INTERNATIONAL ECONOMIC REVIEW
Vol. 9, No. 2, June, 1968

SOME ASPECTS OF THE RANDOM WALK MODEL OF STOCK MARKET PRICES*

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1. INTRODUCTION

THE RANDOM WALK MODEL for stock market price series states that the simple model

$$P_t = P_{t-1} + \varepsilon_t$$

fits such data very well, where P_t is the price series and ε_t is a purely random, white noise series such that ε_t and ε_s are uncorrelated for all $t \neq s$. This model was originally suggested by Bachelier [1] and re-introduced by Osborne [10]. It is now well founded as indicated by many of the papers in the book edited by Cootner [2] and by subsequent papers, such as those by Fama [4] and Godfrey, Granger and Morgenstern [6]. The model has been tested for many series, both indices and individual stocks, for many countries and a variety of sampling units. A number of different statistical techniques have been used, and almost without exception the random walk model has been found to be a very adequate model for the series. On some occasions the price series is replaced by the logarithm of the series, particularly when data over a time-span of several years is used.

The random walk model is, of course, too simple to have been fully accepted and, in fact, a number of modifications to it have been suggested (see [2]). Thus, for example, Granger and Morgenstern [7] found that the long-term movements in the series cannot be fully explained by such a model.

In this paper, two specific aspects of the model will be considered. The first concerns the question of whether or not ε_t is a purely random series if individual transactions data is considered. The second concerns the suggestion by Mandelbrot (in [2]), and supported by Fama (in [2], [5]) that the model is correct but that the variance of ε_t is infinite.

2. TRANSACTIONS DATA

For the New York Stock Exchange, data is available for every single transaction for each stock marketed, the data being the size of the transaction and the price at which it occurs. Let $P_o(t)$ be the opening price on the t-th day and $P_j(t)$ the price at which the j-th transaction occurs on this day. Denote by $P_c(t)$ the closing price for the day. If the price change series is denoted by

$$P_j(t) - P_{j-1}(t) = X_j(t) ,$$

the question arises as to the properties of the series $X_j(t)$.

The study by Godfrey, Granger and Morgenstern [6] suggested that $X_j(t)$

* Manuscript received March 17, 1966, revised October 17, 1966.

was also a purely random series but a more detailed study by Niederhoffer and Osborne [9] using different data came to the conclusion that the series was not white noise, having some significantly non-zero autocorrelations. If this latter conclusion is correct, it is worth enquiring whether or not it is contrary to the random walk model using sampling units of an hour, a day, a week etc.

Suppose, for ease of exposition, that $X_j(t)$ is a Markov series with autocorrelation sequence

$$\text{correlation } (X_j(t), X_{j-\tau}(t)) = \rho^\tau , \qquad j = 0, 1, 2, \cdots ; |\rho| < 1 .$$

Further suppose that σ^2, the variance of $X_j(t)$, is finite and constant for all j, t, in the period considered.

If the mean of $X_j(t)$ is zero for all j, t, then the series of daily closing minus opening prices $P_c(t) - P_o(t)$ will have mean zero and variance $n(t) \sum_\tau (1 - \tau/n(t))\rho^\tau$, where $n(t)$ is the *number* of transactions for the t-th day. This variance is well approximated by $n(t)(1 - \rho)^{-1}$ for $n(t)$ large. Suppose that the series $P_o(t + 1) - P_c(t)$, i.e. the overnight change in price has mean zero and variance $k(t)\sigma^2/(1 - \rho)$, so that there has been the equivalent of $k(t)$ transactions during the night. In [6] it was found that not only was $k(t)$ not zero but was in fact of the same order as $n(t)$.

Thus, the daily change in price, opening to opening, $P_o(t + 1) - P_o(t)$ is the equivalent of the sum of $N(t) = n(t) + k(t)$ transactions.

To make the exposition simpler, consider initially the case where $N(t) = N$ is a constant. Denote the series of changes between the opening prices on adjacent days by

$$C(t) = P_o(t) - P_o(t - 1) .$$

The above assumptions ensure that the mean of $C(t)$ is zero and that the variance is approximately $N\sigma^2/(1 - \rho)$.

The first autocovariance is given by

$$\begin{aligned} &\mathrm{E}[C(t)C(t-1)] \\ &\quad = \sigma^2[\rho + 2\rho^2 + 3\rho^3 + \cdots + N\rho^N + (N-1)\rho^{N+1} + \cdots + 2\rho^{2N-2} + \rho^{2N-1}] . \end{aligned}$$

If ρ is not near one, this can be well approximated by

$$\sigma^2 a(\rho) = \sigma^2 \sum_{k=1}^{\infty} k\rho^k = \sigma^2\left[\frac{\rho}{(1-\rho)^2} - 1\right] .$$

The other autocovariances will be approximately given by

$$\mathrm{E}[C(t)C(t - \tau)] = \sigma^2\rho^{(\tau-1)N}a(\rho) .$$

Thus, the autocorrelation sequence for the series $C(t)$ is approximately

$$r_\tau = \frac{\rho^{(\tau-1)N}a(\rho)(1 - \rho)}{N} , \qquad \tau \neq 0 ,$$

which, for large N, closely approximates the autocorrelation sequence for white noise, which is

$$\rho_\tau = 1 \qquad \tau = 0\,,$$
$$\rho_\tau = 0 \qquad \tau \neq 0\,.$$

The approximations used to achieve this result are generally excellent, for realistically sized N and the various assumptions made can be relaxed without appreciably affecting the result. Thus, for instance, the Markov assumption could be replaced by a series from an autoregressive model by noting that the covariance sequence from such a model may always be written in the form

$$r_\tau = \sum_{j=1}^{m} a_j \rho_j^\tau\,.$$

Provided $\max_j |\rho_j|$ is not near one, the approximation of the r_τ sequence to that of a white noise process still holds. The assumption that $N(t)$ is a constant is also not vital provided that $N(t)$ is always large. This, together with the assumption that the size of variance of $X_j(t)$ does not depend on the size of the particular transaction can be relaxed in realistic ways and only make the working more complicated without in any way changing the result.

Thus, even if the series of price changes for transactions data is not white noise, it will be virtually impossible to distinguish between the random walk model and the truth when using price series with sampling units of an hour or more. In effect, the result states that in markets with many transactions and with a short memory, the price-change series taken over a medium or large sized sampling unit can be expected to appear to fit the random walk model. This might partially explain why the typical spectral shape discussed in [8] is near that of a random walk series.

The results obtained by Niederhoffer and Osborne suggest that the transactions price change series was not a Markov series. They found the first autocorrelation to have a value $\simeq -0.26$ and the second to be $\simeq -0.046$. Both are significantly non-zero. The sign of the first coefficient is particularly interesting and is in agreement with the model proposed by Osborne [10] in which the price series rebounds off semi-barriers formed at whole dollar prices by stop orders.

3. FINITE OR INFINITE VARIANCE?

It is almost universal to assume that the variance of any observed statistical variable is finite. It has, however, been suggested by Mandelbrot (in [2]) that the stock market price change series has infinite variance. A survey of this theory together with some of its implications and evidence in its favor has been provided by Fama [4]. The basic argument is as follows: "The Bachelier-Osborne model begins by assuming that price changes from transaction to transaction in an individual security are independent, identically distributed random variables. It further assumes that transactions are fairly uniformly spread across time, and that the distribution of price changes from transaction to transaction has finite variance. If the number of transactions per day, week, or month is very large, then price changes across these differing intervals will be sums of many independent variables. Under these conditions the central-limit theorems lead us to expect that the daily, weekly and monthly price changes will each have normal or Gaussian distributions." By various tests, it is found that these price changes are not normally distributed. It is then pointed out that the more general central-limit theorem must apply which states that the sum of independent random variables must have a so-called 'stable distribution.' Such distributions are largely characterized by a parameter usually denoted by alpha which must be in the range zero to two. If alpha is equal to two, the distribution is normal and the variance is finite. If alpha takes any other value the stable distribution has infinite variance. For various series alpha has been calculated and it is hardly surprising that the estimated value so found is rarely equal to two. Nevertheless, the evidence found by Fama that the price change series are not normally distributed does need an explanation. The fact that the price change series for individual transactions may not be quite identically distributed or are not independent do not in themselves provide suitable explanations. The central limit theorem still applies to many sums of non-identically distributed variables with finite variance and also to sums of certain non-independent variables (see Diananda [3]).

The argument, however, does not necessarily lead to the conclusion that the series has infinite variance as it misuses the basic central limit theorem. It is the object of this section of the paper to point out that the price series can have finite variance, for the central limit theorem to be applicable and also for a non-normal distribution to be the result.

Consider initially the following simple model: Let the individual transaction price data obey a random walk model

$$Y_j(t) - Y_{j-1}(t) = X_j(t) ,$$

where $X_j(t)$ is a purely random, white noise series with zero mean and finite variance σ^2. Suppose that the number of transactions is $N(t)$ on the t-th day. The daily price change series is defined as the closing price minus the opening price

$$C(t) = Y_c(t) - Y_o(t)$$

and will, according to Mandelbrot's suggestion, have an infinite variance.

Applying the central limit theorem to the model as stated, $C(t)$ will be normally distributed with mean zero and variance $N(t)\sigma^2$. As $N(t)$ is not a constant it is seen that each $C(t)$ comes from a *different* normal distribution. If $N(t)$ is a random variable with frequency function $f(x)$, the sequence of independent terms of $C(t)$ will come from a mixture of normal distributions with overall frequency function

$$f_c(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma}\frac{x^2}{\sqrt{y}}\right) y^{-1/2} f(y) dy ,$$

assuming that $N(t)$ is itself a white noise series. It is thus seen that the distribution of $C(t)$ being a mixture of normal distributions will not necessarily be normal. It is, of course, possible that this distribution will have an infinite variance but not necessarily so, depending on the properties of $f(x)$.

One implication of this argument is that $C(t)/\sqrt{N(t)}$ should be normally distributed, and the results of the previous section can be used to show that this still holds true even if $X_j(t)$ is not a white noise series. The assumption that $N(t)$ was white noise was not found to be correct by Godfrey, Granger and Morgenstern [**6**], but this does not alter any of the basic results.

4. CONCLUSION

Two results have been proved:

(a) price changes for individual transactions data may be autocorrelated but observed changes over longer periods may still appear to obey a random walk model if there is a short memory of price changes, and

(b) the observed non-normality of daily price changes or changes over longer periods is only to be expected due to the non-constancy of the number of transactions during the day, and it does not follow that the series necessarily has infinite variance.

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