Level Sets of Additive Lévy Processes[1]

Davar Khoshnevisan[2]
Department of Mathematics
University of Utah
155 South 1400 East JWB 233
Salt Lake City, UT 84112–0090

Yimin Xiao[3]
Department of Mathematics
University of Utah
155 South 1400 East JWB 233
Salt Lake City, UT 84112–0090
and
Microsoft Corporation
One Microsoft Way
Redmond, WA 98052

January 18, 2001

[1]Research partially supported by grants from NSF and NATO
[2]Email: davar@math.utah.edu, URL: http://www.math.utah.edu/~davar
[3]Email: xiao@math.utah.edu

Abstract

We provide a probabilistic interpretation of a class of natural capacities on Euclidean space in terms of the level sets of a suitably chosen multiparameter additive Lévy process X. We also present several probabilistic applications of the aforementioned potential-theoretic connections. They include areas such as intersections of Lévy processes and level sets, as well as Hausdorff dimension computations.

Keyword and Phrases Additive Stable Processes, Potential Theory
AMS 1991 Subject Classification Primary. 60G60; Secondary. 60J45

Contents

1 Introduction

An N-parameter, $\mathbb{R}^d$-valued, *additive Lévy process* $X = \{X(t);\ t \in \mathbb{R}_+^N\}$ is a multiparameter stochastic process that has the decomposition

$$X = X_1 \oplus \cdots \oplus X_N,$$

where $X_1, \ldots, X_N$ denote independent Lévy processes that take their values in $\mathbb{R}^d$. To put it more plainly,

$$X(t) = \sum_{j=1}^{N} X_j(t_j), \qquad t \in \mathbb{R}_+^N, \tag{1.1}$$

where t_i denotes the ith coordinate of $t \in \mathbb{R}_+^N$ $(i = 1, \ldots, N)$. These random fields naturally arise in the analysis of multiparameter processes such as Lévy's sheets. For example, see Dalang and Mountford [7, 8], Dalang and Walsh [9, 10], Kendall [33], Khoshnevisan [34], Khoshnevisan and Shi [35], Mountford [40] and Walsh [51], to cite only some of the references.

Our interest is in finding connections between the level sets of X and capacity in Euclidean spaces. In order to be concise, we shall next recall some formalism from geometric probability. See Matheron [38] and Stoyan [48] for further information and precise details. To any random set $\mathrm{K} \subset \mathbb{R}^d$, we assign a set function μ_{K} on $\mathbb{R}^d$ as follows:

$$\mu_{\mathrm{K}}(E) = \mathbb{P}\{\mathrm{K} \cap E \neq \varnothing\}, \qquad E \subset \mathbb{R}^d, \text{ Borel}, \tag{1.2}$$

and think of μ_{K} as the *distribution* of the random set K.

Let $X^{-1}(a) = \{t \in \mathbb{R}_+^N \backslash \{0\} :\ X(t) = a\}$ denote the level set of X at $a \in \mathbb{R}^d$. If $a = 0$, $X^{-1}(0)$ is also called the zero set of X. Our intention is to show that under some technical conditions on X, $\mu_{X^{-1}(a)}$ is mutually absolutely continuous with respect to $\mathsf{C}(E)$, where $\mathsf{C}(E)$ is a natural Choquet capacity of E that is explicitly determined by the dynamics of the stochastic process X. Our considerations also determine the Hausdorff dimension $\dim_{\mathsf{H}} X^{-1}(0)$ of the zero set of X, under very mild conditions.

In the one-parameter setting (i.e., when $d = N = 1$), the closure of $X^{-1}(a)$ is the range of a subordinator $S = \{S(t);\ t \geqslant 0\}$; cf. Fristedt [21]. Consequently, in the one-parameter setting, $\mu_{X^{-1}(a)}$ is nothing but the *hitting probability* for S. In particular, methods of probabilistic potential theory can be used to establish capacitary interpretations of the distribution of the level sets of X; see Bertoin [3], Fitzsimmons, Fristedt and Maisonneuve [18], Fristedt [20], and Hawkes [26] for a treatment of this and much more. Unfortunately, when $N > 1$, there are no known connections between $X^{-1}(a)$ and the range of other tractable stochastic processes. Nevertheless, using techniques from the potential theory of multiparameter processes, we show that when some technical conditions are met, the distribution of the level sets of additive Lévy processes do indeed have a potential-theoretic interpretation. Various aspects of the potential theory of multiparameter processes have been treated in Evans [16, 17], Fitzsimmons and Salisbury [19], Hawkes [23, 25], Hirsch [27], Hirsch and Song [28, 29], Khoshnevisan [34], Khoshnevisan and Shi [35], Peres [43], Ren [45] and Salisbury [46].

We conclude the Introduction with the following consequence of our main results that are Theorems 2.9, 2.10 and 2.12 of Section 2.

Theorem 1.1 *Suppose $X_1, \ldots, X_N$ are independent isotropic stable Lévy processes in $\mathbb{R}^d$ with index $\alpha \in]0,2]$ and $X = X_1 \oplus \cdots \oplus X_N$. Then,*

(i) $\mathbb{P}\{X^{-1}(0) \neq \varnothing\} > 0$ *if and only if* $N\alpha > d$; *and*

(ii) if $N\alpha > d$, *then* $\mathbb{P}\big\{\dim_{\mathrm{H}} X^{-1}(0) = N - \frac{d}{\alpha}\big\} > 0$.

Furthermore, for each $M > 1$, there exists a constant $A > 1$, such that simultaneously for all compact sets $E \subset [M^{-1}, M]^N$, and for all $a \in [-M, M]^d$,

$$\frac{1}{A}\,\mathsf{Cap}_{d/\alpha}(E) \leqslant \mu_{X^{-1}(a)}(E) \leqslant A\,\mathsf{Cap}_{d/\alpha}(E),$$

where $\mathsf{Cap}_\beta(E)$ denotes the Riesz–Bessel capacity of E, of index β.

We recall that for all $\beta > 0$,

$$\mathsf{Cap}_\beta(E) = \Big\{ \inf_{\mu \in \mathcal{P}(E)} \int\int |s-t|^{-\beta}\ \mu(ds)\ \mu(dt) \Big\}^{-1}, \tag{1.3}$$

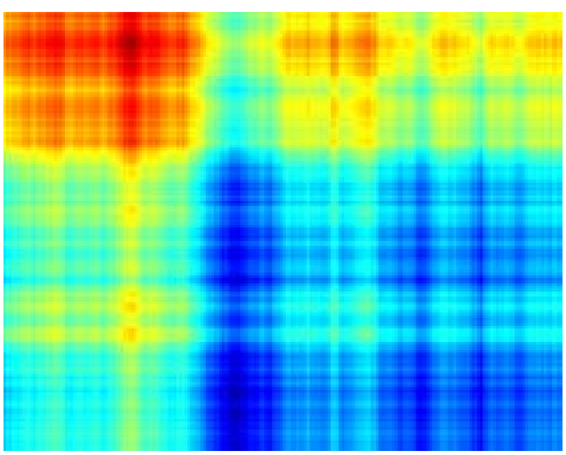

Figure 1: The Zero Set of Additive Brownian Motion

where $\mathcal{P}(E)$ denote the collection of all probability measures on the Borel set $E \subset \mathbb{R}^N_+$ and $|t| = \max_{1 \leqslant j \leqslant N} |t_j|$ denotes the ℓ^∞-norm on $\mathbb{R}^N$. We shall prove this theorem in Section 2 below.

To illustrate some of the irregular features of the level sets in question, we include a simulation of the zero set of $X = X_1 \oplus X_2$, where X_1 and X_2 are independent, linear Brownian motions; cf. Figure 1. In this simulation, the darkest shade of gray represents the set $\{(s,t) : X_1(s) + X_2(t) < 0\}$, while the medium shade of gray represents the collection $\{(s,t) : X_1(s) + X_2(t) > 0\}$. The respective "boundaries" of these two extreme shades together reveal the rather irregular zero set $X^{-1}(0)$.

Throughout this paper, for any $c \in \mathbb{R}_+$, $\mathbf{c}$ denotes the N-dimensional vector $(c, \ldots, c)$ and for any integer $k \geqslant 1$ and any $x \in \mathbb{R}^k$, $|x| = \max_{1 \leqslant \ell \leqslant k} |x_\ell|$ and $\|x\| = \{\sum_{\ell=1}^k x_\ell^2\}^{1/2}$ denote the ℓ^∞ and ℓ^2 norms on $\mathbb{R}^k$, respectively.

The remainder of this paper is organized as follows. In Section 2, after presenting some preliminary results, we state our main Theorems 2.9, 2.10 and 2.12. We then prove the announced Theorem 1.1. Theorem 2.9 is proved in Section 3, while the proof of Theorem 2.10 is given in Section 4. In Section 5 we prove Theorem 2.12. Our main arguments depend heavily upon tools from multiparameter martingale theory. In Section 6, we establish some of the other consequences of Theorems 2.9, 2.12 and 2.10 together with their further connection to the existing literature.

2 Preliminaries and the Statement of the Main Results

Throughout, d and N represent the spatial and temporal dimensions, respectively. The N-dimensional "time" space $\mathbb{R}_+^N$ can be partially ordered in various ways. The most commonly used partial order on $\mathbb{R}_+^N$ is $\preccurlyeq$, where $s \preccurlyeq t$ if and only if $s_i \leqslant t_i$, for all $1 \leqslant i \leqslant N$. This partial order induces a minimum operation: $s \curlywedge t$ denotes the element of $\mathbb{R}_+^N$ whose ith coordinate is $s_i \wedge t_i$, for all $1 \leqslant i \leqslant N$. For $s, t \in \mathbb{R}_+^N$ and $s \preccurlyeq t$, we write $[s, t] = [s_1, t_1] \times \cdots \times [s_N, t_N]$.

Concerning the source of randomness, we let $X_1, \ldots, X_N$ denote N independent $\mathbb{R}^d$-valued Lévy processes and define $X = X_1 \oplus \cdots \oplus X_N$; see Eq. (1.1) for the precise definition. Recall that for each $1 \leqslant j \leqslant N$, the Lévy process X_j is said to be *symmetric*, if $-X_j$ and X_j have the same finite-dimensional distributions. In such a case, by the Lévy–Khintchine formula, there exists a *nonnegative* function (called the *Lévy exponent* of X_j) $\Psi_j : \mathbb{R}^d \to \mathbb{R}_+$, such that for all $t \geqslant 0$,

$$\mathbb{E}\big[\exp\{i\xi \cdot X_j(t)\}\big] = \exp\{-t\Psi_j(\xi)\}, \qquad \xi \in \mathbb{R}^d.$$

In particular, if $\Psi_j(\xi) = \chi_j \|\xi\|^\alpha$ for some constant $\chi_j > 0$, X_j is said to be an *isotropic* stable process with index α.

We say that the process X_j is *absolutely continuous*, if for all $t > 0$, the function $\xi \mapsto e^{-t\Psi_j(\xi)}$ is in $L^1(\mathbb{R}^d)$. In this case, by the inversion formula, the random vector $X_j(t)$ has the following probability density function:

$$p_j(t; x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\xi\cdot x} \exp\big\{-t\Psi_j(\xi)\big\}\, d\xi, \qquad t > 0,\ x \in \mathbb{R}^d.$$

In all but a very few special cases, there are no known explicit formulæ for $p_j(t; x)$. The following folklore lemma gives some information about the behavior of $p_j(t; x)$ and follows immediately from the above representation.

Lemma 2.1 *Suppose X_j is symmetric and absolutely continuous. Let Ψ_j denote the Lévy exponent of X_j and $p_j(t; \bullet)$ the density function of $X_j(t)$. Then,* *(i) for all $t > 0$ and all $x \in \mathbb{R}^d$,*

$$p_j(t;x) \leqslant p_j(t;0) = (2\pi)^{-d} \int_{\mathbb{R}^d} \exp\big\{ - t\Psi_j(\xi)\big\}\ d\xi;$$

(ii) $t \mapsto p_j(t;0)$ is nonincreasing; and

(iv) if $E \subset]0,\infty[$ and $K \subset \mathbb{R}^d$ are both compact, $E \otimes K \ni (t,x) \mapsto p_j(t,x)$ is uniformly continuous.

For each $t \in \mathbb{R}^N_+$, the characteristic function of $X(t)$ is given by

$$\begin{aligned}\mathbb{E}[\exp\{i\xi \cdot X(t)\}] &= \exp\Big\{ - \sum_{j=1}^N t_j \Psi_j(\xi)\Big\} \\ &= \exp\{-t \cdot \Psi(\xi)\}, \qquad \xi \in \mathbb{R}^d.\end{aligned}$$

where $\Psi(\xi) = \Psi_1(\xi) \otimes \cdots \otimes \Psi_N(\xi)$, in tensor notation. We will call $\Psi(\xi)$ the *characteristic exponent* of the additive Lévy process X, and say that the additive Lévy process X is *absolutely continuous* if for each $t \in \mathbb{R}^N_+ \backslash \partial\mathbb{R}^N_+$, where $\partial\mathbb{R}^N_+$ denotes the boundary of $\mathbb{R}^N_+$, the function $\xi \mapsto \exp\big\{ - t \cdot \Psi(\xi)\big\} \in L^1(\mathbb{R}^d)$. In this case, for every $t \in \mathbb{R}^N_+ \backslash \partial\mathbb{R}^N_+$, $X(t)$ has a density function $p(t;\bullet)$ that is given by the formula

$$p(t;x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\xi\cdot x} \exp\Big\{ - \sum_{j=1}^N t_j \Psi_j(\xi)\Big\}\ d\xi, \qquad x \in \mathbb{R}^d. \tag{2.1}$$

Clearly, if at least one of the X_j's are symmetric and absolutely continuous, then the additive Lévy process X is also absolutely continuous. However, there are many examples of non-absolutely continuous Lévy processes $X_1, \ldots, X_N$, such that the associated N-parameter additive Lévy process $X = X_1 \oplus \cdots \oplus X_N$ *is* absolutely continuous. Below, we record the following additive analogue of Lemma 2.1.

Lemma 2.2 *Let $X_1, \ldots, X_N$ be N independent symmetric Lévy processes and let $X = X_1 \oplus \cdots \oplus X_N$. Suppose X is absolutely continuous, and let $p(t;\bullet)$ denote the density of $X(t)$ for each $t \in \mathbb{R}^N_+$. Then,*

(i) for all $t \in \mathbb{R}^N_+ \backslash \partial\mathbb{R}^N_+$ and all $x \in \mathbb{R}^d$, $p(t;x) \leqslant p(t;0)$;

(ii) $t \mapsto p(t;0)$ is nonincreasing with respect to the partial order $\preccurlyeq$; and

(iii) if $E \subset]0,\infty[^N$ and $K \subset \mathbb{R}^d$ are both compact, then $E\otimes K \ni (t,x) \mapsto p(t;x)$ is uniformly continuous.

We say that an $\mathbb{R}^d$-valued random variable Y is *κ-weakly unimodal* if there exists a positive constant κ such that for all $a \in \mathbb{R}^d$ and all $r > 0$,

$$\mathbb{P}\{|Y-a| \leqslant r\} \leqslant \kappa \mathbb{P}\{|Y| \leqslant r\}. \tag{2.2}$$

Throughout much of this paper, we will assume the existence of a fixed κ such that the distribution of $X(t)$ is κ-weakly unimodal for all $t \in \mathbb{R}_+^N \setminus \partial\mathbb{R}_+^N$. If and when this is so, we say that the process X is *weakly unimodal*, for brevity.

We now state some remarks in order to shed some light on this weak unimodality property.

Remark 2.3 By a well known result of Anderson (cf. [1, Th. 1]), if the density function $p(t, x)$ of $X(t)$ $(t \in \mathbb{R}_+^N)$ is symmetric unimodal in the sense that (i) $p(t, x) = p(t, -x)$; and (ii) $\{x : \ p(t, x) \geqslant u\}$ is convex for every u $(0 < u < \infty)$, then, the inequality (2.2) holds with $Y = X(t)$ and $\kappa = 1$. In particular, any nondegenerate, centered Gaussian random vector satisfies these conditions. Using this fact, together with Bochner's subordination, we can deduce that whenever $X = \{X(t); \ t \in \mathbb{R}_+^N\}$ is an additive isotropic stable Lévy process of index $\alpha \in]0, 2]$ (i.e., whenever each X_j is an isotropic stable Lévy process), the density function of $X(t)$ is symmetric unimodal for each $t \in \mathbb{R}_+^N \setminus \{0\}$. In particular, when X is an isotropic stable Lévy process, Eq. (2.2) holds with $\kappa = 1$.□

Remark 2.4 Our definition of weak unimodality is closely related to that of unimodal distribution functions. Recall that a distribution function $F(x)$ on $\mathbb{R}$ is said to be *unimodal* with mode m, if $F(x)$ is convex on $(-\infty, m)$ and concave on (m, ∞). For a multivariate distribution function $F(x), (x \in \mathbb{R}^d)$, there are several different ways of defining unimodality of $F(x)$ such as symmetric unimodality in the sense of Anderson given above and symmetric unimodality in the sense of Kanter; see Kanter [32] or Wolfe [53]. We refer to Wolfe [54] for a survey of the various definitions of unimodality and related results. □

Remark 2.5 Some general conditions for the unimodality of infinitely divisible distributions are known. In this and the next remark (Remark 2.6 below), we cite two of them for the class of self-decomposable distributions.

Recall that a d-dimensional distribution function $F(x)$ is called *self-decomposable,* or of *class L*, if there exists a sequence of independent $\mathbb{R}^d$-valued random variables $\{Y_n\}$ such that for suitably chosen positive numbers $\{a_n\}$ and vectors $\{b_n\}$ in $\mathbb{R}^d$, the distribution functions of the random variables $a_n \sum_{i=1}^n Y_i + b_n$ converge weakly to $F(x)$, and for every $\varepsilon > 0$,

$$\lim_{n \to 0} \max_{1 \leqslant i \leqslant n} \mathbb{P}\big\{a_n |Y_i| \geqslant \varepsilon\big\} = 0.$$

It is well known that $F(x)$ is self-decomposable if and only if for every $a \in (0, 1)$, there exists a distribution function $G_a(x)$ on $\mathbb{R}^d$ such that $\widehat{F}(\xi) = \widehat{F}(a\xi) \, \widehat{G}_a(\xi)$, where $\widehat{H}$ denotes the Fourier transform of H. This result, for $d = 1$, is due to Lévy [36]. It is extended to higher dimensions in Sato [47]; see also Wolfe [53]. From this it follows readily that convolutions of self-decomposable distribution functions are also self-decomposable. Sato [47] also proves in his Theorem 2.3 that all stable distributions on $\mathbb{R}^d$ are self-decomposable. □

Remark 2.6 YAMAZATO [55] proves that all self-decomposable distribution functions on $\mathbb{R}$ are unimodal. For $d > 1$, WOLFE [53] proves that every d-dimensional symmetric self-decomposable distribution function is unimodal in the sense of KANTER [32]. In particular, every symmetric–though not necessarily isotropic–stable distribution on $\mathbb{R}^d$ is symmetric unimodal. We should also mention that MEDGYESSY [39] and WOLFE [52] give a necessary and sufficient condition for symmetric infinitely divisible distributions in $\mathbb{R}$ to be unimodal in terms of their Lévy measures. Their class is strictly larger than the class of self-decomposable distributions. $\square$

We now apply these results to derive weak unimodality for the distribution of a symmetric additive Lévy process $X = \{X(t);\ t \in \mathbb{R}_+^N\}$ in $\mathbb{R}^d$.

Remark 2.7 Suppose that for all $t \in \mathbb{R}_+^N \setminus \partial\mathbb{R}_+^N$, the distribution of $X(t)$ is self-decomposable, e.g., this holds whenever the distribution of $X_j(t_j)$ is self-decomposable for every $j \geqslant 1$ and for all $t_j > 0$. According to Remarks 2.4 and 2.6, the distribution of $X(t)$ is also symmetric unimodal, in the usual sense, when $d = 1$. Furthermore, when $d > 1$, the distribution of $X(t)$ is symmetric unimodal in the sense of Kanter [32]. Now, by the proof of Theorem 1 of ANDERSON [1], we can see that for all $t \in \mathbb{R}_+^N \setminus \partial\mathbb{R}_+^N$, all $a \in \mathbb{R}^d$ and all $r > 0$, (2.2) holds with $\kappa = 1$. In particular, every symmetric–though not necessarily isotropic–additive stable Lévy process $X = X_1 \oplus \cdots \oplus X_N$ satisfies weak unimodality (2.2) with $\kappa = 1$. $\square$

In all cases known to us, weak unimodality holds with $\kappa = 1$; cf. Eq. (2.2). However, it seems plausible that in some cases, Eq. (2.2) holds for some $\kappa > 1$. This might happen when the distribution of the process X is not symmetric unimodal. As we have been unable to resolve when $\kappa > 1$, our formulation of weak unimodality is stated in its current form for maximum generality.

Under the condition of weak unimodality, we can prove the following useful technical lemma.

Lemma 2.8 *Let $X = X_1 \oplus \cdots \oplus X_N$ be an additive, weakly unimodal Lévy process. Then,*

(i) [Weak Regularity] For all $t \in \mathbb{R}_+^N \setminus \partial\mathbb{R}_+^N$ and all $r > 0$,

$$\mathbb{P}\{|X(t)| \leqslant 2r\} \leqslant \kappa 2^d \mathbb{P}\{|X(t)| \leqslant r\};$$

(ii) [Weak Monotonicity] For all $s, t \in \mathbb{R}_+^N \setminus \partial\mathbb{R}_+^N$ with $s \preccurlyeq t$,

$$\mathbb{P}\{|X(t)| \leqslant r\} \leqslant \kappa \mathbb{P}\{|X(s)| \leqslant r\}.$$

In the analysis literature, our notion of weak regularity is typically known as *volume doubling* for the law of $\big|X(t)\big|$.

Proof To prove weak regularity, let $\mathbb{B}(x; r) = \{y \in \mathbb{R}^d : |y - x| \leqslant r\}$ and find $a_1, \ldots, a_{2^d} \in [0, 2r]^d$, such that

(i) the interiors of $\mathbb{B}(a_\ell; r)$'s are disjoint, as ℓ varies in $\{1, \ldots, 2^d\}$; and

(ii) $\cup_{\ell=1}^{2^d} \mathbb{B}(a_\ell; r) = \mathbb{B}(0; 2r)$.

Applying weak unimodality,

$$\begin{aligned}\mathbb{P}\{|X(t)| \leqslant 2r\} &\leqslant \sum_{\ell=1}^{2^d} \mathbb{P}\{|X(t) - a_\ell| \leqslant r\} \\ &\leqslant \kappa 2^d \mathbb{P}\{|X(t)| \leqslant r\}.\end{aligned}$$

To prove weak monotonicity, we fix $s, t \in \mathbb{R}_+^N$ with $s \preccurlyeq t$. Then,

$$\begin{aligned}\mathbb{P}\{|X(t)| \leqslant r\} &= \mathbb{P}\{|X(s) + (X(t) - X(s))| \leqslant r\} \\ &\leqslant \kappa \mathbb{P}\{|X(s)| \leqslant r\},\end{aligned}$$

where the inequality follows from the independence of $X(s)$ and $X(t) - X(s)$ and weak unimodality. This concludes our proof. □

The following function Φ plays a central rôle in our analysis of the process X:

$$\Phi(s) = p(\overline{s}; 0), \qquad s \in \mathbb{R}^N, \tag{2.3}$$

where $\overline{s}$ is the element of $\mathbb{R}_+^N$, whose ith coordinate is $|s_i|$. Clearly, $s \mapsto \Phi(s)$ is nonincreasing in each $|s_i|$ and, equally clearly, $\Phi(0) = +\infty$. We will say that Φ is the *gauge function* for the multiparameter process X. Corresponding to the gauge function Φ, we may define the Φ-*capacity* of a Borel set $E \subset \mathbb{R}_+^N$ as

$$\mathsf{C}_\Phi(E) = \Big\{ \inf_{\mu \in \mathcal{P}(E)} \int\!\!\int \Phi(s-t)\ \mu(ds)\ \mu(dt) \Big\}^{-1}, \tag{2.4}$$

where $\mathcal{P}(E)$ denotes the collection of all probability measures on E. For any $\mu \in \mathcal{P}(E)$, we define the Φ-*energy* of μ by

$$\mathcal{E}_\Phi(\mu) = \int\!\!\int \Phi(s-t)\ \mu(ds)\ \mu(dt). \tag{2.5}$$

Thus, the Φ-capacity of E is defined by the principle of minimum energy:

$$\mathsf{C}_\Phi(E) = \big\{ \inf_{\mu \in \mathcal{P}(E)} \mathcal{E}_\Phi(\mu) \big\}^{-1}.$$

It is not hard to see that C_Φ is a capacity in the sense of G. Choquet; cf. Bass [2] and Dellacherie and Meyer [11].

We are ready to state the main results of this paper. We denote $\mathrm{Leb}(A)$ the d-dimensional Lebesgue measure of the Lebesgue measurable set $A \subset \mathbb{R}^d$.

Theorem 2.9 *Let $X_1, \ldots, X_N$ be N independent symmetric Lévy processes on $\mathbb{R}^d$ and let $X = X_1 \oplus \cdots \oplus X_N$. Suppose X is absolutely continuous and weakly unimodal. If Φ denotes the gauge function of X, the following are equivalent:*

(i) $\Phi \in L^1_{loc}(\mathbb{R}^N)$*;*

(ii) $\mathbb{P}\{\text{Leb}\{X([c,\infty[^N)\} > 0\} = 1$*, for all* $c > 0$*;*

(iii) $\mathbb{P}\{\text{Leb}\{X([c,\infty[^N)\} > 0\} > 0$*, for all* $c > 0$*;*

(iv) $\mathbb{P}\{\text{Leb}\{X([c,\infty[^N)\} > 0\} > 0$*, for some* $c > 0$*;*

(v) $\mathbb{P}\{X^{-1}(0) \cap [c,\infty[^N \neq \varnothing\} > 0$*, for all* $c > 0$*;*

(vi) $\mathbb{P}\{X^{-1}(0) \cap [c,\infty[^N \neq \varnothing\} > 0$*, for some* $c > 0$*.*

When $X^{-1}(0) \neq \varnothing$, it is of interest to determine its Hausdorff dimension. Our next theorem provides upper and lower bounds for $\dim_{\mathsf{H}} X^{-1}(0)$ in terms of the following two indices associated to the gauge function Φ:

$$\overline{\gamma} = \inf\big\{\beta > 0 : \liminf_{s \to 0} \|s\|^{N-\beta}\Phi(s) > 0\big\},$$

$$\gamma = \sup\Big\{\beta > 0 : \int_{[0,1]^N} \frac{1}{\|s\|^\beta}\ \Phi(s)\ ds < \infty\Big\}.$$

It is easy to verify that $0 \leqslant \gamma \leqslant \overline{\gamma} \leqslant N$.

Henceforth, $\|\mathbf{s}\|$ designates the N-dimensional vector $(\|s\|, \ldots, \|s\|)$.

Theorem 2.10 *Given the conditions of Theorem 2.9, for any* $0 < c < C < \infty$*,*

$$\mathbb{P}\big\{\gamma \ \leqslant \dim_{\mathsf{H}}(X^{-1}(0) \cap [c,C]^N) \leqslant \ \overline{\gamma}\big\} > 0. \tag{2.6}$$

Moreover, if there exists a constant $K_1 > 0$ *such that*

$$\Phi(s) \leqslant \Phi(K_1\|\mathbf{s}\|) \qquad \text{for all} \quad s \in [0,1]^N, \tag{2.7}$$

then, $\mathbb{P}\{\dim_{\mathsf{H}}(X^{-1}(0) \cap [c,C]^N) = \gamma\} > 0$*.*

Remark 2.11 Clearly, if $X_1, \ldots, X_N$ have the same Lévy exponent, then (2.7) holds.

Our next theorem further relates the distribution of the level sets of an additive Lévy process to Φ-capacity.

Theorem 2.12 *Given the conditions of Theorem 2.9, for every* $c > 0$*, all compact sets* $E \subset [c,\infty[^N$ *and for all* $a \in \mathbb{R}^d$*,*

$$A_1 \sup_{\mu \in \mathcal{P}(E)} \frac{[\int p(s;a)\ \mu(ds)]^2}{\mathcal{E}_\Phi(\mu)} \leqslant \mu_{X^{-1}\{a\}}(E) \leqslant A_2 \mathsf{C}_\Phi(E), \tag{2.8}$$

where $A_1 = \kappa^{-2}2^{-d}\big\{\Phi(\mathbf{c})\big\}^{-1}$ *and* $A_2 = \kappa^3 2^{5d+3N}\Phi(\mathbf{c})$*.*

The following is an immediate, but useful, corollary.

Corollary 2.13 *Given the conditions and the notation of Theorem 2.12, for all $a \in \mathbb{R}^d$ and for all compact sets $E \subset [c, \infty[^N$,*

$$A_1 \mathsf{C}_\Phi(E) \leqslant \mu_{X^{-1}(a)}(E) \leqslant A_2 \mathsf{C}_\Phi(E),$$

where $A_1 = \kappa^{-2} 2^{-d} \big\{\Phi(\mathbf{c})\big\}^{-1} I_E^2(a)$, $A_2 = \kappa^3 2^{5d+3N} \Phi(\mathbf{c})$ and $I_E(a) = \inf_{s\in E} p(s;a)$.

Applying Lemma 2.2(*iii*), we can deduce that there exists an open neighborhood G of 0 (that may depend on E), such that for all $a \in G$, $I_E(a) > 0$. In particular, $\mu_{X^{-1}(0)}(E)$ is bounded above and below by nontrivial multiples of $\mathsf{C}_\Phi(E)$.

We can now use Theorems 2.9, 2.10 and 2.12 to prove Theorem 1.1.

Proof of Theorem 1.1 Note that for all $t \in \mathbb{R}_+^N$ and all $\xi \in \mathbb{R}^d$,

$$\mathbb{E}\big[\exp\{i\xi \cdot X(t)\}\big] = \exp\Big\{ -\sum_{j=1}^N t_j \ \chi_j \ \|\xi\|^\alpha \Big\}.$$

By (2.1), for all $t \in \mathbb{R}^N$,

$$\begin{aligned}\Phi(t) &= (2\pi)^{-d} \int_{\mathbb{R}^d} \exp\Big\{ -\sum_{j=1}^N |t_j| \ \chi_j \ \|\xi\|^\alpha \Big\} \, d\xi \\ &= \lambda\Big(\sum_{j=1}^N |t_j| \chi_j\Big)^{-d/\alpha},\end{aligned}$$

where $\lambda = (2\pi)^{-d} \int_{\mathbb{R}_+^d} e^{-\|\zeta\|^\alpha} \, d\zeta$. In particular,

$$\lambda N^{-d/\alpha} \overline{\chi}^{\,-d/\alpha} |t|^{-d/\alpha} \leqslant \Phi(t) \leqslant \lambda \underline{\chi}^{-d/\alpha} |t|^{-d/\alpha}, \tag{2.9}$$

where $\overline{\chi} = \max\{\chi_1, \ldots, \chi_N\}$ and $\underline{\chi} = \min\{\chi_1, \ldots, \chi_N\}$, respectively. Consequently, $\Phi \in L^1_{loc}(\mathbb{R}^N)$ if and only if $N\alpha > d$; and $\gamma = \overline{\gamma} = N - d/\alpha$. Hence, the first two assertions of Theorem 1.1 follow from Theorems 2.9 and 2.10 respectively. We also have

$$\lambda^{-1} \underline{\chi}^{d/\alpha} \mathsf{Cap}_{d/\alpha}(E) \leqslant \mathsf{C}_\Phi(E) \leqslant \lambda^{-1} N^{d/\alpha} \overline{\chi}^{\,d/\alpha} \mathsf{Cap}_{d/\alpha}(E).$$

In light of Corollary 2.13, it remains to show that

$$\inf_{a\in[-M,M]^d} \inf_{s\in[M^{-1},M]^N} p(s;a) > 0.$$

This follows from Taylor [50]. ☐

3 Proof of Theorem 2.9

We prove Theorem 2.9 by demonstrating the following Propositions 3.1 and 3.2.

Proposition 3.1 *Under the conditions of Theorem 2.9, the following are equivalent:*

(i) $\mathsf{C}_\Phi([0,1]^N) > 0$;

(ii) $\mathbb{P}\{X^{-1}(0) \cap [c,\infty[^N \neq \varnothing\} > 0$, *for all* $c > 0$;

(iii) $\mathbb{P}\{X^{-1}(0) \cap [c,\infty[^N \neq \varnothing\} > 0$, *for some* $c > 0$;

(vi) $\Phi \in L^1_{loc}(\mathbb{R}^N)$.

Moreover, given any constants $0 < c < C < \infty$, *then for all* $a \in \mathbb{R}^d$,

$$\begin{aligned} \kappa^{-2} 2^{-d} \big\{\Phi(\mathbf{c})\big\}^{-1} \frac{[\int_{[c,C]^N} p(s;a)\ ds]^2}{\mathcal{E}_\Phi(\mathrm{Leb})} &\leqslant \mu_{X^{-1}\{a\}}([c,C]^N) \\ &\leqslant \kappa^5 2^{3d+6N} \Phi(\mathbf{c}) \mathsf{C}_\Phi([c,C]^N). \end{aligned} \tag{3.1}$$

Proposition 3.1 rigorously verifies the folklore statement that the "equilibrium measure" corresponding to sets of the form $[c,C[^N$ is, in fact, the normalized Lebesgue measure. It is sometimes possible to find direct analytical proofs of this statement. For example, suppose that the gauge function Φ is a radial function of form $f(|s-t|)$, where f is decreasing. Then, the analytical method of Pemantle et al. [42] can be used to give an alternative proof that the equilibrium measure is Lebesgue's measure. In general, we only know one probabilistic proof of this fact.

Proposition 3.2 *Under the conditions of Theorem 2.9, the following are equivalent:*

(i) $\mathsf{C}_\Phi([0,1]^N) > 0$;

(ii) $\mathbb{P}\{\mathrm{Leb}\{X([c,\infty[^N)\} > 0\} = 1$, *for all* $c > 0$;

(iii) $\mathbb{P}\{\mathrm{Leb}\{X([c,\infty[^N)\} > 0\} > 0$, *for all* $c > 0$;

(iv) $\mathbb{P}\{\mathrm{Leb}\{X([c,\infty[^N)\} > 0\} > 0$, *for some* $c > 0$.

In order to prove Proposition 3.1, we first prove that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i)$. We call this the *first part* of the proof; this is given in Subsection 3.1. The asserted capacitary estimates in (3.1)–the *second part* of Proposition 3.1–will be demonstrated in Subsection 3.2. All of these are achieved in a sequence of lemmas that we will prove in the next two subsections. Finally, we prove Proposition 3.2 in Subsection 3.3.

We shall have need for some notation. For all $i = 1, \dots, k$, $\mathcal{F}_i = \{\mathcal{F}_i(t);\ t \geqslant 0\}$ denotes the complete, right-continuous filtration generated by the process X_i. We can define the N-parameter filtration $\mathcal{F} = \{\mathcal{F}(t);\ t \in \mathbb{R}_+^N\}$ as

$$\mathcal{F}(t) = \bigvee_{i=1}^{N} \mathcal{F}_i(t_i), \qquad t = (t_1, \dots, t_N) \in \mathbb{R}_+^N.$$

Then, $\mathcal{F} = \{\mathcal{F}(t);\ t \in \mathbb{R}_+^N\}$ satisfies Condition (F4) of Cairoli and Walsh (cf. [6, 51]).

3.1 Proof of the First Part of Proposition 3.1

We now start our proof of the first part by demonstrating that assertion (i) of Proposition 3.1 implies (ii). We first note that $\mathsf{C}_\Phi([0,1]^N) > 0$ implies $\mathsf{C}_\Phi([0,T]^N) > 0$ for all $T > 1$. To see this directly, we assume that σ is a probability measure on $[0,1]^N$ such that

$$\int_{[0,1]^N} \int_{[0,1]^N} \Phi(s-t)\ \sigma(ds)\ \sigma(dt) < \infty, \tag{3.2}$$

and let $\mu = \mu_T$ be the image measure of σ under the mapping $s \mapsto Ts$. Then, μ is a probability measure on $[0,T]^N$. It follows from Lemma 2.2(ii), and from Eq. (3.2), that

$$\int_{[0,T]^N} \int_{[0,T]^N} \Phi(s-t)\ \mu(ds)\ \mu(dt) < \infty. \tag{3.3}$$

Hence, $\mathsf{C}_\Phi([0,T]^N) > 0$. Eq. (3.3) also implies that

$$\sum_{m_1=1}^{\infty} \cdots \sum_{m_N=1}^{\infty} \int_{[0,T]^N} \int_{\mathcal{A}(s)} \Phi(s-t)\ \mu(ds)\ \mu(dt) < \infty,$$

where for all $s \in \mathbb{R}_+^N$, $\mathcal{A}(s)$ designates the annulus,

$$\mathcal{A}(s) = \{t \in \mathbb{R}^N :\ 2^{-m_j} < |t_j - s_j| \leqslant 2^{-m_j+1},\ \text{for all } 1 \leqslant j \leqslant N\}.$$

Thus, for each $j = 1, \dots, N$, we can find an increasing sequence of positive numbers $\{a_{m,j}\}_{m=1}^{\infty}$, such that $\lim_{m\to\infty} a_{m,j} = +\infty$ and

$$\sum_{m_1=1}^{\infty} \cdots \sum_{m_N=1}^{\infty} \prod_{\ell=1}^{N} a_{m_\ell,\ell} \cdot \int_{[0,T]^N} \mu(ds) \int_{\mathcal{A}(s)} \Phi(s-t)\ \mu(dt) < \infty. \tag{3.4}$$

We define N decreasing continuous functions $\varrho_j : (0,\infty) \to [1,\infty)$ such that $\varrho_j(2^{-m}) = a_{m,j}$ and the function $\bar{\varrho} : \mathbb{R}^N \to [1,\infty)$ by $\bar{\varrho}(s) = \prod_{j=1}^N \varrho_j(|s_j|)$. Clearly, for every $s_0 \in \mathbb{R}^N$ with $\bar{s}_0 \in \partial\mathbb{R}_+^N$, we have $\lim_{s\to s_0} \bar{\varrho}(s) = \infty$ and (3.4) implies

$$\int_{[0,T]^N} \int_{[0,T]^N} \bar{\varrho}(s-t)\Phi(s-t)\ \mu(ds)\ \mu(dt) < \infty. \tag{3.5}$$

For each $\varepsilon > 0$, $T > 1$ and for the probability measure μ of Eq. (3.3), we define a random measure $\mathcal{J}_{\varepsilon,T}$ on $[1,T]^N$ by

$$\mathcal{J}_{\varepsilon,T}(B) = (2\varepsilon)^{-d} \int_B \mathbb{1}\big\{|X(s)| \leqslant \varepsilon\big\}\ \mu(ds), \tag{3.6}$$

where $B \subseteq [1,T]^N$ denotes an arbitrary Borel set. (It may help to recall that $|x| = \max_{1 \leqslant j \leqslant d} |x_j|$ denotes the ℓ^∞ norm of $x \in \mathbb{R}^d$.) We will denote the total mass $\mathcal{J}_{\varepsilon,T}([1,T]^N)$ of this random measure by $\|\mathcal{J}_{\varepsilon,T}\|$.

The following lemma is an immediate consequence of Lemma 2.2 and the dominated convergence theorem.

Lemma 3.3 *For any $T > 1$,*

$$\lim_{\varepsilon \to 0+} \mathbb{E}\big\{\|\mathcal{J}_{\varepsilon,T}\|\big\} = \int_{[1,T]^N} \Phi(s)\ \mu(ds).$$

Next, we consider the energy of $\mathcal{J}_{\varepsilon,T}$ with respect to the kernel $\bar{\varrho}$ and state a second moment bound for $\|\mathcal{J}_{\varepsilon,T}\|$.

Lemma 3.4 *Suppose $K : \mathbb{R}_+^N \times \mathbb{R}_+^N \to \mathbb{R}_+$ is a measurable function. For any $T > 1$ and for all $\varepsilon > 0$,*

$$\begin{aligned} &\mathbb{E}\Big\{\int_{[1,T]^N}\int_{[1,T]^N} K(s,t)\ \mathcal{J}_{\varepsilon,T}(ds)\ \mathcal{J}_{\varepsilon,T}(dt)\Big\} \\ &\qquad \leqslant \kappa^2 \Phi(\mathbf{1}) \varepsilon^{-d} \times \\ &\qquad\qquad \times \int_{[1,T]^N}\int_{[1,T]^N} K(s,t) \mathbb{P}\{|X(s) - X(t)| \leqslant \varepsilon\}\ \mu(ds)\ \mu(dt). \end{aligned} \tag{3.7}$$

In particular,

$$\begin{aligned} &\mathbb{E}\Big\{\int_{[1,T]^N}\int_{[1,T]^N} K(s,t)\ \mathcal{J}_{\varepsilon,T}(ds)\ \mathcal{J}_{\varepsilon,T}(dt)\Big\} \\ &\qquad \leqslant \kappa^2 2^d \Phi(\mathbf{1}) \cdot \int_{[1,T]^N}\int_{[1,T]^N} K(s,t) \Phi(s-t)\ \mu(ds)\ \mu(dt). \end{aligned} \tag{3.8}$$

Proof Recalling Lemma 2.2(i), and the fact that $|x| = \max_j |x_j|$, we obtain

$$\mathbb{P}\{|X(s)| \leqslant \varepsilon\} \leqslant (2\varepsilon)^d \Phi(s).$$

Thus, Eq. (3.8) indeed follows from (3.7). Hence, we only need to verify (3.7). By Fubini's Theorem,

$$\begin{aligned} &\mathbb{E}\Big\{\int_{[1,T]^N}\int_{[1,T]^N} K(s,t)\ \mathcal{J}_{\varepsilon,T}(ds)\ \mathcal{J}_{\varepsilon,T}(dt)\Big\} \\ &\quad = (2\varepsilon)^{-2d} \int_{[1,T]^N}\int_{[1,T]^N} K(s,t) \mathbb{P}\{|X(s)| \leqslant \varepsilon\,,\ |X(t)| \leqslant \varepsilon\}\ \mu(ds)\ \mu(dt). \end{aligned}$$

We define

$$\begin{aligned} Z_1 &= X(s) - X(s \curlywedge t) \\ Z_2 &= X(t) - X(s \curlywedge t). \end{aligned}$$

Clearly,

$$\begin{aligned} &\mathbb{P}\{|X(s)| \leqslant \varepsilon\,,\, |X(t)| \leqslant \varepsilon\} \\ &\quad = \mathbb{P}\{|X(s \curlywedge t) + Z_1| \leqslant \varepsilon\,,\, |X(s \curlywedge t) + Z_2| \leqslant \varepsilon\} \\ &\quad \leqslant \mathbb{P}\{|X(s \curlywedge t) + Z_1| \leqslant \varepsilon\,,\, \big|Z_1 - Z_2| \leqslant 2\varepsilon\}. \end{aligned} \tag{3.9}$$

Elementary properties of Lévy processes imply that $X(s \curlywedge t)$, Z_1 and Z_2 are three independent random vectors in $\mathbb{R}^d$. Moreover, by the weak unimodality of the distribution of $X(s \curlywedge t)$ and by Lemma 2.2,

$$\begin{aligned} \mathbb{P}\{|X(s \curlywedge t) + Z_1| \leqslant \varepsilon \mid Z_1, Z_2\} &\leqslant \kappa \mathbb{P}\{|X(s \curlywedge t)| \leqslant \varepsilon\} \\ &\leqslant \kappa (2\varepsilon)^d \Phi(s \curlywedge t) \\ &\leqslant \kappa (2\varepsilon)^d \Phi(\mathbf{1}). \end{aligned}$$

Since $Z_1 - Z_2 = X(s) - X(t)$, Eq. (3.9) implies

$$\mathbb{P}\{|X(s)| \leqslant \varepsilon\,,\, |X(t)| \leqslant \varepsilon\} \leqslant \kappa (2\varepsilon)^d \cdot \Phi(\mathbf{1}) \mathbb{P}\{|X(s) - X(t)| \leqslant 2\varepsilon\}.$$

Thus, we have demonstrated that

$$\begin{aligned} &\mathbb{E}\Big\{ \int_{[1,T]^N} \int_{[1,T]^N} K(s,t) \mathcal{J}_{\varepsilon,T}(ds) \mathcal{J}_{\varepsilon,T}(dt) \Big\} \\ &\quad \leqslant \kappa (2\varepsilon)^{-d} \Phi(\mathbf{1}) \times \\ &\qquad \times \int_{[1,T]^N} \int_{[1,T]^N} K(s,t) \mathbb{P}\{|X(s) - X(t)| \leqslant 2\varepsilon\}\ \mu(ds)\ \mu(dt). \end{aligned}$$

The lemma follows from this and weak regularity; cf. Lemma 2.8. □

Remark 3.5 A little thought shows that we can apply Lemma 3.4 with $K(s,t) \equiv 1$ to obtain

$$\begin{aligned} \mathbb{E}\{\|\mathcal{J}_{\varepsilon,T}\|^2\} &\leqslant \kappa^2 \Phi(\mathbf{1}) \varepsilon^{-d} \times \\ &\quad \times \int_{[1,T]^N} \int_{[1,T]^N} \mathbb{P}\{|X(s) - X(t)| \leqslant \varepsilon\}\ \mu(ds)\ \mu(dt). \end{aligned} \tag{3.10}$$

In particular,

$$\mathbb{E}\{\|\mathcal{J}_{\varepsilon,T}\|^2\} \leqslant \kappa^2 2^d \Phi(\mathbf{1}) \cdot \int_{[1,T]^N} \int_{[1,T]^N} \Phi(s - t)\ \mu(ds)\ \mu(dt). \tag{3.11}$$

If μ is chosen to be the N-dimensional Lebesgue measure on $[1,T]^N$, then, by the symmetry of Lévy processes X_j $(j=1,\ldots,N)$, Eq. (3.10) becomes

$$\mathbb{E}\{\|\mathcal{J}_{\varepsilon,T}\|^2\} \leqslant \kappa^2 2^N (T-1)^N \Phi(\mathbf{1})\varepsilon^{-d} \cdot \int_{[0,T-1]^N} \mathbb{P}\{|X(s)| \leqslant \varepsilon\}\, ds. \tag{3.12}$$

That is, Lemma 3.4 implies an energy estimate. □

We can now prove

Lemma 3.6 *In Proposition 3.1, (i)⇒(ii).*

Proof Upon changing the notation of the forthcoming proof only slightly, we see that it is sufficient to prove that

$$\mathbb{P}\big\{0 \in X([1,2]^N)\big\} > 0.$$

We will prove this by constructing a random Borel measure on the zero set $X^{-1}(0) \cap [1,2]^N$. Let $\{\mathcal{J}_{\varepsilon,2}\}$ be the family of random measures on $[1,2]^N$ defined by (3.6). Lemmas 3.3 and 3.4 with $K(s,t) = \bar{\varrho}(s-t)$ and Eq. (3.11), together with a second moment argument (see Kahane [31, pp.204–206], or LeGall, Rosen and Shieh [37, pp.506–507]), imply that there exists a subsequence $\{\mathcal{J}_{\varepsilon_n,2}\}$ that converges weakly to a random measure $\mathcal{J}_2$ such that

$$\begin{aligned}\mathbb{E}\Big\{\int_{[1,2]^N}\int_{[1,2]^N} \bar{\varrho}(s-t)\mathcal{J}_2(ds)\mathcal{J}_2(dt)\Big\} \\ \leqslant \kappa^2 2^d \Phi(\mathbf{1}) \cdot \int_{[1,2]^N}\int_{[1,2]^N} \bar{\varrho}(s-t)\Phi(s-t)\ \mu(ds)\ \mu(dt).\end{aligned} \tag{3.13}$$

Moreover, letting $A = \kappa^{-2}2^{-d}\big\{\Phi(\mathbf{1})\big\}^{-1}$, we have

$$\begin{aligned}\mathbb{P}\{\|\mathcal{J}_2\| > 0\} \quad &\geqslant A \cdot \Big\{\int_{[1,2]^N} \Phi(s)\ \mu(ds)\Big\}^2 \times \\ &\times \Big\{\int_{[1,2]^N}\int_{[1,2]^N} \Phi(s-t)\ \mu(ds)\ \mu(dt)\Big\}^{-1},\end{aligned} \tag{3.14}$$

which is positive. The first integral is clearly positive and the second is finite, thanks to Eq. (3.3).

It remains to prove that the random measure $\mathcal{J}_2$ is supported on $X^{-1}(0) \cap [1,2]^N$. To this end, it is sufficient to show that for each $\delta > 0$, $\mathcal{J}_2(D(\delta)) = 0$, a.s., where $D(\delta) = \{s \in [1,2]^N : |X(s)| > \delta\}$. We employ an argument that is similar, in spirit, to that used by LeGall, Rosen and Shieh [37, pp. 507–508].

Since the sample functions of each Lévy process X_j $(j=1,\ldots,N)$ are right continuous and have left limit everywhere, the limit $\lim_{t \xrightarrow{(A)} s^-} X(t)$ exists for every $A \in \Pi$ and $s \in \mathbb{R}_+^N$, where $t \xrightarrow{(A)} s^-$ means $t_j \uparrow s_j$ for $j \in A$ and $t_j \downarrow s_j$ for $j \in A^{\complement}$. Note that $\lim_{t \xrightarrow{(\varnothing)} s^-} X(t) = X(s)$. Let

$$D_1(\delta) = \Big\{s \in [1,2]^N : \ |\lim_{t \xrightarrow{(A)} s^-} X(t)| > \delta \ \text{ for all } \ A \in \Pi\Big\},$$

and

$$D_2(\delta)=\big\{s\in[1,2]^N:\ |X(s)|>\delta\ \text{ and }\ |\lim_{t\xrightarrow{(A)}s^-}X(t)|\leqslant\delta\ \text{ for some }\ A\in\Pi\big\}.$$

Then, we have the decomposition: for all $\delta>0$,

$$D(\delta)\setminus D_1(\delta)\subseteq D_2(\delta).$$

We observe that $D_1(\delta)$ is open in $[1,2]^N$, and $D_2(\delta)$ is contained in a countable union of hyperplanes of form $\{t\in[1,2]^N: t_j=a\ \text{ for some } j\}$, for various values of a. These hyperplanes are solely determined by the discontinuities of X_i's.

Directly from the definition of $\mathcal{J}_{\varepsilon,2}$, we can deduce that for all $\varepsilon>0$ small enough, $\mathcal{J}_{\varepsilon,2}(D_1(\delta))=0$. Hence, $\mathcal{J}_2(D_1(\delta))=0$, almost surely. On the other hand, Eq.'s (3.5) and (3.13) together imply that the following holds with probability one:

$$\mathcal{J}_2\{t\in[1,2]^N: t_j=a\ \text{ for some } j\}=0,\qquad \forall a\in\mathbb{R}_+.$$

Consequently, $\mathcal{J}_2(D_2(\delta))=0$, a.s., for each $\delta>0$. We have proved that with positive probability, $0\in X([1,2]^N)$, which verifies the lemma. $\square$

In Proposition 3.1, the implications of $(ii)\Rightarrow(iii)$ and $(iv)\Rightarrow(i)$ are obvious. To prove $(iii)\Rightarrow(iv)$, we define the N-parameter process $M=\{M(t);\ t\in\mathbb{R}_+^N\}$ by

$$M(t)=\mathbb{E}\{\|\mathcal{J}_{\varepsilon,3}\|\mid\mathcal{F}(t)\},\qquad t\in\mathbb{R}_+^N,\tag{3.15}$$

where $\mathcal{J}_{\varepsilon,3}$ is described in Eq. (3.6) with μ replaced by the N-dimensional Lebesgue measure. Clearly, M is an N-parameter martingale in the sense of Cairoli [6]. We shall tacitly work with Doob's separable version modification of M.

Lemma 3.7 *Suppose $t\in[1,2]^N$ and $\varepsilon>0$. Then, a.s.,*

$$\mathbb{1}\{|X(t)|\leqslant\frac{\varepsilon}{2}\}\leqslant\kappa(4\varepsilon)^d\cdot M(t)\Big[\int_{[0,1]^N}\mathbb{P}\{|X(s)|\leqslant\varepsilon\}\ ds\Big]^{-1}.$$

Proof Clearly, for $t\preccurlyeq s$, $X(s)-X(t)$ is independent of $\mathcal{F}(t)$. Hence

$$\begin{aligned}M(t)\ &\geqslant\ (2\varepsilon)^{-d}\int_{[1,3]^N}\mathbb{1}\{s\succcurlyeq t\}\ \mathbb{P}\{|X(s)-X(t)|\leqslant\frac{\varepsilon}{2}\}\ ds\cdot\mathbb{1}\big\{|X(t)|\leqslant\frac{\varepsilon}{2}\big\}\\ &=\ (2\varepsilon)^{-d}\int\mathbb{1}\{0\preccurlyeq r\preccurlyeq\mathbf{3}-t\}\mathbb{P}\{|X(r)|\leqslant\frac{\varepsilon}{2}\}\ ds\cdot\mathbb{1}\big\{|X(t)|\leqslant\frac{\varepsilon}{2}\big\},\end{aligned}$$

In particular, for all $t\in[1,2]^N$,

$$M(t)\geqslant(2\varepsilon)^{-d}\int_{[0,1]^N}\mathbb{P}\big\{|X(r)|\leqslant\frac{\varepsilon}{2}\big\}\ dr\cdot\mathbb{1}\big\{|X(t)|\leqslant\frac{\varepsilon}{2}\big\}.$$

The lemma follows from weak regularity; cf. Lemma 2.8. $\square$

The last link in our proof of the first part of Proposition 3.1 is given by the following lemma.

Lemma 3.8 *In Proposition 3.1, (iii) ⇒ (iv).*

Proof Upon squaring both sides of the inequality of Lemma 3.7, and after taking the supremum over $[1,2]^N \cap \mathbb{Q}^N$ and taking expectations, we obtain

$$\begin{aligned}
&\mathbb{P}\Big\{|X(t)| \leqslant \frac{\varepsilon}{2} \text{ for some } t \in [1,2]^N \cap \mathbb{Q}^N \Big\} \\
&\qquad \leqslant \kappa^2 (4\varepsilon)^{2d} \cdot \mathbb{E}\Big\{ \sup_{t\in[1,2]^N \cap \mathbb{Q}^N} |M(t)|^2 \Big\} \cdot \ \Big[\int_{[0,1]^N} \mathbb{P}\{|X(r)| \leqslant \varepsilon\} \ dr \Big]^{-2}.
\end{aligned}$$

By Cairoli's maximal inequality (Cairoli [6, Th. 1])

$$\begin{aligned}
\mathbb{E}\Big\{ \sup_{t\in[1,2]^N \cap \mathbb{Q}^N} |M(t)|^2 \Big\} &\leqslant 4^N E\big\{|M(\mathbf{2})|^2\big\} \\
&\leqslant 4^N \mathbb{E}\{\|\mathcal{J}_{\varepsilon,3}\|^2\}.
\end{aligned}$$

We now apply (3.12) to obtain

$$\begin{aligned}
&\mathbb{P}\Big\{|X(t)| \leqslant \frac{\varepsilon}{2} \text{ for some } t \in [1,2]^N \cap \mathbb{Q}^N \Big\} \\
&\qquad \leqslant \kappa^4 2^{4d+4N} \Phi(\mathbf{1}) \varepsilon^d \int_{[0,2]^N} \mathbb{P}\big\{|X(r)| \leqslant \varepsilon\big\} \ dr \ \Big[\int_{[0,1]^N} \mathbb{P}\{|X(r)| \leqslant \varepsilon\} \ dr \Big]^{-2} \\
&\qquad \leqslant \kappa^5 2^{4d+5N} \Phi(\mathbf{1}) \varepsilon^d \ \Big[\int_{[0,1]^N} \mathbb{P}\{|X(r)| \leqslant \varepsilon\} \ dr \Big]^{-1},
\end{aligned}$$

where the last inequality follows from

$$\int_{[0,2]^N} \mathbb{P}\{|X(r)| \leqslant \varepsilon\} \ dr \leqslant \kappa 2^N \int_{[0,1]^N} \mathbb{P}\{|X(r)| \leqslant \varepsilon\} \ dr.$$

We have used the weak monotonicity property given by Lemma 2.8. By the general theory of Lévy processes, we can assume $t \mapsto X(t)$ to be right-continuous with respect to the partial order $\preccurlyeq$; cf. Bertoin [4]. Consequently,

$$\begin{aligned}
&\mathbb{P}\Big\{|X(t)| \leqslant \frac{\varepsilon}{2} \text{ for some } t \in [1,2]^N \Big\} \\
&\qquad \leqslant \kappa^5 2^{4d+5N} \Phi(\mathbf{1}) \varepsilon^d \ \Big[\int_{[0,1]^N} \mathbb{P}\{|X(r)| \leqslant \varepsilon\} \ dr \Big]^{-1}.
\end{aligned}$$

By Fatou's lemma,

$$\liminf_{\varepsilon \to 0+} (2\varepsilon)^{-d} \int_{[0,1]^N} \mathbb{P}\{|X(r)| \leqslant \varepsilon\} \ dr \geqslant \int_{[0,1]^N} \Phi(s) \ ds.$$

Thus, by the mentioned sample right-continuity,

$$\mathbb{P}\{X(t) = 0 \text{ for some } t \in [1,2]^N\} \leqslant \kappa^5 2^{3d+5N} \Phi(\mathbf{1}) \Big[\int_{[0,1]^N} \Phi(s) \ ds \Big]^{-1}.$$

In fact, this proof shows that for any $c > 0$, $u \in [c, \infty)^N$ and $h > 0$,

$$\begin{aligned} \mathbb{P}\{X(t) = 0 \quad & \text{for some } t \in [u, u + \mathbf{h}]^N\} \\ & \leqslant \kappa^5 2^{3d+5N} h^N \Phi(\mathbf{c}) \Big[\int_{[0,h]^N} \Phi(s)\ ds \Big]^{-1}. \end{aligned} \tag{3.16}$$

This proves $(iii) \Rightarrow (iv)$, and concludes our proof of the first part of Proposition 3.1. $\square$

Remark 3.9 Proposition 3.1 implies that if $\mathsf{C}_\Phi([0,1]^N) > 0$, then $\mathsf{C}_\Phi([0,T]^N) > 0$ for all $T > 0$. $\square$

3.2 Proof of the Second Part of Proposition 3.1

The arguments leading to the second part of our proof are similar to those of the first part. As such, we only sketch a proof.

For any $\varepsilon > 0$, $a \in \mathbb{R}^d$ and $T > 1$, define a random measure $\mathfrak{I}_{a;\varepsilon,T}$ on $[1,T]^N$ by

$$\mathfrak{I}_{a;\varepsilon,T}(B) = (2\varepsilon)^{-d} \int_B \mathbb{1}\Big\{ \big|X(s) - a\big| \leqslant \varepsilon \Big\}\ ds, \tag{3.17}$$

where $B \subseteq [1,T]^N$ designates an arbitrary Borel set. Similar arguments that lead to Lemmas 3.3 and 3.4 can be used to deduce the following.

Lemma 3.10 *For any $a \in \mathbb{R}^d$,*

$$\lim_{\varepsilon \to 0+} \mathbb{E}\big\{ \|\mathfrak{I}_{a;\varepsilon,T}\| \big\} = \int_{[1,T]^N} p(s; a)\ ds. \tag{3.18}$$

Moreover, if $K : \mathbb{R}_+^N \times \mathbb{R}_+^N \to \mathbb{R}_+$ is a measurable function, for any $T > 1$ and for all $\varepsilon > 0$,

$$\begin{aligned} \mathbb{E}\Big\{ \int_{[1,T]^N} \int_{[1,T]^N} & K(s,t) \mathfrak{I}_{a;\varepsilon,T}(ds) \mathfrak{I}_{a;\varepsilon,T}(dt) \Big\} \\ & \leqslant \kappa^2 \Phi(\mathbf{1}) \varepsilon^{-d} \times \\ & \quad \times \int_{[1,T]^N} \int_{[1,T]^N} K(s,t) \mathbb{P}\Big\{ \big|X(s) - X(t)\big| \leqslant \varepsilon \Big\}\ ds\ dt. \end{aligned} \tag{3.19}$$

In particular,

$$\mathbb{E}\big\{ \|\mathfrak{I}_{a;\varepsilon,T}\|^2 \big\} \leqslant \kappa^2 \Phi(\mathbf{1}) \varepsilon^{-d} \cdot \int_{[1,T]^N} \int_{[1,T]^N} \mathbb{P}\Big\{ \big|X(t) - X(s)\big| \leqslant \varepsilon \Big\}\ ds\ dt. \tag{3.20}$$

We are ready for

Proof of Eq. (3.1) Without loss of generality, we may and will assume that $c = 1$ and $C = 2$. The lower bound in (3.1) follows from the second moment argument of Lemma 3.6, using Eq.'s (3.18), (3.19) and (3.20) of Lemma 3.10 with $T = 2$; see Eq. (3.14).

To prove the upper bound in (3.1), we follow the lines of proof of Lemma 3.8 and define $M_{a;\varepsilon,3} = \{M_{a;\varepsilon,3}(t);\ t \in \mathbb{R}_+^N\}$ by

$$M_{a;\varepsilon,3}(t) = \mathbb{E}\big[\|\mathcal{J}_{a;\varepsilon,3}\| \bigm| \mathcal{F}(t)\big], \qquad t \in \mathbb{R}_+^N.$$

This is the analogue of (3.15) and is always an N-parameter martingale. As in Lemma 3.7, for all $t \in [1,2]^N$ and $\varepsilon > 0$,

$$\mathbb{1}\Big\{|X(t) - a| \leqslant \frac{\varepsilon}{2}\Big\} \leqslant \kappa(4\varepsilon)^d M(t) \Big[\int_{[0,1]^N} \mathbb{P}\{|X(s)| \leqslant \varepsilon\}\ ds\Big]^{-1}.$$

The presented proof of Lemma 3.8 can be adapted, using Eq. (3.20) with $T = 3$ in place of Eq. (3.12), to yield

$$\mathbb{P}\big\{X(t) = a \text{ for some } t \in [1,2]^N\big\} \leqslant \kappa^5 2^{3d+5N} \Phi(\mathbf{1}) \Big[\int_{[0,1]^N} \Phi(s)\ ds\Big]^{-1}.$$

Since

$$\int_{[1,2]^N} \int_{[1,2]^N} \Phi(s-t) ds\ dt \leqslant 2^N \int_{[0,1]^N} \Phi(s)\ ds,$$

this proves the upper bound in (3.1). $\square$

3.3 Proof of Proposition 3.2

In order to prove $(i) \Rightarrow (ii)$, we use the Fourier-analytic ideas of Kahane [31, Theorem 2, Ch. 14] to show that for every $c > 0$

$$\mathbb{P}\Big\{ \operatorname{Leb}\{X([c,2c]^N)\} > 0\Big\} = 1. \tag{3.21}$$

Suppose assertion (i) holds. Then, by Proposition 3.1, $\Phi \in L^1_{loc}(\mathbb{R}^N)$. We denote by σ the image measure of the restriction of Lebesgue's measure on $[c,2c]^N$ under X. The Fourier transform of σ is

$$\widehat{\sigma}(u) = \int_{[c,2c]^N} \exp\big\{iu \cdot X(t)\big\}\ dt.$$

By Fubini's Theorem,

$$\begin{aligned}
\mathbb{E}\big(|\widehat{\sigma}(u)|^2\big) &= \int_{[c,2c]^N} \int_{[c,2c]^N} \exp\Big\{-\sum_{j=1}^N |t_j - s_j| \Psi_j(u)\Big\}\ ds\ dt \\
&\leqslant (2c)^N \int_{[0,\ c]^N} \exp\Big\{-\sum_{j=1}^N t_j \Psi_j(u)\Big\}\ dt\ .
\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{E}\int_{\mathbb{R}^d}|\widehat{\sigma}(u)|^2du &\leqslant (2c)^N\int_{[0,\ c]^N}\int_{\mathbb{R}^d}\exp\Big\{-\sum_{j=1}^N t_j\Psi_j(u)\Big\}\ du\ dt\\ &= 2^{N+d}\pi^d c^N\int_{[0,\ c]^N}\Phi(t)\ dt,\end{aligned}$$

which is finite, since $\Phi\in L^1_{loc}(\mathbb{R}^N)$. Consequently, $\widehat{\sigma}\in L^2(\mathbb{R}^d)$, a.s. By the Riesz–Fischer theorem and/or Plancherel's theorem, σ is a.s. absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ and its density is a.s. in $L^2(\mathbb{R}^d)$. This proves Eq. (3.21); assertion (ii) of Proposition 3.2 follows suit.

The implications $(ii)\Rightarrow(iii)\Rightarrow(iv)$ being immediate, we will show $(iv)\Rightarrow(i)$. Assuming that (iv) holds, there exists a constant $c_0>c$ and an open set $G\subset\mathbb{R}^d$ such that

$$\mathbb{P}\Big\{\operatorname{Leb}\{X([c,C]^N)\cap G\}>0\Big\}>0,\qquad \text{for all } C\geqslant c_0.$$

It follows from Eq. (3.1) that for all $a\in\mathbb{R}^d$,

$$\mathbb{P}\{a\in X([c,C]^N)\}\leqslant\kappa^5 2^{3d+6N}\Phi(\mathbf{c})\mathsf{C}_\Phi([c,C]^N).$$

By Fubini's theorem, we obtain

$$\mathbb{E}\Big(\operatorname{Leb}\{X([c,C]^N)\cap G\}\Big)\leqslant\kappa^5 2^{3d+6N}\Phi(\mathbf{c})\operatorname{Leb}(G)\mathsf{C}_\Phi([c,C]^N).$$

Hence, $\mathsf{C}_\Phi([c,C]^N)>0$, which implies $\mathsf{C}_\Phi([0,1]^N)>0$. This completes our proof of Proposition 3.2. □

4 Proof of Theorem 2.10

Without loss of generality, we will assume that $c=1$ and $C=2$. With this in mind, we first prove the lower bound in (2.6). For any $\beta<\gamma$,

$$\int_{[0,1]^N}\|s\|^{-\beta}\Phi(s)ds<\infty.$$

This implies that for any $T>0$,

$$\int_{[0,T]^N}\int_{[0,T]^N}\|s-t\|^{-\beta}\Phi(s-t)\ ds\ dt<\infty. \tag{4.1}$$

We let $\mathcal{J}_2$ denote the random measure constructed in the presented proof of Lemma 3.6 with μ being Lebesgue's measure on $[1,2]^N$. We have already proved that $\mathcal{J}_2$ is supported on $X^{-1}(0)\cap[1,2]^N$ and that it is positive with some probability $\eta > 0$, which is independent of β; see Eq. (3.14). On the other hand, Lemma 3.4 with $K(s,t) = \|s-t\|^{-\beta}$ and Eq. (4.1), together imply

$$\mathbb{E}\Big\{ \int_{[1,2]^N} \int_{[1,2]^N} \|s-t\|^{-\beta} \; \mathcal{J}_2(ds) \; \mathcal{J}_2(dt) \Big\} < \infty.$$

Hence, $\mathbb{P}\{\dim_{\mathsf{H}}(X^{-1}(0) \cap [1,2]^N) \geqslant \beta\} > \eta$. Letting $\beta \uparrow \gamma$ along a rational sequence, we obtain the lower bound in (2.6).

Next, we will use the hitting probability estimate (3.16) and a covering argument to prove the upper bound in (2.6). For any $\beta' > \overline{\gamma}$, we choose $\beta \in (\overline{\gamma}, \beta')$. Then

$$\Phi(s) \geqslant \frac{1}{\|s\|^{N-\beta}} \quad \text{ for all } \; s \text{ near } 0.$$

Hence, there exists a constant $K_2 > 0$ such that for all $h > 0$ small enough

$$\int_{[0,h]^N} \Phi(s) ds \geqslant K_2 h^{\beta}. \tag{4.2}$$

Now, we can take n large enough and divide $[1,2]^N$ into n^N subcubes $\{C_{n,i}\}_{i=1}^{n^N}$, each of which has side $1/n$. Let us now define a covering $\mathbf{C}_{n,1}, \ldots, \mathbf{C}_{n,n^N}$ of $X^{-1}(0) \cap [1,2]^N$ by

$$\mathbf{C}_{n,i} = \begin{cases} C_{n,i} & \text{if } X^{-1}(0) \cap C_{n,i} \neq \varnothing \\ \varnothing & \text{otherwise} \end{cases} .$$

It follows from (3.16) and (4.2) that for each $C_{n,i}$,

$$\mathbb{P}\big\{X^{-1}(0) \cap C_{n,i} \neq \varnothing\big\} \leqslant K_3 \big(\frac{1}{n}\big)^{N-\beta},$$

where K_3 is a positive and finite constant. Hence, with the covering $\{\mathbf{C}_{n,i}\}_{i=1}^{n^N}$ in mind,

$$\begin{aligned} &\mathbb{E}\Big[\mathsf{H}_{\beta'}\big(X^{-1}(0) \cap [1,2]^N\big)\Big] \\ &\quad \leqslant \liminf_{n \to \infty} \sum_{i=1}^{n^N} (\sqrt{N} n^{-1})^{\beta'} \, \mathbb{P}\big\{X^{-1}(0) \cap C_{n,i} \neq \varnothing\big\} \\ &\quad \leqslant \liminf_{n \to \infty} K_3 \sqrt{N}^{\beta'} \; n^{\beta - \beta'} = 0, \end{aligned}$$

where $\mathsf{H}_{\beta'}(E)$ denotes the β'-dimensional Hausdorff measure of E. This proves $\dim_{\mathsf{H}}(X^{-1}(0) \cap [1,2]^N) \leqslant \beta'$ a.s. and hence the upper bound in (2.6).

To prove the second assertion of Theorem 2.10, it suffices to show that under Condition (2.7), $\dim_{\mathsf{H}}(X^{-1}(0) \cap [1,2]^N) \leqslant \gamma$, a.s. This can be done by combining the above first moment argument and the following lemma. We omit the details. □

Lemma 4.1 *Under Condition (2.7), for any $\beta > 0$,*

$$\int_{[0,1]^N} \|s\|^{-\beta}\ \Phi(s)\ ds = \infty \tag{4.3}$$

implies that for any $u \in [1,2]^N$ and any $\beta' > \beta$,

$$\liminf_{h \to 0} h^{\beta' - N} \mathbb{P}\big\{X^{-1}(0) \cap [u, u + \mathbf{h}] \neq \varnothing\big\} = 0. \tag{4.4}$$

Proof Under Condition (4.3), for any $\varepsilon > 0$ we must have

$$\limsup_{s \to 0} \|s\|^{N-\beta-\varepsilon} \Phi(s) = \infty.$$

This and Eq. (2.7), together imply that

$$\limsup_{h \to 0+} h^{N-\beta-\varepsilon} \Phi(\mathbf{h}) = \infty. \tag{4.5}$$

On the other hand, it is not hard to see that $\Phi(s) \geqslant \Phi(\|\mathbf{s}\|)$ and that $\Phi(\mathbf{h})$ is nonincreasing in h. Hence, for $h > 0$,

$$\int_{[0,h]^N} \Phi(s)\ ds \geqslant K_4 \Phi(\mathbf{h}) h^N, \tag{4.6}$$

for some positive constant K_4. It follows from (3.16) and (4.6) that

$$\mathbb{P}\big\{X^{-1}(0) \cap [u, u + \mathbf{h}] \neq \varnothing\big\} \leqslant K_5\ \frac{1}{\Phi(\mathbf{h})}. \tag{4.7}$$

Eq. (4.4) follows from Eq.'s (4.5) and (4.7), upon taking $\varepsilon \in (0, \beta' - \beta)$. This finishes our proof of the Lemma. ☐

5 Proof of Theorem 2.12

Theorem 2.12 is divided into two parts: an upper bound (on the hitting probability), as well as a corresponding lower bound. The latter is simple enough to prove: the proof of the lower bound in Eq. (2.8) uses Lemma 3.10 and follows the second moment argument of Lemma 3.6 closely; we omit the details.

Regarding the proof of the upper bound, while we sincerely believe that it should be a mere abstraction of the corresponding upper bound in Proposition 3.1, the only justification that we can devise is much more complicated and requires that we first prove a somewhat different theorem. Interestingly enough, this (somewhat different) theorem completes a circle of ideas in the literature that is sometimes referred to as Kahane's problem and is introduced is Subsection 5.1. The remaining Subsections 5.2, 5.3 and 5.4 prove Kahane's problem and also derive the hard part of Theorem 2.12, in succession.

5.1 Lebesgue's Measure of Stochastic Images

We now intend to demonstrate the following result on Lebesgue's measure of the image of a compact set under the random function X. Throughout this section, Leb denotes Lebesgue's measure on $\mathbb{R}^d$.

Theorem 5.1 *Let $X_1, \ldots, X_N$ be N independent symmetric Lévy processes on $\mathbb{R}^d$ and let $X = X_1 \oplus \cdots \oplus X_N$. Suppose that X is absolutely continuous, weakly unimodal and has gauge function Φ. Then, for any compact set $E \subset \mathbb{R}_+^N$,*

$$\kappa^{-1} 2^{-d} \mathsf{C}_\Phi(E) \leqslant \mathbb{E}\big\{ \operatorname{Leb}[X(E)] \big\} \leqslant 2^{5d+3N} \kappa^3 \mathsf{C}_\Phi(E).$$

The following is an immediate corollary.

Corollary 5.2 *In the setting of Theorem 5.1, for any compact set $E \subset \mathbb{R}_+^N$,*

$$\mathbb{E}\big\{ \operatorname{Leb}[X(E)] \big\} > 0 \iff \mathsf{C}_\Phi(E) > 0.$$

Remark 5.3 To the knowledge of the authors, this result is new at this level of generality, even for Lévy processes, i.e. $N = 1$. Special cases of this one-parameter problem have been treated in HAWKES [24, Th. 5] (for Brownian motion); see also KAHANE [31, Ch. 16, 17]. □

Now suppose $X_1, \ldots, X_N$ are i.i.d. isotropic stable Lévy processes all with $\alpha \in]0, 2]$. In this case, the above completes a program initiated by J.-P. Kahane who has shown that for $N = 1, 2$,

$$\mathsf{Cap}_{d/\alpha}(E) > 0 \implies \mathbb{E}\big\{ \operatorname{Leb}[X(E)] \big\} > 0 \implies \mathsf{H}_{d/\alpha}(E) > 0, \tag{5.1}$$

where H_β denotes the β-dimensional Hausdorff measure on $\mathbb{R}_+^N$. See KAHANE [30, 31] for this and for a discussion of the history of this result, together with interesting applications to harmonic analysis. A combination of Corollary 5.2 and Eq. (2.9) yields the following that completes Eq. (5.1) by essentially closing the "hard half".

Corollary 5.4 *Suppose $X_1, \ldots, X_N$ are i.i.d. isotropic stable Lévy processes all with the same index $\alpha \in]0, 2]$. If $X = X_1 \oplus \cdots \oplus X_N$ and if $E \subset \mathbb{R}_+^N$ is compact,*

$$\mathbb{E}\big\{ \operatorname{Leb}[X(E)] \big\} > 0 \iff \mathsf{Cap}_{d/\alpha}(E) > 0.$$

Once again, the proof of Theorem 5.1 is divided in two main parts: an *upper bound* (on $\mathbb{E}\{\cdots\}$) and a *lower bound* (on $\mathbb{E}\{\cdots\}$). The latter is more or less standard and will be verified first in §5.2 below. The former is the "hard half" and is proved in §5.3.

5.2 Proof of Theorem 5.1: Lower Bound

For the purposes of exposition, it is beneficial to work on a canonical probability space. Recall the space $\mathcal{D}(\mathbb{R}_+)$ of all functions $f : \mathbb{R}_+ \to \mathbb{R}^d$ that are right continuous and have left limits. As usual, $\mathcal{D}(\mathbb{R}_+)$ is endowed with Skorohod's topology. Define $\Omega = \mathcal{D}(\mathbb{R}_+) \oplus \cdots \oplus \mathcal{D}(\mathbb{R}_+)$, and let it inherit the topology from $\mathcal{D}(\mathbb{R}_+)$. That is, $f \in \Omega$ if and only if there are $f_1, \ldots, f_N \in \mathcal{D}(\mathbb{R}_+)$ such that $f = f_1 \oplus \cdots \oplus f_N$. Moreover, as $n \to \infty$, $f^n \to f^\infty$ in Ω, if and only if for all $\ell = 1, \ldots, N$, $\lim_n f_\ell^n = f_\ell^\infty$ in $\mathcal{D}(\mathbb{R}_+)$, where $f^n = f_1^n \oplus \cdots \oplus f_N^n$ for all $1 \leqslant n \leqslant \infty$.

Let $X = \{X(t);\ t \in \mathbb{R}_+^N\}$ denote the canonical coordinate process on Ω. That is, for all $\omega \in \Omega$ and all $t \in \mathbb{R}_+^N$, $X(t)(\omega) = \omega(t)$. Also, let $\mathcal{F}$ denote the collection of all Borel subsets of Ω. In a completely standard way, one can construct a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$, such that under the measure $\mathbb{P}$, X has the same finite-dimensional distributions as the process of Theorem 5.1. In fact, one can do more and define for all $x \in \mathbb{R}^d$ a probability measure $\mathbb{P}_x$ on $(\Omega, \mathcal{F})$ as follows: for all $G \in \mathcal{F}$,

$$\mathbb{P}_x\{G\} = \mathbb{P}_x\{\omega \in \Omega:\ \omega \in G\} = \mathbb{P}\{\omega \in \Omega:\ x + \omega \in G\},$$

where the function $x + \omega$ is, as usual, defined pointwise by $(x+\omega)(t) = x + \omega(t)$ for all $t \in \mathbb{R}_+^N$. The corresponding expectation operator is denoted by $\mathbb{E}_x$. Moreover, $\mathbb{P}_{\mathrm{Leb}}$ ($\mathbb{E}_{\mathrm{Leb}}$, resp.) refers to the σ-finite measure $\int \mathbb{P}_x(\bullet)\, dx$ (linear operator $\int \mathbb{E}_x(\bullet)\, dx$, resp.).

It is easy to see that the σ-finite measures $\mathbb{P}_{\mathrm{Leb}}$ have a similar structure as $\mathbb{P}$; one can define conditional expectations, (multi-)parameter martingales, etc. We will use the (probability) martingale theory that is typically developed for $\mathbb{P}$, and apply it to that for $\mathbb{P}_{\mathrm{Leb}}$. It is completely elementary to see that the theory extends easily and naturally. In a one-parameter, discrete setting, the details can be found in Dellacherie and Meyer [12, Eq. (40.2), p. 34]. One generalizes this development to our present multiparameter setting by applying the arguments of R. Cairoli; cf. Walsh [51].

The above notation is part of the standard notation of the theory of Markov processes and will be used throughout the remainder of this section. In order to handle the measurability issues, the σ-field $\mathcal{F}$ will be assumed to be complete with respect to the measure $\mathbb{P}_{\mathrm{Leb}}$. This can be assumed without any loss in generality, for otherwise, $(\Omega, \mathcal{F})$ will be replaced by its $\mathbb{P}_{\mathrm{Leb}}$-completion throughout with no further changes.

Our proof of Theorem 5.1 relies on the following technical lemma.

Lemma 5.5 *Under the σ-finite measure $\mathbb{P}_{\mathrm{Leb}}$, for each $t \in \mathbb{R}_+^N$ the law of $X(t)$ is Lebesgue's measure on $\mathbb{R}^d$. Moreover, for all $n \geqslant 1$, all $\varphi_j \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, all $s^j, t \in \mathbb{R}_+^N$ $(j = 1, \cdots, n)$ and for* Leb*-almost all $z \in \mathbb{R}^d$,*

$$\mathbb{E}_{\mathrm{Leb}}\Big[\prod_{j=1}^n \varphi_j\big(X(s^j)\big)\ \Big|\ X(t) = z\Big] = \mathbb{E}\Big[\prod_{j=1}^n \varphi_j\big(X(s^j) - X(t) + z\big)\Big]. \qquad (5.2)$$

Proof The condition that $\varphi_j \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ for all $j = 1, \ldots, n$, implies that $\prod_{j=1}^n \varphi_j(X(s^j)) \in L^1(\mathbb{P}_{\mathrm{Leb}})$. Moreover, for any bounded measurable function $g : \mathbb{R}^d \to \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}_{\mathrm{Leb}}\Big[g\big(X(t)\big) \prod_{j=1}^n \varphi_j\big(X(s^j)\big)\Big] &= \int_{\mathbb{R}^d} \mathbb{E}\Big[g\big(X(t)+x\big) \cdot \prod_{j=1}^n \varphi_j\big(X(s^j)+x\big)\Big]\, dx \\
&= \int_{\mathbb{R}^d} g(y)\ \mathbb{E}\Big[\prod_{j=1}^n \varphi_j\big(X(s^j) - X(t) + y\big)\Big]\, dy.
\end{aligned}$$

Set $\varphi_1 = \varphi_2 = \cdots \equiv 1$ to see that the $\mathbb{P}_{\mathrm{Leb}}$ distribution of $X(t)$ is Leb. Since the displayed equation above holds true for all measurable g, we have verified Eq. (5.2). $\square$

Remarks

(i) Equality (5.2) can also be established using regular conditional (σ-finite) probabilities.

(ii) There are no conditions imposed on s^j $(j = 1, \cdots, n)$ and t.

The second, and final, lemma used in our proof of the lower bound is a joint density function estimate.

Lemma 5.6 *For all $\varepsilon > 0$ and all $s, t \in \mathbb{R}_+^N$,*

$$\kappa^{-1} 2^{-d} \varepsilon^d \leqslant \frac{\mathbb{P}_{\mathrm{Leb}}\big\{|X(s)| \leqslant \varepsilon\,,\ |X(t)| \leqslant \varepsilon\big\}}{\mathbb{P}\big\{|X(t) - X(s)| \leqslant \varepsilon\big\}} \leqslant \kappa (4\varepsilon)^d,$$

where $0 \div 0 = 1$.

Proof We will verify the asserted lower bound on the probability. The upper bound is proved by similar arguments that we omit.

$$\begin{aligned}
&\mathbb{P}_{\mathrm{Leb}}\{|X(s)| \leqslant \varepsilon\,,\ |X(t)| \leqslant \varepsilon\} \\
&\qquad \geqslant \mathbb{P}_{\mathrm{Leb}}\Big\{|X(s)| \leqslant \varepsilon\,,\ |X(t)| \leqslant \frac{\varepsilon}{2}\Big\} \\
&\qquad \geqslant \mathbb{P}_{\mathrm{Leb}}\Big\{|X(t)| \leqslant \frac{\varepsilon}{2}\Big\} \inf_{z \in \mathbb{R}^d:\ |z| \leqslant \varepsilon/2} \mathbb{P}_{\mathrm{Leb}}\big\{|X(s)| \leqslant \varepsilon \mid X(t) = z\big\}.
\end{aligned}$$

By Lemma 5.5, the first term equals ε^d and the second is bounded below by $\mathbb{P}\{|X(t) - X(s)| \leqslant \frac{1}{2}\varepsilon\}$. The lower bound on the probability follows from weak regularity; cf. Lemma 2.8. $\square$

Proof of Theorem 5.1: Lower Bound For any $\mu \in \mathcal{P}(E)$ and all $\varepsilon > 0$, define

$$J = (2\varepsilon)^{-d} \int \mathbb{1}_{\{|X(s)| \leqslant \varepsilon\}}\ \mu(ds). \tag{5.3}$$

By Lemmas 5.5 and 5.6,

$$\begin{aligned} \mathbb{E}_{\rm Leb}\{J\} &= 1 \\ \mathbb{E}_{\rm Leb}\{J^2\} &\leqslant \kappa\varepsilon^{-d}\int\int \mathbb{P}\{|X(t)-X(s)|\leqslant\varepsilon\}\ \mu(ds)\ \mu(dt). \end{aligned} \tag{5.4}$$

Thus, by the Paley-Zygmund inequality applied to the σ-finite measure $\mathbb{P}_{\rm Leb}$,

$$\begin{aligned} &\mathbb{P}_{\rm Leb}\{\exists s\in E:\ |X(s)|\leqslant\varepsilon\} \\ &\quad\geqslant \mathbb{P}_{\rm Leb}\{J>0\} \\ &\quad\geqslant \Big[\kappa\varepsilon^{-d}\int\int \mathbb{P}\{|X(t)-X(s)|\leqslant\varepsilon\}\ \mu(ds)\ \mu(dt)\Big]^{-1}; \end{aligned}$$

cf. Kahane [31] for the latter inequality. Let $\varepsilon\to 0^+$ and use Fatou's lemma to conclude that

$$\mathbb{P}_{\rm Leb}\{0\in\overline{X(E)}\}\geqslant \kappa^{-1}2^{-d}\big[\mathcal{E}_\Phi(\mu)\big]^{-1}.$$

On the other hand,

$$\mathbb{P}_{\rm Leb}\{0\in\overline{X(E)}\} = \int \mathbb{P}\{x\in\overline{X(E)}\}\ dx = \mathbb{E}\{\mathrm{Leb}[X(E)]\}.$$

Since $\mu\in\mathcal{P}(E)$ is arbitrary, the lower bound follows. □

5.3 Proof of Theorem 5.1: Upper Bound

The verification of the upper bound of Theorem 5.1 is made particularly difficult, due to the classical fact that the parameter space $\mathbb{R}^N_+$ can not be well ordered in such a way that the ordering respects the Markovian structure of $\mathbb{R}^N_+$. (Of course, $\mathbb{R}^N_+$ can *always* be well ordered under the influence of the axiom of choice, thanks to a classical theorem of Zermelo.) This difficulty is circumvented by the introduction of 2^N partial orders that are conveniently indexed by the power set of $\{1,\dots,N\}$ as follows: let Π denote the collection of all subsets of $\{1,\dots,N\}$ and for all $A\in\Pi$, define the partial order $\overset{(A)}{\preccurlyeq}$ on $\mathbb{R}^N$ as

$$s\overset{(A)}{\preccurlyeq} t \iff \begin{cases} s_i\leqslant t_i, & \text{for all } i\in A \\ s_i\geqslant t_i, & \text{for all } i\notin A \end{cases}.$$

The key idea behind this definition is that the collection $\{\overset{(A)}{\preccurlyeq};\ A\in\Pi\}$ of partial orders totally orders $\mathbb{R}^N$ in the sense that given any two points $s,t\in\mathbb{R}^N$, there exists $A\in\Pi$, such that $s\overset{(A)}{\preccurlyeq} t$. By convention, $s\overset{(A)}{\preccurlyeq} t$ is written in its equivalent form $t\overset{(A)}{\succcurlyeq} s$ and these two ways of writing the same thing are used interchangeably throughout.[1] Corresponding to each $A \in \Pi$, one defines an N-parameter filtration $\mathcal{F}^A = \{\mathcal{F}^A_t;\ t \in \mathbb{R}^N_+\}$ by defining $\mathcal{F}^A_t$ to be the σ-field generated by the collection $\{X(r);\ r \overset{(A)}{\preccurlyeq} t\}$, for all $t \in \mathbb{R}^N_+$. The following is proved along the lines of Khoshnevisan and Shi [35, Lemma 2.1]; see also Khoshnevisan [34, Lemma 4.1].

Lemma 5.7 *For each $A \in \Pi$, $\mathcal{F}^A$ is a commuting N-parameter filtration.*

In other words, when $s \overset{(A)}{\preccurlyeq} t$ are both in $\mathbb{R}^N_+$, $\mathcal{F}^A_s \subset \mathcal{F}^A_t$. Moreover, $\mathcal{F}^A$ satisfies condition (F4) of Cairoli and Walsh; see [51].

The following important proposition is an analogue of the Markov property for additive Lévy processes, with respect to the σ-finite measure $\mathbb{P}_{\mathrm{Leb}}$.

Proposition 5.8 (The Markov Property) *For each fixed $A \in \Pi$, $s, t \in \mathbb{R}^N_+$ with $t \overset{(A)}{\preccurlyeq} s$, $\mathcal{F}^A_t$ and $X(s)$ are conditionally independent under $\mathbb{P}_{\mathrm{Leb}}$, given $X(t)$. That is, for all $\psi \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $\mathbb{P}_{\mathrm{Leb}}$ almost surely*

$$\mathbb{E}_{\mathrm{Leb}}\big[\psi(X(s)) \bigm| \mathcal{F}^A_t\big] = \mathbb{E}_{\mathrm{Leb}}\big[\psi(X(s)) \bigm| X(t)\big].$$

Proof It is sufficient to prove that for all $n \geqslant 1$, all $\varphi_j \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and all $r^j \in \mathbb{R}^N_+$ with $r^j \overset{(A)}{\preccurlyeq} t$ $(j = 1, \cdots, n)$,

$$\begin{aligned} &\mathbb{E}_{\mathrm{Leb}}\Big[\psi(X(s)) \cdot \prod_{j=1}^n \varphi_j(X(r^j)) \Bigm| X(t)\Big] \\ &\quad = \mathbb{E}_{\mathrm{Leb}}\big[\psi(X(s)) \bigm| X(t)\big] \cdot \mathbb{E}_{\mathrm{Leb}}\Big[\prod_{j=1}^n \varphi_j(X(r^j)) \Bigm| X(t)\Big]. \end{aligned} \tag{5.5}$$

To this end, consider any bounded measurable function $g : \mathbb{R}^d \to \mathbb{R}$. Then,

$$\begin{aligned} &\mathbb{E}_{\mathrm{Leb}}\Big[\psi(X(s)) \cdot g(X(t)) \cdot \prod_{j=1}^n \varphi_j(X(r^j))\Big] \\ &\quad = \mathbb{E}\Big\{\int_{\mathbb{R}^d} \psi(X(s)+x) \cdot g(X(t)+x) \cdot \prod_{j=1}^n \varphi_j(X(r^j)+x)\, dx\Big\} \\ &\quad = \mathbb{E}\Big\{\int_{\mathbb{R}^d} \psi(X(s)-X(t)+y) \cdot g(y) \cdot \prod_{j=1}^n \varphi_j(X(r^j)-X(t)+y)\, dy\Big\} \\ &\quad = \int_{\mathbb{R}^d} \mathbb{E}\big\{\psi(X(s)-X(t)+y)\big\} \cdot g(y) \cdot \mathbb{E}\Big\{\prod_{j=1}^n \varphi_j(X(r^j)-X(t)+y)\Big\}\, dy. \end{aligned}$$

[1] It is worth noting that there are some redundancies in this definition. While Π has 2^N elements, one only needs 2^{N-1} partial orders to totally order $\mathbb{R}^N$. This distinction will not affect our applications and, as such, not deemed important to this discussion.

In the last step, we have used Fubini's theorem, together with the independence of $X(s) - X(t)$ and $\{X(r^j) - X(t);\ j = 1, \ldots, n\}$ under $\mathbb{P}$. By Lemma 5.5, the $\mathbb{P}_{\mathrm{Leb}}$-distribution of $X(t)$ is Leb. This proves (5.5) and, hence, the proposition.□

The last important step in the proof of the upper bound of Theorem 5.1 is the following proposition. Roughly speaking, it states that for each $t \in \mathbb{R}^N_+$, $\mathbb{1}_{\{X(t)=0\}}$ is comparable to a collection of reasonably nice N-parameter martingales, not with respect to probability measures $\mathbb{P}$, but with respect to the σ-finite measure $\mathbb{P}_{\mathrm{Leb}}$.

Proposition 5.9 *Let $\varepsilon > 0$ and $\mu \in \mathcal{P}(E)$ be fixed and recall J from (5.3). Then, for every $A \in \Pi$ and for all $t \in \mathbb{R}^N_+$,*

$$\mathbb{E}_{\mathrm{Leb}}\{J \mid \mathcal{F}^A_t\} \geqslant (4\varepsilon)^{-d}\kappa^{-1} \int_{s \overset{(A)}{\succcurlyeq} t} \mathbb{P}\{|X(t) - X(s)| \leqslant \varepsilon\}\ \mu(ds) \cdot \mathbb{1}_{\{|X(t)| \leqslant \varepsilon/2\}},$$

$\mathbb{P}_{\mathrm{Leb}}$*-almost surely.*

It is *very* important to note that the conditional expectation on the left hand side is computed under the σ-finite measure $\mathbb{P}_{\mathrm{Leb}}$, under which the above holds a.s., while the probability term in the integral is computed with respect to the measure $\mathbb{P}$.

Proof Clearly, for all fixed $t \in \mathbb{R}^N_+$,

$$\begin{aligned} \mathbb{E}_{\mathrm{Leb}}\{J \mid \mathcal{F}^A_t\} &\geqslant (2\epsilon)^{-d}\mathbb{E}_{\mathrm{Leb}}\Big\{ \int_{s \overset{(A)}{\succcurlyeq} t} \mathbb{1}_{\{|X(s)| \leqslant \varepsilon\}}\ \mu(ds)\ \Big|\ \mathcal{F}^A_t\Big\} \\ &= (2\epsilon)^{-d} \int_{s \overset{(A)}{\succcurlyeq} t} \mathbb{P}_{\mathrm{Leb}}\{|X(s)g| \leqslant \varepsilon \mid \mathcal{F}^A_t\}\ \mu(ds), \end{aligned}$$

$\mathbb{P}_{\mathrm{Leb}}$-almost surely. It follows from Proposition 5.8 that

$$\begin{aligned} \mathbb{E}_{\mathrm{Leb}}\{J \mid \mathcal{F}^A_t\} &\geqslant (2\varepsilon)^{-d} \int_{s \overset{(A)}{\succcurlyeq} t} \mathbb{P}_{\mathrm{Leb}}\{|X(s)| \leqslant \varepsilon \mid X(t)\}\ \mu(ds) \\ &\geqslant (2\varepsilon)^{-d} \int_{s \overset{(A)}{\succcurlyeq} t} \mathbb{P}_{\mathrm{Leb}}\{|X(s)| \leqslant \varepsilon \mid X(t)\}\ \mu(ds) \cdot \mathbb{1}_{\{|X(t)| \leqslant \varepsilon/2\}}, \end{aligned}$$

$\mathbb{P}_{\mathrm{Leb}}$-almost surely. On the other hand, for almost all $z \in \mathbb{R}^d$ with $|z| \leqslant \varepsilon/2$,

$$\begin{aligned} \mathbb{P}_{\mathrm{Leb}}\{|X(s)| \leqslant \varepsilon \mid X(t) = z\} &= \mathbb{P}\{|X(t) - X(s) + z| \leqslant \varepsilon\} \\ &\geqslant \mathbb{P}\{|X(t) - X(s)| \leqslant \tfrac{1}{2}\varepsilon\} \\ &\geqslant \kappa^{-1}2^{-d}\mathbb{P}\{|X(t) - X(s)| \leqslant \varepsilon\}. \end{aligned}$$

The first line follows from Lemma 5.5 and the last from weak unimodality. This proves the proposition. □

Proof of Theorem 5.1: Upper Bound Without loss of generality, we may assume that $\mathbb{E}\{\operatorname{Leb} X(E)\} > 0$, for, otherwise, there is nothing to prove. Equivalently, we may assume that

$$\mathbb{P}_{\text{Leb}}\{0 \in \overline{X(E)}\} > 0;$$

cf. the proof of the lower bound of Theorem 5.1.

Since E is compact, it has a countable dense subset, that we assume to be $\mathbb{Q}_+^N$, to keep our notation from becoming overtaxing. Fix $\varepsilon > 0$ and let T_ε denote any measurable selection of $t \in E \cap \mathbb{Q}_+^N$ for which $|X(t)| \leqslant \varepsilon/2$. If such a t does not exist, define $\mathrm{T}_\varepsilon = \Delta$, where $\Delta \in \mathbb{Q}_+^N \setminus E$ but is otherwise chosen quite arbitrarily. It is clear that T_ε is a random vector in $\mathbb{Q}_+^N \cup \Delta$. Define μ_ε by

$$\mu_\varepsilon(\bullet) = \mathbb{P}_{\text{Leb}}\{\mathrm{T}_\varepsilon \in \bullet \mid \mathrm{T}_\varepsilon \in E\}.$$

Clearly, μ_ε is a measure on E. Let L_n denote the restriction of Leb to $[-n,n]^d$. It is not hard to check that for every Borel set $B \subset E$,

$$\mu_\varepsilon(B) = \lim_{n\to\infty} \mathbb{P}_{L_n}\{\mathrm{T}_\varepsilon \in B \mid \mathrm{T}_\varepsilon \in E\},$$

where $\mathbb{P}_{L_n}\{\bullet\} = \int_{\mathbb{R}^d} \mathbb{P}_x\{\bullet\}\, L_n(dx)$. In particular, we have the important observation that $\mu_\varepsilon \in \mathcal{P}(E)$. It is clear that Proposition 5.9 holds simultaneously for all $t \in \mathbb{Q}_+^N$, $\mathbb{P}_{\text{Leb}}$-almost surely. Consequently, since $\mathrm{T}_\varepsilon \in \mathbb{Q}_+^N$, Proposition 5.9 can be applied with $t = \mathrm{T}_\varepsilon$ and $\mu = \mu_\varepsilon$ to yield

$$\begin{aligned}
&\mathbb{E}_{\text{Leb}}\Big\{\Big[\sup_{t\in\mathbb{Q}_+^N} \mathbb{E}_{\text{Leb}}\{J \mid \mathcal{F}_t^A\}\Big]^2\Big\} \\
&\quad\geqslant (4\varepsilon)^{-2d}\kappa^{-2} \int \Big[\int_{s \overset{(A)}{\succcurlyeq} t} \mathbb{P}\{|X(t)-X(s)| \leqslant \varepsilon\}\ \mu_\varepsilon(ds)\Big]^2\ \mu_\varepsilon(dt)\ \times \\
&\qquad\qquad \times \mathbb{P}_{\text{Leb}}\{\mathrm{T}_\varepsilon \in E\} \\
&\quad\geqslant (4\varepsilon)^{-2d}\kappa^{-2}\Big[\int\int_{s \overset{(A)}{\succcurlyeq} t} \mathbb{P}\{|X(t)-X(s)| \leqslant \varepsilon\}\ \mu_\varepsilon(ds)\ \mu_\varepsilon(dt)\Big]^2 \times \\
&\qquad\qquad \times \mathbb{P}_{\text{Leb}}\{\mathrm{T}_\varepsilon \in E\},
\end{aligned}$$

by the Cauchy–Schwarz inequality. By Lemma 5.7, the N-parameter process $t \mapsto \mathbb{E}_{\text{Leb}}\{J \mid \mathcal{F}_t^A\}$ is an N-parameter martingale with respect to the N-parameter, commuting filtration $\mathcal{F}^A$. As such, by the $L^2(\mathbb{P}_{\text{Leb}})$-maximal inequality of Cairoli (cf. Walsh [51]),

$$\mathbb{E}_{\text{Leb}}\Big\{\Big[\sup_{t\in\mathbb{Q}_+^N} \mathbb{E}_{\text{Leb}}\{J \mid \mathcal{F}_t^A\}\Big]^2\Big\} \leqslant 4^N \sup_{t\in\mathbb{R}_+^N} \mathbb{E}_{\text{Leb}}\Big\{\big[\mathbb{E}_{\text{Leb}}\{J \mid \mathcal{F}_t^A\}\big]^2\Big\},$$

which is bounded above by $4^N\mathbb{E}_{\rm Leb}\{J^2\}$, by the Cauchy–Schwarz inequality for conditional expectation under $\mathbb{P}_{\rm Leb}$.[2] Combining this with Eq. (5.4) yields

$$\begin{aligned}
&4^{N+2d}\kappa^3 \int\int \mathbb{P}\big\{|X(s)-X(t)|\leqslant\varepsilon\big\}\ \mu_\varepsilon(ds)\ \mu_\varepsilon(dt)\\
&\geqslant(2\varepsilon)^{-d}\Big[\int\int_{s \overset{(A)}{\succcurlyeq} t} \mathbb{P}\big\{|X(t)-X(s)|\leqslant\varepsilon\big\}\ \mu_\varepsilon(ds)\ \mu_\varepsilon(dt)\Big]^2\times\\
&\qquad\times\mathbb{P}_{\rm Leb}\big\{\mathrm{T}_\varepsilon\in E\big\}.
\end{aligned}$$

For all nonnegative sequences $\{x_A;\ A\in\Pi\}$, $\sum_{A\in\Pi}x_A^2$ is bounded below by $2^{-N}[\sum_{A\in\Pi}x_A]^2$. Thus, one can sum the above displayed inequality over all $A\in\Pi$ and obtain

$$\mathbb{P}_{\rm Leb}\{\mathrm{T}_\varepsilon\in E\}\leqslant\frac{2^{N+d}4^{N+2d}\kappa^3}{(2\varepsilon)^{-d}\int\int\mathbb{P}\big\{|X(s)-X(t)|\leqslant\varepsilon\big\}\ \mu_\varepsilon(ds)\ \mu_\varepsilon(dt)}.$$

As $\varepsilon\to0^+$, the left hand side converges to $\mathbb{P}_{\rm Leb}\{0\in\overline{X(E)}\}=\mathbb{E}\big\{\ {\rm Leb}[X(E)]\big\}$. On the other hand, since $\mu_\varepsilon\in\mathcal{P}(E)$ and since E is compact, by Prohorov's theorem, μ_ε has a subsequential weak limit $\mu_0\in\mathcal{P}(E)$. Consequently, by Fatou's lemma,

$$\mathbb{E}_{\rm Leb}\big\{\ {\rm Leb}[X(E)]\big\}\leqslant\frac{2^{N+d}4^{N+2d}\kappa^3}{\mathcal{E}_\Phi(\mu_0)};$$

see Billingsley [5, Ch 1.6]. This proves Theorem 5.1. □

5.4 Conclusion of the Proof of Theorem 2.12

It suffices to show the upper bound. Suppose there exists $\eta\in]0,1[$ such that $E\subset[\eta,\eta^{-1}]^N$. Then,

$$\begin{aligned}
\mathbb{P}\big\{X^{-1}(0)\cap E\neq\varnothing\big\} &= \int\mathbb{P}\{X(\boldsymbol{\eta})\in dx\}\ \mathbb{P}_x\big\{X^{-1}(0)\cap(E\ominus\boldsymbol{\eta})\neq\varnothing\big\}\\
&\leqslant \Phi(\boldsymbol{\eta})\int\mathbb{P}_x\big\{X^{-1}(0)\cap(E\ominus\boldsymbol{\eta})\neq\varnothing\big\}\ dx\\
&= \Phi(\boldsymbol{\eta})\mathbb{P}_{\rm Leb}\big\{X^{-1}(0)\cap(E\ominus\boldsymbol{\eta})\neq\varnothing\big\}\\
&= \Phi(\boldsymbol{\eta})\mathbb{E}\big\{\ {\rm Leb}[X(E\ominus\boldsymbol{\eta})]\big\},
\end{aligned}$$

where $E\ominus\boldsymbol{\eta}=\{x-\boldsymbol{\eta}:x\in E\}$. The main theorem finally follows from Theorem 5.1 and the simple fact that C_Φ is translation invariant. □

[2] As mentioned earlier, some care is needed. The theory of martingales, as well as that of multiparameter martingales, is often stated with respect to probability measures. However, our intended applications of the theory go through with no essential changes for $\mathbb{P}_{\rm Leb}$.

6 Consequences

In this section, we present some applications of Theorems 2.9 and 2.12. One could also apply the arguments of this section, in conjunction with Theorem 2.10, in order to compute the Hausdorff dimension of the intersection of zero sets and the intersection times of independent additive Lévy processes. We make one such calculation in Example 6.3 below.

6.1 Intersections of Zero Sets

Let $L_1, \ldots, L_k$ denote the zero sets of k independent N-parameter additive Lévy processes. We shall assume that the latter processes are symmetric, absolutely continuous and weakly unimodal in the sense of §2. Let $\Phi_1, \ldots, \Phi_k$ designate their corresponding gauge functions; cf. (2.3). $\square$

Theorem 6.1 *Given the above conditions, the following are equivalent:*

(i) $\mathbb{P}\{L_1 \cap \cdots \cap L_k \cap [c, \infty[^N \neq \varnothing\} > 0$, *for all* $c > 0$*;*

(ii) $\mathbb{P}\{L_1 \cap \cdots \cap L_k \cap [c, \infty[^N \neq \varnothing\} > 0$, *for some* $c > 0$*; and*

(iii) $\prod_{\ell=1}^{k} \Phi_\ell \in L^1_{loc}(\mathbb{R}^N)$.

Moreover, for any $M > 1$*, there exists a constant* $A > 1$*, such that for all compact sets* $E \subset [M^{-1}, M]^N$*,*

$$\frac{1}{A}\, \mathsf{C}_{\prod_{\ell=1}^{k} \Phi_\ell}(E) \leqslant \mu_{\cap_{\ell=1}^{k} L_\ell}(E) \leqslant A\, \mathsf{C}_{\prod_{\ell=1}^{k} \Phi_\ell}(E).$$

Remark 6.2 In the special case $d = N = 1$, one can use the connections to subordinators (mentioned earlier) to show this result; see Bertoin [3] for this and more. In the more general case where $N \geqslant 1$, $\Phi(t) = f(|t|)$ and where f is monotone, one can combine our Theorem 2.9 together with Peres [44, Cor. 15.4] to provide an alternative proof of the first part of Theorem 6.1 above. In the following, our proof of the first part is based on Theorem 2.9 alone.

Proof We need some notation for this proof. For any $1 \leqslant \ell \leqslant k$, let $X_1^\ell, \ldots, X_N^\ell$ denote N independent Lévy processes on $\mathbb{R}^d$ and define $\mathsf{X}_\ell = X_1^\ell \oplus \cdots \oplus X_N^\ell$. By choosing the appropriate X_j^ℓ's, we can ensure that $L_\ell = \mathsf{X}_\ell^{-1}\{0\}$ for all $1 \leqslant \ell \leqslant k$. Let $\mathsf{Y} = \{\mathsf{Y}(t);\ t \in \mathbb{R}_+^N\}$ be the $\mathbb{R}^{dk}$-valued stochastic process defined by

$$\mathsf{Y}(t) = \mathsf{X}_1(t) \otimes \cdots \otimes \mathsf{X}_k(t), \qquad t \in \mathbb{R}_+^N,$$

in tensor notation. For each $a \in \mathbb{R}^{dk}$ we write it in tensor notation as $a = a^1 \otimes \cdots \otimes a^k$, where $a^\ell \in \mathbb{R}^d$, for all $1 \leqslant \ell \leqslant k$. Suppose the Lévy exponent of X_j^ℓ is denoted by Ψ_j^ℓ. Then, the characteristic exponent of X_ℓ is $\Psi^\ell = \Psi_1^\ell \otimes \cdots \otimes \Psi_N^\ell$ and the characteristic exponent of $\mathsf{Y}(t)$ is $\sum_{\ell=1}^{k} \Psi^\ell$. It should now be clear that

Y is a symmetric, absolutely continuous additive Lévy process; it takes its values in $\mathbb{R}^{dk}$, and the density function $p(t;\bullet)$ of $\mathsf{Y}(t)$ is

$$p(t;x) = (2\pi)^{-dk} \int_{\mathbb{R}^{dk}} e^{-ix\cdot\xi} \prod_{\ell=1}^{k} \mathbb{E}\big[\exp\{i\xi^\ell \cdot \mathsf{X}_\ell(t)\}\big]\, d\xi, \qquad t \in \mathbb{R}_+^N,$$

where $\xi = \xi^1 \otimes \cdots \otimes \xi^k \in \mathbb{R}^{dk}$, in tensor notation. In particular, if Φ_ℓ denotes the gauge function for X_ℓ and Φ denotes the gauge function for Y, then $\Phi(t) = \prod_{\ell=1}^k \Phi_\ell(t)$, for all $t \in \mathbb{R}_+^N$. It remains to verify weak unimodality. For any $t \in \mathbb{R}_+^N \setminus \partial\mathbb{R}_+^N$, $a = a^1 \otimes \cdots \otimes a^k \in \mathbb{R}^{dk}$ and any $r > 0$, we have

$$\begin{aligned}
\mathbb{P}\{|Y(t) - a| \leqslant r\} &\leqslant \mathbb{P}\{|\mathsf{X}_1(t) - a^1| \leqslant r, \ldots, |\mathsf{X}_k(t) - a^k| \leqslant rr\} \\
&\leqslant \kappa^k \prod_{\ell=1}^{k} \mathbb{P}\{|\mathsf{X}_\ell(t)| \leqslant r\} \\
&= \kappa^k \mathbb{P}\{|Y(t)| \leqslant r\}.
\end{aligned}$$

Therefore, Theorem 2.9 implies the equivalence of (i)–(iii). To prove the asserted inequality, we can apply Corollary 2.13, and note that by Lemma 2.2(ii), $\inf_{s\in[M^{-1},M]} p(s;0) > 0$. □

Example 6.3 As an instructive example, let us consider $L_1, \ldots, L_k$ to be the zero sets of k independent processes of the type considered in Theorem 1.1. Let $\alpha_1, \ldots, \alpha_k \in]0, 2]$ denote the corresponding stable indices. Our proof of the latter theorem shows us that the Lévy exponent, $\Psi_\ell(t)$, of the ℓth process is bounded above and below by a constant multiple of $|t|^{-d/\alpha_\ell}$. By Theorem 6.1 and by Lemma 2.2(i), $\cap_{\ell=1}^k L_\ell$ is not a.s. empty if and only if $\int_{|t|\leqslant 1} |t|^{-d\sum_{\ell=1}^k \alpha_\ell^{-1}}\, dt < \infty$, which, upon calculating in polar coordinates, is seen to be equivalent to the condition: $N > d\sum_{\ell=1}^k \frac{1}{\alpha_\ell}$. Moreover, if $\alpha_\ell = \alpha$ for $\ell = 1, \ldots, k$ and $N > kd/\alpha$, then by Theorem 2.10,

$$\mathbb{P}\big\{\dim_{\mathsf{H}}(\cap_{\ell=1}^k L_\ell \cap [c, C]^N) = N - \tfrac{1}{\alpha} kd\big\} > 0,$$

for all $0 < c < C < \infty$. □

6.2 Intersections of the Sample Paths

In this subsection, we apply Theorem 2.9 to study the intersections of the sample paths of k independent N-parameter additive Lévy processes. We will use the same notations as in Subsection 6.1.

Let $\mathsf{X}_1, \ldots, \mathsf{X}_k$ be k independent N-parameter absolutely continuous additive Lévy processes in $\mathbb{R}^d$. Recall that for each $1 \leqslant \ell \leqslant k$, $\mathsf{X}_\ell = X_1^\ell \oplus \cdots \oplus X_N^\ell$, where X_j^ℓ's are independent symmetric $\mathbb{R}^d$-valued Lévy processes with exponents Ψ_j^ℓ, respectively. We will also need the additive Lévy process Z in the proof of Theorem 6.4 to be weakly unimodal. This follows, for example, if for all $1 \leqslant \ell \leqslant k$ and $t \in \mathbb{R}_+^N \setminus \partial\mathbb{R}_+^N$, the distribution of $\mathsf{X}_\ell(t)$ is self-decomposable. For $\widetilde{s} \in \mathbb{R}^{kN}$, we write $\widetilde{s} = s^1 \otimes \cdots \otimes s^k$, where $s^\ell \in \mathbb{R}^N$ for all $1 \leqslant \ell \leqslant k$. For all $\widetilde{s} \in \mathbb{R}^{kN}$, we define

$$\overline{\Phi}(\widetilde{s}) = (2\pi)^{-d(k-1)} \times \tag{6.1}$$
$$\times \int_{\mathbb{R}^{d(k-1)}} \exp\Big\{ - \sum_{j=1}^N |s_j^1| \Psi_j^1 (\sum_{\ell=1}^{k-1} v^\ell) - \sum_{j=1}^N \sum_{\ell=1}^{k-1} |s_j^{\ell+1}| \Psi_j^{\ell+1}(v^\ell) \Big\} \, d\widetilde{v}.$$

Theorem 6.4 *Under the above conditions, the sample paths of* $\mathsf{X}_1, \ldots, \mathsf{X}_k$ *intersect with positive probability if and only if* $\overline{\Phi} \in L^1_{loc}(\mathbb{R}^{kN})$.

Proof Let $\mathsf{Z} = \big\{ \mathsf{Z}(\widetilde{t});\ \widetilde{t} \in \mathbb{R}_+^{kN} \big\}$ be the stochastic process defined by

$$\mathsf{Z}(\widetilde{t}) = \big(\mathsf{X}_2(t^2) - \mathsf{X}_1(t^1) \big) \otimes \cdots \otimes \big(\mathsf{X}_k(t^k) - \mathsf{X}_{k-1}(t^{k-1}) \big), \qquad \widetilde{t} \in \mathbb{R}_+^{kN}.$$

We observe that the sample paths of $\mathsf{X}_1, \ldots, \mathsf{X}_k$ intersect if and only if $\mathsf{Z}^{-1}(0)$ is nonempty. We now relate the zero set of Z to our previous theorems.

It is not hard to see that Z is a symmetric additive Lévy process. Indeed, $\mathsf{Z}(\widetilde{t})$ equals

$$\big(-\mathsf{X}_1(t^1), 0, \ldots, 0 \big) + \big(\mathsf{X}_2(t^2), -\mathsf{X}_2(t^2), 0, \ldots, 0 \big) + \cdots + \big(0, \ldots, 0, \mathsf{X}_k(t^k) \big),$$

which is a sum of k independent, symmetric and self-decomposable $\mathbb{R}^{d(k-1)}$-valued random vectors. Hence, Z is weakly unimodal. Moreover, since direct sums of independent additive Lévy processes are themselves additive Lévy processes, Z is a symmetric, weakly unimodal additive Lévy process. Finally, a direct calculation reveals that Z is absolutely continuous. Moreover, $\mathsf{Z}(\widetilde{t})$ has a continuous density for each $\widetilde{t} \in \mathbb{R}_+^{kN} \backslash \partial \mathbb{R}_+^{kN}$ and the gauge function $\overline{\Phi}$ of Z is given by (6.1). Hence, Theorem 6.4 follows from Theorem 2.9. □

When $\mathsf{X}_1, \ldots, \mathsf{X}_k$ are k independent N-parameter additive stable Lévy processes, Theorem 6.4 implies the following corollary.

Corollary 6.5 *Let* $\mathsf{X}_1, \ldots, \mathsf{X}_k$ *be k independent N-parameter additive isotropic stable Lévy processes in* $\mathbb{R}^d$ *with indices* $\alpha_\ell \in (0, 2]$ $(\ell = 1, \ldots, k)$*, respectively. Then, the sample paths of* $\mathsf{X}_1, \ldots, \mathsf{X}_k$ *intersect with positive probability if and only if for every* $1 \leqslant j \leqslant k$, $N \sum_{\ell=1}^j \alpha_\ell > d(j-1)$.

Proof Recall that $\Psi_j^\ell(v^\ell) = \chi_j^\ell \|v^\ell\|^{\alpha_\ell}$, where $\chi_j^\ell > 0$ are constants. For simplicity of notations, we assume that $\chi_j^\ell = 1$ for all ℓ and j. It follows from Fubini's theorem that for any constant $T > 0$,

$$\int_{[0,T]^{kN}} \overline{\Phi}(\widetilde{s}) d\widetilde{s} = (2\pi)^{-d(k-1)} \times$$
$$\times \int_{\mathbb{R}^{d(k-1)}} \frac{1}{\| \sum_{\ell=1}^{k-1} v^\ell \|^{\alpha_1 N}} \Big(1 - e^{-T \| \sum_{\ell=1}^{k-1} v^\ell \|^{\alpha_1}} \Big)^N \times \tag{6.2}$$
$$\times \prod_{\ell=1}^{k-1} \frac{1}{\|v^\ell\|^{\alpha_{\ell+1} N}} \Big(1 - e^{-T \|v^\ell\|^{\alpha_{\ell+1}}} \Big)^N \, d\widetilde{v}.$$

If there exists a $j \leqslant k$ such that $N\sum_{\ell=1}^{j}\alpha_\ell \leqslant d(j-1)$, we write the integral on the right hand side of (6.2) as

$$\begin{aligned}\int_{\mathbb{R}^{d(k-j)}} \prod_{\ell=j}^{k-1} \frac{1}{\|v^\ell\|^{\alpha_{\ell+1}N}} \Big(1-e^{-T\|v^\ell\|^{\alpha_{\ell+1}}}\Big)^N dv^j \cdots dv^{k-1} \times \\ \times \int_{\mathbb{R}^{d(j-1)}} \frac{1}{\|\sum_{\ell=1}^{k-1} v^\ell\|^{\alpha_1 N}} \Big(1-e^{-T\|\sum_{\ell=1}^{k-1} v^\ell\|^{\alpha_1}}\Big)^N \times \\ \times \prod_{\ell=1}^{j-1} \frac{1}{\|v^\ell\|^{\alpha_{\ell+1}N}} \Big(1-e^{-T\|v^\ell\|^{\alpha_{\ell+1}}}\Big)^N \, dv^1 \cdots\, dv^{j-1}.\end{aligned} \tag{6.3}$$

By using spherical coordinates, we see that for every $(v^j, \ldots, v^{k-1}) \in \mathbb{R}^{d(k-j)}$, the inside integral in (6.3) is infinite. Hence, Theorem 6.4 implies that almost surely the sample paths of $\mathsf{X}_1, \ldots, \mathsf{X}_k$ do not intersect.

Now we assume that $N\sum_{\ell=1}^{j}\alpha_\ell > d(j-1)$ for $j = 1, \ldots, k$. In order to show the integral in (6.2) is finite, we first note that if $N\alpha_j > d$ for some $j \leqslant k$ (say $N\alpha_k > d$) then Theorem 1.1 implies that X_k hits every fixed point with positive probability and, hence, it will also hit the intersection points of $\mathsf{X}_1, \ldots, \mathsf{X}_{k-1}$ (when the latter is not empty) with positive probability. Therefore, without loss of generality, we may and will assume $N\alpha_j \leqslant d$ for $j = 1, \ldots, k$.

In addition, we will make use of the following generalized Hölder's inequality: If h_j $(j = 1, 2, \ldots, k)$ are nonnegative functions on $\mathbb{R}^m$ and $p_j > 1$ $(j = 1, 2, \ldots, k)$ such that $\sum_{j=1}^{k} 1/p_j = 1$, then

$$\int_{\mathbb{R}^m} \prod_{j=1}^{k} h_j(x)dx \;\leqslant\; \prod_{j=1}^{k} \Big[\int_{\mathbb{R}^m} h_j(x)^{p_j}\, dx\Big]^{1/p_j}.$$

For each $j = 1, \ldots, k$, denote

$$\begin{aligned}\beta_j &= \frac{N}{k-1}\Big(\sum_{\ell=1}^{k}\alpha_\ell - (k-1)\alpha_j\Big)\\ p_j &= \frac{\sum_{\ell=1}^{k}\alpha_\ell}{\sum_{\ell=1}^{k}\alpha_\ell - (k-1)\alpha_j}.\end{aligned}$$

Since $N\alpha_j \leqslant d$, $\beta_j > 0$, for each $1 \leqslant j \leqslant k$. Moreover, $p_j > 1$ and

$$\sum_{\ell \neq j} \beta_\ell = N\alpha_j, \qquad\qquad \sum_{\ell=1}^{k} \frac{1}{p_\ell} = 1.$$

Hence, we can write the integrand on the right hand side of (6.2) as

$$\prod_{j=1}^{k} \prod_{\ell \neq j} \frac{1}{\|u^\ell\|^{\beta_j}} \big(1-e^{-T\|u^\ell\|^{\alpha_\ell}}\big)^{\beta_j/\alpha_\ell},$$

where $u^1 = \sum_{\ell=1}^{k-1} v^\ell,\ u^\ell = v^{\ell+1}$ for $\ell = 1, \ldots, k-1$. Hence, by the generalized Hölder's inequality, we see that the integral in (6.2) is bounded above by

$$\begin{aligned}
&\prod_{j=1}^{k}\Big[\int_{\mathbb{R}^{d(k-1)}} \prod_{\ell\neq j} \frac{1}{\|u^\ell\|^{\beta_j p_j}}\ \big(1-e^{-T\|u^\ell\|^{\alpha_\ell}}\big)^{\beta_j p_j/\alpha_\ell} d\widetilde{v}\Big]^{1/p_j} \\
&\quad = \prod_{j=1}^{k}\Big[\int_{\mathbb{R}^{d(k-1)}} \prod_{\ell\neq j} \frac{1}{\|u^\ell\|^{\beta_j p_j}}\ \big(1-e^{-T\|u^\ell\|^{\alpha_\ell}}\big)^{\beta_j p_j/\alpha_\ell} d\widetilde{u}\Big]^{1/p_j}.
\end{aligned} \tag{6.4}$$

The last equality follows from the fact that for each j, the linear operator $(v^1, \ldots, v^{k-1}) \mapsto (u^\ell,\ \ell \neq j)$ on $\mathbb{R}^{d(k-1)}$ is nonsingular with Jacobian 1. Since $\beta_j p_j > d$ for each $j = 1, \ldots, k$, we see that all the integrals in (6.4) are finite. This proves that $\overline{\Phi} \in L^1_{loc}(\mathbb{R}^{kN})$, and Corollary 6.5 follows. □

Remark 6.6 When $N = 1$, Theorem 6.4 describes the following necessary and sufficient condition for the intersections of k independent, symmetric, absolutely continuous, self-decomposable Lévy processes in terms of their Lévy exponents Ψ^ℓ ($\ell = 1, \ldots, k$): there exists some $T > 0$, for which the following integral is finite

$$\int_{\mathbb{R}^{d(k-1)}} \frac{1}{\Psi^1(\sum_{\ell=1}^{k-1} v^\ell)} \left(1 - e^{-T\Psi^1(\sum_{\ell=1}^{k-1} v^\ell)}\right) \cdot \prod_{\ell=1}^{k-1} \frac{1}{\Psi^{\ell+1}(v^\ell)} \left(1 - e^{-T\Psi^{\ell+1}(v^\ell)}\right) d\widetilde{v}.$$

Since the Ψ^ℓ's are nonnegative, we can use the monotone convergence theorem and conclude that k independent, symmetric and absolutely continuous Lévy processes with exponents $\Psi^1, \ldots, \Psi^k$ intersect if and only if

$$\int_{\mathbb{R}^{d(k-1)}} \frac{1}{1 + \Psi^1(\sum_{\ell=1}^{k-1} v^\ell)} \cdot \prod_{\ell=1}^{k-1} \frac{1}{1 + \Psi^{\ell+1}(v^\ell)}\ d\widetilde{v} < \infty. \tag{6.5}$$

When $N = 1$, Eq. (6.5) provides a necessary and sufficient condition for intersections of k independent symmetric, absolutely continuous, weakly unimodal Lévy processes. That is, when $N = 1$, our condition (6.5) agrees with the necessary and sufficient conditions of Fitzsimmons and Salisbury [19], Hirsch [27] and Hirsch and Song [28, 29], specialized to the Lévy processes of the type considered in this paper. For earlier (partial) results, when $N = 1$, see LeGall, Rosen and Shieh [37] and Evans [16].

In the special case of k independent isotropic stable Lévy processes with indices $\alpha_\ell \in (0, 2]$ ($\ell = 1, \ldots, k$), respectively, and $\alpha_1 \leqslant \cdots \leqslant \alpha_k$, Corollary 6.5 implies that their sample paths intersect with positive probability if and only if $\sum_{\ell=1}^{j} \alpha_\ell > d(j-1)$ for every $1 \leqslant j \leqslant k$. This result was essentially proved by Taylor in [49] and by Fristedt [22] for k independent isotropic stable Lévy processes with the *same* index $\alpha \in]0, 2]$. □

We conclude this subsection with the following simple example.

Example 6.7 Consider 2 independent, isotropic stable Lévy processes on $\mathbb{R}^d$: $X_1 = \{X_1(t);\ t \geqslant 0\}$ and $X_2 = \{X_2(t);\ t \geqslant 0\}$. Let α_i denote the index of X_i, where $i = 1, 2$. We define the 2-parameter additive process $X = \{X(t);\ t \in \mathbb{R}^2_+\}$ by $X(t) = X_1(t_1) - X_2(t_1)$. By symmetry, this is a special case of (1.1). Clearly,

$$X^{-1}\{0\} = \{(s,t) \in \mathbb{R}^2_+ :\ X_1(s) = X_2(t)\},$$

is the collection of all intersection times for X_1 and X_2. Thus, the paths of X_1 and X_2 intersect nontrivially (i.e., at points other than the origin) if and only if $\alpha_1 + \alpha_2 > d$. To specialize further, choose $\alpha_1 = \alpha_2 = 2$ to recover the classical fact that two independent Brownian paths in $\mathbb{R}^d$ cross if and only if $d < 4$; see Dvoretzky, Erdős and Kakutani [13] and Dvoretzky, Erdős, Kakutani and Taylor [14]. Next, consider an independent copy Y of X. Another application of Example 6.3 above shows that $X^{-1}\{0\} \cap Y^{-1}\{0\}$ is nonvoid if and only if $d < 2$. That is, while in dimensions 2 and 3, two Brownian paths intersect, their intersection points are too thin to hit an independent copy of themselves. □

6.3 Lebesgue's Measure

Let $X = \{X(t);\ t \in \mathbb{R}^N_+\}$ denote an N-parameter stochastic process that takes its values in $\mathbb{R}^d$. The following question has a long history:

> "*Given that $N > 1$, when is it possible that* $\mathrm{Leb}\{X(E)\} > 0$?".

Some results related to this question can be found in Evans [15], Kahane [31, Th. 5, §6, Ch. 16] and Mountford [41] and their combined references. In the special case when $N = 2$ and X is additive Brownian motion, the above question is answered in the affirmative by Khoshnevisan [34]. We can apply Theorem 5.1 to give a comprehensive and immediate answer to the mentioned question for any $N \geqslant 1$, in case X is any of the additive Lévy processes of the present paper.

Corollary 6.8 *Suppose X is an N-parameter, $\mathbb{R}^d$-valued, symmetric, weak unimodal and absolutely continuous additive Lévy process with gauge function Φ. Then, for any given compact set $E \subset \mathbb{R}^N_+$, the following are equivalent,*

(i) $\mathbb{P}[\mathrm{Leb}\{X(E)\} > 0] = 1$;

(ii) $\mathbb{P}[\mathrm{Leb}\{X(E)\} > 0] > 0$;

(iii) $\mathsf{C}_\Phi(E) > 0$.

By symmetrization, one also obtains the following extension of the results of Evans [15] and Mountford [41] to the multiparameter setting. For simplicity, we will assume the distributions of $X_1, \ldots, X_N$ to be self-decomposable.

Corollary 6.9 *Suppose $X_1, \ldots, X_N$ are $\mathbb{R}^d$-valued self-decomposable Lévy processes such that the N-parameter additive Lévy process X given by (1.1) is absolutely continuous. The following are equivalent:*

(i) for all Borel measurable functions $f : \mathbb{R}_+^N \to \mathbb{R}^d$,

$$\mathbb{P}\big[\mathrm{Leb}\{(X+f)([c,\infty[^N)\} > 0\big] = 1, \quad \text{for all} \quad c > 0;$$

(ii) for all Borel measurable functions $f : \mathbb{R}_+^N \to \mathbb{R}^d$,

$$\mathbb{P}\big[\mathrm{Leb}\{(X+f)([c,\infty[^N)\} > 0\big] > 0, \quad \text{for all} \quad c > 0;$$

(iii) $\mathsf{C}_\Phi([0,1]^N) > 0$, where Φ denotes the gauge function for $X - X'$, where X' is an independent copy of X. That is, if Ψ_j denotes the Lévy exponent of X_j, then for all $s \in \mathbb{R}^N$,

$$\Phi(s) = (2\pi)^{-d} \int_{\mathbb{R}^d} \exp\Big\{ -2 \sum_{\ell=1}^N |s_\ell| \ \mathrm{Re}\Psi_\ell(\xi)\Big\} \ d\xi.$$

Proof The proof is very similar to that of Evans [15]: using symmetrization and Theorem 6.8 for (*ii*) $\Rightarrow$ (*iii*) and Kahane's argument for (*iii*) $\Rightarrow$ (*i*). We omit the details.

References

[1] T. W. Anderson (1955). The integral of a symmetric unimodal functions over a symmetric convex set and some probability inequalities, *Proc. Amer. Math. Soc.* **6**, 170–176.

[2] R. F. Bass (1995). *Probabilistic Techniques in Analysis*, Springer–Verlage

[3] J. Bertoin (1999). *Subordinators: Examples and Applications*, Ecole d'été de Probabilités de St.-Flour, 1997, Springer, Berlin (To appear).

[4] J. Bertoin (1996). *Lévy Processes*, Cambridge Tracts in Mathematics, Cambridge, U.K.

[5] P. Billingsley (1968). *Convergence of Probability Measures*, J. Wiley & Sons, Inc. , New York.

[6] R. Cairoli (1968). Une inegalité pour martingales à indices multiples et ses applications, *Séminaire de Probabilités IV*, 1–27.

[7] R. C. Dalang and T. S. Mountford (1997). Points of increase of the Brownian sheet, *Prob. Th. Rel. Fields*, **108**(1), 1–27.

[8] R. C. Dalang and T. S. Mountford (1996). Nondifferentiability of curves on the Brownian sheet, *Ann. Probab.*, **24**(1), 182–195.

[9] R. C. Dalang and J. B. Walsh (1993). Geography of the level sets of the Brownian sheet, *Prob. Th. Rel. Fields*, **96**(2), 153–176.

[10] R. C. Dalang and J. B. Walsh (1993). The structure of a Brownian bubble, *Prob. Th. Rel. Fields*, **96**(4), 475–501.

[11] C. Dellacherie and P.-A. Meyer (1975). *Probabilities and Potential.* North–Holland Publishing Co., Amsterdam.

[12] C. Dellacherie and P.-A. Meyer (1982). *Probabilities and Potential, B.* Theory of Martingales. North–Holland Publishing Co., Amsterdam. (J. P. Wilson, Trans.)

[13] A. Dvoretzky, P. Erdős and S. Kakutani (1950). Double points of paths of Brownian motion in n-space, *Acta. Sci. Math. Szeged*, **12**, 75–81.

[14] A. Dvoretzky, P. Erdős, S. Kakutani and S. J. Taylor (1957). Triple points of Brownian motion in 3-space, *Proc. Camb. Phil. Soc.*, **53**, 856–862.

[15] S. N. Evans (1989). The range of a perturbed Lévy process, *Prob. Th. Rel. Fields*, **81**, 555–557.

[16] S. N. Evans (1987). Multiple points in the sample paths of a Lévy process, *Prob. Th. Rel. Fields*, **76**(3), 359–367.

[17] S. N. Evans (1987). Potential Theory for a family of several Markov processes, *Ann. Inst. Henri Poincaré Prob. Stat.*, **23**(3), 499–530.

[18] P. J. Fitzsimmons, B. Fristedt and B. Maisonneuve (1985). Intersections and limits of regenerative sets, *Z. Wahr. Verw. Geb.*, **70**(2), 157–173.

[19] P. J. Fitzsimmons and T. S. Salisbury (1989). Capacity and energy for multiparameter processes, *Ann. Inst. Henri Poincaré Prob. Stat.*, **25**, 325–350.

[20] B. Fristedt (1996). Intersections and limits of regenerative sets, *Random Discrete Structures*, 121–151, IMA Vol. Math. Appl., **76**, Springer, New York.

[21] B. Fristedt (1974). Sample functions of stochastic processes with stationary, independent increments, *Advances in Probability and Related Topics*, **3**, 241–396, Dekker, New York.

[22] B. Fristedt (1967). An extension of a theorem of S. J. Taylor concerning the multiple points of the symmetric stable process, *Z. Wahr. verw. Geb.*, **9**, 62–64.

[23] J. Hawkes (1978). Multiple points for symmetric Lévy processes, *Math. Proc. Camb. Phil. Soc.*, **83**, 83–90.

[24] J. Hawkes (1977). Local Properties of some Gaussian Processes, *Z. Wahr. verw. Geb.*, **40**, 309–315.

[25] J. Hawkes (1977). Intersections of Markov Processes, *Z. Wahr. verw. Geb.*, **37**, 243–251.

[26] J. Hawkes (1975). On the potential theory of subordinators, *Z. Wahr. verw. Geb.*, **33**, 113–132.

[27] F. Hirsch (1995). Potential Theory related to some multiparameter processes, *Potential Analysis*, **4**, 245–267.

[28] F. Hirsch and S. Song (1995). Symmetric Skorohod topology on n–variable functions and hierarchical Markov properties of n-parameter processes, *Prob. Th. Rel. Fields*, **103**(1), 25–43.

[29] F. Hirsch and S. Song (1995). Markov properties of multiparameter processes and capacities, *Prob. Th. Rel. Fields*, **103**(1), 45–71.

[30] J.-P. Kahane (1983). Points multiples des processus de Lévy symmetriques stables restreints á Un ensemble de valeurs du temps, *Publ. Math. Orsay* **(83-02)**, 74–105.

[31] J.-P. KAHANE (1985). *Some Random Series of Functions*, Cambridge Studies in Advanced Mathematics, Cambridge, U.K.

[32] M. KANTER (1977). Unimodality and dominance of symmetric random vectors, *Trans. Amer. Math. Soc.* **229**, 65–86.

[33] W. S. KENDALL (1980). Contours of Brownian process with several–dimensional times, *Z. Wahr. Verw. Geb.*, **52**, 267–276.

[34] D. KHOSHNEVISAN (1999). Brownian sheet and Bessel-Riesz capacity, *Trans. Amer. Math. Soc.*, **351**(7), 2607–2622.

[35] D. KHOSHNEVISAN AND Z. SHI (1999). Brownian sheet and capacity, *The Annals of Prob.* (To appear).

[36] P. LÉVY (1954). *Théorie de L'addition des Variables Aléatoires.* Gauthier–Villars, Paris.

[37] J.-F. LEGALL, J. ROSEN AND N.-R. SHIEH (1989) Multiple points of Lévy processes, *Ann. Probab.* **17**, 503–515.

[38] G. MATHERON (1975). *Random Sets and Integral Geometry*, J. Wiley and Sons, New York.

[39] P. MEDGYESSY (1967). On a new class of unimodal infinitely divisible distribution functions and related topics, *Stud. Sci. Math. Hungar.* **2**, 441–446.

[40] T. S. MOUNTFORD (1993). Estimates of the Hausdorff dimension of the boundary of positive Brownian sheet components, *Séminaire de Probabilités, XXVII*, 233–255, Lecture Notes in Math., #1557, Springer, Berlin.

[41] T. S. MOUNTFORD (1989). Time inhomogeneous Markov processes and the polarity of single points, *Ann. Probab.*, **17**(2), 573–585.

[42] R. PEMANTLE, Y. PERES AND J. SHAPIRO (1996) The trace of spatial Brownian motion is capacity-equivalent to the unit square, *Probab. Theory and Related Fields*, **106**, 379– 399.

[43] Y. PERES (1996). Intersection–equivalence of Brownian paths and certain branching processes, *Comm. Math. Phys.*, **177**, 99–114.

[44] Y. PERES (1999). *Probability on Trees: An Introductory Climb*, Ecole d'été de Probabilités de St.-Flour, 1997, Springer, Berlin (To appear).

[45] J. REN (1990). Topologie p-fine sure l'espace de Wiener et théoréme des fonctions implicites, *Bull. Sci. math.*, 2^{e} séries, **114**, 99-114.

[46] T. S. SALISBURY (1996). Energy, and intersection of Markov chains, *Random Discrete Structures*, 213–225, IMA Vol. Math. Appl., **76**, Springer, Berlin.

[47] K.-I. SATO (1980). Class L of multivariate distributions and its subclasses, *J. Multivar. Anal.* **10**, 207–232.

[48] D. STOYAN (1998). Random sets: models and statistics, *International Statistical Review*, Vol. 66(1), 1–28.

[49] S. J. TAYLOR (1966). Multiple points for the sample paths of the symmetric stable processes, *Z. Wahr. Verw. Geb.*, **5**, 247–264.

[50] S. J. TAYLOR (1967). Sample path properties of a transient stable processes, *J. Math. Mech.* **16**, 1229–1246.

[51] J. B. Walsh (1986). *Martingales with a Multidimensional Parameter and Stochastic Integrals in the Plane*, Lecture Notes in Math. #1215, Springer, Berlin, Ed's: G. del Piño and R. Robodello.

[52] S. J. Wolfe (1978). On the unimodality of infinitely divisible distribution functions, *Z. Wahr. Verw. Geb.* **45**, 329–335.

[53] S. J. Wolfe (1978). On the unimodality of multivariate symmetric distribution functions of class L, *J. Multivar. Anal.* **8**, 141–145.

[54] S. J. Wolfe (1981). On the unimodality of infinitely divisible distribution functions II, In: *Analytical Methods in Probability Theory (Oberwolfach, 1980)*, *Lect. Notes in Math.* **861**, 178–183, Springer-Verlag, Berlin.

[55] M. Yamazato (1978). Unimodality of infinitely divisible distribution functions of class L, *Ann. of Probab.* **6**, 523–531.