.0007102
General Motors	.4716	.7455	.8859	.9571	.9792	.9910	0.995851	0.9979253	.0020741
Goodyear	.4638	.7487	.8898	.9509	.9854	.9914	0.996558	0.9982788	.0017212
International Harvester	.4408	.7450	.8967	.9475	.9750	.9875	0.996667	0.9991667	.0008338
International Nickel	.4722	.7635	.8833	.9413	.9686	.9871	0.995173	1.0000000	.0000000
International Paper	.4444	.7498	.8742	.9433	.9758	.9869	0.996545	1.0000000	.0000000
Johns Manville	.4365	.7377	.8730	.9485	.9809	.9909	0.997510	0.9991701	.0008299
Owens Illinois	.4778	.7389	.8909	.9466	.9717	.9838	0.997575	0.9991916	.0008084
Procter & Gamble	.5017	.7706	.8887	.9378	.9710	.9862	0.995853	0.9986178	.0013822
Sears	.5388	.7856	.9021	.9490	.9701	.9830	0.993528	0.9959547	.0040453
Standard Oil (Calif.)	.4584	.7348	.8724	.9439	.9764	.9917	0.997047	0.9994093	.0005907
Standard Oil (N.J.)	.5035	.7751	.8953	.9559	.9766	.9896	0.997405	0.9982699	.0017301
Swift & Co.	.4647	.7476	.8817	.9405	.9703	.9875	0.997234	1.0000000	.0000000
Texaco	.4599	.7282	.8697	.9517	.9750	.9879	0.998274	1.0000000	.0000000
Union Carbide	.4168	.7191	.8783	.9401	.9785	.9946	0.999106	1.0000000	.0000000
United Aircraft	.4583	.7483	.8858	.9500	.9808	.9908	0.997500	0.9991667	.0008333
U.S. Steel	.4125	.6933	.8758	.9508	.9817	.9933	0.999167	1.0000000	.0000000
Westinghouse	.4392	.7320	.8847	.9503	.9765	.9903	0.997928	0.9986188	.0013812
Woolworth	0.4969	0.7668	0.8844	0.9474	0.9737	0.9841	0.996540	0.9986159	0.0013841
Averages	0.4667	0.7469	0.8847	0.9478	0.9756	0.9886	0.996959	0.9988368	0.0011632

given interval. For example, the number in column (5) opposite Allied Chemical implies that about 1.21 per cent less total frequency is within 2.5 standard deviations of the mean than would be expected under the Gaussian hypothesis. This means there is about twice as much frequency beyond 2.5 standard deviations than would be expected if the distribution were normal.

The most striking feature of the tables is the presence of some degree of leptokurtosis for every stock. In every case the empirical distributions are more peaked in the center and have longer tails than the normal distribution. The pattern is best illustrated in Table 2. In columns (1), (2), and (3) all the numbers are positive, implying that in the empirical distributions there are more observations within 0.5, 1.0, and 1.5 standard deviations than would be expected under the Gaussian hypothesis. In columns (4) through (8) the overwhelming preponderance of negative numbers indicates that there is a general deficiency of relative frequency within any interval 2 to 5 standard deviations from the mean and thus a general excess of relative frequency beyond these points. In column (9) twenty-two out of thirty of the numbers are positive, pointing to a general excess of relative frequency greater than five standard deviations from the mean.

TABLE 2

COMPARISON OF EMPIRICAL FREQUENCY DISTRIBUTIONS WITH UNIT NORMAL

Stock	Intervals								
	0.5 S (1)	1.0 S (2)	1.5 S (3)	2.0S (4)	2.5 S (5)	3.0 S (6)	4.0 S (7)	5.0 S (8)	>5.0 S (9)
Allied Chemical	0.0765	0.0623	0.0118	0.0005	—0.0121	—0.0104	—0.003209	—0.0016347	0.0016347
Alcoa	.0548	.0434	.0042	— .0125	— .0111	— .0032	.000062	.0000006	— .0000006
American Can	.1108	.0669	.0319	— .0054	— .0204	— .0129	— .004860	— .0024604	.0024604
A.T.&T.	.1994	.1336	.0573	.0037	— .0081	— .0112	— .007321	— .0049215	.0049215
American Tobacco	.1564	.0992	.0229	— .0083	— .0172	— .0129	— .005394	— .0031171	.0031171
Anaconda	.0470	.0249	.0121	— .0023	— .0119	— .0040	— .000776	.0000006	— .0000006
Bethlehem Steel	.0962	.0524	.0186	— .0062	— .0126	— .0098	— .003271	— .0008327	.0008327
Chrysler	.0520	.0438	.0130	— .0059	— .0095	— .0068	— .002302	— .0005904	.0005904
Du Pont	.0506	.0431	.0161	— .0076	— .0101	— .0037	— .002351	— .0008039	.0008039
Eastman Kodak	.0580	.0646	.0116	— .0078	— .0142	— .0078	— .001553	— .0016149	.0016149
General Electric	.0801	.0634	.0107	— .0118	— .0100	— .0103	— .002891	— .0005901	.0005901
General Foods	.0659	.0667	.0207	— .0078	— .0125	— .0129	— .002069	— .0007096	.0007096
General Motors	.0886	.0629	.0195	.0026	— .0083	— .0063	— .004087	— .0020749	.0020741
Goodyear	.0808	.0661	.0234	— .0035	— .0022	— .0059	— .003380	— .0017206	.0017206
International Harvester	.0578	.0624	.0303	— .0070	— .0126	— .0098	— .003271	— .0008327	.0008327
International Nickel	.0892	.0809	.0169	— .0132	— .0190	— .0102	— .004765	.0000006	— .0000006
International Paper	.0614	.0672	.0078	— .0112	— .0118	— .0104	— .003393	.0000006	— .0000006
Johns Manville	.0535	.0551	.0066	— .0059	— .0067	— .0064	— .002428	— .0008293	.0008293
Owens Illinois	.0948	.0563	.0245	— .0078	— .0159	— .0135	— .002363	— .0008078	.0008078
Procter & Gamble	.1187	.0880	.0223	— .0167	— .0166	— .0111	— .004084	— .0013822	.0013822
Sears	.1558	.1030	.0537	— .0055	— .0175	— .0143	— .006411	— .0040447	.0040447
Standard Oil (Calif.)	.0754	.0522	.0060	— .0106	— .0112	— .0056	— .002891	— .0005901	.0005901
Standard Oil (N.J.)	.1204	.0925	.0289	.0014	— .0109	— .0077	— .002533	— .0017295	.0017295
Swift & Co.	.0817	.0650	.0153	— .0140	— .0173	— .0097	— .002704	.0000006	— .0000006
Texaco	.0769	.0456	.0033	— .0028	— .0126	— .0094	— .001664	.0000006	— .0000006
Union Carbide	.0338	.0365	.0119	— .0144	— .0091	— .0027	— .000832	.0000006	— .0000006
United Aircraft	.0753	.0657	.0194	— .0045	— .0068	— .0065	— .002438	— .0008327	.0008327
U.S. Steel	.0295	.0107	.0094	— .0037	— .0059	— .0040	— .000771	.0000006	— .0000006
Westinghouse	.0562	.0494	.0183	— .0042	— .0111	— .0070	— .002010	— .0013806	.0013806
Woolworth	0.1139	0.0842	0.0180	—0.0071	—0.0139	—0.0132	—0.003398	—0.0013835	0.0013835
Averages	0.0837	0.0636	0.0183	—0.0066	—0.0120	—0.0086	—0.002979	—0.0011632	—0.0011632

At first glance it may seem that the absolute size of the deviations from normality reported in Table 2 is not important. For example, the last line of the table tells us that the excess of relative frequency beyond five standard deviations from the mean is, on the average, about 0.12 per cent. This is misleading, however, since under the Gaussian hypothesis the total predicted relative frequency beyond five standard deviations is 0.00006 per cent. Thus the actual *excess* frequency is 2,000 times larger than the total expected frequency.

Figure 1 provides a better insight into the nature of the departures from normality in the empirical distributions. The dashed curve represents the unit normal density function, whereas the solid curve represents the general shape of the empirical distributions. A consistent departure from normality is the excess of observations within one-half standard deviation of the mean. On the average there is 8.4 per cent too much relative frequency in this interval. The curves of the empirical density functions are above the curve for the normal distribution. Before 1.0 standard deviation from the mean, however, the empirical curves cut down through the normal curve from above. Although there is a general excess of relative frequency within 1.0 standard deviation, in twenty-four out of thirty cases the excess is not as great as that within one-half standard deviation. Thus the empirical relative frequency between 0.5 and 1.0 standard deviations must be less than would be expected under the Gaussian hypothesis.

Somewhere between 1.5 and 2.0 standard deviations from the mean the empirical curves again cross through the normal curve, this time from below. This is indicated by the fact that in the empirical distributions there is a consistent deficiency of relative frequency within 2.0, 2.5, 3.0, 4.0, and 5.0 standard deviations, implying that there is too much relative frequency beyond these intervals. This is, of course, what is meant by long tails.

The results in Tables 1 and 2 can be cast into a different and perhaps more illuminating form. In sampling from a normal distribution the probability that an observation will be more than two standard deviations from the mean is 0.04550. In a sample of size N the expected number of observations more than two standard deviations from the mean is $N \times 0.04550$. Similarly, the expected numbers greater than three, four, and five standard deviations from the mean are, respectively, $N \times 0.0027$, $N \times 0.000063$, and $N \times 0.0000006$. Following this procedure Table 3 shows for each

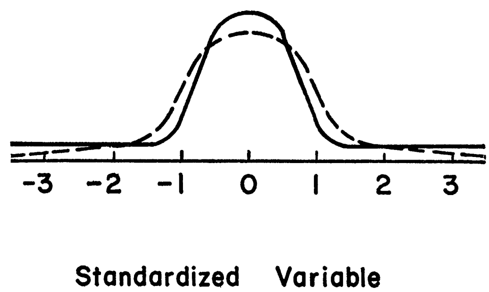

FIG. 1.—Comparison of empirical and unit normal probability distributions.

stock the expected and actual *numbers* of observations greater than 2, 3, 4, and 5 standard deviations from their means.

The results are consistent and impressive. Beyond three standard deviations there should only be, on the average, three to four observations per security. The actual numbers range from six to twenty-three. Even for the sample sizes under consideration the expected number of observations more than four standard deviations from the mean is only about 0.10 per security. In fact for all stocks but one there is at least one observation greater than four standard deviations from the mean, with one stock having as many as nine observations in this range.

In simpler terms, if the population of price changes is strictly normal, on the average for any given stock we would expect an observation greater than 4 standard deviations from the mean about once every fifty years. In fact observations this extreme are observed about four times in every five-year period. Similarly, under the Gaussian hypothesis for any given stock an observation more than five standard deviations from the mean should be observed about once every 7,000 years. In fact such observations seem to occur about once every three to four years.

These results can be put into the form of a significance test. Tippet [44] in 1925 calculated the distribution of the largest value in samples of size 3–1,000 from a normal population. In Table 4 his results for N = 1,000 have been used to find the approximate significance levels of the most extreme positive and negative first differences of log price for each stock. The significance levels are only approximate because the actual sample sizes are greater than 1,000. The effect of this is

TABLE 3

ANALYSIS OF EXTREME TAIL AREAS IN TERMS OF NUMBER OF OBSERVATIONS RATHER THAN RELATIVE FREQUENCIES

Stock	N*	Interval							
		>2 S		>3 S		>4 S		>5 S	
		Expected No.	Actual No.	Expected No.	Actual No.	Expected No.	Actual No.	Expected No.	Actual No.
Allied Chemical	1,223	55.5	55	3.3	16	0.08	4	0.0007	2
Alcoa	1,190	54.1	69	3.2	7	.07	0	.0007	0
American Can	1,219	55.5	62	3.3	19	.08	6	.0007	3
A.T.&T.	1,219	55.5	51	3.3	17	.08	9	.0007	6
American Tobacco	1,283	58.4	69	3.5	20	.08	7	.0008	4
Anaconda	1,193	54.3	57	3.2	8	.08	1	.0007	0
Bethlehem Steel	1,200	54.6	62	3.2	15	.08	4	.0007	1
Chrysler	1,692	77.0	87	4.6	16	.11	4	.0010	1
Du Pont	1,243	56.6	66	3.4	8	.08	3	.0007	1
Eastman Kodak	1,238	56.3	66	3.3	13	.08	2	.0007	2
General Electric	1,693	77.0	97	4.6	22	.11	5	.0010	1
General Foods	1,408	64.1	75	3.8	22	.09	3	.0008	1
General Motors	1,446	65.8	62	3.9	13	.09	6	.0009	3
Goodyear	1,162	52.9	57	3.1	10	.07	4	.0007	2
International Harvester	1,200	54.6	63	3.2	15	.08	4	.0007	1
International Nickel	1,243	56.5	73	3.4	16	.08	6	.0007	0
International Paper	1,447	65.8	82	3.9	19	.09	5	.0009	0
Johns Manville	1,205	54.8	62	3.2	11	.08	3	.0007	1
Owens Illinois	1,237	56.3	66	3.3	20	.08	3	.0007	1
Procter & Gamble	1,447	65.8	90	3.9	20	.09	6	.0009	2
Sears	1,236	56.2	63	3.3	21	.08	8	.0007	5
Standard Oil (Calif.)	1,693	77.0	95	4.6	14	.11	5	.0010	1
Standard Oil (N.J.)	1,156	52.5	51	3.1	12	.07	3	.0007	2
Swift & Co.	1,446	65.8	86	3.9	18	.09	4	.0009	0
Texaco	1,159	52.7	56	3.1	14	.07	2	.0007	0
Union Carbide	1,118	50.9	67	3.0	6	.07	1	.0007	0
United Aircraft	1,200	54.6	60	3.2	11	.08	3	.0007	1
U.S. Steel	1,200	54.6	59	3.2	8	.08	1	.0007	0
Westinghouse	1,448	65.9	72	3.9	14	.09	3	.0009	2
Woolworth	1,445	65.7	78	3.9	23	0.09	5	0.0009	2
Totals	....	1,787.4	2,058	105.8	448	2.51	120	0.0233	45

* Total sample size.

to overestimate the significance level, since in samples of 1,300 an extreme value greater than a given size is more probable than in samples of 1,000. In most cases, however, the error introduced in this way will affect at most the third decimal place and hence is negligible in the present context.

Columns (1) and (4) in Table 4 show the most extreme negative and positive changes in log price for each stock. Columns (2) and (5) show these values measured in units of standard deviations from their means. Columns (3) and (6) show the significance levels of the extreme values. The significance levels should be interpreted as follows: in samples of 1,000 observations from a normal population on the average in a proportion P of all samples, the most extreme value of a given tail would be smaller in absolute value than the extreme value actually observed.

As would be expected from previous discussions, the significance levels in Table 4 are very high, implying that the observed extreme values are much more extreme than would be predicted by the Gaussian hypothesis.

D. NORMAL PROBABILITY GRAPHS

Another sensitive tool for examining departures from normality is probability graphing. If u is a Gaussian random variable with mean μ and variance σ^2, the standardized variable

$$z = \frac{u - \mu}{\sigma} \qquad (2)$$

TABLE 4

SIGNIFICANCE TESTS FOR EXTREME VALUES

Stock	Smallest Value (1)	Standardized Variable (2)	P (3)	Largest Value (4)	Standardized Variable (5)	P (6)
Allied Chemical	−0.07178	− 5.012	0.99971	0.08377	5.820	0.99999
Alcoa	− .05314	− 3.381	0.71391	.06188	3.945	0.95304
American Can	− .06230	− 5.446	0.99997	.06752	5.853	0.99999
A.T.&T.	− .10376	−10.342	1.00000	.09890	9.724	1.00000
American Tobacco	− .08004	− 6.678	1.00000	.07238	5.949	0.99999
Anaconda	− .05733	− 3.851	0.93020	.06004	4.015	0.96882
Bethlehem Steel	− .07250	− 5.571	0.99999	.06195	4.748	0.99870
Chrysler	− .08049	− 4.660	0.99870	.10085	5.853	0.99999
Du Pont	− .05990	− 5.843	0.99999	.05148	4.950	0.99952
Eastman Kodak	− .04434	− 3.399	0.71391	.07788	5.832	0.99999
General Electric	− .06466	− 5.135	0.99983	.05647	4.456	0.99460
General Foods	− .04683	− 3.937	0.95304	.06246	5.065	0.99983
General Motors	− .09764	− 7.761	1.00000	.08292	6.547	1.00000
Goodyear	− .09459	− 5.919	0.99999	.17435	10.879	1.00000
International Harvester	− .08701	− 6.290	0.99999	.06870	4.880	0.99952
International Nickel	− .05917	− 4.917	0.99952	.05670	4.628	0.99789
International Paper	− .05072	− 4.219	0.98674	.05327	4.454	0.99460
Johns Manville	− .06868	− 4.386	0.99460	.11935	7.575	1.00000
Owens Illinois	− .06372	− 5.195	0.99990	.06062	4.881	0.99952
Procter & Gamble	− .06351	− 5.504	0.99998	.06560	5.559	0.99998
Sears	− .10728	− 9.338	1.00000	.06062	5.148	0.99983
Standard Oil (Calif.)	− .06333	− 4.793	0.99921	.06738	5.056	0.99983
Standard Oil (N.J.)	− .10318	− 9.275	1.00000	.10073	9.013	1.00000
Swift & Co.	− .06752	− 4.761	0.99921	.06283	4.418	0.99460
Texaco	− .05932	− 4.650	0.99789	.05476	4.193	0.98674
Union Carbide	− .04556	− 4.396	0.99460	.03943	3.783	0.93020
United Aircraft	− .15234	− 8.878	1.00000	.08490	4.939	0.99952
U.S. Steel	− .05386	− 3.968	0.96882	.05550	4.091	0.97955
Westinghouse	− .08037	− 5.415	0.99997	.08630	5.808	0.99999
Woolworth	−0.06744	− 5.890	0.99999	0.08961	7.743	1.00000

will be unit normal. Since z is just a linear transformation of u, the graph of z against u is just a straight line

The relationship between z and u can be used to detect departures from normality in the distribution of u. If u_i, $i = 1, \ldots, N$ are N sample values of the variable u arranged in ascending order, then a particular u_i is an estimate of the f fractile of the distribution of u, where the value of f is given by[20]

$$f = \frac{(3i - 1)}{(3N + 1)}. \qquad (3)$$

Now the exact value of z for the f fractile of the unit normal distribution need not be estimated from the sample data. It can be found easily either in any standard table or (much more rapidly) by computer. If u is a Gaussian random variable, then a graph of the *sample* values of u against the values of z derived from the *theoretical* unit normal cumulative distribution function (c.d.f.) should be a straight line. There may, of course, be some departures from linearity due to sampling error. If the departures from linearity are extreme, however, the Gaussian hypothesis for the distribution of u should be questioned.

The procedure described above is called normal probability graphing. A normal probability graph has been constructed for each of the stocks used in this report, with u equal, of course, to the daily first difference of log price. The graphs are found in Figure 2.

The scales of the graphs in Figure 2 are determined by the two most extreme values of u and z. The origin of each graph is the point $(u_{\min}, z_{\min})$, where $u_{\min}$ and $z_{\min}$ are the minimum values of u and z for the particular stock. The last point in the upper right-hand corner of each graph is $(u_{\max}, z_{\max})$. Thus if the Gaussian hypothesis is valid, the plot of z against u should for each security approximately trace a 45° straight line from the origin.[21]

Several comments concerning the graphs can be made immediately. First, probability graphing is just another way of examining an empirical frequency distribution, and there is a direct relationship between the frequency distributions examined earlier and the normal probability graphs. When the tails of empirical frequency distributions are longer than those of the normal distribution, the slopes in the extreme tail areas of the normal probability graphs should be lower than those in the central parts of the graphs, and this is in fact the case. That is, the graphs in general take the shape of an elongated S with the curvature at the top and bottom varying directly with the excess of relative frequency in the tails of the empirical distribution.

Second, this tendency for the extreme tails to show lower slopes than the main portions of the graphs will be accentuated by the fact that the central bells of the empirical frequency distributions are higher than those of a normal distribution. In this situation the central portions of the normal probability graphs should be steeper than would be the case

[20] This particular convention for estimating f is only one of many that are available. Other popular conventions are $i/(N+1)$, $(i - \frac{3}{8})/(N + \frac{1}{4})$, and $(i - \frac{1}{2})/N$. All four techniques give reasonable estimates of the fractiles, and with the large samples of this report, it makes very little difference which specific convention is chosen. For a discussion see E. J. Gumbel [20, p. 15] or Gunnar Blom [8, pp. 138–46].

[21] The reader should note that the origin of every graph is an actual sample point, even if it is not always visible in the graphs because it falls at the point of intersection of the two axes. It is probably of interest to note that the graphs in Figure 2 were produced by the cathode ray tube of the University of Chicago's I.B.M. 7094 computer.

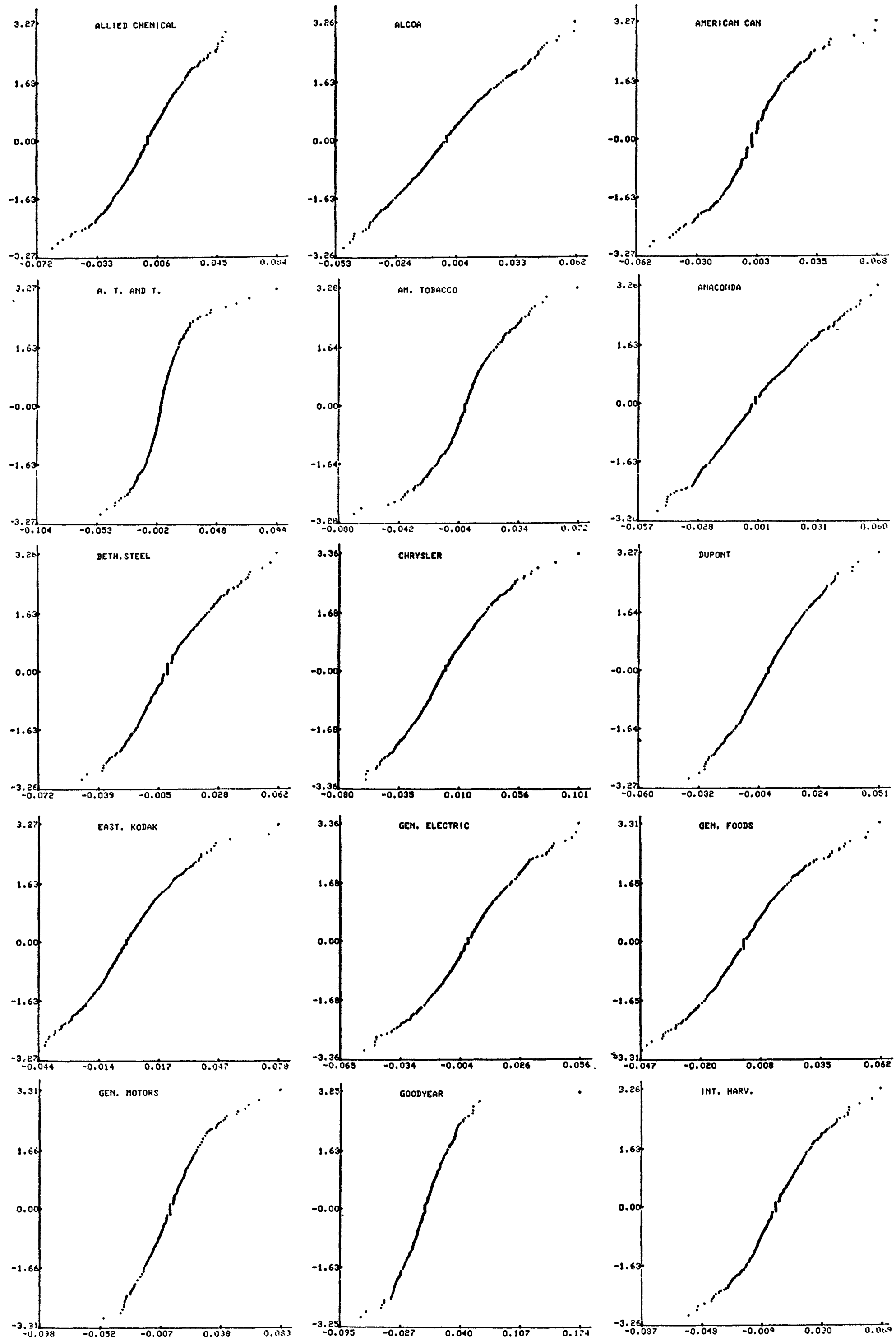

FIG. 2.—Normal probability graphs for daily changes in log price of each security. Horizontal axes of graphs show u, values of the daily changes in log price; vertical axes show z, values of the unit normal variable at different estimated fractile points.

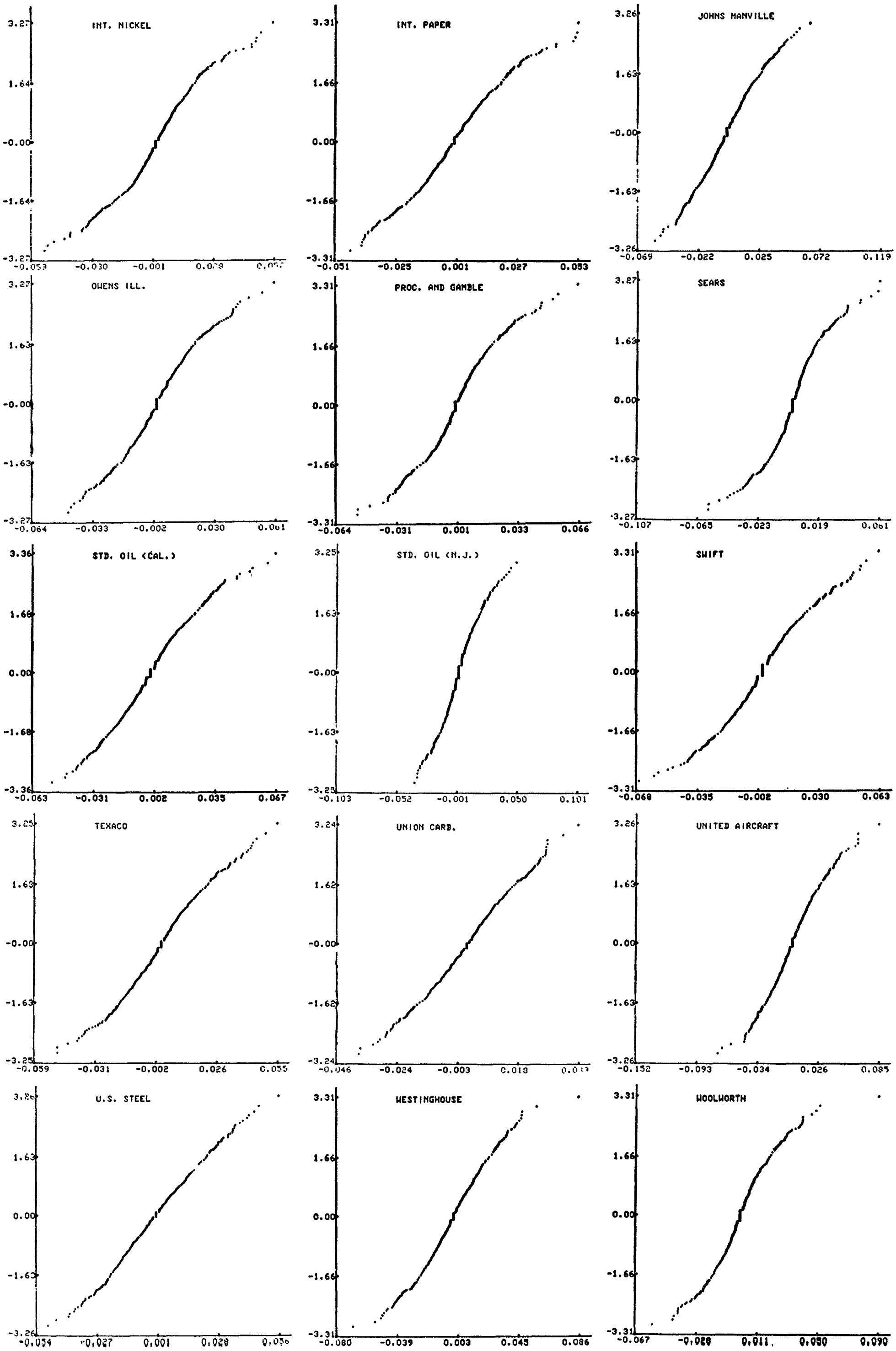

FIG. 2.—*Continued*

if the underlying distributions were strictly normal. This sort of departure from normality is evident in the graphs.

Finally, before the advent of the Mandelbrot hypothesis, some of our normal probability graphs would have been considered acceptable within a hypothesis of "approximate" normality. This is true, for example, for Anaconda and Alcoa. It is not true, however, for most of the graphs. The tail behavior of stocks such as American Telephone and Telegraph and Sears is clearly inconsistent with any simple normality hypothesis. The emphasis is on the word simple. The natural next step is to consider complications of the Gaussian model that could give rise to departures from normality of the type encountered.

E. TWO POSSIBLE ALTERNATIVE EXPLANATIONS OF DEPARTURES FROM NORMALITY

1. MIXTURE OF DISTRIBUTIONS

Perhaps the most popular approach to explaining long-tailed distributions has been to hypothesize that the distribution of price changes is actually a mixture of several normal distributions with possibly the same mean, but substantially different variances. There are, of course, many possible variants of this line of attack, and little can be done to test them unless the investigator is prepared to specify some details of the mechanism instead of merely talking vaguely of "contamination." One such plausible mechanism is the following suggested by Lawrence Fisher of the Graduate School of Business, University of Chicago.

It is possible that the relevant unit of time for the generation of information bearing on stock prices is the chronological day rather than the trading day. Political and economic news, after all, occurs continuously, and if it is assimilated continuously by investors, the variance of the distribution of price changes between two points in time would possibly be proportional to the actual number of days elapsed rather than to the number of trading days. Thus in our tests a mixture of distributions would be produced by the fact that changes in log price from Friday (close) to Monday (close) involve three chronological days while the changes during the week involve only one chronological day.

To test this hypothesis, eleven stocks were randomly chosen from the sample of thirty, and for each stock two arrays were set up. One array contained changes involving only one chronological day. These are, of course, the daily changes from Monday to Friday of each week. The other array contained changes involving more than one chronological day. These include Friday-to-Monday changes and changes across holidays

Table 5 gives a comparison of the total variances for each type of price change. Column (1) shows the variances for changes involving one chronological day. Column (2) contains the variances for weekend and holiday changes. Column (3) shows the ratio of column (2) to column (1). If the chronological day rather than the trading day were the relevant unit of time, then, according to the well-known law for the variance of sums of independent variables, the variance of the weekend and holiday changes should be a little less than three times the variance of the day-to-day changes within the week. It should be a little less than three because three days pass between Friday (close) and Monday (close), but holidays normally involve a lapse of only two days. Actually, however, it turns out that the weekend and holiday variance is not three times but only about 22 per cent greater than the within week variance—a rather small discrepancy.[22]

However, for the moment let us continue under the assumption that the weekend and holiday changes and the changes within the week come from different normal distributions. This implies that the normal probability graphs for the weekend and daily changes should each be straight lines, even though the combined distributions plot as elongated S's. In fact when the within-week and weekend changes were plotted separately, the graphs turned out to be of exactly the same form as the graph for the two distributions combined. The same departures from normality were present and the same elongated S shapes occurred.

As an example, Figure 3 shows three normal probability graphs for Procter and Gamble.[23] The first shows the graph of the first differences of log price for daily changes within the week. The second is the graph of Friday-to-Monday changes and of changes across holidays. The third is the combined graph for changes where the differencing interval is the trading day and chronological time is ignored.

The conclusion drawn from the above discussion is that it makes no substantial difference whether weekend and holiday changes are considered separately or together with the daily changes within the week. The nature of the tails of the distribution seems the same under each type of analysis.

TABLE 5

VARIANCE COMPARISON OF DAILY AND WEEKEND CHANGES

Stock	Daily Variance (1)	Weekend Variance (2)	Weekend Variance/ Daily Variance (3)
Alcoa	0.000247	0.000252	1.020
A.T.&T.	.000091	.000105	1.154
Anaconda	.000212	.000252	1.189
Chrysler	.000278	.000363	1.306
International Harvester	.000186	.000226	1.215
International Nickel	.000146	.000145	0.993
Procter & Gamble	.000125	.000178	1.424
Standard Oil (Calif.)	.000162	.000215	1.327
Standard Oil (N.J.)	.000114	.000153	1.342
Texaco	.000153	.000209	1.366
U.S. Steel	0.000176	0.000198	1.125

2. CHANGING PARAMETERS

Another popular explanation of long-tailed empirical distributions is non-stationarity. It may be that the distribution of price changes at any point in time is normal, but across time the parameters

[22] The relative unimportance of the weekend effect is also documented, in a different way, by Godfrey, Granger, and Morgenstern [18.]

[23] The reader will note that the normal probability graphs of Figure 3 (and also Figure 4) follow the more popular convention of showing the c.d.f. on the vertical axis rather than the standardized variable z. Since there is a one-to-one correspondence between values of z and points on the c.d.f., from a theoretical standpoint it is a matter of indifference as to which variable is shown on the vertical axis. From a practical standpoint, however, when the graphs are done by hand it is easier to use "probability paper" and the c.d.f. When the graphs are done by computer, it is easier to use the standardized variable z.

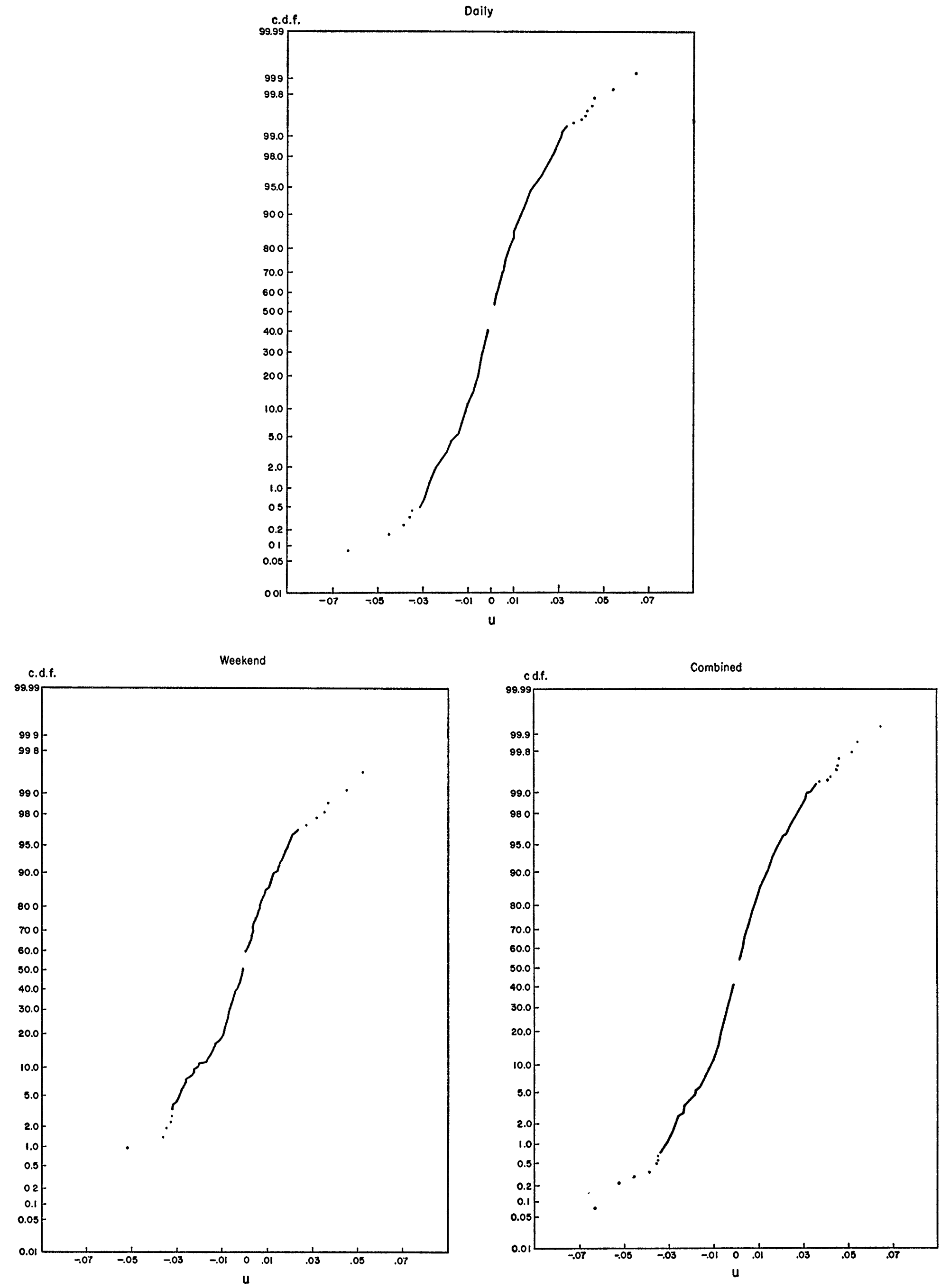

FIG. 3.—Daily, weekend, and combined normal probability graphs for Procter & Gamble. Horizontal axes show u, values of the daily changes in log price; vertical axes show fractiles of the c.d.f.

of the distribution change. A company may become more or less risky, and this may bring about a shift in the variance of the first differences. Similarly, the mean of the first differences can change across time as the company's prospects for future profits follow different paths. This paper will consider only changes in the mean.

If a shift in the mean change in log price of a daily series is to persist for any length of time, it must be small, unless the eventual change in price is to be astronomical. For example, a stock's price will double in less than four months if the mean of the daily changes in log price shifts from zero to 0.01. It is not that large changes in the mean are uninteresting. It is just that unless the eventual price change is to be phenomenal, a large change in the mean will not persist long enough to be identified. The basic problem is one of identification. "Trends" that do not last very long are numerous. It is usually difficult to explain these short "trends" plausibly whether the eventual price change is large or small. On the other hand, changes in the mean that persist are presumably identifiable by their very persistence. It is not particularly unreasonable to treat a period of, say, a year or more that shows a fairly steady trend differently from other periods.

In an effort to test the non-stationarity hypothesis, five stocks were chosen which seemed to show changes in trend that persisted for rather long periods of time during the period covered by this study.[24] "Trends" were "identified" simply by examining a graph of the stock's price during the sampling period. The procedure, though widely practiced, is of course completely arbitrary.

The results, however, are quite interesting. For each stock, normal probability graphs were constructed for each separate trend period. In all cases the results were the same; each of the subperiods of different apparent trend showed exactly the same type of tail behavior as the total sample of price changes for the stock for the entire sampling period.

As an example three normal probability graphs for American Telephone and Telegraph are presented in Figure 4. The first covers the time period November 25, 1957–December 11, 1961, when the mean of the distribution of first differences of log price was 0.00107. The second covers the period December 11, 1961–September 24, 1962, when the mean was —0.00061. The third is the graph of the total sample with over-all mean 0.000652. As was typical of all the stocks the graphs are extremely similar. The same type of elongated S appears in all three.

Thus it seems that the behavior of the distribution in the tails is independent of the mean. This is not really a very unusual result. A change in the mean, if it is to persist, must be rather small. In particular the shift is small relative to the largest values of a random variable from a long-tailed distribution.

It is true that we have only considered changes in the mean that persist for fairly long periods of time, and this is a possible shortcoming of the preceding tests. It is also true, however, that any distribution, no matter how wild, can be represented as a mixture of normals if one is willing to postulate many short-lived periods of non-stationarity. One of the main sources of appeal of Mandelbrot's model, however, is that it is capable of explaining both periods of turbu-

[24] The stocks chosen were American Can, American Telephone and Telegraph, American Tobacco, Procter and Gamble, and Sears.

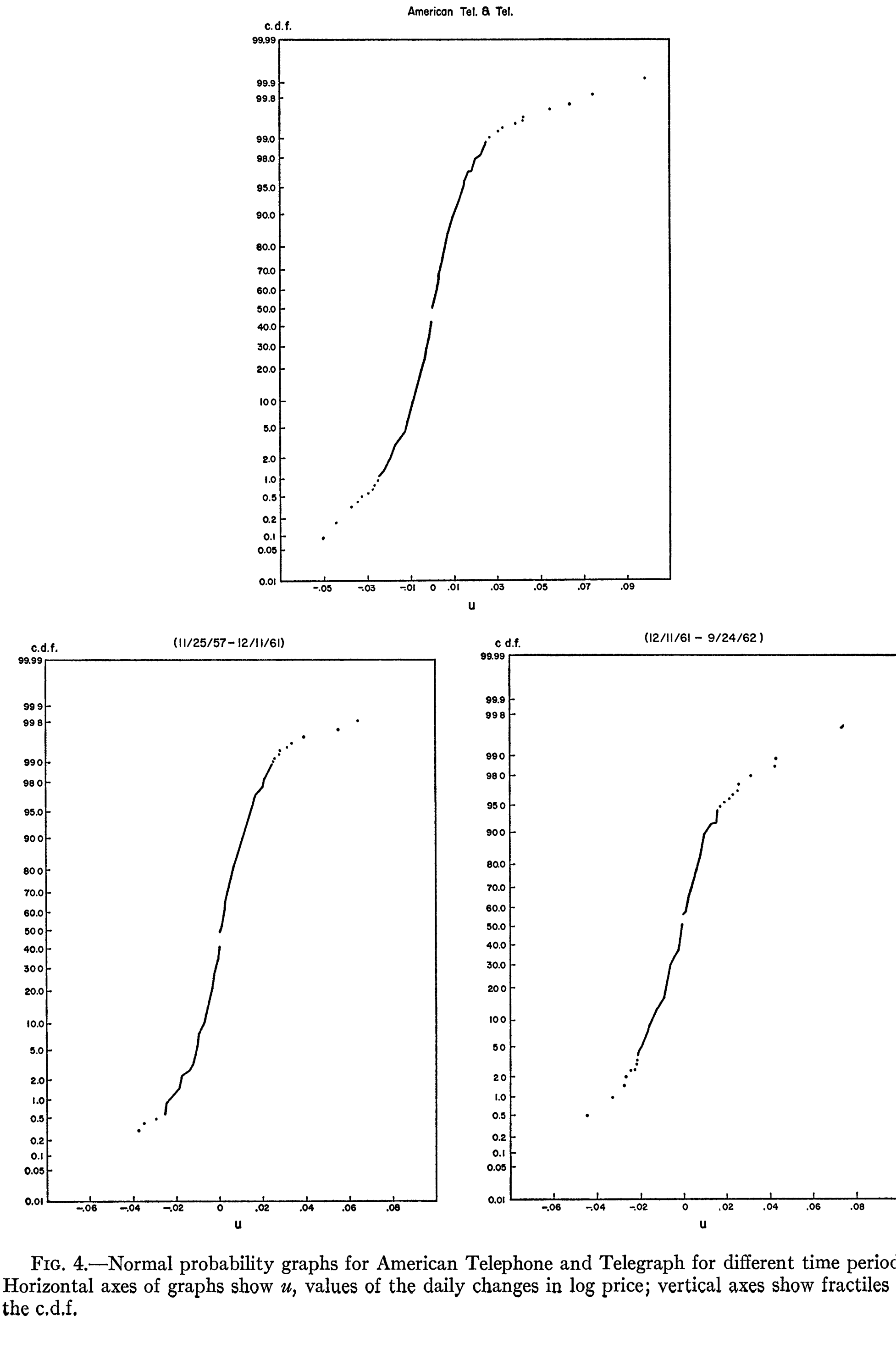

FIG. 4.—Normal probability graphs for American Telephone and Telegraph for different time periods. Horizontal axes of graphs show u, values of the daily changes in log price; vertical axes show fractiles of the c.d.f.

lence and periods of calm, without resorting to non-stationarity arguments.

F. CONCLUSION

The main result of this section is that the departures from normality in the distributions of the first differences of the logarithms of stock prices are in the direction predicted by the Mandelbrot hypothesis. Moreover, the two more complicated versions of the Gaussian model that were examined are incapable of explaining the departures. In the next section further tests will be used to decide whether the departures from normality are sufficient to warrant rejection of the Gaussian hypothesis.

IV. A Closer Look at the Empirical Distributions

The first step in this section will be to test whether the distributions of price changes have the crucial property of stability. If stability seems to hold, the problem will have been reduced to deciding whether the characteristic exponent α of the underlying stable Paretian process is less than 2, as assumed by the Mandelbrot hypothesis, or equal to 2 as assumed by the Gaussian hypothesis.

A. STABILITY

By definition, stable Paretian distributions are stable or invariant under addition. That is, except for origin and scale, sums of independent, identically distributed, stable Paretian variables have the same distribution as the individual summands. Hence, if successive daily changes in stock prices follow a stable Paretian distribution, changes across longer intervals such as a week or a month will follow stable Paretian distributions of exactly the same form.[25] Most simply this means that the characteristic exponent α of the weekly and monthly distributions will be the same as the characteristic exponent of the distribution of the daily changes.

Thus the most direct way to test stability would be to estimate α for various differencing intervals to see if the same value holds in each case. Unfortunately, this direct approach is not feasible. We shall see later that in order to make reasonable estimates of α very large samples are required. Though the samples of *daily* price changes used in this report will probably be sufficiently large, the sampling period covered is not long enough to make reliable estimates of α for differencing intervals longer than a single day.

The situation is not hopeless, however. We can develop an alternative, though cruder and more indirect, way of testing stability by making use of certain properties of the parameter α. The characteristic exponent α of a stable Paretian distribution determines the length or height of the extreme tails of the distribution. Thus, if α has the same value for different distributions, the behavior of the extreme tails of the distributions should be at least roughly similar.

A sensitive technique for examining the tails of distributions is normal probability graphing. As explained in Section III, the normal probability plot of ranked values of a Gaussian variable will be a straight line. Since the Gaussian distribution is stable, sums of Gaussian variables will also plot as a straight line on a normal probability graph. A stable Paretian distribution with $\alpha < 2$ has longer tails than a Gaussian distribution, however, and thus its normal probability graph will have the appearance of an elongated S, with the degree of curvature in the extreme tails larger the smaller the value of α. Sums of such variables

[25] Weekly and monthly changes in log price are, of course, just sums of daily changes.

should also plot as elongated S's with roughly the same degree of curvature as the graph of the individual summands.

Thus if successive daily changes in log price for a given security follow a stable Paretian distribution with characteristic exponent $\alpha < 2$, the normal probability graph for the changes should have the appearance of an elongated S. Since, by the property of stability, the value of α will be the same for distributions involving differencing intervals longer than a single day, the normal probability graphs for these longer differencing intervals should also have the appearance of elongated S's with about the same degree of curvature in the extreme tails as the graph for the daily changes.

A normal probability graph for the distribution of changes in log price across successive, non-overlapping periods of four trading days has been plotted for each stock. The graphs for four companies (American Tobacco, Eastman Kodak, International Nickel, and Woolworth) are shown in Figure 5. In each case the graph for the four-day changes in Figure 5 seems, except for scale, almost indistinguishable from the corresponding graph for the daily changes in Figure 2. On this basis we conclude that the assumption of stability seems to be justified. The problem in the remainder of Section IV will be to decide whether the underlying stable Paretian process has characteristic exponent less than 2, as proposed by the Mandelbrot hypothesis, or equal to 2, as proposed by the Gaussian hypothesis.

Unfortunately, however, estimation of α is not a simple problem. In most cases there are no known explicit density functions for the stable Paretian distributions, and thus there is virtually no sam-

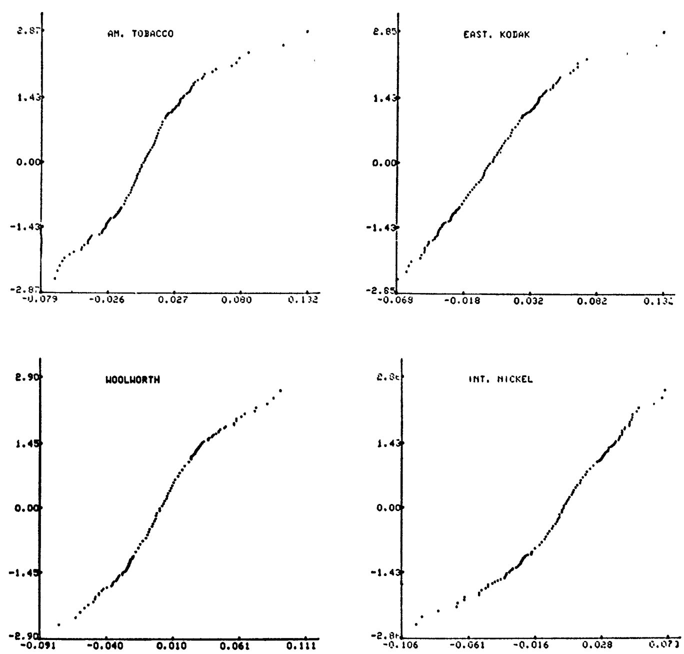

FIG. 5.—Normal probability graphs for price changes across four trading days. Horizontal axes show u, values of the changes in log price; vertical axes show z, the values of the unit normal variable at different estimated fractile points.

pling theory available. Because of this the best that can be done is to make as many different estimates of α as possible in an attempt to bracket the true value. In the remainder of Section IV three different techniques will be used to estimate α. First, each technique will be examined in detail, and then a comparison of the results will be made.

B. ESTIMATING α FROM DOUBLE-LOG AND PROBABILITY GRAPHS

If the distribution of the random variable u is stable Paretian with characteristic exponent $0 < \alpha < 2$, its tails follow an asymptotic form of the law of Pareto such that

$$\begin{aligned} Pr(u > \hat{u}) &\rightarrow (\hat{u}/U_1)^{-\alpha}, \ \hat{u} > 0 \,, \quad \text{and} \\ Pr(u < \hat{u}) &\rightarrow (|\hat{u}|/U_2)^{-\alpha}, \quad \hat{u} < 0 \,, \end{aligned} \tag{4}$$

where U_1 and U_2 are constants and the symbol $\rightarrow$ means that the ratio[26]

$$\frac{Pr(u > \hat{u})}{(u/U_1)^{-\alpha}} \rightarrow 1 \qquad \text{as} \qquad \hat{u} \rightarrow \infty \,.$$

Taking logarithms in expression (4) we have,

$$\log Pr(u > \hat{u}) \rightarrow -\alpha(\log \hat{u} - \log U_1) \,,$$

$$\text{and} \qquad \log Pr(u < \hat{u}) \tag{5}$$

$$\rightarrow -\alpha(\log |u| - \log U_2) \,.$$

Expression (5) implies that if $Pr(u > \hat{u})$ and $Pr(u < \hat{u})$ are plotted against $|\hat{u}|$ on double-log paper, the two curves should become asymptotically straight and have slope that approaches $-\alpha$ as $|\hat{u}|$ approaches infinity. Thus double-log graphing is one technique for estimating α. Unfortunately it is not very powerful if α is close to 2.[27] If the distribution is normal (i.e., $\alpha = 2$), $Pr(u > \hat{u})$ decreases faster than $|u|$ increases, and the slope of the graph of $\log Pr(u > \hat{u})$ against $\log |\hat{u}|$ will approach $-\infty$. Thus the law of Pareto does not hold even asymptotically for the normal distribution.

When α is less than 2 the law of Pareto will hold, but on the double-log graph the true asymptotic slope will only be observed within a tail area containing total probability $p_0(\alpha)$ that is smaller the larger the value of α. This is demonstrated in Figure 6[28] which shows plots of log

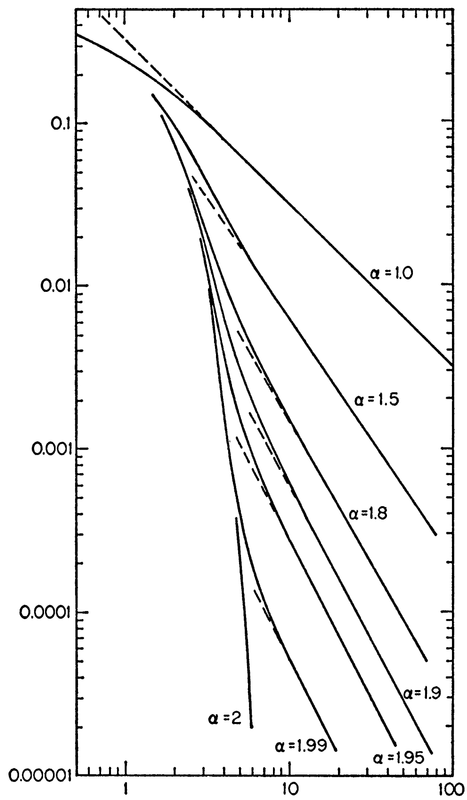

FIG. 6.—Double-log graphs for symmetric stable Paretian variables with different values of α. The various lines are double-log plots of the symmetric stable Paretian probability distributions with $\delta = 0$, $\gamma = 1$, $\beta = 0$ and various values of α. Horizontal axis shows $\log u$; vertical axis shows $\log Pr(u > u)$ $\log Pr(u < -u)$. Taken from Mandelbrot [37, p. 402].

[26] Thus we see that the name stable Paretian for these distributions arises from the property of stability and the asymptotically Paretian nature of the extreme tail areas.

[27] Cf. Mandelbrot [35].

[28] Taken from Mandelbrot [37], p. 402.

$Pr(u > \hat{u})$ against log $|\hat{u}|$ for values of α from one to two, and where the location, skewness, and scale parameters are given the values $\delta = 0$, $\beta = 0$, and $\gamma = 1$. When α is between 1.5 and 2, the absolute value of the slope in the middle of the double-log graph is greater than the true asymptotic slope, which is not reached until close to the bottom of the graph. For example, when $\alpha = 1.5$, the asymptotic slope is closely attained only when $Pr(u > \hat{u}) \leq 0.015$, so that $p_0(\alpha) = 0.015$; and when $\alpha = 1.8$, $p_0(\alpha) = 0.0011$.

If, on the average, the asymptotic slope can be observed only in a tail area containing total probability $p_0(\alpha)$, it will be necessary to have more than $N_0(\alpha) = 1/p_0(\alpha)$ observations before the slope of the graph will even *begin* to approach $-\alpha$. When α is close to 2, extremely large samples are necessary before the asymptotic slope becomes observable.

As an illustration Table 6 shows $p_0(\alpha)$ and $N_0(\alpha)$ for different values of α. The most important feature of the table is the rapid increase of $N_0(\alpha)$ with α. On the average, the double-log graph will *begin* to approach its asymptotic slope in samples of less than 100 only if α is 1.5 or less. If the true value of α is 1.80, usually the graph will only begin to approach its asymptotic slope for sample sizes greater than 909. For higher values of α the minimum sample sizes become almost unimaginable by most standards.

Moreover, the expected number of extreme values which will exhibit the true asymptotic slope is $Np_0(\alpha)$, where N is the size of the sample. If, for example, the true value of α is 1.8 and the sample contains 1,500 observations, on the average the asymptotic slope will be observable only for the largest one or two observations in each tail. Clearly, for large values of α double-log graphing puts much too much weight on the one or two largest observations to be a good estimation procedure. We shall see later that the values of α for the distributions of daily changes in log price of the stocks of the DJIA are definitely greater than 1.5. Thus for our data double-log graphing is not a good technique for estimating α.

The situation is not hopeless, however, the asymptotically Paretian nature of the extreme tails of stable Paretian distributions can be used, in combination with *probability* graphing, to estimate the characteristic exponent α. Looking back at Figure 6, we see that the theoretical double-log graph for the case $\alpha = 1.99$ breaks away from the double-log graph for $\alpha = 2$ at about the point where $Pr(u > \hat{u}) = 0.001$. Similarly, the double-log plot for $\alpha = 1.95$ breaks away from the double-log plot for $\alpha = 1.99$ at about the point where $Pr(u > \hat{u}) = 0.01$. From the point of view of the normal-probability graphs this means that, if α is between 1.99 and 2, we should begin to observe curvature in the graphs somewhere beyond the point where $Pr(u > \hat{u}) = 0.001$. Similarly, if the true value of α is between 1.95 and 1.99, we should observe that the normal-probability graph begins to show curvature somewhere between the point where $Pr(u > \hat{u}) = 0.01$ and the point where $Pr(u > \hat{u}) = 0.001$.

This relationship between the theo-

TABLE 6

α	$p_0(\alpha)$	$N_0(\alpha)$
1.00	0.13000	8
1.50	.01500	67
1.80	.00110	909
1.90	.00050	2,000
1.95	.00030	3,333
1.99	.00006	16,667
2.00	0	....

retical double-log graphs for different values of α and the normal-probability graphs provides a natural procedure for estimating α. Continuing the discussion of the previous paragraph, we see in Figure 6 that the double-log plot for $\alpha = 1.90$ breaks away from the plot for $\alpha = 1.95$ at about the point where $Pr(u > u) = 0.05$. Thus, if a particular normal-probability graph for some stock begins to show curvature somewhere between the points where $Pr(u > \hat{u}) = 0.05$ and $Pr(u > \hat{u}) = 0.01$, we would estimate that α is probably somewhere in the interval $1.90 \leq \alpha \leq 1.95$. Similarly, if the curvature in the normal-probability graphs begins to become evident somewhere between the points where $Pr(u > \hat{u}) = 0.10$ and $Pr(u > \hat{u}) = 0.05$, we shall say that α is probably somewhere in the interval $1.80 \leq \alpha \leq 1.90$. If none of the normal-probability graph is even vaguely straight, we shall say that α is probably somewhere in the interval $1.50 \leq \alpha \leq 1.80$.

Thus we have a technique for estimating α which combines properties of the normal-probability graphs with properties of the double-log graphs. The estimates produced by this procedure are found in column (1) of Table 9. Admittedly the procedure is completely subjective. In fact, the best we can do with it is to try to set *bounds* on the true value of α. The technique does not readily lend itself to point estimation. It is better than just the double-log graphs alone, however, since it takes into consideration more of the total tail area.

C. ESTIMATING α BY RANGE ANALYSIS

By definition, sums of independent, identically distributed, stable Paretian variables are stable Paretian with the same value of the characteristic exponent α as the distribution of the individual summands. The process of taking sums, however, does change the scale of the distribution. In fact it is shown in the appendix that the scale of the distribution of sums is $n^{1/\alpha}$ times the scale of the distribution of the individual summands, where n is the number of observations in each sum.

This property can be used as the basis of a procedure for estimating α. Define an interfractile range as the difference between the values of a random variable at two different fractiles of its distribution. The interfractile range, R_n, of the distribution of sums of n independent realizations of a stable Paretian variable as a function of the same interfractile range, R_1, of the distribution of the individual summands is given by

$$R_n = n^{1/\alpha} R_1 . \qquad (6)$$

Solving for α, we have

$$\alpha = \frac{\log n}{\log R_n - \log R_1} . \qquad (7)$$

By taking different summing intervals (i.e., different values of n), and different interfractile ranges, (7) can be used to get many different estimates of α from the same set of data.

Range analysis has one important drawback, however. If successive price changes *in the sample* are not independent, this procedure will produce "biased" estimates of α. If there is positive serial dependence in the first differences, we should expect that the interfractile range of the distribution of sums will be more than $n^{1/\alpha}$ times the fractile range of the distribution of the individual summands. On the other hand, if there is negative serial dependence in the first differences, we should expect that the interfractile range of the distribution of sums will be less than $n^{1/\alpha}$ times that of the individual summands. Since the range of the sums comes into the denominator of (7), these biases will work in the opposite direction in the estimation of the characteristic exponent α. Positive dependence will produce downward biased estimates of α, while the estimates will be upward biased in the case of negative dependence.[29]

We shall see in Section V, however, that there is, in fact, no evidence of important dependence in successive price changes, at least for the sampling period covered by our data. Thus it is probably safe to say that dependence will not have important effects on any estimates of α produced by the range analysis technique.

Range analysis has been used to compute fifteen different estimates of α for each stock. Summing intervals of four, nine, and sixteen days were used; and for each summing interval separate estimates of α were made on the basis of interquartile, intersextile, interdecile, 5 per cent, and 2 per cent ranges.[30] The procedure can be clarified by adding a superscript to the formula for α as follows:

$$\alpha = \log n / (\log R_n^i - \log R_1^i) \,, \qquad (8)$$
$$n = 4, 9, 16, \quad \text{and} \quad i = 1, \ldots, 5 \,,$$

[29] It must be emphasized that the "bias" depends on the serial dependence shown by the sample and not the true dependence in the population. For example, if there is positive dependence in the sample, the interfractile range of the sample sums will usually be more than $n^{1/\alpha}$ times the interfractile range of the individual summands, even if there is no serial dependence in the population. In this case the nature of the sample dependence allows us to pinpoint the direction of the *sampling error* of the estimate of α. On the other hand, when the sample dependence is indicative of true dependence in the population, the error in the estimate of α is a genuine *bias* rather than just sampling error. This distinction, however, is irrelevant for present purposes.

[30] The ranges are defined as follows:

Interquartile	= 0.75 fractile − 0.25 fractile;
Intersextile	= 0.83 fractile − 0.17 fractile;
Interdecile	= 0.90 fractile − 0.10 fractile;
5 per cent	= 0.95 fractile − 0.05 fractile;
2 per cent	= 0.98 fractile − 0.02 fractile.

where n refers to the summing interval and i refers to a particular fractile range. For each value of n there are five different values of i, the different fractile ranges.

Column (2) of Table 9 shows the average values of α computed for each stock by the range analysis technique. The number for a given stock is the average of the fifteen different values of α computed for the stock.

D. ESTIMATING α FROM THE SEQUENTIAL VARIANCE

Although the *population* variance of a stable Paretian process with characteristic exponent $\alpha < 2$ is infinite, the variance computed from any *sample* will always be finite. If the process is truly stable Paretian, however, as the sample size is increased, we should expect to see some upward growth or trend in the sample variance. In fact the appendix shows that, if u_t is an independent stable Paretian variable generated in time series, then the median of the distribution of the cumulative sample variance of u_t at time t_1, as a function of the sample variance at time t_0, is given by

$$S_1^2 = S_0^2 \left(\frac{n_1}{n_0}\right)^{-1+2/\alpha} , \qquad (9)$$

where n_1 is the number of observations in the sample at time t_1, n_0 is the number at time t_0, and S_1^2 and S_0^2 are the cumulative sample variances. Solving equation (9) for α we get,

$$\alpha = \frac{2(\log n_1 - \log n_0)}{2 \log S_1 - 2 \log S_0 + \log n_1 - \log n_0} \,. \qquad (10)$$

It is easy to see that estimates of α from equation (10) will depend largely on the difference between the values of the sample variances at times t_0 and t_1. If S_1^2 is greater than S_0^2, then the estimate of α will be less than 2. If the sample variance has declined between t_0 and t_1, then the estimate of α will be more than 2.

Now equation (10) can be used to obtain many estimates of α for each stock. This is done by varying the starting point n_0 and the ending point n_1 of the interval of estimation. For this study starting points of from $n_0 = 200$ to $n_0 = 800$ observations by jumps of 100 observations were used. Similarly, for each value of n_0, α was computed for values of $n_1 = n_0 + 100$, $n_1 = n_0 + 200$, $n_1 = n_0 + 300, \ldots$, and $n_1 = N$, where N is the total number of price changes for the given security. Thus, if the sample of price changes for a stock contains 1,300 observations, the sequential variance procedure of expression (10) would be used to compute fifty-six different estimates of α. For each stock the median of the different estimates of α produced by the sequential variance procedure was computed. These median values of α are shown in column (3) of Table 9.

We must emphasize, however, that, of the three procedures for estimating α used in this report, the sequential-variance technique is probably the weakest. Like probability graphing and range analysis, its theoretical sampling behavior is unknown, since explicit expressions for the density functions of stable Paretian distributions are unknown. In addition, however, the sequential-variance procedure depends on the properties of sequential estimates of a sample parameter. Sampling theory for sequential parameter estimates is not well developed even for cases where an explicit expression for the density function of the basic variable is known. Thus we may know that in general the sample sequential variance grows proportionately to $(n_1/n_0)^{-1+2/\alpha}$ but we do not know how large the sample must be before this growth tendency can be used to make meaningful estimates of α.

TABLE 7

ESTIMATES OF α FOR AMERICAN TOBACCO BY THE SEQUENTIAL-VARIANCE PROCEDURE

n_0	n_1										
	300	400	500	600	700	800	900	1,000	1,100	1,200	$N=$ 1,283
200	18.54	2.64	2.49	2.39	2.23	1.63	1.63	1.62	1.61	1.42	1.32
300		1.19	1.47	1.58	1.57	1.18	1.22	1.24	1.25	1.12	1.05
400			2.11	2.05	1.87	1.18	1.23	1.26	1.27	1.10	1.02
500				1.99	1.74	0.97	1.06	1.11	1.14	0.98	0.91
600					1.52	0.74	0.88	0.96	1.01	0.87	0.80
700						0.46	0.69	0.83	0.91	0.77	0.72
800							1.65	1.59	1.52	0.99	0.85

The problems in estimating α by the sequential variance procedure are illustrated in Table 7 which shows all the different estimates for American Tobacco. The estimates are quite erratic. They range from 0.46 to 18.54. Reading across any line in the table makes it clear that the estimates are highly sensitive to the ending point (n_1) of the interval of estimation. Reading down any column, one sees that they are also extremely sensitive to the starting point (n_0).

By way of contrast, Table 8 shows the different estimates of α for American Tobacco that were produced by the range analysis procedure. Unlike the sequential-variance estimates, the estimates in Table 8 are relatively stable. They range from 1.67 to 2.06. Moreover, the results for American Tobacco are quite representative. For each stock the estimates produced by the sequential-variance procedure show much greater dispersion than do the estimates produced by range analysis. It seems safe to conclude, therefore, that range analysis is a much more precise estimation procedure than sequential-variance analysis.

TABLE 8

ESTIMATES OF α FOR AMERICAN TOBACCO BY RANGE-ANALYSIS PROCEDURE

Range	Summing Interval (Days)		
	Four	Nine	Sixteen
Interquartile	1.98	1.99	1.67
Intersextile	1.99	1.87	1.70
Interdecile	1.80	2.02	1.87
5 per cent	1.86	1.99	2.06
2 per cent	1.80	1.89	1.70

E. COMPARISON OF THE THREE PROCEDURES FOR ESTIMATING α

Table 9 shows the estimates of α given by the three procedures discussed above. Column (1) shows the estimates produced by the double-log–normal-probability graphing procedure. Because of the subjective nature of this technique,

TABLE 9

COMPARISON OF ESTIMATES OF THE CHARACTERISTIC EXPONENT

Stock	Double-Log–Normal-Probability Graphs (1)	Range Analysis (2)	Sequential Variance (3)
Allied Chemical	1.99–2.00	1.94	1.40
Alcoa	1.95–1.99	1.80	2.05
American Can	1.85–1.90	2.10	1.71
A.T.&T.	1.50–1.80	1.77	1.07
American Tobacco	1.85–1.90	1.88	1.24
Anaconda	1.95–1.99	2.03	2.55
Bethlehem Steel	1.90–1.95	1.89	1.85
Chrysler	1.90–1.95	1.95	1.36
Du Pont	1.90–1.95	1.88	1.65
Eastman Kodak	1.90–1.95	1.92	1.76
General Electric	1.80–1.90	1.95	1.57
General Foods	1.85–1.90	1.87	1.86
General Motors	1.95–1.99	2.05	1.44
Goodyear	1.80–1.95	2.06	1.39
International Harvester	1.85–1.90	2.06	2.22
International Nickel	1.90–1.95	1.77	2.80
International Paper	1.90–1.95	1.87	1.95
Johns Manville	1.85–1.90	2.08	1.75
Owens Illinois	1.85–1.90	1.95	2.06
Procter & Gamble	1.80–1.90	1.84	1.70
Sears	1.85–1.90	1.75	1.66
Standard Oil (Calif.)	1.95–1.99	2.08	2.41
Standard Oil (N.J.)	1.90–1.95	2.02	2.09
Swift & Co.	1.85–1.90	1.99	1.87
Texaco	1.90–1.95	1.85	1.76
Union Carbide	1.80–1.90	1.75	1.56
United Aircraft	1.80–1.90	1.93	1.13
U.S. Steel	1.95–1.99	1.96	1.78
Westinghouse	1.95–1.99	2.10	1.35
Woolworth	1.80–1.99	1.93	1.02
Averages	1.87–1.94	1.93	1.73

the best that can be done is to estimate the interval within which the true value appears to fall. Column (2) shows the estimates of α based on range analysis, while column (3) shows the estimates based on the sequential variance procedure.

The reasons why different techniques for estimating α are used, as well as the shortcomings of each technique, are fully discussed in preceding sections. At this point we merely summarize the previous discussions.

First of all, since explicit expressions for the density functions of stable Paretian distributions are, except for certain very special cases, unknown sampling theory for the parameters of these distributions is practically non-existent. Since it is not possible to make firm statements about the sampling error of any given estimator, the only alternative is to use many different estimators of the same parameter in an attempt at least to bracket the true value.

In addition to the lack of sampling theory, each of the techniques for estimating α has additional shortcomings. For example, the procedure based on properties of the double-log and normal-probability graphs is entirely subjective. The range procedure, on the other hand, may be sensitive to whatever serial dependence is present in the sample data. Finally, the sequential-variance technique produces estimates which are erratic and highly dependent on the time interval chosen for the estimation.

It is not wholly implausible, however, that the errors and biases in the various estimators may, to a considerable extent, be offsetting. Each of the three procedures represents a radically different approach to the estimation problem. Therefore there is good reason to expect the results they produce to be independent. At the very least, the three different estimating procedures should allow us to decide whether α is strictly less than 2, as proposed by the Mandelbrot hypothesis, or equal to 2, as proposed by the Gaussian hypothesis.

Even a casual glance at Table 9 is sufficient to show that the estimates of α produced by the three different procedures are consistently less than 2. In combination with the results produced by the frequency distributions and the normal-probability graphs, this would seem to be conclusive evidence in favor of the Mandelbrot hypothesis.

F. CONCLUSION

In sum, the results of Sections III and IV seem to indicate that the daily changes in log price of stocks of large mature companies follow stable Paretian distributions with characteristic exponents close to 2, but nevertheless less than 2. In other words, the Mandelbrot hypothesis seems to fit the data better than the Gaussian hypothesis. In Section VI the implications of this conclusion will be examined from many points of view. In the next section we turn our attention to tests of the independence assumption of the random-walk model.

V. Tests for Dependence

In this section, three main approaches to testing for dependence will be followed. The first will be a straightforward application of the usual serial correlation model; the second will make use of a new approach to the theory of runs; while the third will involve Alexander's [1], [2] well-known filter technique.

Throughout this section we shall be interested in independence from two points of view, the statistician's and the investor's. From a statistical standpoint we are interested in determining whether the departures from normality in the distributions of price changes are due to patterns of dependence in successive changes. That is, we wish to determine whether dependence in successive price changes accounts for the long tails that have been observed in the empirical distributions. From the investor's point of view, on the other hand, we are interested in testing whether there are dependencies in the series that he can use to increase his expected profits.

A. SERIAL CORRELATION

1. THE MODEL

The serial correlation coefficient (r_τ) provides a measure of the relationship between the value of a random variable in time t and its value τ periods earlier. For example, for the variable u_t, defined as the change in log price of a given security from the end of day $t - 1$ to the end of day t, the serial correlation coefficient for lag τ is

$$r_\tau = \frac{\text{covariance}(u_t, u_{t-\tau})}{\text{variance}(u_t)}. \quad (11)$$

If the distribution of u_t has finite variance, then in very large samples the standard error of r_τ will be given by

$$\sigma(r_\tau) = \sqrt{1/(N - \tau)}, \quad (12)$$

where N is the sample size (cf. Kendall [25]).

Previous sections have suggested, however, that the distribution of u_t is stable Paretian with characteristic exponent α less than 2. Thus the assumption of finite variance is probably not valid, and as a result equation (12) is not a precise measure of the standard error of r_τ, even for extremely large samples. Moreover, since the variance of u_t comes into the denominator of the expression for r_τ, it would seem questionable whether serial correlation analysis is an adequate tool for examining our data.

Wise [49] has shown, however, that as long as the characteristic exponent α of the underlying stable Paretian process is greater than 1, the statistic r_τ is a consistent and unbiased estimate of the true serial correlation in the population. That is, the sample estimate of r_τ is unbiased and converges in probability to its population value as the sample size approaches infinity.[31]

In order to shed some light on the convergence rate of r_τ when $\alpha < 2$, the serial correlation coefficient for lag $\tau = 1$ has been computed sequentially for each stock on the basis of randomized first differences. The purpose of randomization was to insure that the expectation of the serial coefficient would be zero. The procedure was first to reorder randomly the array of first differences for each stock and then to compute the cumulative sample serial correlation coefficient for samples of size $n = 5, 10, \ldots,$ N. Thus, except for five additional observations, each sample contains the same values of u as the preceding one.

Although the estimator r_1 is consistent and unbiased, we should expect that, when $\alpha < 2$, the variability of the sample serial correlation coefficients will be greater than if the distribution of u_t had finite variance. The estimates, however, should converge to the true value, zero, as the sample size is increased. In order to judge the variability of the sample

[31] What Wise actually shows is that the least-squares estimate of b_τ in the regression equation,

$$u_t = a + b_\tau u_{t-\tau} + \xi_t,$$

is consistent and unbiased as long as the characteristic exponent α of the distribution of ξ_t is greater than 1. Since the least squares estimate of b_τ is identical to the estimate of r_τ, however, this is equivalent to proving that the estimate of r_τ is also consistent and unbiased.

coefficients two σ control limits were computed by means of the formula

$$r_1 \pm 2\,\sigma(r_1) = 0 \pm 2\sqrt{1/(n-1)}\,,$$
$$n = 5\,, 10, \ldots, N\,.$$

Although the results must be judged subjectively, the sample serial correlation coefficients for the randomized first differences appear to break through their control limits only slightly more often than would be the case if the underlying distribution of the first differences had finite variance. From the standpoint of consistency the most important feature of the sample coefficients is that for every stock the serial correlation coefficient is very close to the true value, zero, for samples with more than, say, three hundred observations. In addition, the sample coefficient stays close to zero thereafter.

For purposes of illustration graphs of the sequential randomized serial correlation coefficients for Goodyear and U.S. Steel are presented in Figure 7. The ordinates of the graphs show the values of the sequential serial correlation coefficents, while the abscissas show sequential sample size. The irregular lines on the graphs show the path of the coefficent while the smooth curves represent the two σ control limits. The striking feature of both graphs is the quickness with which the sample coefficient settles down to its true value, zero, and stays close to the true value thereafter. On the basis of this evidence we conclude that, for large samples and for the values of α observed for our stocks, the sample serial correlation coefficient seems to be an effective tool in testing for serial independence.

2. COEFFICIENTS FOR DAILY CHANGES

Using the data as they were actually generated in time, the sample serial correlation coefficient for daily changes in log price has been computed for each stock for lag τ of from 1 to 30 days. The results for $\tau = 1, 2, \ldots, 10$ are shown in Table 10. Essentially the sample coefficients in the table tell us whether any of the price changes for the last ten days are likely to be of much help in predicting tomorrow's change.

All the sample serial correlation coefficients in Table 10 are quite small in absolute value. The largest is only .123. Although eleven of the coefficients for lag $\tau = 1$ are more than twice their computed standard errors, this is not regarded as important in this case. The standard errors are computed according to equation (12); and, as we saw earlier, this formula underestimates the true variability of the coefficient when the underlying variable is stable Paretian with characteristic exponent $\alpha < 2$. In addition, for our large samples the standard error of the serial correlation coefficient is very small. In most cases a coefficient as small as .06 is more than twice its standard error. "Dependence" of such a small order of magnitude is, from a practical point of view, probably unimportant for both the statistician and the investor.

3. COEFFICIENTS FOR FOUR-, NINE-, AND SIXTEEN-DAY CHANGES

Although the sample serial correlation coefficients for the daily changes are all very small, it is possible that price changes across longer differencing intervals would show stronger evidence of dependence. To test this, serial correlation coefficients for lag $\tau = 1, 2, \ldots, 10$ were computed for each stock for non-overlapping differencing intervals of four, nine, and sixteen days. The results for $\tau = 1$ are shown in Table 11.[32]

[32] Of course, in taking longer differencing intervals the sample size is considerably reduced. The

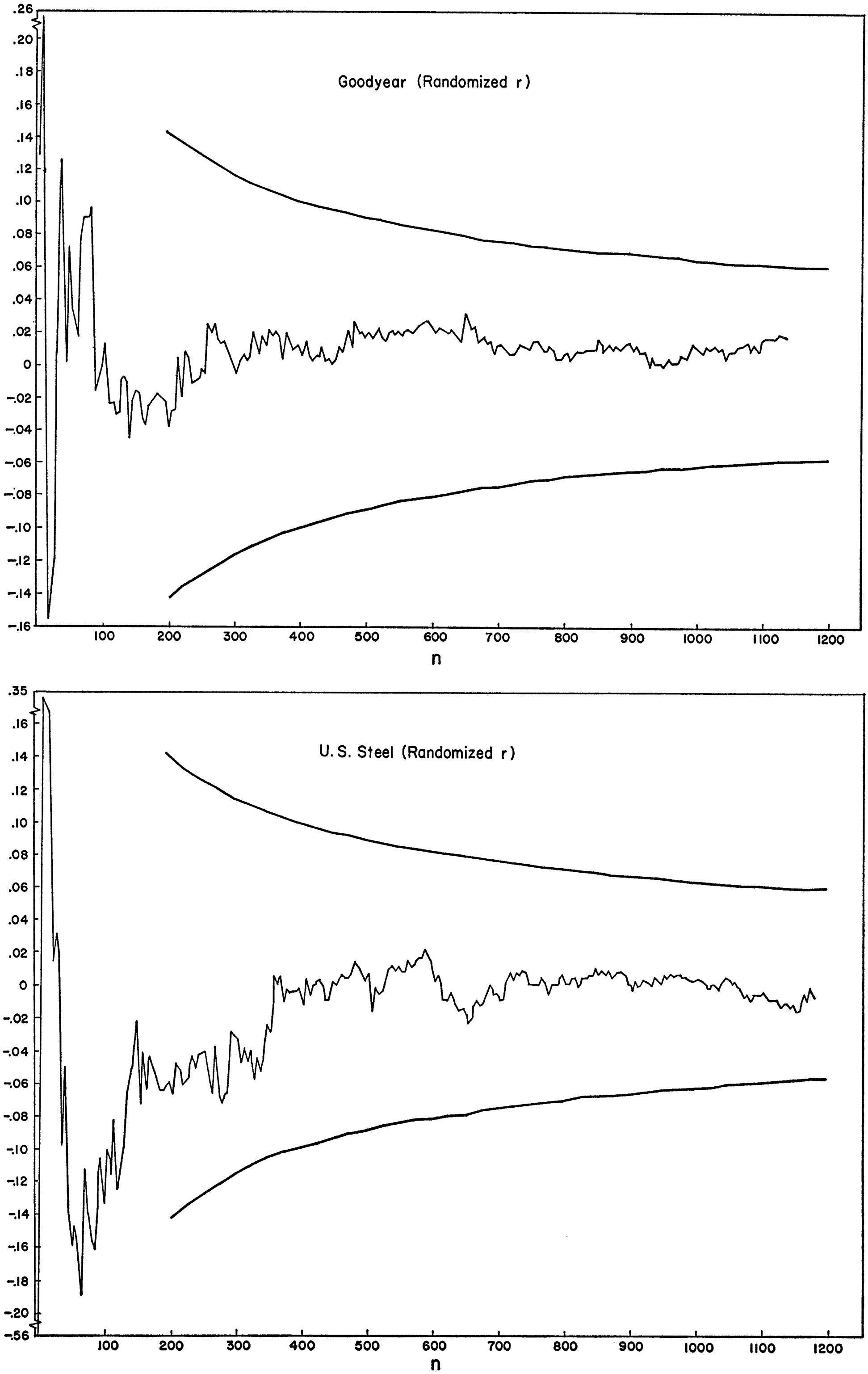

FIG. 7.—Randomized sequential serial correlation coefficients

Again, all the sample serial correlation coefficients are quite small. In general, the absolute size of the coefficients seems to increase with the differencing interval. This does not mean, however, that price changes over longer differencing intervals show more dependence, since we know that the variability of r is inversely related to the sample size. In fact the average size of the coefficients relative to their standard errors decreases with the differencing interval. This is demonstrated by the fact that for four-, nine-, and sixteen-day differencing intervals there are, respectively, five, two, and one coefficients greater than twice their standard errors in Table 11.

sample for the four-day changes is only one-fourth as large as the sample for the daily changes. Similarly, the samples for the nine- and sixteen-day changes are only one-ninth and one-sixteenth as large as the corresponding samples for the daily changes.

An interesting feature of Tables 10 and 11 is the pattern shown by the signs of the serial correlation coefficients for lag $\tau = 1$. In Table 10 twenty-three out of thirty of the first-order coefficients for the daily differences are positive, while twenty-one and twenty-four of the coefficients for the four- and nine-day differences are negative in Table 11. For

TABLE 10

DAILY SERIAL CORRELATION COEFFICIENTS FOR LAG $\tau = 1, 2, \ldots, 10$

Stock	Lag									
	1	2	3	4	5	6	7	8	9	10
Allied Chemical	.017	−.042	.007	−.001	.027	.004	−.017	−.026	−.017	−.007
Alcoa	.118*	.038	−.014	.022	−.022	.009	.017	.007	−.001	−.033
American Can	−.087*	−.024	.034	−.065*	−.017	−.006	.015	.025	−.047	−.040
A.T.&T.	−.039	−.097*	.000	.026	.005	−.005	.002	.027	−.014	.007
American Tobacco	.111*	−.109*	−.060*	−.065*	.007	−.010	.011	.046	.039	.041
Anaconda	.067*	−.061*	−.047	−.002	.000	−.038	.009	.016	−.014	−.056
Bethlemen Steel	.013	−.065*	.009	.021	−.053	−.098*	−.010	.004	−.002	−.021
Chrysler	.012	−.066*	−.016	−.007	−.015	.009	.037	.056*	−.044	.021
Du Pont	.013	−.033	.060*	.027	−.002	−.047	.020	.011	−.034	.001
Eastman Kodak	.025	.014	−.031	.005	−.022	.012	.007	.006	.008	.002
General Electric	.011	−.038	−.021	.031	−.001	.000	−.008	.014	−.002	.010
General Foods	.061*	−.003	.045	.002	−.015	−.052	−.006	−.014	−.024	−.017
General Motors	−.004	−.056*	−.037	−.008	−.038	−.006	.019	.006	−.016	.009
Goodyear	−.123*	.017	−.044	.043	−.002	−.003	.035	.014	−.015	.007
International Harvester	−.017	−.029	−.031	.037	−.052	−.021	−.001	.003	−.046	−.016
International Nickel	.096*	−.033	−.019	.020	.027	.059*	−.038	−.008	−.016	.034
International Paper	.046	−.011	−.058*	.053*	.049	−.003	−.025	−.019	−.003	−.021
Johns Manville	.006	−.038	−.027	−.023	−.029	−.080*	.040	.018	−.037	.029
Owens Illinois	−.021	−.084*	−.047	.068*	.086*	−.040	.011	−.040	.067*	−.043
Procter & Gamble	.099*	−.009	−.008	.009	−.015	.022	.012	−.012	−.022	−.021
Sears	.097*	.026	.028	.025	.005	−.054	−.006	−.010	−.008	−.009
Standard Oil (Calif.)	.025	−.030	−.051*	−.025	−.047	−.034	−.010	.072*	−.049*	−.035
Standard Oil (N.J.)	.008	−.116*	.016	.014	−.047	−.018	−.022	−.026	−.073*	.081*
Swift & Co.	−.004	−.015	−.010	.012	.057*	.012	−.043	.014	.012	.001
Texaco	.094*	−.049	−.024	−.018	−.017	−.009	.031	.032	−.013	.008
Union Carbide	.107*	−.012	.040	.046	−.036	−.034	.003	−.008	−.054	−.037
United Aircraft	.014	−.033	−.022	−.047	−.067*	−.053	.046	.037	.015	−.019
U.S. Steel	.040	−.074*	.014	.011	−.012	−.021	.041	.037	−.021	−.044
Westinghouse	−.027	−.022	−.036	−.003	.000	−.054*	−.020	.013	−.014	.008
Woolworth	.028	−.016	.015	.014	.007	−.039	−.013	.003	−.088*	−.008

* Coefficient is twice its computed standard error.

the sixteen-day differences the signs are about evenly split. Seventeen are positive and thirteen are negative.

The preponderance of positive signs in the coefficients for the daily changes is consistent with Kendall's [26] results for weekly changes in British industrial share prices. On the other hand, the results for the four- and nine-day differences are in agreement with those of Cootner [10] and Moore [41], both of whom found a preponderance of negative signs in the serial correlation coefficients of weekly changes in log price of stocks on the New York Stock Exchange.

Given that the absolute size of the serial correlation coefficients is always quite small, however, agreement in sign among the coefficients for the different securities is ***not necessarily*** evidence for consistent patterns of dependence. King [27] has shown that the price changes for different securities are related (although not all to the same extent) to the behavior of a "market" component common to all securities. For any given sampling period the serial correlation coefficient for a given security will be partly determined by the serial behavior of this market component and partly by the serial behavior of factors peculiar to that security and perhaps also to its industry.

TABLE 11

FIRST-ORDER SERIAL CORRELATION COEFFICIENTS FOR FOUR-, NINE-, AND SIXTEEN-DAY CHANGES

STOCK	DIFFERENCING INTERVAL (DAYS)		
	Four	Nine	Sixteen
Allied Chemical	.029	−.091	−.118
Alcoa	.095	−.112	−.044
American Can	−.124*	−.060	.031
A.T. & T.	−.010	−.009	−.003
American Tobacco	−.175*	.033	.007
Anaconda	−.068	−.125	.202
Bethlehem Steel	−.122	−.148	.112
Chrysler	.060	−.026	.040
Du Pont	.069	−.043	−.055
Eastman Kodak	−.006	−.053	−.023
General Electric	.020	−.004	.000
General Foods	−.005	−.140	−.098
General Motors	−.128*	.009	−.028
Goodyear	.001	−.037	.033
International Harvester	−.068	−.244*	.116
International Nickel	.038	.124	.041
International Paper	.060	−.004	−.010
Johns Manville	−.068	−.002	.002
Owens Illinois	−.006	.003	−.022
Procter & Gamble	−.006	.098	.076
Sears	−.070	−.113	.041
Standard Oil (Calif.)	−.143*	−.046	.040
Standard Oil (N.J.)	−.109	−.082	−.121
Swift & Co.	−.072	.118	−.197
Texaco	−.053	−.047	−.178
Union Carbide	.049	−.101	.124
United Aircraft	−.190*	−.192*	−.040
U.S. Steel	−.006	−.056	.236*
Westinghouse	−.097	−.137	.067
Woolworth	−.033	−.112	.040

* Coefficient is twice its computed standard error.

Since the market component is common to all securities, however, its behavior during the sampling period may tend to produce a common sign for the serial correlation coefficients of all the different securities. Thus, although both the market component and the factors peculiar to individual firms and industries may be characterized by serial independence, the *sample* behavior of the market component during any given time period may be expected to produce agreement among the signs of the sample serial correlation coefficients for different securities. The fact that this agreement in sign is caused by pure sampling error in a random component common to all securities is evidenced by the small absolute size of the sample coefficients. It is also evidenced by the fact that, although different studies have invariably found some sort of consistency in sign, the actual direction of the "dependence" varies from study to study.[33]

In sum, the evidence produced by the serial-correlation model seems to indicate that dependence in successive price changes is either extremely slight or completely non-existent. This conclusion should be regarded as tentative, however, until further results, to be provided by the runs tests of the next section, are examined.

B. THE RUNS TESTS

1. INTRODUCTION

A run is defined as a sequence of price changes of the same sign. For example, a plus run of length i is a sequence of i consecutive positive price changes preceded and followed by either negative or zero changes. For stock prices there are three different possible types of price changes and thus three different types of runs.

The approach to runs-testing in this section will be somewhat novel. The differences between expected and actual numbers of runs will be analyzed in three different ways, first by totals, then by sign, and finally by length. First, for each stock the difference between the total actual number of runs, irrespective of sign, and the total expected number will be examined. Next, the total expected and actual numbers of plus, minus, and no-change runs will be studied. Finally, for runs of each sign the expected and actual numbers of runs of each length will be computed.

2. TOTAL ACTUAL AND EXPECTED NUMBER OF RUNS

If it is assumed that the sample proportions of positive, negative, and zero price changes are good estimates of the population proportions, then under the

[33] The model, in somewhat oversimplified form, is as follows. The change in log price of stock j during day t is a linear function of the change in a market component, I_t, and a random error term, ξ_{tj}, which expresses the factors peculiar to the individual security. The form of the function is $u_{tj} = b_j I_t + \xi_{tj}$, where it is assumed that the I_t and ξ_{tj} are both serially independent and that ξ_{tj} is independent of current and past values of I_t. If we further assume, solely for simplicity, that $E(\xi_{tj}) = E(I_t) = 0$ for all t and j, we have

$$\begin{aligned}\mathrm{cov}\,(u_{tj}, u_{t-\tau,\,j}) &= E[b_j I_t + \xi_{tj})(b_j I_{t-\tau} \\ &\quad + \xi_{t-\tau,\,j})] = b_j^2\,\mathrm{cov}\,(I_t, I_{t-\tau}) \\ &\quad + b_j\,\mathrm{cov}\,(I_t, \xi_{t-\tau,\,j}) \\ &\quad + b_j\,\mathrm{cov}\,(I_{t-\tau}, \xi_{tj}) + \mathrm{cov}\,(\xi_{tj}, \xi_{t-\tau,\,j})\,.\end{aligned}$$

Although the expected values of the covariances on the right of the equality are all zero, their sample values for any given time period will not usually be equal to zero. Since cov $(I_t, I_{t-\tau})$ will be the same for all j, it will tend to make the signs of cov $(u_{tj}, u_{t-\tau,\,j})$ the same for different j. Essentially we are saying that the serial correlation coefficients for different securities for given lag and time period are not independent of each other. Thus we should not be surprised when we find a preponderance of signs in one direction or the other.

hypothesis of independence the total expected number of runs of all signs for a stock can be computed as

$$m = \left[N(N+1) - \sum_{i=1}^{3} n_i^2\right] \Big/ N, \quad (13)$$

where N is the total number of price changes, and the n_i are the numbers of price changes of each sign. The standard error of m is

$$\sigma_m = \left(\frac{\sum_{i=1}^{3} n_i^2\left[\sum_{i=1}^{3} n_i^2 + N(N+1)\right] - 2N\sum_{i=1}^{3} n_i^3 - N^3}{N^2(N-1)} \right)^{1/2}, \quad (14)$$

and for large N the sampling distribution of m is approximately normal.[34]

Table 12 shows the total expected and actual numbers of runs for each stock for

[34] Cf. Wallis and Roberts [48], pp. 569–72. It should be noted that the asymptotic properties of the sampling distribution of m do not depend on the assumption of finite variance for the distribution of price changes. We saw previously that this is not true for the sampling distribution of the serial correlation coefficient. In particular, except for the properties of consistency and unbiasedness, we

TABLE 12

TOTAL ACTUAL AND EXPECTED NUMBERS OF RUNS FOR ONE-, FOUR-, NINE-, AND SIXTEEN-DAY DIFFERENCING INTERVALS

Stock	Daily		Four-Day		Nine-Day		Sixteen-Day	
	Actual	Expected	Actual	Expected	Actual	Expected	Actual	Expected
Allied Chemical	683	713.4	160	162.1	71	71.3	39	38.6
Alcoa	601	670.7	151	153.7	61	66.9	41	39.0
American Can	730	755.5	169	172.4	71	73.2	48	43.9
A.T.&T.	657	688.4	165	155.9	66	70.3	34	37.1
American Tobacco	700	747.4	178	172.5	69	72.9	41	40.6
Anaconda	635	680.1	166	160.4	68	66.0	36	37.8
Bethlehem Steel	709	719.7	163	159.3	80	71.8	41	42.2
Chrysler	927	932.1	223	221.6	100	96.9	54	53.5
Du Pont	672	694.7	160	161.9	78	71.8	43	39.4
Eastman Kodak	678	679.0	154	160.1	70	70.1	43	40.3
General Electric	918	956.3	225	224.7	101	96.9	51	51.8
General Foods	799	825.1	185	191.4	81	75.8	43	40.5
General Motors	832	868.3	202	205.2	83	85.8	44	46.8
Goodyear	681	672.0	151	157.6	60	65.2	36	36.3
International Harvester	720	713.2	159	164.2	84	72.6	40	37.8
International Nickel	704	712.6	163	164.0	68	70.5	34	37.6
International Paper	762	826.0	190	193.9	80	82.8	51	46.9
Johns Manville	685	699.1	173	160.0	64	69.4	39	40.4
Owens Illinois	713	743.3	171	168.6	69	73.3	36	39.2
Procter & Gamble	826	858.9	180	190.6	66	81.2	40	42.9
Sears	700	748.1	167	172.8	66	70.6	40	34.8
Standard Oil (Calif.)	972	979.0	237	228.4	97	98.6	59	54.3
Standard Oil (N.J.)	688	704.0	159	159.2	69	68.7	29	37.0
Swift & Co.	878	877.6	209	197.2	85	83.8	50	47.8
Texaco	600	654.2	143	155.2	57	63.4	29	35.6
Union Carbide	595	620.9	142	150.5	67	66.7	36	35.1
United Aircraft	661	699.3	172	161.4	77	68.2	45	39.5
U.S. Steel	651	662.0	162	158.3	65	70.3	37	41.2
Westinghouse	829	825.5	198	193.3	87	84.4	41	45.8
Woolworth	847	868.4	193	198.9	78	80.9	48	47.7
Averages	735.1	759.8	175.7	175.8	74.6	75.3	41.6	41.7

one-, four-, nine-, and sixteen-day price changes. For the daily changes the actual number of runs is less than the expected number in twenty-six out of thirty cases. This agrees with the results produced by the serial correlation coefficients. In Table 10, twenty-three out of thirty of the first-order serial correlation coefficients are positive. For the four- and nine-day differences, however, the results of the runs tests do not lend support to the results produced by the serial correlation coefficients. In Table 11 twenty-one and twenty-four of the serial correlation coefficients for four- and nine-day changes are negative. To be consistent with negative dependence, the actual numbers of runs in Table 12 should be greater than the expected numbers for these differencing intervals. In fact, for the four-day changes the actual number of runs is greater than the expected number for only thirteen of the thirty stocks, and for the nine-day changes the actual number is greater than the expected number in only twelve cases. For the sixteen-day differences there is no evidence for dependence of any form in either the serial correlation coefficients or the runs tests.

For most purposes, however, the absolute amount of dependence in the price changes is more important than whether the dependence is positive or negative. The amount of dependence implied by the runs tests can be depicted by the size of the differences between the total actual numbers of runs and the total expected numbers. In Table 13 these differences are standardized in two ways.

For large samples the distribution of the total number of runs is approximately normal with mean m and standard error σ_m as defined by equations (13) and (14). Thus the difference between the actual number of runs, R, and the expected number can be expressed by means of the usual standardized variable,

$$K = \frac{(R + \frac{1}{2}) - m}{\sigma_m}, \qquad (15)$$

where the $\frac{1}{2}$ in the numerator is a discontinuity adjustment. For large samples K will be approximately normal with mean 0 and variance 1. The columns labeled K in Table 13 show the standardized variable for the four differencing intervals. In addition, the columns labeled $(R - m)/m$ show the differences between the actual and expected numbers of runs as proportions of the expected numbers.

For the daily price changes the values of K show that for eight stocks the actual number of runs is more than two standard errors less than the expected number. Caution is required in drawing conclusions from this result, however. The expected number of runs increases about proportionately with the sample size, while its standard error increases proportionately with the square root of the sample size. Thus a constant but small *percentage* difference between the expected and actual number of runs will produce higher and higher values of the standardized variable as the sample size is increased. For example, for General Foods the actual number of runs is about 3 per cent less than the expected number for both the daily and the four-day changes. The standardized variable, however, goes from -1.46 for the daily changes to -0.66 for the four-day changes.

In general, the percentage differences between the actual and expected numbers of runs are quite small, and this is

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know very little about the distribution of the serial correlation coefficient when the price changes follow a stable Paretian distribution with characteristic exponent $\alpha < 2$. From this standpoint at least, runs-testing is, for our purposes, a better way of testing independence than serial correlation analysis.

probably the more relevant measure of dependence.

3. ACTUAL AND EXPECTED NUMBERS OF RUNS OF EACH SIGN

If the signs of the price changes are generated by an independent Bernoulli process with probabilities $P(+)$, $P(-)$, and $P(0)$ for the three types of changes, for large samples the expected number of plus runs of length i in a sample of N changes[35] will be approximately

$$NP(+)^i[1 - P(+)]^2 . \quad (16)$$

The expected number of plus runs of all lengths will be

$$\sum_{i=1}^{\infty} NP(+)^i[1-P(+)]^2 = NP(+)[1-P(+)] . \quad (17)$$

Similarly the expected numbers of minus and no-change runs of all lengths will be

$$NP(-)[1 - P(-)] \quad \text{and} \quad NP(0)[1 - P(0)] . \quad (18)$$

For a given stock, the sum of the expected numbers of plus, minus, and no-change runs will be equal to the total expected number of runs of all signs, as defined in the previous section. Thus the

[35] Cf. Hald [21], pp. 342–53.

TABLE 13

RUNS ANALYSIS: STANDARDIZED VARIABLES AND PERCENTAGE DIFFERENCES

Stock	Daily		Four-Day		Nine-Day		Sixteen-Day	
	K	$(R-m)/m$	K	$(R-m)/m$	K	$(R-m)/m$	K	$(R-m)/m$
Allied Chemical	−1.82	−0.043	−0.19	−0.013	0.04	−0.004	0.21	0.011
Alcoa	−4.23	−.104	−.26	−.018	−0.95	−.089	0.60	.052
American Can	−1.54	−.034	−.35	−.020	−0.30	−.030	1.16	.090
A.T.&T.	−1.88	−.046	1.14	.058	−0.71	−.060	−0.65	−.083
American Tobacco	−2.80	−.063	.70	.032	−0.63	−.054	0.22	.010
Anaconda	−2.75	−.066	.73	.035	0.44	.030	−0.30	−.047
Bethlehem Steel	−0.63	−.015	.50	.023	1.57	.114	−0.16	−.028
Chrysler	−0.24	−.006	.19	.006	0.54	.032	0.20	.010
Du Pont	−1.32	−.033	−.16	−.012	1.16	.086	0.93	.090
Eastman Kodak	−0.03	−.002	−.64	−.038	0.06	−.002	0.77	.066
General Electric	−1.94	−.040	.08	.001	0.68	.042	−0.06	−.015
General Foods	−1.46	−.032	−.66	−.033	0.99	.068	0.71	.061
General Motors	−2.02	−.042	−.30	−.016	−0.37	−.032	−0.50	−.061
Goodyear	0.59	.013	−.75	−.042	−0.83	−.080	0.05	−.008
International Harvester	0.45	.010	−.58	−.032	2.16	.156	0.67	.059
International Nickel	−0.49	−.012	−.06	−.006	−0.35	−.036	−0.75	−.096
International Paper	−3.53	−.077	−.38	−.020	−0.38	−.034	0.98	.087
Johns Manville	−0.83	−.020	1.62	.081	−0.89	−.078	−0.22	−.035
Owens Illinois	−1.81	−.041	.34	.014	−0.68	−.059	−0.65	−.082
Procter & Gamble	−1.82	−.038	−1.13	−.056	−2.51	−.188	−0.59	−.068
Sears	−2.94	−.064	−.66	−.034	−0.79	−.066	1.69	.149
Standard Oil (Calif.)	−0.33	−.007	.92	.038	−0.16	−.016	1.03	.086
Standard Oil (N.J.)	−0.98	−.023	.03	−.001	0.15	.005	−1.78	−.216
Swift & Co.	0.05	.000	1.34	.060	0.28	.015	0.58	.045
Texaco	−3.33	−.083	−1.43	−.078	−1.08	−.101	−1.51	−.186
Union Carbide	−1.60	−.042	−.99	−.056	0.16	.005	0.33	.024
United Aircraft	−2.32	−.055	1.33	.066	1.63	.128	1.42	.140
U.S. Steel	−0.63	−.017	.49	.023	−0.85	−.075	−0.90	−.102
Westinghouse	0.22	.004	.56	.024	0.51	.031	−0.92	−.105
Woolworth	−1.18	−0.025	−0.59	−0.030	−0.38	−0.035	0.17	0.006
Averages	−1.44	−0.033	0.03	−0.001	−0.05	−0.010	0.09	0.005

above expressions give the breakdown of the total expected number of runs into the expected numbers of runs of each sign.

For present purposes, however, it is not desirable to compute the breakdown by sign of the total *expected* number of runs. This would blur the results of this section, since we know that for some differencing intervals there are consistent discrepancies between the total actual numbers of runs of all signs and the total expected numbers. For example, for twenty-six out of thirty stocks the total expected number of runs of all signs for the daily differences is greater than the total actual number. If the total expected number of runs is used to compute the expected numbers of runs of each sign, the expected numbers by sign will tend to be greater than the actual numbers. And this will be the case even if the breakdown of the total actual number of runs into the actual number of runs of each sign is proportional to the expected breakdown.

This is the situation we want to avoid in this section. What we will examine here are discrepancies between the *expected* breakdown by sign of the total *actual* number of runs and the actual breakdown. To do this we must now define a method of computing the expected breakdown by sign of the total actual number of runs.

The probability of a plus run can be expressed as the ratio of the expected number of plus runs in a sample of size N to the total expected number of runs of all signs, or as

$$P(+\text{ run}) = NP(+)[1 - P)(+)]/m . \quad (19)$$

Similarly, the probabilities of minus and no-change runs can be expressed as

$$P(-\text{ run}) = NP(-)[1 - P(-)]/m , \quad \text{and} \quad (20)$$

$$P(0\text{ run}) = NP(0)[1 - P(0)]/m . \quad (21)$$

The expected breakdown by sign of the total actual number of runs (R) is then given by

$$\bar{R}(+) = R[P(+\text{ run})] ,$$
$$\bar{R}(-) = R[P(-\text{ run})] , \quad \text{and} \quad (22)$$
$$\bar{R}(0) = R[P(0\text{ run})] ,$$

where $\bar{R}(+)$, $\bar{R}(-)$, and $\bar{R}(0)$ are the expected numbers of plus, minus, and no-change runs. These formulas have been used to compute the expected numbers of runs of each sign for each stock for differencing intervals of one, four, nine, and sixteen days. The actual numbers of runs and the differences between the actual and expected numbers have also been computed. The results for the daily changes are shown in Table 14. The results for the four-, nine-, and sixteen-day changes are similar, and so they are omitted.

The differences between the actual and expected numbers of runs are all very small. In addition there seem to be no important patterns in the signs of the differences. We conclude, therefore, that the actual breakdown of runs by sign conforms very closely to the breakdown that would be expected if the signs were generated by an independent Bernoulli process.

4. DISTRIBUTION OF RUNS BY LENGTH

In this section the expected and actual distributions of runs by length will be examined. As in the previous section, an effort will be made to separate the analysis from the results of runs tests discussed previously. To accomplish this, the discrepancies between the total actual and expected numbers of runs and those between the actual and expected numbers of runs of each sign will be taken as given. Emphasis will be placed on the *expected* *distributions* by length of the total *actual* number of runs of each sign.

As indicated earlier, the expected number of plus runs of length i in a sample of N price changes is $NP(+)^i[1 - P(+)]^2$, and the total expected number of plus runs is $NP(+)[1 - P(+)]$. Out of the total expected number of plus runs, the expected proportion of plus runs of length i is

$$NP(+)^i[1 - P(+)]^2/NP(+) \times [1 - P(+)] = P(+)^{i-1}[1 - P(+)]. \quad (23)$$

This proportion is equivalent to the conditional probability of a plus run of length i, given that a plus run has been observed. The sum of the conditional probabilities for plus runs of all lengths is one. The analogous conditional probabilities for minus and no-change runs are

$$P(-)^{i-1}[1 - P(-)] \quad \text{and} \quad P(0)^{i-1}[1 - P(0)]\,. \quad (24)$$

These probabilities can be used to compute the expected distributions by length of the total actual number of runs of each sign. The formulas for the expected numbers of plus, minus, and no-change runs of length i, $i = 1, \ldots, \infty$, are

$$\bar{R}_i(+) = R(+)\,P(+)^{i-1}[1 - P(+)]\,,$$
$$\bar{R}_i(-) = R(-)\,P(-)^{i-1} \times [1 - P(-)]\,, \quad (25)$$
$$\bar{R}_i(0) = R(0)\,P(0)^{i-1}[1 - P(0)]\,,$$

TABLE 14

RUNS ANALYSIS BY SIGN (DAILY CHANGES)

Stock	Positive			Negative			No Change		
	Actual	Expected	Actual-Expected	Actual	Expected	Actual-Expected	Actual	Expected	Actual-Expected
Allied Chemical	286	290.1	− 4.1	294	290.7	3.3	103	102.2	0.8
Alcoa	265	264.4	0.6	262	266.5	− 4.5	74	70.1	3.9
American Can	289	290.2	− 1.2	285	284.6	0.4	156	155.2	0.8
A.T.&T.	290	291.2	− 1.2	285	285.3	− 0.3	82	80.5	1.5
American Tobacco	296	300.2	− 4.2	295	294.0	1.0	109	105.8	3.2
Anaconda	271	272.9	− 1.9	276	278.8	− 2.8	88	83.3	4.7
Bethlehem Steel	282	286.4	− 4.4	300	294.6	5.4	127	128.0	−1.0
Chrysler	417	414.9	2.1	421	421.1	− 0.1	89	91.0	−2.0
Du Pont	293	300.3	− 7.3	305	299.2	5.8	74	72.5	1.5
Eastman Kodak	306	308.6	− 2.6	312	308.7	3.3	60	60.7	−0.7
General Electric	404	404.5	− 0.5	401	404.7	− 3.7	113	108.8	4.2
General Foods	346	340.8	5.2	320	331.3	−11.3	133	126.9	6.1
General Motors	340	342.7	− 2.7	339	340.3	− 1.3	153	149.0	4.0
Goodyear	294	291.9	2.1	292	293.0	− 1.0	95	96.1	−1.1
International Harvester	303	300.1	2.9	301	298.8	2.2	116	121.1	−5.1
International Nickel	312	307.0	5.0	296	301.9	− 5.9	96	95.1	0.9
International Paper	322	330.2	− 8.2	338	333.2	4.8	102	98.6	3.4
Johns Manville	293	292.6	0.4	296	293.5	2.5	96	98.9	−2.9
Owens Illinois	297	293.7	3.3	295	291.2	3.8	121	128.1	−7.1
Procter & Gamble	343	346.4	− 3.4	342	340.3	1.7	141	139.3	1.7
Sears	291	289.3	1.7	265	271.3	− 6.3	144	139.4	4.6
Standard Oil (Calif.)	406	417.9	−11.9	427	416.6	10.4	139	137.5	1.5
Standard Oil (N.J.)	272	277.3	− 5.3	281	277.9	3.1	135	132.8	2.2
Swift & Co.	354	354.3	− 0.3	355	356.9	− 1.9	169	166.8	2.2
Texaco	266	265.6	0.4	258	263.6	− 5.6	76	70.8	5.2
Union Carbide	266	268.1	− 2.1	265	265.6	− 0.6	64	61.3	2.7
United Aircraft	281	280.4	0.6	282	282.2	− 0.2	98	98.4	−0.4
U.S. Steel	292	293.5	− 1.5	296	295.2	0.8	63	62.3	0.7
Westinghouse	359	361.3	− 2.3	364	362.1	1.9	106	105.6	0.4
Woolworth	349	348.7	0.3	350	345.9	4.1	148	152.4	−4.4

where $\bar{R}_i(+)$, $\bar{R}_i(-)$, and $\bar{R}_i(0)$ are the expected numbers of plus, minus, and no-change runs of length i, while $R(+)$, $R(-)$, and $R(0)$ are the total actual numbers of plus, minus, and no-change runs. Tables showing the expected and actual distributions of runs by length have been computed for each stock for differencing intervals of one, four, nine, and sixteen days. The tables for the daily changes of three randomly chosen securities are found together in Table 15. The tables show, for runs of each sign, the probability of a run of each length and the expected and actual numbers of runs of each length. The question answered by the tables is the following: Given the total actual number of runs of each sign, how would we *expect* the totals to be distributed among runs of different lengths and what is the actual distribution?

For all the stocks the expected and actual distributions of runs by length turn out to be extremely similar. Impressive is the fact that there are very few long runs, that is, runs of length longer than seven or eight. There seems to be no tendency for the number of long runs to be higher than expected under the hypothesis of independence.

5. SUMMARY

There is little evidence, either from the serial correlations or from the various runs tests, of any large degree of dependence in the daily, four-day, nine-day, and sixteen-day price changes. As far as these tests are concerned, it would seem that any dependence that exists in these series is not strong enough to be used either to increase the expected profits of the trader or to account for the departures from normality that have been observed in the empirical distribution of price changes. That is, as far as these tests are concerned, there is no evidence of important dependence from either an investment or a statistical point of view.

We must emphasize, however, that although serial correlations and runs tests are the common tools for testing dependence, there are situations in which they do not provide an adequate test of either practical or statistical dependence. For example, from a practical point of view the chartist would not regard either type of analysis as an *adequate* test of whether the past history of the series can be used to increase the investor's expected profits. The simple linear relationships that underlie the serial correlation model are much too unsophisticated to pick up the complicated "patterns" that the chartist sees in stock prices. Similarly, the runs tests are much too rigid in their approach to determining the duration of upward and downward movements in prices. In particular, a run is terminated whenever there is a change in sign in the sequence of price changes, regardless of the size of the price change that causes the change in sign. A chartist would like to have a more sophisticated method for identifying movements—a method which does not always predict the termination of the movement simply because the price level has temporarily changed direction. One such method, Alexander's filter technique, will be examined in the next section.

On the other hand, there are also possible shortcomings to the serial correlation and runs tests from a statistical point of view. For example, both of these models only test for dependence which is present all through the data. It is possible, however, that price changes are dependent only in special conditions. For example, although small changes may be independent, large changes may tend to be followed consistently by large changes of the same sign, or perhaps by large changes of the opposite sign. One version of this hypothesis will also be tested later.

C. ALEXANDER'S FILTER TECHNIQUE

The tests of independence discussed thus far can be classified as primarily statistical. That is, they involved computation of sample estimates of certain statistics and then comparison of the results with what would be expected under the assumption of independence of successive price changes. Since the sample estimates conformed closely to the values that would be expected by an independent model, we concluded that the independence assumption of the random-walk model was upheld by the data. From this we then *inferred* that there are probably no mechanical trading rules based solely on properties of past histories of price changes that can be used to make the expected profits of the trader greater than they would be under a simple buy-and-hold rule. We stress, however, that until now this is just an *inference;* the actual profitability of mechanical trading rules has not yet been directly tested. In this section one such trading rule, Alexander's filter technique [1], [2], will be discussed.

An x per cent filter is defined as follows. If the daily closing price of a particular security moves up at least x per cent, buy and hold the security until its price moves down at least x per cent from a subsequent high, at which time simultaneously sell and go short. The short position is maintained until the daily closing price rises at least x per cent above a subsequent low, at which time one should simultaneously cover and buy. Moves less than x per cent in either direction are ignored.

In his earlier article [1, Table 7] Alexander reported tests of the filter technique for filters ranging in size from 5 per cent to 50 per cent. The tests covered different time periods from 1897 to 1959 and involved closing "prices" for two *indexes*, the Dow-Jones Industrials from 1897 to 1929 and Standard and Poor's Industrials from 1929 to 1959. Alexander's results indicated that, in general, filters of all different sizes and for all the different time periods yield substantial profits—indeed, profits significantly greater than those earned by a simple buy-and-hold policy. This led him to conclude that the independence assumption of the random-walk model was not upheld by his data.

Mandelbrot [37], however, discovered a flaw in Alexander's computations which led to serious overstatement of the profitability of the filters. Alexander assumed that his hypothetical trader could always buy at a price exactly equal to the low plus x per cent and sell at a price exactly equal to the high minus x per cent. There is, of course, no assurance that such prices ever existed. In fact, since the filter rule is defined in terms of a trough plus *at least* x per cent or a peak minus *at least* x per cent, the purchase price will usually be something higher than the low plus x per cent, while the sale price will usually be below the high minus x per cent.

In a later paper [2, Table 1], however, Alexander derived a bias factor and used it to correct his earlier work. With the corrections for bias it turned out that the filters only rarely compared favorably with buy-and-hold, even though the higher broker's commissions incurred under the filter rule were ignored. It would seem, then, that at least for the purposes of the individual investor Alexander's filter results tend to support the independence assumption of the random walk model.

In the later paper [2, Tables 8, 9, 10,

TABLE 15—EXPECTED AND ACTUAL DISTRIBUTIONS OF RUNS BY LENGTH

LENGTH	PLUS RUNS			MINUS RUNS			NO-CHANGE RUNS		
	Probability	Expected No.	Actual No.	Probability	Expected No.	Actual No.	Probability	Expected No.	Actual No.
	American Tobacco								
1	0.52221	154.58	133	0.57521	169.69	164	0.90257	98.38	94
2	.24951	73.85	80	.24434	72.08	66	.08794	9.58	14
3	.11921	35.29	40	.10379	30.62	34	.00857	0.93	1
4	.05696	16.86	25	.04409	13.01	19	.00083	0.09	0
5	.02721	8.06	9	.01873	5.52	3	.00008	0.01	0
6	.01300	3.85	8	.00796	2.35	7	.00001	0.00	0
7	.00621	1.84	1	.00338	1.00	2	.00000	0.00	0
8	.00297	0.88	0	.00144	0.42	0	.00000	0.00	0
9	.00142	0.42	0	.00061	0.18	0	.00000	0.00	0
10	.00068	0.20	0	.00026	0.08	0	.00000	0.00	0
11	.00032	0.10	0	.00011	0.03	0	.00000	0.00	0
12	.00015	0.05	0	.00005	0.01	0	.00000	0.00	0
13	.00007	0.02	0	.00002	0.01	0	.00000	0.00	0
14	.00004	0.01	0	.00001	0.00	0	.00000	0.00	0
15	0.00003	0.01	0	0.00001	0.00	0	0.00000	0.00	0
Totals	....	296.00	296	....	295.00	295	....	109.00	109
	Bethlehem Steel								
1	0.59000	166.38	159	0.53333	160.00	155	0.87667	111.34	107
2	.24190	68.22	73	.24889	74.67	79	.10812	13.73	19
3	.09918	27.97	29	.11615	34.84	37	.01334	1.69	1
4	.04066	11.47	11	.05420	16.26	16	.00164	0.21	0
5	.01667	4.70	6	.02529	7.59	9	.00020	0.03	0
6	.00684	1.93	2	.01180	3.54	2	.00003	0.00	0
7	.00280	0.79	2	.00551	1.65	1	.00000	0.00	0
8	.00115	0.32	0	.00257	0.77	1	.00000	0.00	0
9	.00047	0.13	0	.00120	0.36	0	.00000	0.00	0
10	.00019	0.05	0	.00056	0.17	0	.00000	0.00	0
11	.00008	0.02	0	.00026	0.08	0	.00000	0.00	0
12	.00003	0.01	0	.00012	0.04	0	.00000	0.00	0
13	.00001	0.00	0	.00006	0.02	0	.00000	0.00	0
14	.00001	0.00	0	.00003	0.01	0	.00000	0.00	0
15	0.00000	0.00	0	0.00002	0.01	0	0.00000	0.00	0
Totals	....	282.00	282	....	300.00	300	....	127.00	127
	International Harvester								
1	0.55167	167.15	171	0.56083	168.81	168	0.88750	102.95	98
2	.24733	74.94	75	.24630	74.14	82	.09984	11.58	17
3	.11089	33.60	33	.10817	32.56	27	.01123	1.30	1
4	.04971	15.06	8	.04750	14.30	14	.00126	0.15	0
5	.02229	6.75	13	.02086	6.28	5	.00014	0.02	0
6	.00999	3.03	1	.00916	2.76	3	.00002	0.00	0
7	.00448	1.36	1	.00402	1.21	1	.00000	0.00	0
8	.00201	0.61	1	.00177	0.53	1	.00000	0.00	0
9	.00090	0.27	0	.00078	0.23	0	.00000	0.00	0
10	.00040	0.12	0	.00034	0.10	0	.00000	0.00	0
11	.00018	0.05	0	.00015	0.05	0	.00000	0.00	0
12	.00008	0.02	0	.00007	0.02	0	.00000	0.00	0
13	.00004	0.01	0	.00003	0.01	0	.00000	0.00	0
14	.00002	0.00	0	.00001	0.00	0	.00000	0.00	0
15	0.00001	0.00	0	0.00001	0.00	0	0.00000	0.00	0
Totals	....	303.00	303	....	301.00	301	....	116.00	116

and 11], however, Alexander goes on to test various other mechanical trading techniques, one of which involved a simplified form of the Dow theory. It turns out that most of these other techniques provide better profits than his filter technique, and indeed better profits than buy-and-hold. This again led him to conclude that the independence assumption of the random-walk model had been overturned.

Unfortunately a serious error remains, even in Alexander's latest computations. The error arises from the fact that he neglects dividends in computing profits for all of his mechanical trading rules. This tends to overstate the profitability of these trading rules relative to buy-and-hold. The reasoning is as follows. Under the buy-and-hold method the total profit is the price change for the time period plus any dividends that have been paid. Thus dividends act simply to increase the profitability of holding stock. All of Alexander's more complicated trading rules, however, involve short sales. In a short sale the borrower of the securities is usually required to reimburse the lender for any dividends that are paid while the short position is outstanding. Thus taking dividends into consideration will always tend to reduce the profitability of a mechanical trading rule relative to buy-and-hold. In fact, since in Alexander's computations the more complicated techniques are not substantially better than buy-and-hold, we would suspect that in most cases proper adjustment for dividends would probably completely turn the tables in favor of the buy-and-hold method.

The above discussion would seem to raise grave doubts concerning the validity of Alexander's most recent empirical results and thus of the conclusions he draws from these results. Because of the complexities of the issues, however, these doubts cannot be completely or systematically resolved within the confines of this paper. In a study now in progress various mechanical trading rules will be tested on data for individual securities rather than price indices. We turn now to a discussion of some of the preliminary results of this study.

Alexander's filter technique has been applied to the price series for the individual securities of the Dow-Jones Industrial Average used throughout this report. Filters from 0.5 per cent to 50 per cent were used. All profits were computed on the basis of a trading block of 100 shares, taking proper account of dividends. That is, if an ex-dividend date occurs during some time period, the amount of the dividend is added to the net profits of a long position open during the period, or subtracted from the net profits of a short position. Profits were also computed gross and net of broker's commissions, where the commissions are the exact commissions on lots of 100 shares at the time of transaction. In addition, for purposes of comparison the profits before commissions from buying and holding were computed for each security.

The results are shown in Table 16. Columns (1) and (2) of the table show average profits per filter, gross and net of commissions. Column (3) shows profits from buy-and-hold. Although they must be regarded as very preliminary, the results are nevertheless impressive. We see in column (2) that, when commissions are taken into account, profits per filter are positive for only four securities. Thus, from the point of view of the average investor, the results produced by the filter technique do not seem to invalidate the independence assumption of the random-walk model. In practice the largest profits under the filter technique would seem to be those of the broker.

A comparison of columns (1) and (3) also yields negative conclusions with respect to the filter technique. Even excluding commissions, in only seven cases are the profits per filter greater than those of buy-and-hold. Thus it would seem that even for the floor trader, who of course avoids broker commissions, the filter technique cannot be used to make expected profits greater than those of buy-and-hold. It would seem, then, that from the trader's point of view the independence assumption of the random-walk model is an adequate description of reality.

Although in his later article [2] Alexander seems to accept the validity of the independence assumption for the purposes of the investor or the trader, he argues that, from the standpoint of the academician, a stronger test of independence is relevant. In particular, he argues

TABLE 16

SUMMARY OF FILTER PROFITABILITY IN RELATION TO NAÏVE BUY-AND-HOLD TECHNIQUE*

Stock	Profits per Filter†		
	Without Commissions (1)	With Commissions (2)	Buy-and-Hold (3)
Allied Chemical	648.37	−10,289.33	2,205.00
Alcoa	3,207.40	− 3,929.42	− 305.00
American Can	− 844.32	− 5,892.85	1,387.50
A.T.&T.	16,577.26	4,912.84	20,005.00
American Tobacco	8,342.61	− 1,467.71	7,205.00
Anaconda	− 28.26	− 7,145.82	862.50
Bethlehem Steel	− 837.94	− 6,566.80	652.50
Chrysler	− 954.68	−12,258.61	− 1,500.00
Du Pont	6,564.21	− 465.35	9,550.00
Eastman Kodak	6,584.95	− 5,926.10	11,860.50
General Electric	− 107.06	− 8,601.28	2,100.00
General Foods	11,370.33	2,266.89	11,420.00
General Motors	− 1,099.40	− 8,440.42	2,025.00
Goodyear	− 2,241.28	−17,323.20	2,920.70
International Harvester	− 735.95	− 7,444.92	3,045.00
International Nickel	5,231.25	− 3,509.97	5,892.50
International Paper	2,266.82	− 7,976.68	− 278.10
Johns Manville	− 1,090.22	− 8,368.44	1,462.50
Owens Illinois	727.27	− 5,960.05	3,437.50
Procter & Gamble	12,202.83	4,561.52	8,550.00
Sears	4,871.36	408.65	5,195.00
Standard Oil (Calif.)	− 3,639.79	−21,055.08	5,326.50
Standard Oil (N.J.)	− 1,416.48	− 6,208.68	1,380.00
Swift & Co.	− 923.07	− 8,161.76	552.50
Texaco	2,803.98	− 5,626.11	6,546.50
Union Carbide	3,564.02	− 1,612.83	1,592.50
United Aircraft	− 1,190.10	− 8,369.88	562.50
U.S. Steel	1,068.23	− 5,650.03	475.00
Westinghouse	− 338.85	−12,034.56	745.00
Woolworth	4,190.78	− 2,403.34	3,225.00

* All figures are computed on the basis of 100 shares. Column (1) is total profits minus total losses on all filters, divided by the number of different filters tried on the security. Column (2) is the same as column (1) except that total profits and losses are computed net of commissions. Column (3) is last price plus any dividends paid during the period, minus the initial price for the period.

† The different filters are from 0.5 per cent to 5 per cent by steps of 0.5 per cent; from 6 per cent to 10 per cent by steps of 1 per cent; from 12 per cent to 20 per cent by steps of 2 per cent; and then 25 per cent, 30 per cent, 40 per cent and 50 per cent.

that the academic researcher is not interested in whether the dependence in series of price changes can be used to increase expected profits. Rather, he is primarily concerned with determining whether the independence assumption is an *exact* description of reality. In essence he proposes that we treat independence as a extreme null hypothesis and test it accordingly.

At this time we will ignore important counterarguments as to whether a strict test of an extreme null hypothesis is likely to be meaningful, given that for practical purposes the hypothesis would seem to be a valid approximation to reality for *both* the statistician and the investor. We simply note that a signs test applied to the profit figures in column (1) of Table 16 would not reject the extreme null hypothesis of independence for any of the standard significance levels. Sixteen of the profit figures in column (1) are positive and fourteen are negative, which is not very far from the even split that would be expected under a pure random model without trends in the price levels. If we allowed for the long-term upward bias of the market, the results would conform even more closely to the predictions of the strict null hypothesis. Thus the results produced by the filter technique do not seem to overturn the independence assumption of the random-walk model, regardless of how strictly that assumption is interpreted.

Finally, we emphasize again that these results must be regarded as preliminary. Many more complicated analyses of the filter technique are yet to be completed. For example, although average profits per filter do not compare favorably with buy-and-hold, there may be particular filters which are consistently better than buy-and-hold for all securities. We prefer, however, to leave such issues to a later paper. For now suffice it to say that preliminary results seem to indicate that the filter technique does not overturn the independence assumption of the random-walk model.

D. DISTRIBUTION OF SUCCESSORS TO LARGE VALUES

Mandelbrot [37, pp. 418–19] has suggested that one plausible form of dependence that could partially account for the long tails of empirical distributions of price changes is the following: Large changes may tend to be followed by large changes, but of random sign, whereas small changes tend to be followed by small changes.[36] The economic rationale for this type of dependence hinges on the nature of the information process in a world of uncertainty. The hypothesis implicitly assumes that when important new information comes into the market, it cannot always be evaluated precisely. Sometimes the immediate price change caused by the new information will be too large, which will set in motion forces to produce a reaction. In other cases the immediate price change will not fully discount the information, and impetus will be created to move the price again in the same direction.

The statistical implication of this hypothesis is that the conditional probability that tomorrow's price change will be large, given that today's change has been large, is higher than the unconditional probability of a large change. To test this, empirical distributions of the immediate successors to large price changes have been computed for the daily differ-

[36] Although the existence of this type of price behavior could not be used by the investor to increase his expected profits, the behavior does fit into the statistical definition of dependence. That is, knowledge of today's price change does condition our prediction of the *size*, if not the *sign*, of tomorrow's change.

ences of ten stocks. Six of the stocks were chosen at random. They include Allied Chemical, American Can, Eastman Kodak, Johns Manville, Standard Oil of New Jersey, and U.S. Steel. The other four were chosen because they showed longer than average tails in the tests of Sections III and IV. A large daily price change was defined as a change in log price greater than 0.03 in absolute value.

The results of the computations are shown in Table 17. The table is arranged to facilitate a direct comparison between the frequency distributions of successors to large daily price changes and the frequency distributions of all price changes. It shows for each stock the number and relative frequency of observations in the distribution of successors within given ranges of the distribution of all price changes. For example, the number in column (1) opposite Allied Chemical indicates that there are twenty-seven observations in the distribution of successors to large values that fall within the intersextile range of the distribution of all price changes for Allied Chemical. The number in column (6) opposite Allied Chemical indicates that twenty-seven observations are 55.1 per cent of

TABLE 17

DISTRIBUTIONS OF SUCCESSORS TO LARGE VALUES*

Stock	Intersextile (1)	2 Per Cent (2)	1 Per Cent (3)	> 1 Per Cent (4)	Total (5)
	Number				
Allied Chemical	27	46	48	1	49
American Can	13	26	27	5	32
A.T.&T.	4	12	14	2	16
Eastman Kodak	25	35	39	5	44
Goodyear	40	66	66	4	70
Johns Manville	38	62	63	3	66
Sears	14	25	28	3	31
Standard Oil (N.J.)	11	18	18	2	20
United Aircraft	49	78	84	4	88
U.S. Steel	14	27	31	5	36
	Frequency				
	(6)	(7)	(8)	(9)	
Expected frequency	0.6667	0.9600	0.9800	0.0200	
Allied Chemical	.5510	.9388	.9796	.0204	
American Can	.4063	.8125	.8438	.1562	
A.T.&T.	.2500	.7500	.8750	.1250	
Eastman Kodak	.5682	.7955	.8864	.1136	
Goodyear	.5714	.9429	.9429	.0571	
Johns Manville	.5758	.9394	.9545	.0455	
Sears	.4516	.8065	.9032	.0968	
Standard Oil (N.J.)	.5500	.9000	.9000	.1000	
United Aircraft	.5568	.8864	.9545	.0455	
U.S. Steel	0.3889	0.7500	0.8611	0.1389	

* Number and frequency of observations in the distributions of successors within given ranges of the distributions of all changes. The ranges are defined as follows: Intersextile = 0.83 fractile —0.17 fractile; 2 per cent = 0.98 fractile —0.02 fractile; 1 per cent = 0.99 fractile —0.01 fractile. The fractiles are the fractiles of the distributions of all price changes and not of the distributions of successors to large changes.

the total number of successors to large values, whereas the distribution of all price changes contains, by definition, 66.7 per cent of its observations within its intersextile range. Similarly, the number in column (9) opposite Goodyear indicates that in the distribution of successors 5.7 per cent of the observations fall outside of the 1 per cent range, whereas by definition only 2 per cent of the observations in the distribution of all changes are outside this range.

It is evident from Table 17 that the distributions of successors are flatter and have longer tails than the distributions of all price changes. This is best illustrated by the relative frequencies. In every case the distribution of successors has less relative frequency within each fractile range than the distribution of all changes, which implies that the distribution of successors has too much relative frequency outside these ranges.

These results can be presented graphically by means of simple scatter diagrams. This is done for American Telephone and Telegraph and Goodyear in Figure 8. The abscissas of the graphs show X_1, the value of the large price change. The ordinates show X_2, the price change on the day immediately following a large change. Though it is difficult to make strong statements from such graphs, as would be expected in light of Table 17, it does seem that the successors do not concentrate around the abscissas of the graphs as much as would be expected if their distributions were the same as the distributions of all changes. Even a casual glance at the graphs shows, however, that the signs of the successors do indeed seem to be random. Moreover, these statements hold for the graphs of the securities not included in Figure 8.

In sum, there is evidence that large changes tend to be followed by large changes, but of random sign. However, though there does seem to be more bunching of large values than would be predicted by a purely independent model, the tendency is not very strong. In Table 17 most of the successors to large observations do fall within the intersextile range even though more of the successors fall into the extreme tails than would be expected in a purely random model.

E. SUMMARY

None of the tests in this section give evidence of any important dependence in the first differences of the logs of stock prices. There is some evidence that large changes tend to be followed by large changes of either sign, but the dependence from this source does not seem to be too important. There is no evidence at all, however, that there is any dependence in the stock-price series that would be regarded as important for investment purposes. That is, the past history of the series cannot be used to increase the investor's expected profits.

It must be emphasized, however, that, while the observed departures from independence are extremely slight, this does not mean that they are unimportant for every conceivable purpose. For example, the fact that large changes tend to be followed by large changes may not be information which yields profits to chart readers; but it may be very important to the economist seeking to understand the process of price determination in the capital market. The importance of any observed dependence will always depend on the question to be answered.

VI. Conclusion

The purpose of this paper has been to test empirically the random-walk model of stock price behavior. The model makes

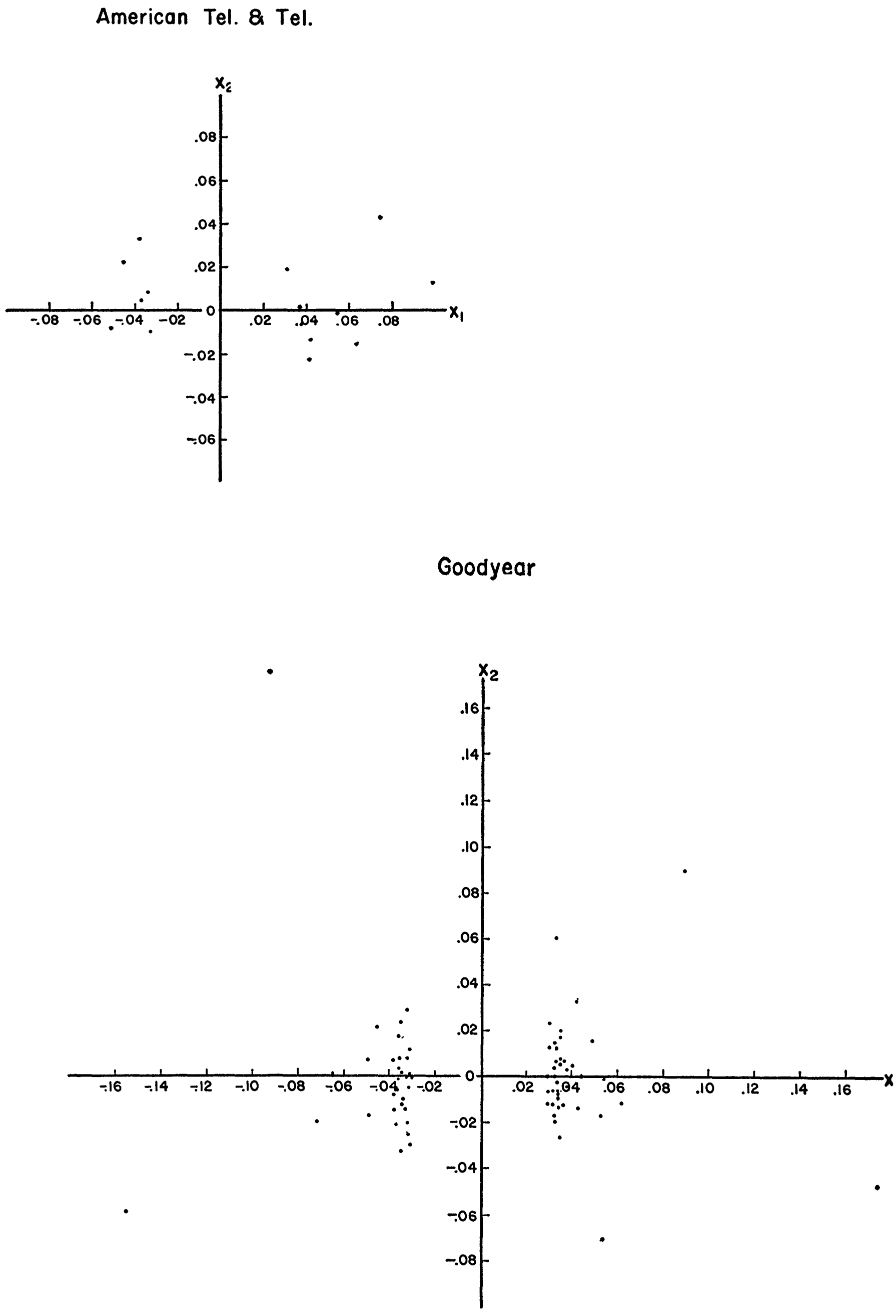
American Tel. & Tel.
X_2
.08
.06
.04
.02
0
-.02
-.04
-.06
-.08
-.06
-.04
-.02
.02
.04
.06
.08
X_1
Goodyear
X_2
.16
.14
.12
.10
.08
.06
.04
.02
0
-.02
-.04
-.06
-.08
-.16
-.14
-.12
-.10
-.08
-.06
-.04
-.02
.02
.04
.06
.08
.10
.12
.14
.16
X_1

FIG. 8

two basic assumptions: (1) successive price changes are independent, and (2) the price changes conform to some probability distribution. We begin this section by summarizing the evidence concerning these assumptions. Then the implications of the results will be discussed from various points of view.

A. DISTRIBUTION OF PRICE CHANGES

In previous research on the distribution of price changes the emphasis has been on the general shape of the distribution, and the conclusion has been that the distribution is approximately Gaussian or normal. Recent findings of Benoit Mandelbrot, however, have raised serious doubts concerning the validity of the Gaussian hypothesis. In particular, the Mandelbrot hypothesis states that empirical distributions of price changes conform better to stable Paretian distributions with characteristic exponents less than 2 than to the normal distribution (which is also stable Paretian but with characteristic exponent exactly equal to 2). The conclusion of this paper is that Mandelbrot's hypothesis does seem to be supported by the data. This conclusion was reached only after extensive testing had been carried out. The results of this testing will now be summarized.

If the Mandelbrot hypothesis is correct, the empirical distributions of price changes should have longer tails than does the normal distribution. That is, the empirical distributions should contain more relative frequency in their extreme tails than would be expected under a simple Gaussian hypothesis. In Section III frequency distributions were computed for the daily changes in log price of each of the thirty stocks in the sample. The results were quite striking. The empirical distribution for *each* stock contained more relative frequency in its central bell than would be expected under a normality hypothesis. More important, however, in *every* case the extreme tails of the distributions contained more relative frequency than would be expected under the Gaussian hypothesis. As a further test of departures from normality, a normal probability graph for the price changes of each stock was also exhibited in Section III. As would be expected with long-tailed frequency distributions, the graphs generally assumed the shape of an elongated S.

In an effort to explain the departures from normality in the empirical frequency distributions, two simple complications of the Gaussian model were discussed and tested in Section III. One involved a variant of the mixture of distributions approach and suggested that perhaps weekend and holiday changes come from a normal distribution, but with a higher variance than the distribution of daily changes within the week. The empirical evidence, however, did not support this hypothesis. The second approach, a variant of the non-stationarity hypothesis, suggested that perhaps the leptokurtosis in the empirical frequency distributions is due to changes in the mean of the daily differences across time. The empirical tests demonstrated, however, that the extreme values in the frequency distributions are so large that reasonable shifts in the mean cannot adequately explain them.

Section IV was concerned with testing the property of stability and developing estimates of the characteristic exponent α of the underlying stable Paretian process. It was emphasized that rigorously established procedures for estimating the parameters of stable Paretian distributions are practically unknown because for most values of the characteristic exponent there are no known, explicit expressions for the density functions. As a result there is virtually no sampling theory available. It was concluded that at present the only way to get satisfactory estimates of the characteristic exponent is to use more than one estimating procedure. Thus three different techniques for estimating α were discussed, illustrated, and compared. The techniques involved double-log–normal-probability graphing, sequential computation of variance, and range analysis. In a very few cases α seemed to be so close to 2 that it was indistinguishable from 2 in the estimates. In the vast majority of cases, however, the estimated values were less than 2, with some dispersion about an average value close to 1.90. On the basis of these estimates of α and the results produced by the frequency distributions and normal probability graphs, it was concluded that the Mandelbrot hypothesis fits the data better than the Gaussian hypothesis.

B. INDEPENDENCE

Section V of this paper was concerned, with testing the validity of the independence assumption of the random-walk model on successive price changes for differencing intervals of one, four, nine, and sixteen days. The main techniques used were a serial correlation model, runs analysis, and Alexander's filter technique. For all tests and for all differencing intervals the amount of dependence in the data seemed to be either extremely slight or else non-existent. Finally, there was some evidence of bunching of large values in the daily differences, but the degree of bunching seemed to be only slightly greater than would be expected in a purely random model. On the basis of all these tests it was concluded that the independence assumption of the random-walk model seems to be an adequate description of reality.

C. IMPLICATIONS OF INDEPENDENCE

We saw in Section II that a situation where successive price changes are independent is *consistent with* the existence of an "efficient" market for securities, that is, a market where, given the available information, actual prices at every point in time represent very good estimates of intrinsic values. We also saw that two factors that could possibly contribute toward establishing independence are (1) the existence of many sophisticated chart readers actively competing with each other to take advantage of any dependencies in series of price changes, and (2) the existence of sophisticated analysts, where sophistication implies an ability both to *predict* better the occurrence of economic and political events which have a bearing on prices and to evaluate the eventual effects of such events on prices.

If his activities succeed in helping to establish independence of successive price changes, then the sophisticated chart reader has defeated his own purposes. When successive price changes are independent, there can be no chart-reading technique which makes the expected profits of the investor greater than they would be under a naïve buy-and-hold model. Such dogmatic statements do not apply to superior intrinsic value analysis, however. People who can consistently predict the occurrence of important events and evaluate their effects on prices will usually make larger profits than people who do not have this talent. The fact that the activities of these superior analysts help to make successive price changes independent does *not* imply that their expected profits cannot be greater than those of the investor who follows a buy-and-hold policy.

Of course, in practice, identifying people who qualify as superior analysts is not an easy task. The simple criterion put forth in Section II was the following: A superior analyst is one whose gains over many periods of time are *consistently* greater than those of the market. There are many institutions and individuals that claim to meet this criterion. In a separate paper their claims will be systematically tested. We present here some of the preliminary results for open-end mutual funds.[37]

In their appeals to the public, mutual funds usually make two basic claims: (1) because it pools the resources of many individuals, a fund can diversify much more effectively than the average small investor; and (2) because of its management's closeness to the market, the fund is better able to detect "good buys" in individual securities. In most cases the first claim is probably true. The second, however, implies that mutual funds provide returns higher than those earned by the market as a whole. It is this second claim that we now wish to test.

The return earned by the "market" during any time period can be measured in various ways. One possibility has been extensively explored by Fisher and Lorie [16] in a recent issue of this *Journal*. The basic assumption in all their computations is that at the beginning of each period studied the investor puts an equal amount of money in each common stock listed at that time on the New York Stock Exchange. Different rates of return for the period are then computed for different possible tax brackets of the investor, first under the assumption that all dividends are reinvested in the month paid and then under the assumption that dividends are not reinvested. All computations include the relevant brokers' commissions. Following the Lorie-Fisher procedure, a tax-exempt investor who initially entered the market at the end of 1950 and reinvested subsequent dividends in the securities paying them would have made a compound annual rate of return of 14.7 per cent upon disinvesting his entire portfolio at the end of 1960.

Similar computations have been carried out for thirty-nine open-end mutual funds. The funds studied have been chosen on the following basis: (1) the fund was operating during the entire period from the end of 1950 through the end of 1960; and (2) no more than 5 per cent of its total assets were invested in bonds at the end of 1960. It was assumed that the investor put $10,000 into each fund at the end of 1950, reinvested all subsequent dividend distributions, and then cashed in his portfolio at the end of 1960. It was also assumed, for simplicity, that the investor was tax exempt.

For our purposes, two different types of rates of return are of interest, gross and net of any loading charges. Most funds have a loading charge of about 8 per cent on new investment. That is, on a gross investment of $10,000 the investor receives only about $9,200 worth of the fund's shares. The remainng $800 is usually a straight salesman's commission and is not available to the fund's management for investment. From the investor's point of view the relevant rate of return on mutual funds to compare with the "market" rate is the return gross of loading charges, since the gross sum is the amount that the investor allocates to the funds. It is also interesting, however, to compute the yield on mutual funds net of any loading changes, since the net sum is the amount actually available to management. Thus the net return is the relevant measure of management's performance in relation to the market.

For the period 1950–60 our mutual-fund investments had a gross return of

[37] The preliminary results reported below were prepared as an assigned term paper by one of my students, Gerhard T. Roth. The data source for all the calculations was Wiesenberger [24].

14.1 per cent which is below the 14.7 per cent earned by the "market," as defined by Fisher and Lorie. The return, net of loading charges, on the mutual funds was 14.9 per cent, slightly but not significantly above the "market" return. Thus it seems that, at least for the period studied, mutual funds in general did not do any better than the market.

Although mutual funds taken together do no better than the market, in a world of uncertainty, during any given time period some funds will do better than the market and some will do worse. When a fund does better than the market during some time period, however, this is not necessarily evidence that the fund's management has knowledge superior to that of the average investor. A good showing during a particular period may merely be a chance result which is, in the long run, balanced by poor showings in other periods. It is only when a fund *consistently* does better than the market that there is any reason to feel that its higher than average returns may not be the work of lady luck.

In an effort to examine the consistency of the results obtained by different funds across time two separate tests were carried out. First, the compound rate of return, net of loading charges, was computed for each fund for the entire 1950–60 period. Second, the return for each fund for each year was computed according to the formula

$$r_{j,\ t+1}=\frac{p_{j,\ t+1}+d_{j,\ t+1}-p_{jt}}{p_{jt}},$$

$$t=1950,\ldots,1959$$

where P_{jt} is the price of a share in fund j at the end of year t, $p_{j,\ t+1}$ is the price at the end of year $t+1$, and $d_{j,\ t+1}$ are the dividends per share paid by the fund during year $t+1$. For each year the returns on the different funds were then ranked in ascending order, and a number from 1 to 39 was assigned to each.

The results are shown in Table 18. The order of the funds in the table is according to the return, net of loading charges, shown by the fund for the period 1950–60. This net return is shown in column (1). Columns (2)–(11) show the relative rankings of the year-by-year returns of each fund.

The most impressive feature of Table 18 is the *inconsistency* in the rankings of year-by-year returns for any given fund. For example, out of thirty-nine funds, *no* single fund consistently had returns high enough to place it among the top twenty funds for every year in the time period. On the other hand *no* single fund had returns low enough to place it among the bottom twenty of each year. Only two funds, Selected American and Equity, failed to have a return among the top ten for some year, and only three funds, Investment Corporation of America, Founders Mutual, and American Mutual, do not have a return among the bottom ten for some year. Thus funds in general seem to do no better than the market; in addition, individual funds do not seem to outperform consistently their competitors.[38] Our conclusion, then, must be that so far the sophisticated analyst has escaped detection.

D. IMPLICATIONS OF THE MANDELBROT HYPOTHESIS

The main conclusion of this paper with respect to the distribution of price changes is that a stable Paretian distribution with characteristic exponent α less than 2 seems to fit the data better

[38] These results seem to be in complete agreement with those of Ira Horowitz [22] and with the now famous "Study of Mutual Funds," prepared for the Securities and Exchange Commission by the Wharton School, University of Pennsylvania (87th Cong., 2d sess. [Washington, D.C.: Government Printing Office, 1962]).

than the normal distribution. This conclusion has implications from two points of view, economic and statistical, which we shall now discuss in turn.

1. ECONOMIC IMPLICATIONS

The important difference between a market dominated by a stable Paretian process with characteristic exponent $\alpha < 2$ and a market dominated by a Gaussian process is the following. In a Gaussian market, if the sum of a large number of price changes across some long time period turns out to be very large, chances are that each individual price change during the time period is negligible when compared to the total change. In a market that is stable Paretian with $\alpha < 2$,

TABLE 18

YEAR-BY-YEAR RANKING OF INDIVIDUAL FUND RETURNS

Fund	Return on Net	Year									
		1951	1952	1953	1954	1955	1956	1957	1958	1959	1960
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
Keystone Lower Price	18.7	29	1	38	5	3	8	35	1	1	36
T Rowe Price Growth	18.7	1	33	2	8	14	15	2	25	7	4
Dreyfuss	18.4	37	37	14	3	7	11	3	2	3	7
Television Electronic	18.4	21	4	9	2	33	20	16	2	4	20
National Investors Corp.	18.0	3	35	4	19	27	4	5	5	8	1
De Vegh Mutual Fund	17.7	32	4	1	8	14	4	8	15	23	36
Growth Industries	17.0	7	34	14	17	9	9	20	5	6	11
Massachusetts Investors Growth	16.9	5	36	31	11	9	1	23	4	9	4
Franklin Custodian	16.5	26	2	4	13	33	20	16	5	9	4
Investment Co. of America	16.0	21	15	14	11	17	15	23	15	15	15
Chemical Fund, Inc.	15.6	1	39	14	27	3	33	1	27	4	23
Founders Mutual	15.6	21	13	25	8	2	20	16	11	13	28
Investment Trust of Boston	15.6	6	3	25	3	14	26	31	20	29	20
American Mutual	15.5	14	13	4	22	14	13	16	25	25	4
Keystone Growth	15.3	29	15	25	1	1	1	39	11	18	38
Keystone High	15.2	10	7	3	27	23	36	5	27	25	11
Aberdeen Fund	15.1	32	23	9	25	9	7	10	27	7	30
Massachusetts Investors Trust	14.8	8	9	14	16	9	15	20	18	32	28
Texas Fund, Inc.	14.6	3	15	9	32	23	26	5	27	37	7
Eaton & Howard Stock	14.4	14	9	4	17	20	15	13	37	29	17
Guardian Mutual	14.4	21	26	25	34	31	29	13	20	15	2
Scudder, Stevens, Clark	14.3	14	23	14	19	27	15	29	9	15	30
Investors Stock Fund	14.2	8	28	21	22	27	20	23	5	29	23
Fidelity Fund, Inc.	14.1	21	6	31	6	23	29	33	11	25	23
Fundamental Inv.	13.8	14	15	31	15	9	11	31	18	25	30
Century Shares	13.5	14	28	35	25	3	20	23	31	34	2
Bullock Fund Ltd.	13.5	29	9	21	19	14	9	20	34	34	20
Financial Industries	13.0	26	15	31	13	19	29	34	20	9	35
Group Common Stock	13.0	38	8	25	27	27	33	8	20	34	17
Incorporated Investors	12.9	14	13	37	6	3	13	37	11	18	39
Equity Fund	12.9	14	27	21	32	31	33	13	31	18	23
Selected American Shares	12.8	21	15	21	31	23	20	23	15	32	30
Dividend Shares	12.7	32	7	14	34	20	32	4	37	37	11
General Capital Corp.	12.4	10	28	9	38	35	39	23	34	13	23
Wisconsin Fund	12.3	32	26	4	37	35	38	10	34	18	7
International Resources	12.3	10	37	39	22	35	1	37	39	1	11
Delaware Fund	12.1	36	23	25	27	39	26	29	9	23	30
Hamilton Fund	11.9	38	28	9	34	35	36	10	31	18	17
Colonial Energy	10.9	10	15	35	39	20	4	36	20	39	10

however, the size of the total will more than likely be the result of a few very large changes that took place during much shorter subperiods. In other words, whereas the path of the price level of a given security in a Gaussian market will be fairly continuous, in a stable Paretian market with $\alpha < 2$ it will usually be discontinuous. More simply, in a stable Paretian market with $\alpha < 2$, the price of a security will often tend to jump up or down by very large amounts during very short time periods.[39]

When combined with independence of successive price changes, the discontinuity of price levels in a stable Paretian market may provide important insights into the nature of the process that generates changes in intrinsic values across time. We saw earlier that independence of successive price changes is consistent with an "efficient" market, that is, a market where prices at every point in time represent best estimates of intrinsic values. This implies in turn that, when an intrinsic value changes, the actual price will adjust "instantaneously," where instantaneously means, among other things, that the actual price will initially overshoot the new intrinsic value as often as it will undershoot it.

In this light the combination of independence and a *Gaussian* distribution for the price changes would imply that intrinsic values do not very often change by large amounts. On the other hand, the combination of independence and a *stable Paretian* distribution with $\alpha < 2$ for the price changes would imply that intrinsic values often change by large amounts during very short periods of time—a situation quite consistent with a dynamic economy in a world of uncertainty.

[39] For a proof of these statements see Darling [13] or Anov and Bobnov [4].

The discontinuous nature of a stable Paretian market has some more practical implications, however. The fact that there are a large number of abrupt changes in a stable Paretian market means that such a market is inherently more risky than a Gaussian market. The variability of a given expected yield is higher in a stable Paretian market than it would be in a Gaussian market, and the probability of large losses is greater.

Moreover, in a stable Paretian market with $\alpha < 2$ speculators cannot usually protect themselves from large losses by means of such devices as "stop-loss" orders. If the price level is going to fall very much, the total decline will probably be accomplished very rapidly, so that it may be impossible to carry out many "stop-loss" orders at intermediate prices.

Finally, in some cases it may be possible a posteriori to find "causal explanations" for specific large price changes in terms of more basic economic variables. If the behavior of these more basic variables is itself largely unpredictable, however, the "causal explanation" will not be of much help in forecasting the appearance of large changes in the future. In addition it must be kept in mind that in the series we have been studying, there are very many large changes and the "explanations" are far from obvious. For example, the two largest changes in the Dow-Jones Industrial Average during the period covered by the data occurred on May 28 and May 29, 1962. Market analysts are still trying to find plausible "explanations" for these two days.

2. STATISTICAL IMPLICATIONS

The statistical implications of the Mandelbrot hypothesis follow mostly from the absence of a finite variance for stable Paretian distributions with characteristic exponents less than 2. In practical terms "infinite" variance means that the sample variance and standard deviation of a stable Paretian process with $\alpha < 2$ will show extremely erratic behavior even for very large samples. That is, for larger and larger sample sizes the variability of the sample variance and standard deviation will not tend to dampen nearly as much as would be expected with a Gaussian process. Because of their extremely erratic behavior, the sample variance and standard deviation are not meaningful measures of the variability inherent in a stable Paretian process with $\alpha < 2$.

This does not mean, however, that we are helpless in describing the dispersion of such a process. There are other measures of variability, such as interfractile ranges and the mean absolute deviation, which have both finite expectation and much less erratic sampling behavior than the variance and standard deviation.[40]

Figure 9 presents a striking demonstration of these statements. It shows the path of the sequential sample standard deviation and the sequential mean absolute deviation for four securities.[41] The upper set of points on each graph represents the path of the standard deviation, while the lower set represents the sample sequential mean absolute deviation. In every case the sequential mean absolute deviation shows less erratic behavior as the sample size is increased than does the sequential standard deviation. Even for very large samples the sequential standard deviation often shows very large discrete jumps, which are of course due to the occurrence of extremely large price changes in the data. As the sample size is increased, however, these same large price changes do not have nearly as strong an effect on the sequential mean absolute deviation. This would seem to be strong evidence that for distributions of price changes the mean absolute deviation is a much more reliable estimate of variability than the standard deviation.

In general, when dealing with stable Paretian distributions with characteristic exponents less than 2, the researcher should avoid the concept of variance both in his empirical work and in any economic models he may construct. For example, from an empirical point of view, when there is good reason to believe that the distribution of residuals has infinite variance, it is not very appealing to use a regression technique that has as its criterion the minimization of the sum of squared residuals from the regression line, since the expectation of that sum will be infinite.

This does not mean, however, that we are helpless when trying to estimate the parameters of a linear model if the variables of interest are subject to stable Paretian distributions with infinite variances. For example, an alternative technique, absolute-value regression, involves minimizing the sum of the absolute values of the residuals from the regression line. Since the expectation of the absolute value of the residual will be finite as long as the characteristic exponent α of the distribution of residuals is greater than 1, this minimization criterion is meaning-

[40] The mean absolute deviation is defined as

$$|D| = \sum_{i=1}^{N} \frac{|x_i - \bar{x}|}{N},$$

where x is the variable and N is the total sample size.

[41] Sequential computation of a parameter means that the *cumulative* sequential sample value of the parameter is recomputed at fixed intervals subsequent to the beginning of the sampling period. Each new computation of the parameter in the sequence contains the same values of the random variable as the computation immediately preceding it, plus any new values of the variable that have since been generated.

ful for a wide variety of stable Paretian processes.[42]

A good example of an economic model which uses the notion of variance in situations where there is good reason to believe that variances are infinite is the classic Markowitz [39] analysis of efficient portfolios. In Markowitz' terms, efficient portfolios are portfolios which have maximum expected return for given variance of expected return. If yields on securities follow distributions with infinite variances, however, the expected yield of a diversified portfolio will also follow a

[42] For a discussion of the technique of absolute value regression see Wagner [46], [47]. Wise [49] has shown that when the distribution of residuals has characteristic exponent $1 < \alpha < 2$, the usual least squares estimators of the parameters of a regression equation are consistent and unbiased. He has further shown, however, that when $\alpha < 2$, the least squares estimators are not the most efficient linear estimators, i.e., there are other techniques for which the sampling distributions of the regression parameters have lower dispersion than the sampling distributions of the least squares estimates. Of course it is also possible that some non-linear technique, such as absolute value regression, provides even more efficient estimates than the most efficient linear estimators.

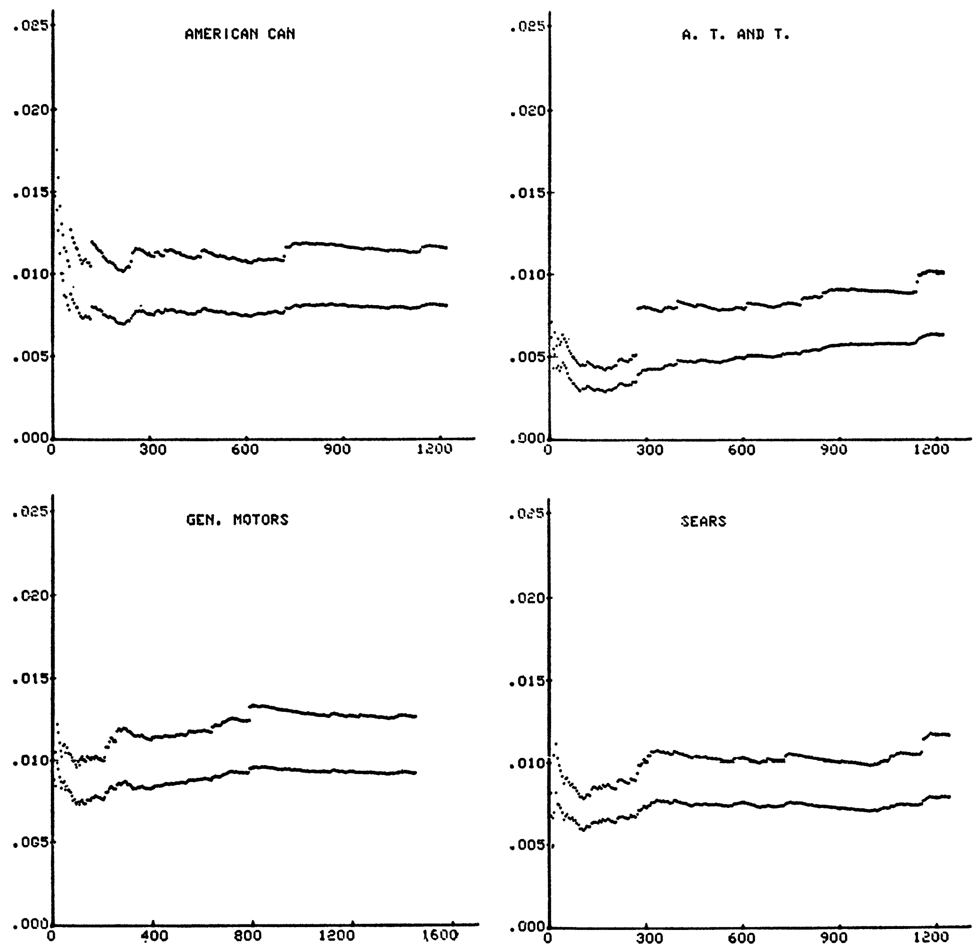

FIG. 9.—Sequential standard deviations and sequential mean absolute deviations. Horizontal axes show sequential sample sizes; vertical axes show parameter estimates.

distribution with an infinite variance. In this situation the mean-variance concept of an efficient portfolio loses its meaning.

This does *not* mean, however, that diversification is a meaningless concept in a stable Paretian market, or that it is impossible to develop a model for portfolio analysis. In a separate paper [15] this author has shown that, if concepts of variability other than the variance are used, it is possible to develop a model for portfolio analysis in a stable Paretian market. It is also possible to define the conditions under which increasing diversification has the effect of reducing the dispersion of the distribution of the return on the portfolio, even though the variance of that distribution may be infinite.

Finally, although the Gaussian or normal distribution does not seem to be an adequate representation of distributions of stock price changes, it is not necessarily the case that stable Paretian distributions with infinite variances provide the only alternative. It is possible that there are long-tailed distributions with finite variances that could also be used to describe the data.[43] We shall now argue, however, that one is forced to accept many of the conclusions discussed above, regardless of the position taken with respect to the finite-versus-infinite-variance argument.

For example, although one may feel that it is nonsense to talk about infinite variances when dealing with real-world variables, one is nevertheless forced to admit that for distributions of stock price changes the sampling behavior of the standard deviation is much more erratic than that of alternative dispersion parameters such as the mean absolute deviation. For this reason it may be better to use these alternative dispersion parameters in empirical work even though one may feel that in fact all variances are finite.

Similarly, the asymptotic properties of the parameters in a classical least-squares regression analysis are strongly dependent on the assumption of finite variance in the distribution of the residuals. Thus, if in some practical situation one feels that this distribution, though long-tailed, has finite variance, in principle one may feel justified in using the least-squares technique. If, however, one observes that the sampling behavior of the parameter estimates produced by the least-squares technique is much more erratic than that of some alternative technique, one may be forced to conclude that for reasons of efficiency the alternative technique is superior to least squares.

The same sort of argument can be applied to the portfolio-analysis problem. Although one may feel that in principle real-world distributions of returns must have finite variances, it is well known that the usual Markowitz-type efficient set analysis is highly sensitive to the estimates of the variances that are used. Thus, if it is difficult to develop good estimates of variances because of erratic sampling behavior induced by long-tailed distributions of returns, one may feel forced to use an alternative measure of dispersion in portfolio analyses.

Finally, from the point of view of the individual investor, the name that the researcher gives to the probability distribution of the return on a security is irrelevant, as is the argument concerning whether variances are finite or infinite. The investor's sole interest is in the *shape* of the distribution. That is, the only information he needs concerns the proba-

[43] It is important to note, however, that stable Paretian distributions with characteristic exponents less than 2 are the only long-tailed distributions that have the crucial property of stability or invariance under addition.

bility of gains and losses greater than given amounts. As long as two different hypotheses provide adequate descriptions of the relative frequencies, the investor is indifferent as to whether the researcher tells him that distributions of returns are stable Paretian with characteristic exponent $\alpha < 2$ or just long-tailed but with finite variances.

In essence, all of the above arguments merely say that, given the long-tailed empirical frequency distributions that have been observed, in most cases one's subsequent behavior in light of these results will be the same whether one leans toward the Mandelbrot hypothesis or toward some alternative hypothesis involving other long-tailed distributions. For most purposes the implications of the empirical work reported in this paper are independent of any conclusions concerning the name of the hypothesis which the data seem to support.

E. POSSIBLE DIRECTIONS FOR FUTURE RESEARCH

It seems safe to say that this paper has presented strong and voluminous evidence in favor of the random-walk hypothesis. In business and economic research, however, one can never claim to have established a hypothesis beyond question. There are always additional tests which would tend either to confirm the validity of the hypothesis or to contradict results previously obtained. In the final paragraphs of this paper we wish to suggest some possible directions which future research on the random-walk hypothesis could take.

1. ADDITIONAL POSSIBLE TESTS OF DEPENDENCE

There are two different approaches to testing for independence. First, one can carry out purely statistical tests. If these tend to support the assumption of independence, one may then infer that there are probably no mechanical trading rules based on patterns in the past history of price changes which will make the profits of the investor greater than they would be under a buy-and-hold policy. Second, one can proceed by directly testing different mechanical trading rules to see whether or not they do provide profits greater than buy-and-hold. The serial-correlation model and runs tests discussed in Section V are representative of the first approach, while Alexander's filter technique is representative of the second.

Academic research to date has tended to concentrate on the statistical approach. This is true, for example, of the extremely sophisticated work of Granger and Morgenstern [19], Moore [41], Kendall [26], and others. Aside from Alexander's work [1], [2], there has really been very little effort by academic people to test directly the various chartist theories that are popular in the financial world. Systematic validation or invalidation of these theories would represent a real contribution.

2. POSSIBLE RESEARCH ON THE DISTRIBUTION OF PRICE CHANGES

There are two possible courses which future research on the distribution of price changes could take. First, until now most research has been concerned with simply finding statistical distributions that seem to coincide with the empirical distributions of price changes. There has been relatively little effort spent in exploring the more basic processes that give rise to the empirical distributions. In essence, there is as yet no general model of price formation in the stock market which explains price levels and distributions of price changes in terms of the behavior of more basic economic variables. Developing and testing such a model would contribute greatly toward establishing sound theoretical foundations in this area.

Second, if distributions of price changes are truly stable Paretian with characteristic exponent $\alpha < 2$, then it behooves us to develop further the statistical theory of stable Paretian distributions. In particular, the theory would be much advanced by evidence concerning the sampling behavior of different estimators of the parameters of these distributions. Unfortunately, rigorous analytical sampling theory will be difficult to develop as long as explicit expressions for the density functions of these distributions are not known.

Using Monte Carlo techniques, however, it is possible to develop an approximate sampling theory, even though explicit expressions for the density functions remain unknown. In a study now under way the series-expansion approximation to stable Paretian density functions derived by Bergstrom [7] is being used to develop a stable Paretian random numbers generator. With such a random numbers generator it will be possible to examine the behavior of different estimators of the parameters of stable Paretian distributions in successive random samples and in this way to develop an approximate sampling theory. The same procedure can be used, of course, to develop sampling theory for many different types of statistical tools.

In sum, it has been demonstrated that first differences of stock prices seem to follow stable Paretian distributions with characteristic exponent $\alpha < 2$. An important step which remains to be taken is the development of a broad range of statistical tools for dealing with these distributions.

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APPENDIX

STATISTICAL THEORY OF STABLE PARETIAN DISTRIBUTIONS

A. STABLE PARETIAN DISTRIBUTIONS: DEFINITION AND PARAMETERS

The stable Paretian family of distributions is defined by the logarithm of its characteristic function which has the general form

$$\begin{aligned}\log f(t) &= \log E(e^{iut}) \\ &= i\delta t - \gamma|t|^{\alpha}[1 + i\beta(t/|t|)w(t, \alpha)]\,,\end{aligned} \tag{A1}$$

where u is the random variable, t is any real number, i is $\sqrt{-1}$, and

$$w(t, \alpha) = \begin{cases} \tan\dfrac{\pi\alpha}{2}, & \text{if } \alpha \neq 1\,, \\[2ex] \dfrac{2}{\pi}\log|t|, & \text{if } \alpha = 1\,. \end{cases} \tag{A2}$$

Stable Paretian distributions have four parameters, α, β, δ, and γ. The parameter α is called the characteristic exponent of the distribution. It determines the height of, or total probability contained in, the extreme tails of the distribution and can take any value in the interval $0 < \alpha \leq 2$. When $\alpha = 2$, the relevant stable Paretian distribution is the normal distribution.[44] When α is in the interval $0 < \alpha < 2$, the extreme tails of the stable Paretian distributions are higher than those of the normal distribution, and the total probability in the extreme tails is larger the smaller the value of α. The most important consequence of this is that the variance exists (i.e., is finite) only in the limiting case $\alpha = 2$. The mean, however, exists as long as $\alpha > 1$.[45]

The parameter β is an index of skewness which can take any value in the interval $-1 \leq \beta \leq 1$. When $\beta = 0$, the distribution is symmetric. When $\beta > 0$, the distribution is skewed right (i.e., has a long tail to the right), and the degree of right skewness is larger the larger the value of β. Similarly when $\beta < 0$ the distribution is skewed left, and the degree of left skewness is larger the smaller the value of β.

The parameter δ is the location parameter of the stable Paretian distribution. When the characteristic exponent α is greater than 1, δ is the expected value or mean of the distribution. When $\alpha \leq 1$, however, the mean of the distribution is not defined. In this case δ will be some other parameter (e.g., the median when $\beta = 0$), which will describe the location of the distribution.

Finally, the parameter γ defines the scale of a stable Paretian distribution. For example, when $\alpha = 2$ (the normal distribution), γ is one-half the variance. When $\alpha < 2$, however, the variance of the stable Paretian distribution is infinite. In this case there will be a finite parameter γ which defines the scale of the distri-

[44] The logarithm of the characteristic function of a normal distribution is $\log f_j(t) = i\mu t - (\sigma^2/2)t^2$. This is the log characteristic function of a stable Paretian distribution with parameters $\alpha = 2$, $\delta = \mu$, and $\gamma = \sigma^2/2$. The parameters μ and σ^2 are, of course, the mean and variance of the normal distribution.

[45] For a proof of these statements see Gnedenko and Kolmogorov [17], pp. 179–83.

bution, but it will not be the variance. For example, when $\alpha = 1, \beta = 0$ (which is the Cauchy infinite. In this case there will be a finite parameter γ which defines the scale of the distri- distribution), γ is the semi-interquartile range (i.e., one-half of the 0.75 fractile minus the 0.25 fractile).

B. KEY PROPERTIES OF STABLE PARETIAN DISTRIBUTIONS

The three most important properties of stable Paretian distributions are (1) the asymptotically Paretian nature of the extreme tail areas, (2) stability or invariance under addition, and (3) the fact that these distributions are the only possible limiting distributions for sums of independent, identically distributed, random variables.

1. *The law of Pareto.*—Lévy [29] has shown that the tails of stable Paretian distributions follow a weak or asymptotic form of the law of Pareto. That is,

$$Pr(u > \hat{u}) \to (\hat{u}/U_1)^{-\alpha} \quad \text{as} \quad \hat{u} \to \infty\ , \tag{A3}$$

and

$$Pr(\hat{u} < \hat{u}) \to (|\hat{u}|/U_2)^{-\alpha} \quad \text{as} \quad \hat{u} \to -\infty\ , \tag{A4}$$

where u is the random variable, and the constants U_1 and U_2 are defined by[46]

$$\beta = \frac{U_1^\alpha - U_2^\alpha}{U_1^\alpha + U_2^\alpha}. \tag{A5}$$

From expressions (A3) and (A4) it is possible to define approximate densities for the extreme tail areas of stable Paretian distributions. If a new function $P(u)$ for the tail probabilities is defined by expressions (A3) and (A4), the density functions for the asymptotic portions of the tails are given by

$$p(u) \approx -d\,P(u)/du \approx \alpha(U_1)^\alpha\, u^{-(\alpha+1)}, \quad u \to \infty\ , \tag{A6}$$

$$p(u) \approx \alpha(U_2)^\alpha |u|^{-(\alpha+1)}, \quad u \to -\infty\ . \tag{A7}$$

Although it has been proven that stable Paretian distributions are unimodal,[47] closed expressions for the densities of the *central areas* of these distributions are known for only three cases, the Gaussian ($\alpha = 2$), the Cauchy ($\alpha = 1, \beta = 0$), and the well-known coin-tossing case ($\alpha = \frac{1}{2}, \beta = 1, \delta = 0$ and $\gamma = 1$). At this point this is probably the greatest weakness in the theory. Without density functions it is very difficult to develop sampling theory for the parameters of stable Paretian distributions. The importance of this limitation has been stressed throughout this paper.[48]

2. *Stability or invariance under addition.*—By definition, a stable Paretian distribution is any distribution that is stable or invariant under addition. That is, the distribution of sums of independent, identically distributed, stable Paretian variables is itself stable Paretian and has the same form as the distribution of the individual summands. The phrase "has the same form" is, of course, an imprecise verbal expression for a precise mathematical property. A more rigorous definition of stability is given by the logarithm of the characteristic function of sums of independent, identically distributed, stable Paretian variables. The expression for this function is

$$n \log f(t) = i(n\delta)t - (n\gamma)\,|t|^\alpha\left[1 + i\beta \frac{t}{|t|} w(t, \alpha)\right], \tag{A8}$$

where n is the number of variables in the sum and $\log f(t)$ is the logarithm of the characteristic function for the distribution of the individual summands. Expression (A8) is the same as (A1), the expression for $\log f(t)$, except that the parameters δ (location) and γ (scale) are multiplied by n. That is, except for origin and scale,

[46] The constants U_1 and U_2 can be regarded as scale parameters for the positive and negative tails of the distribution. The relative size of these two constants determines the value of β and thus the skewness of the distribution. If U_2 is large relative to U_1, the distribution is skewed left (i.e., $\beta < 0$), and skewed right when U_1 is large relative to U_2 (i.e., $\beta > 0$). When U_1 is zero the distribution has maximal left skewness. When U_2 is zero, the distribution has maximal right skewness. These two limiting cases correspond, of course, to values of β of -1 and 1. When $U_1 = U_2$, $\beta = 0$, and the distribution is symmetric.

[47] Ibraginov and Tchernin [23].

[48] It should be noted, however, that Bergstrom [7] has developed a series expansion to approximate the densities of stable Paretian distributions. The potential use of the series expansion in developing sampling theory for the parameters by means of Monte Carlo methods is discussed in Section VI of this paper.

the distribution of the sums is exactly the same as the distribution of the individual summands. More simply, stability means that the values of the parameters α and β remain constant under addition.

The definition of stability is always in terms of independent, identically distributed random variables. It will now be shown, however, that any linear weighted sum of independent, stable Paretian variables with the same characteristic exponent α will be stable Paretian with the same value of α. In particular, suppose we have n independent, stable Paretian variables, u_j, $j = 1, \ldots, n$. Assume further that the distributions of the various u_j have the same characteristic exponent α, but possibly different location, scale, and skewness parameters (δ_j, γ_j, and β_j, $j = 1, \ldots, n$). Let us now form a new variable, V, which is a weighted sum of the u_j with constant weights p_j, $j = 1, \ldots, n$. The log characteristic function of V will then be

$$\log F(t) = \sum_{j=1}^{n} \log f_j(p_j t)$$

$$= i\left(\sum_{j=1}^{n} p_j\delta_j\right)t - \left(\sum_{j=1}^{n} \gamma_j |p_j|^{\alpha}\right) \quad \text{(A9)}$$

$$\times |t|^{\alpha}\left[1 + i\bar{\beta}\frac{t}{|t|}w(t, \alpha)\right],$$

where

$$\bar{\beta} = \frac{\sum_{j=1}^{n} \gamma_j |p_j|^{\alpha}\beta_j}{\sum_{j=1}^{n} \gamma_j |p_j|^{\alpha}}, \quad \text{(A10)}$$

and $\log f_j(t)$ is the log characteristic function of u_j. Expression (A9) is the log characteristic function of a stable Paretian distribution with characteristic exponent α and with location, scale, and skewness parameters that are weighted sums of the location, scale, and skewness parameters of the distributions of the u_j.

3. *Limiting distributions.*—It can be shown that stability or invariance under addition leads to a most important corollary property of stable Paretian distributions; they are the only possible limiting distributions for sums of independent, identically distributed, random variables.[49] It is well known that if such variables have finite variance the limiting distribution for their sum will be the normal distribution. If the basic variables have infinite variance, however, and if their sums follow a limiting distribution, the limiting distribution must be stable Paretian with $0 < \alpha < 2$.

It has been proven independently by Gnedenko and Doeblin that, in order for the limiting distribution of sums to be stable Paretian with characteristic exponent $\alpha(0 < \alpha < 2)$, it is necessary and sufficient that[50]

$$\frac{F(-u)}{1-F(u)} \rightarrow \frac{C_1}{C_2} \quad \text{as} \quad u \rightarrow \infty, \quad \text{(A11)}$$

and for every constant $k > 0$,

$$\frac{1-F(u)+F(-u)}{1-F(ku)+F(-ku)} \rightarrow k^{\alpha} \quad \text{(A12)}$$

$$\text{as} \quad u \rightarrow \infty,$$

where F is the cumulative distribution function of the random variable u and C_1 and C_2 are constants. Expressions (A11) and (A12) will henceforth be called the conditions of Doeblin and Gnedenko.

It is clear that any variable that is asymptotically Paretian (regardless of whether it is also stable) will satisfy these conditions. For such a variable, as $u \rightarrow \infty$,

$$\frac{F(-u)}{1-F(u)} \rightarrow \left[\frac{(|-u|/U_2)}{(u/U_1)}\right]^{-\alpha} = \frac{U_2^{\alpha}}{U_1^{\alpha}},$$

and

$$\frac{1-F(u)+F(-u)}{1-F(ku)+F(-ku)}$$

$$\rightarrow \frac{(u/U_1)^{-\alpha} + (|-u|/U_2)^{-\alpha}}{(ku/U_1)^{-\alpha} + (|-ku|/U_2)^{-\alpha}} = k^{\alpha},$$

and the conditions of Doeblin and Gnedenko are satisfied.

To the best of my knowledge non-stable, asymptotically Paretian variables with exponent $\alpha < 2$ are the only known variables of infinite variance that satisfy conditions (A11) and (A12). Thus they are the only known non-stable variables whose sums approach stable Paretian limiting distributions with characteristic exponents less than 2.

[49] For a proof see Gnedenko and Kolmogorov [17], pp. 162–63.

[50] For a proof see Gnedenko and Kolmogorov [17], pp. 175–80.

C. PROPERTIES OF RANGES OF SUMS OF STABLE PARETIAN VARIABLES

By the definition of stability, sums of independent realizations of a stable Paretian variable are stable Paretian with the same value of the characteristic exponent α as the distribution of the individual summands. The process of taking sums does, of course, change the scale or unit of measurement of the distribution.

Let us now pose the problem of finding a constant by which to weight each variable in the sum so that the scale parameter of the distribution of sums is the same as that of the distribution of the individual summands. This amounts to finding a constant, a, such that

$$n\gamma|at|^{\alpha} = \gamma|t|^{\alpha} . \quad \text{(A13)}$$

Solving this expression for a we get

$$a = n^{-1/\alpha} , \quad \text{(A14)}$$

which implies that each of the summands must be divided by $n^{1/\alpha}$ if the scale, or unit of measurement, of the distribution of sums is to be the same as that of the distribution of the individual summands. The converse proposition, of course, is that the scale of the distribution of unweighted sums is $n^{1/\alpha}$ times the scale of the distribution of the individual summands. Thus, for example, the intersextile range of the distribution of sums of n independent realizations of a stable Paretian variable will be $n^{1/\alpha}$ times the intersextile range of the distribution of the individual summands. This property provides the basis of the range analysis approach to estimating α discussed in Section IV, C of this paper.[51]

D. PROPERTIES OF THE SEQUENTIAL VARIANCE OF A STABLE PARETIAN VARIABLE

Let u be a stable Paretian random variable with characteristic exponent $\alpha < 2$, and with location, scale, and skewness parameters δ, γ, and β. Define a new variable, $y = u - \delta$, whose distribution is exactly the same as that of u, except that the location parameter has been set equal to 0.

Suppose now that we are interested in the probability distribution of y^2. The positive tail of the distribution of y^2 is related to the tails of the distribution of y in the following way:

$$Pr(y^2 > \hat{y}) = Pr(y > \hat{y}^{1/2}) + Pr(y < -[\hat{y}^{1/2}]) , \quad \hat{y} > 0 . \quad \text{(A15)}$$

But since the tails of the distribution of y follow an asymptotic form of the law of Pareto, for very large values of y this is just

$$Pr(y^2 > \hat{y}) \rightarrow (\hat{y}^{1/2}/U_1)^{-\alpha} + (\hat{y}^{1/2}/U_2)^{-\alpha} , \quad \hat{y} \rightarrow \infty . \quad \text{(A16)}$$

Substituting $C_1 = U_1^{\alpha}$ and $C_2 = U_2^{\alpha}$ into expression (A16) and simplifying we get

$$Pr(y^2 > \hat{y}) \rightarrow (C_1 + C_2)\, \hat{y}^{-(\alpha/2)} , \quad \text{(A17)}$$

which is a Paretian expression with exponent $\alpha' = \alpha/2$ and scale parameter $C'_1 = C_1 + C_2$.

The tail probabilities for the negative tail of the distribution of y^2 are, of course, all identically zero. This is equivalent to saying that the scale parameter, C'_2, in the Paretian expression for the negative tail of the distribution of y^2 is zero.

Let us now turn our attention to the distribution of sums of independent realizations of the variable y^2. Since y^2 is asymptotically Paretian, it satisfies the conditions of Doeblin and Gnedenko, and thus sums of y^2 will approach a stable Paretian distribution with characteristic exponent $\alpha' = \alpha/2$ and skewness

$$\beta' = \frac{C'_1 - C'_2}{C'_1 + C_2} = 1 . \quad \text{(A18)}$$

We know from previous discussions that, if the scale of the distribution of sums is to be the same as that of the distribution of $\hat{y}^2$, the sums must be scaled by $n^{-1/\alpha'} = n^{-2/\alpha}$, where n is the number of summands. Thus the distributions of

$$y^2 \text{ and } n^{-2/\alpha} \sum_{i=1}^{n} y^2 \quad \text{(A19)}$$

will be identical.

This discussion provides us with a way to analyze the distribution of the sample variance of the stable Paretian variable u. For values

[51] It is worth noting that although the scale of the distribution of sums expands with n at the rate $n^{1/\alpha}$, the scale parameter γ expands directly with n. Thus γ itself represents some more basic scale parameter raised to the power of α. For example, in the normal case ($\alpha = 2$) γ is related to the variance, but the variance is just the square of the standard deviation. The standard deviation, of course, is the more direct measure of the scale of the normal distribution.

of α less than 2, the population variance of the random variable u is infinite. The sample variance of n independent realizations of u is

$$S^2 = n^{-1} \sum_{i=1}^{n} y_i^2 . \qquad \text{(A20)}$$

This can be multiplied by $n^{-2/\alpha + 2/\alpha} = 1$ with the result

$$S^2 = n^{-1+2/\alpha} \left(n^{-2/\alpha} \sum_{i=1}^{n} y_i^2 \right) . \qquad \text{(A21)}$$

Now we know that the distribution of

$$n^{-2/\alpha} \sum_{i=1}^{n} y_i^2$$

is stable Paretian and independent of n. In particular, the median (or any other fractile) of this distribution has the same value for all n. This is not true, however, for the distribution of S^2. The median or any other fractile of the distribution of S^2 will grow in proportion to $n^{-1+2/\alpha}$. For example, if u_t is an independent, stable Paretian variable generated in time series, then the $.f$ fractile of the distribution of the cumulative sample variance of u_t at time t_1, as a function of the $.f$ fractile of the distribution of the sample variance at time t_0 is given by

$$S_1^2 = S_0^2 \left(\frac{n_1}{n_0} \right)^{-1+2/\alpha} , \qquad \text{(A22)}$$

where n_1 is the number of observations in the sample at time t_1, n_0 is the number at t, and S_1^2 and S_0^2 are the $.f$ fractiles of the distributions of the cumulative sample variances.

This result provides the basis for the sequential variance approach to estimating α discussed in Section IV, D of this paper.