COMPUTING MAXIMUM SMOOTHNESS FORWARD RATE CURVES

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1. INTRODUCTION

Adams and Deventer (1994) presents a new approach[1] to yield curve smoothing that provides a sound basis for implementing many of the no-arbitrage term structure models such as Vasicek (1977), Heath, Jarrow and Morton (1992), Hull and White (1993) and so on. By carefully defining the criterion for the best fitting yield curve to have maximum smoothness for the forward rate curve, they arrive at a simple but powerful method providing a closed-form solution for a yield curve that fits all observed yields. A key result in that paper that enables this method or procedure is that the smoothest forward rate curves are produced by a fourth-degree polynomial with the cubic term missing. In this paper we show that excluding the cubic term is incorrect and sup-optimal. We provide a correct proof of the result that the smoothest forward rate curves are produced by an unconstrained fourth-degree polynomial. We also provide the numerical algorithm to compute the corresponding yield and the forward rate curves.

2. MAXIMUM SMOOTHNESS FORWARD RATE CURVE

Let P_i be the price of a zero coupon bond with par value 1 and time to maturity t_i. Suppose there are m such observations: P_1 , P_2 , …… , P_m . Let the instantaneous spot yield and forward rates be y(t) and f(t) respectively. Then,

$$\int_0^{t_i} f(s)ds = -\ln P_i \qquad , \ i=1, 2,\ldots, m. \qquad (1)$$

The relationship between the forward rates and the yield is thus

$$\int_0^{t_i} f(s)ds = y(t_i)\,t_i \qquad , \ i=1, 2,\ldots, m. \qquad (2)$$

Given the m observed P_i's, we can find the smoothest forward rate curve such that the m constraints in (1) are satisfied. Then, by definition we would also have satisfied all the observed yields as in (2). From the forward rate curve f(s), for $0 \le s \le t_m$, we can obtain via (2) the yield curve. This yield curve should fit all the m observed yields.

A maximum smoothness forward rate curve is one that minimizes the total curvature of the curve and that fits the observed yields exactly. Assuming $f(\cdot) \in C^2[0, t_m]$, $t_m \le T$, where f(.) is of a polynomial form, the general criterion for the total curvature is the integral of the squared second-order differential function

$$Z = \int_0^T f''^2(s)ds . \tag{3}$$

Maximum smoothness obtains when Z is minimized subject to (1) and to some auxiliary conditions ensuring continuity of the curve and its derivatives. The estimation of the polynomial function coefficients becomes an optimization problem

$$\min_f \int_0^T f''^2(s)ds$$

$$\text{s.t.} \int_0^{t_i} f(s)ds = -\ln P_i \quad , \text{ i=1, 2,…, m.}$$

$$f(0) = r_0 , \tag{4}$$

$$f'(t_m) = 0 . \tag{5}$$

Assumption (4) basically fixes the instantaneous spot rate at r_0. Assumption (5) follows that in Adams and Deventer (1994). It can be suitably generalized to $f'(t_m) = r_m$ where r_m is a non-zero slope.

Integrating f(.), we obtain

$$\int_0^t f(s)ds = f(t)t - \int_0^t sf'(s)ds . \tag{6}$$

Since $\int_0^t sf'(s)ds = t^2 f'(t) - \int_0^t sf'(s)ds - \int_0^t s^2 f''(s)ds$,

therefore $\int_0^t sf'(s)ds = \frac{1}{2}[t^2 f'(t) - \int_0^t s^2 f''(s)ds]$. (7)

Substituting (7) into (6), we obtain

$$\int_0^t f(s)ds = f(t)t - \tfrac{1}{2}t^2 f'(t) + \tfrac{1}{2}\int_0^t s^2 f''(s)ds . \quad (8)$$

Putting $g(t) = f''(t)$, $0 \le t \le T$,

we have $f'(t) = \int_0^t g(s)ds + f'(0) = \int_0^t g(s)ds$ (9)

and $f(t) = \int_0^t f'(s)ds + r_0 = \int_0^t \int_0^u g(v)dvdu + r_0$. (10)

Substituting (9) and (10) into (8), the constraint becomes

$$(\int_0^{t_i} \int_0^s g(v)dvds + r_0)t_i - \tfrac{1}{2}t_i^{\,2} \int_0^{t_i} g(s)ds + \tfrac{1}{2}\int_0^{t_i} s^2 g(s)ds = -\ln P_i , \quad \text{for} \quad i = 1,\ldots,m \quad (11)$$

We use the indicator function

$$u(t) = \begin{cases} 1, & t \ge 0 \\ 0, & t < 0 \end{cases} .$$

The constraint is then expressed as

$$Q_i + \tfrac{1}{2}\int_0^t s^2 u(t_i - s)g(s)ds = -\ln P_i ,$$

for $i = 1,\ldots,m$ (12)

where

$$Q_i = t_i \int_0^t u(t_i - s)\int_0^s g(v)dvds - \tfrac{1}{2}t_i^{\,2}\int_0^t u(t_i - s)g(s)ds + r_0 t_i . \quad (13)$$

Equations in (12) are identical to equations in (A7) in Adams and Deventer (1994). However, the term Q_i is now expressed explicitly in terms of integrals of g(s) above in (13). It is important to recognize that Q_i is a function of g(s).

Let λ_i for $i = 1,...,m$ be the Lagrange multipliers corresponding to the constraints in (13). The objective function becomes

$$\min_{g(.),\lambda_i} Z[g,\lambda] = \int_0^T g^2(s)ds \quad +$$

$$\int_0^T \left[\sum_{i=1}^m \lambda_i u(t_i - s)[t_i \int_0^s g(v)dv - \tfrac{1}{2} t_i^{\,2} u(t_i - s) g(s) + \tfrac{1}{2} s^2 g(s)] \right] ds \quad +$$

$$\sum_{i=1}^m \lambda_i (r_0 t_i + \ln P_i) \,. \tag{14}$$

Suppose g^* is the solution of the optimization problem, then

$$\frac{d}{d\varepsilon} Z[g^* + \varepsilon\, h]\bigg|_{\varepsilon=0} = 0$$

for any continuous function $h(.)$ defined on $[0,T]$ such that h(s)=g(s)-g*(s), where g*(s) is optimal. From (14) we have

$$\frac{d}{d\varepsilon} Z[g^* + \varepsilon\, h]\bigg|_{\varepsilon=0} = 0$$

$$= \int_0^T \left[2g^*(s)h(s) + \sum_{i=1}^m \lambda_i u(t_i - s)[t_i \int_0^s h(v)dv + \tfrac{1}{2}(s^2 - t_i^2)h(s)] \right] ds$$

$$= \int_0^T \left[2g^*(s) + \sum_{i=1}^m \lambda_i u(t_i - s) \tfrac{1}{2}(s^2 - t_i^2) \right] \cdot h(s)ds$$

$$+ \int_0^T [\sum_{i=1}^m \lambda_i u(t_i - s) t_i][\int_0^s h(v)dv] ds = 0. \tag{15}$$

We shall now prove a lemma that we will use in solving for the optimal g*(.).

Lemma

Given A(.) and h(.) are continuous functions and B(.) is integrable, then

$$\int_a^b A(u)h(u)du + \int_a^b B(u)\cdot[\int_a^u h(v)dv]\cdot du = 0 \qquad \text{(L1)}$$

if and only if

$$A(u) = -\int_u^b B(v)dv \qquad \text{for all a} \le \text{u} \le \text{b}. \qquad \text{(L2)}$$

Proof

(L1) can be written as

$$\int_a^b A(v)h(v)dv + \int_a^b h(u)[\int_u^b B(v)dv]du$$

$$= \int_a^b A(v)h(v)dv + \int_a^b h(v)[\int_v^b B(u)du]dv$$

$$= \int_a^b h(v)\,[A(v) + \int_v^b B(u)du]\,dv \quad = 0$$

For any continuous h(.), therefore (L1) obtains if and only if (L2) holds.

Q.E.D.

Applying the above lemma, (15) equals to zero if and only if

$$2g^*(s) + \sum_{i=1}^m \lambda_i u(t_i - s)\tfrac{1}{2}(s^2 - t_i^2) = -\int_s^T \sum_{i=1}^m \lambda_i u(t_i - v)t_i dv \qquad (16)$$

for all $0 \le s \le T$. Thus

$$g^*(t) = -\tfrac{1}{4}\sum_{j=i+1}^m \lambda_j (t_j - t)^2 \qquad \text{when} \quad t_i < t \le t_{i+1},\ i = 0,...,m-1.$$

Thus $g^*(t)$ is a continuous function in second-order polynomial form in each interval $[t_k, t_{k+1}]$.

By integrating $g^*(t)$, we can obtain

$$f(t) = a_i t^4 + b_i t^3 + c_i t^2 + d_i t + e_i \quad \text{when} \ t_{i-1} < t \le t_i,\ i = 1,...,m. \qquad (17)$$

The maximum smoothness forward curve f(.) is thus an unconstrained fourth-order polynomial function in each segment and is second-order continuously differentiable. This is summarized in the following proposition.

Proposition

When $f(0)=r_0$ is known and $f'(t_m)=0$, the function of the maximum smoothness forward curve is a fourth-order polynomial spline and is second-order continuously differentiable in the range $(0,T)$.

Proposition 1 shows that it is sub-optimal to omit the cubic term as in Adams and Deventer (1994). They had incorrectly ignored the fact that Q_i is a function of g(s) and thus omitted some terms in the differentiation step as in (15). Indeed it is easy to see that if we constrain $b_i = c_i = 0$ in (17), we arrive exactly at the solution (A14) in their paper. Clearly, avoiding these constraints would produce a superior optimal solution, i.e. larger smoothness in our case.

3. ALGORITHM

In this section we show how to implement the optimization solution to obtain the maximum smoothness forward rate curve and also the corresponding yield curve. The program can be transformed to a quadratic form. We then obtain an explicit solution by the Lagrange-multiplier method. From the forward rate function in (17), by integrating

$$\int_{t_{i-1}}^{t_i} f''^2(t)dt = \int_{t_{i-1}}^{t_i} (12a_i t^2 + 6b_i t + 2c_i)^2\, dt$$

$$= \tfrac{144}{5}\Delta_i^5 a_i^2 + 36\Delta_i^4 a_i b_i + 12\Delta_i^3 b_i^2 + 16\Delta_i^3 a_i c_i + 12\Delta_i^2 b_i c_i + 4\Delta_i^1 c_i^2$$

$$= \mathbf{x_i^T h_i x_i}$$

where $\mathbf{x_i} = \begin{bmatrix} a_i \\ b_i \\ c_i \\ d_i \\ e_i \end{bmatrix}$, $\mathbf{h_i} = \begin{bmatrix} \frac{144}{5}\Delta_i^5 & 18\Delta_i^4 & 8\Delta_i^3 & 0 & 0 \\ 18\Delta_i^4 & 12\Delta_i^3 & 6\Delta_i^2 & 0 & 0 \\ 8\Delta_i^3 & 6\Delta_i^2 & 4\Delta_i^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$,

$$\Delta_i^l = t_i^l - t_{i-1}^l, \quad l = 1,\ldots,5.$$

Then the object function becomes

$$\min_{\mathbf{x}} \mathbf{x}^{\mathbf{T}}\mathbf{h}\mathbf{x}$$

where $\mathbf{x} = \begin{bmatrix} \mathbf{x_1} \\ \vdots \\ \mathbf{x_m} \end{bmatrix}$, $\mathbf{h} = \begin{bmatrix} \mathbf{h_1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{h_m} \end{bmatrix}$

The constraints are as follows.

(a) Fitting the observed points:

$$\tfrac{1}{5}\Delta_i^5 a_i + \tfrac{1}{4}\Delta_i^4 b_i + \tfrac{1}{3}\Delta_i^3 c_i + \tfrac{1}{2}\Delta_i^2 d_i + \Delta_i^1 e_i - \ln\left(\frac{P_i}{P_{i-1}}\right), \quad i = 1,\ldots,m \tag{18}$$

(b) Continuity of the spline function at the joints:

$$(a_{i+1} - a_i)t_i^4 + (b_{i+1} - b_i)t_i^3 + (c_{i+1} - c_i){t_i}^2 + (d_{i+1} - d_i)t_i + (e_{i+1} - e_i) = 0\,,\ i = 1,\ldots,m-1 \tag{19}$$

(c) Continuity of the first-order differential of the spline function

$$4(a_{i+1} - a_i)t_i^3 + 3(a_{i+1} - a_i)t_i^2 + 2(c_{i+1} - c_i)t_i + (d_{i+1} - d_i) = 0\,, \qquad i = 1,\ldots,m-1 \tag{20}$$

(d) Continuity of the second-order differential of the spline function

$$12(a_{i+1} - a_i)t_i^2 + 6(b_{i+1} - b_i)t_i + 2(c_{i+1} - c_i) = 0 \qquad i = 1,\ldots,m-1 \tag{21}$$

and

(e) Boundary conditions

$f(0) = r_0$; $\qquad e_1 = r_0$ (22)

$f'(t_m) = 0$; $\qquad d_1 = 0$ (23)

All the constraints (18) to (23) are linear with respect to $\mathbf{x}$. We can write the constraints in matrix form $\mathbf{Ax} = \mathbf{B}$, where $\mathbf{A}$ is a 4m-1 × 5m matrix, and $\mathbf{B}$ is a 4m-1 × 1 vector. Let $\lambda = [\lambda_1, \lambda_2 ..., \lambda_{4m-1}]^T$ be the corresponding Lagrange multiplier vector to the constraints. The object function becomes

$$\min_{\mathbf{x},\boldsymbol{\lambda}} Z(\mathbf{x},\boldsymbol{\lambda}) = \mathbf{x}^{\mathbf{T}}\mathbf{hx} + \boldsymbol{\lambda}^{\mathbf{T}}(\mathbf{Ax} - \mathbf{B})$$

If $[\mathbf{x_0}, \boldsymbol{\lambda_0}]$ is the solution, we should have

$$\frac{\partial}{\partial \mathbf{x}} Z(\mathbf{x},\boldsymbol{\lambda})\bigg|_{x_0,\lambda_0} = 2\mathbf{hx_0} + \mathbf{A}^{\mathbf{T}}\boldsymbol{\lambda_0} = 0$$

$$\frac{\partial}{\partial \boldsymbol{\lambda}} Z(\mathbf{x},\boldsymbol{\lambda})\bigg|_{x_0,\lambda_0} = \mathbf{Ax_0} - \mathbf{B} = 0$$

or $$\begin{bmatrix} \mathbf{2h} & \mathbf{A}^{\mathbf{T}} \\ \mathbf{A} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x_0} \\ \boldsymbol{\lambda_0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{B} \end{bmatrix}.$$

then $$\mathbf{x_0} = [\mathbf{I}_{5m\times 5m} \quad \mathbf{0}_{5m\times(4m-1)}] \cdot \begin{bmatrix} \mathbf{2h} & \mathbf{A}^{\mathbf{T}} \\ \mathbf{A} & \mathbf{0} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \mathbf{0} \\ \mathbf{B} \end{bmatrix}$$

The explicit solution of the parameter vector is thus obtained.

4. YIELD CURVE

The connection with the yield curve is established in this section. The maximum smoothness forward rate curve is obtained using the following fourth degree polynomial.

$$f(t) = a_i t^4 + b_i t^3 + c_i t^2 + d_i t + e_i \quad \text{when } t_{i-1} < t \le t_i,\ i = 1,...,m.$$

From (2), $\int_0^{t_i} f(s)ds = y(t_i)t_i$, i=1, 2,…, m.

We may write this as

$$\begin{aligned} y(t_i) &= \frac{1}{t_i}\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}(a_j s^4 + b_j s^3 + c_j s^2 + d_j s + e_j)ds \\ &= \frac{1}{t_i}\sum_{j=1}^{i}(\tfrac{1}{5}a_j\Delta_j^{\,5} + \tfrac{1}{4}b_j\Delta_j^{\,4} + \tfrac{1}{3}c_j\Delta_j^{\,3} + \tfrac{1}{2}d_j\Delta_j^{\,2} + e_j\Delta_j^{\,1}). \end{aligned} \tag{24}$$

Thus (24) allows the computation of the yield curve based on the maximum smoothness forward rate curve where the coefficients **x** are determined.

From (2), we can also show that for each segment i=1,2,…,m, the maximum forward rate curve smoothness is equivalent to

$$\min_{x_i}\int_{t_{i-1}}^{t_i} f''^2(s)ds = \min_{x_i}\int_{t_{i-1}}^{t_i}\left(3y''(s) + sy'''(s)\right)^2 ds .$$

Thus in general, maximum smoothness of forward rate curve will be different from maximum smoothness of yield curve. However, if y''' is small, then the maximizing of forward rate curve smoothness will also result in a smooth yield curve. Obviously the continuity conditions for the forward rate curve at each segment will be different from those of the yield curve segments, although this may not have a major impact if the adjacent coefficients $\mathbf{x}_\mathbf{i}$ and $\mathbf{x}_{\mathbf{i-1}}$ are close in values.

5. CONCLUSIONS

This paper introduces a simple method to estimate the maximum smoothness forward rate curve and the corresponding yield curve. The integral of the squared second-order derivative of the curve function is defined as a measure of smoothness. We provide a correct proof of the result that the smoothest forward rate curves are produced by an unconstrained fourth-degree polynomial. We also provide the numerical algorithm to compute the corresponding yield and the forward rate curves.

Increasingly many of the interest rate derivatives pricing and hedging depend on the availability of many points on the forward rate curve or the yield curve than is provided by the observed finite number of points in the market. The ability to fit an intuitively appropriate maximum smoothness forward rate curve allows for such pricing and hedging.

ENDNOTES

1. Other approaches include Schwartz (1989)'s cubic splines and McCulloch (1975)'s polynomial spline functions to fit the observed data. However, the use of polynomial function to fit the entire yield curve may lead to unacceptable yield patterns. Vasicek and Fong (1982) use the exponential spline for the discount function, and choose the cubic form as the lowest odd order form from continuous derivatives. Delbaen and Lorimier (1992) introduce the discrete approach to estimate the yield curve by minimizing the difference between two adjacent forward rates. Frishling and Yamamula (1996) use a similar approach on the coupon bond prices.

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