CEIOPS

QIS 5 Risk-free interest rates – Extrapolation method

Table of contents

Introduction

For liabilities expressed in any of the EEA currencies, Japanese yen, Swiss franc, Turkish lira or USA dollar, QIS5 provides to participants risk-free interest rate term structures.

The appropriate risk-free interest rate term structure is in practice constructed from a finite number of data points. Therefore, both interpolation between these data points and extrapolation beyond the last available data point of sufficient liquidity is required.

The purpose of this paper is to describe how the extrapolated part of the interest rate term structures for currencies where the relevant risk-free interest rate term structures are provided in the spreadsheet included in QIS5 package was set up.

Liquid points of risk-free interest rate curve

The non-extrapolated part of the risk-free interest rate curve for QIS5 purposes was delivered by the industry. The aspects of the risk-free interest rate term structure that had to be considered were the selection of the basic risk-free interest rate term structure, the method for adjusting inter-bank swaps for credit risk, the assessment of the last liquid point to enter the yield curve extrapolation and the derivation of the liquidity premium. Details on the determination of these can be found in the documentation given by CFO and CRO Forum.

This data basis consists of continuously compounded spot rates. Furthermore, the rates have been derived from market data by fitting a smoothed regression spline to market swap rates using Barrie & Hibbert's yield curve fitting methodology.

To use these rates in the extrapolation tool, they had to be converted into spot rates with annual compounding, as the extrapolation tool expects spot rates with annual compounding as input and delivers spot rates with annual compounding as output. Furthermore, the Smith-Wilson approach achieves both interpolation (for maturities in the liquid end of the term structure where risk-free zero coupon rates are missing) and extrapolation. A consistent approach for inter- and extrapolation would be preferred by CEIOPS and we would thus recommend not to use already smoothed market data as a starting point.

Extrapolation method

For QIS5, macroeconomic extrapolation techniques are used to derive the extrapolation beyond the last available data point. The overall aim is to construct a stable and robust extrapolated yield curve which reflects current market conditions and at the same time embodies economical views on how unobservable long term rates are expected to behave. Macroeconomic extrapolation techniques assume a long-term equilibrium interest rate. A transition of observed interest rates of short-term maturities to the assessed equilibrium interest rate of long-term maturities takes place within a certain maturity spectrum.

Valuation of technical provisions and the solvency position of an insurer or reinsurer shall not be heavily distorted by strong fluctuations in the short-term interest rate. This is particularly important for currencies where liquid reference rates are only available for short term maturities and simple extrapolation of these short term interest rates may cause excessive volatility. A macro-economic model meets the demands on a model that ensures relatively stable results in the long term.

There are some considerations that have to be faced when specifying the macro-economic extrapolation method for QIS5 purposes. These are examined further in the following sections.

Determination of ultimate forward rate

A central feature is the definition of an unconditional ultimate long-term forward rate (UFR) for infinite maturity and for all practical purposes for very long maturities. The UFR has to be determined for each currency. While being subject to regular revision, the ultimate long term forward rate should be stable over time and only change due to fundamental changes in long term expectations. The unconditional ultimate long-term forward rate is determined for each currency by macro-economic methods.

Common principles governing the methods of calculation should ensure a level playing field between the different currencies. For all currencies interest rates beyond the last observable maturity - where no market prices exist - are needed.

The most important economic factors explaining long term forward rates are long-term expected inflation and expected real interest rates. From a theoretical point of view it can be argued that there are at least two more components: the expected long-term nominal term premium and the long-term nominal convexity effect.

The term premium represents the additional return an investor may expect on risk-free long dated bonds relative to short dated bonds, as compensation for the longer term investment. This factor can have both a positive and a negative value, as it depends on liquidity considerations and on preferred investor habitats. As no empirical data on the term premium for ultra-long maturities exists, a practical estimation of the term premium is not undertaken for QIS5 purposes.

The convexity effect arises due to the non-linear (convex) relationship between interest rates and the bond prices used to estimate the interest rates. This is a purely technical effect and always results in a negative component.

In order to have a robust and credible estimate for the UFR the assessment is based on the estimates of the expected inflation and the expected short term real rate only.

Making assumptions about expectations this far in the future for each economy is difficult. However, in practice a high degree of convergence in forward rates can be expected when extrapolating at these long-term horizons. From a macro economical point of view it seems consistent to expect broadly the same value for the UFR around the world in 100 years. Nevertheless, where the analysis of expected long term inflation or real rate for a currency indicates significant deviations, an adjustment to the long term expectation and thus the UFR has to be applied. Therefore, three categories are established capturing the medium UFR as well as deviations up or down.

Thus, the macro economically assessed UFR for use in the QIS5 is set to 4.2 per cent (+/-1 percentage points) per anno. This value is assessed as the sum of the expected inflation rate of annually 2 per cent (+/- 1 percentage points) and of an expected short term return on risk free bonds of 2.2 per cent per anno. Further details on the estimation of expected inflation rate and expected real rate can be found in Appendix A.

For QIS5 the following UFR are used:

Category	Currencies	Macro economically assessed UFR
1	**JPY, CHF**	**3.2**
2	**Euro, SEK, NOK, DKK, GBP, USD, CZK, BGN, LVL, LTL, EEK, PLN, RON, HUF, ISK**	**4.2**
3	**TRY**	**5.2**

Transition to the equilibrium rate

This paragraph considers the issue of how to extrapolate between the estimated forward rates and the unconditional ultimate forward rate.

Technique for transition

For QIS5 the Smith-Wilson method will be used. If applied to observed zero coupon bond prices from the liquid market, this method ensures that the term structure is fitted exactly to all observed zero coupon bond prices, i.e. all liquid market data points are used without smoothing. If applied to the already smoothed market data that the industry has delivered for the liquid part of the term structure, the extrapolated term structure will pass through all zero coupon market rates that are given as input.

Furthermore, with the Smith-Wilson approach both interpolation (for maturities in the liquid end of the term structure, if risk-free zero coupon rates are missing) and extrapolation can be achieved. The industry has already delivered risk-free rates for all maturities from one year up to the last liquid maturity (given in whole years) for the currencies in question. Thus, for these currencies, the Smith-Wilson approach will be used only for interpolation in cases of non-integer maturities and for the extrapolation beyond the last liquid data point.

It is a sophisticated approach that is still easy to use, and gives both a relative smooth forward rate and a smooth spot rate curve in the extrapolated part. Further details on the Smith-Wilson technique can be found in Appendix B.

Nevertheless, the linear method has been also run in order to provide a kind of cross-checking, avoiding a full reliance in a single method and enhancing the robustness of results provided by the Smith-Wilson approach.

Speed of transition

The speed of transition towards the UFR can be specified by the maturity T2 at which the forward rate curve "reaches" the UFR. A range for T2 between 70 and 120 years is considered appropriate. The forward rate curve is deemed to reach the UFR at T2, if the spread between the UFR and the annual forward rate at T2 - in absolute values - lies within predefined limits. These limits are chosen as a threshold of 3 BP for QIS 5 purposes.

The choice for the maturity at which the ultimate forward rate will be "reached" between 70 or 120 years has an impact on the stability of the yield curve over time. On the one hand the yield curve follows a quite flat course beyond the maturity the UFR is "reached". Therefore, the earlier the UFR is "reached" the more stable is the yield curve for long maturities. On the other hand, it has to be considered that the earlier the UFR is "reached", the more sensitive will the yield curve be towards changes in the choice of the UFR.

In light of this, CEIOPS provides a set of extrapolated interest rate curves for different choices of T2, namely 70, 90 and 120 years.

Nevertheless, there is no fixed, predefined maturity where the UFR is deemed to be arrived at in the Smith-Wilson approach, but the speed of convergence to the unconditional ultimate forward rates has to be set. Therefore, in the Smith-Wilson technique the speed of transition as defined above has to be translated into a convergence parameter α (alpha). Thomas[1], who was looking at Australian term structures, fitted this parameter empirically to $\alpha = 0.1$ as it ensured sensible results and economically appropriate curves in most cases.

For QIS5, in a first step the default value for alpha is 0.1. Only if the extrapolated rates deviate from the UFR at the predefined time T2 by more than 0.03% (i.e. 3 BP), then alpha is recalibrated in a second step, such that the spread lies within this threshold. This ensures that the extrapolated curve is sufficiently close to the chosen UFR at T2.

Allowance for liquidity premium

For every currency, the liquidity premium is allowed for in the risk-free interest rate curve up to a cut-off point. Past that cut-off point a phasing-out period of 5 years for the LP is applied.

There are two alternatives for the application of the liquidity premium to the basic risk-free interest rate curve. The first alternative is to interpret the LP as a spot rate and adjust the risk-free basic term structure (i.e. the basic zero coupon rates) with this premium. The second alternative would be to interpret the LP as a forward rate and adjust the forward rates with the LP. An example of these two possible options can be found in Appendix C.

The main difference between adjusting the spot rates and adjusting the forward rates is the impact on discounting cash flows beyond maturities for which a liquidity premium in assets is deemed to exist and is deemed to be measurable:

1. In the first alternative, the LP is restricted to the discounting of cash flows of maturities for which a liquidity premium for assets can be assessed.

- The method therefore fulfils the requirement that no liquidity premium should be included in the extrapolated part of the interest rate curve, as formulated in point I–6 c) (page 18) of the Task Force report and as reiterated in the Commission draft implementing provisions, Article IR6(3): *"No illiquidity premium shall be applied to the extrapolated part of the relevant basic risk-free interest rate term structure."*
- Furthermore, the method is in line with the LP measurement methods as presented in the Task Force report. The common feature in all three methods mentioned is that the LP is estimated from the spreads of the yields on corporate bonds over the yields of government bonds for given maturities. The CDS Negative-Basis method deduces from these spreads the prices of the Credit Default Swaps on the corporate bonds in order to get the LP. In the Structural Model Method the actual computed spreads are decreased by theoretical credit spreads, computed with methods from option pricing theory. In the Covered Bond Method the LP is computed directly as spread between the yields of two bonds (other instruments) which are equal in all, except liquidity.

[1] Michael Thomas, Eben Maré: Long Term Forecasting and Hedging of the South African Yield Curve, Presentation at the 2007 Convention of the Actuarial Society of South Africa

- Nevertheless, this method induces forward rates (implied by the adjusted risk-free term structure) that might not behave very smooth in the 5 year phasing-out period. For a high LP and high maturities, the forward rates implied by the adjusted risk-free term structure can even become negative during this period.

2. In the second alternative the LP has an impact on the discounting of cash flows of all maturities even beyond maturities for which a liquidity premium in assets is deemed to exist. This is due to the fact that spot rates are a kind of average of forward rates, and thus spot rates implicitly contain the liquidity adjustments on the forward rates that enter the average.

- For this method the term structure is smoother, and the implied forward rates stay positive during the 5 year phasing-out period.
- Nevertheless, this method is not consistent with the requirement that no liquidity premium should be included in the extrapolated part of the interest rate curve, as formulated in point I – 6 c) of the Task Force Report and as reiterated in the Commission draft implementing provisions Article IR6(3).

After having discussed the pros and cons of the issue, the extrapolation team decided to attach the utmost weight to the condition that no liquidity premium should be included in the extrapolated part of the interest rate curve, as required in point I–6 c) (page 18) of the Task Force report, and hence to implement the LP as an adjustment of the spot rate for QIS5. In this case no liquidity premium would be allowed for in the discounting of the cash flows beyond maturities for which a liquidity premium in assets is deemed to exist.

Appendix A

Estimation of expected long term inflation rate

The expected inflation should not solely be based on historical averages of observed data, as the high inflation rates of the past century do not seem to be relevant for the future. The fact is that in the last 15-20 years many central banks have set an inflation target or a range of inflation target levels and have been extremely successful in controlling inflation, compared to previous periods.

Barrie Hibbert[2] propose to assess the inflation rate as 80 per cent of the globally prevailing inflation target of 2 per cent per anno and 20 per cent of an exponentially weighted average of historical CPI inflations when modelling the term structure in their Economic Scenario Generator. When they assess the historical inflation average of the main economies they still compute a high level as of December 2007 (they assess an expected global inflation rate of 2.4 per cent per anno) but with a strong downward trend over the sample of data they considered.

In order to have a robust and credible estimate for the UFR, the standard expected long term inflation rate is set to 2 per cent per anno, consistently to the explicit target for inflation most central banks operate with[3].

Nevertheless, based on historical data for the last 10-15 years and current inflation, two additional categories are introduced to capture significant deviations either up or down in the expected long term inflation rate for certain countries. Table 1 shows inflation data for the OECD-countries in the period 1994 – 2009.

Table 1: Inflation 1994 – 2009 OECD Countries

Price indices (MEI) i: Consumer prices - Annual inflation

Data extracted on 15 Mar 2010 13:35 UTC (GMT) from OECD.Stat

Frequency	Annual															
Time	1994	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
Country																
Australia	i 1.9	4.6	2.6	0.3	0.9	1.5	4.5	4.4	3.0	2.8	2.3	2.7	3.5	2.3	4.4	1.8
Austria	i 3.0	2.2	1.9	1.3	0.9	0.6	2.3	2.7	1.8	1.4	2.1	2.3	1.4	2.2	3.2	0.5
Belgium	i 2.4	1.5	2.1	1.6	0.9	1.1	2.5	2.5	1.6	1.6	2.1	2.8	1.8	1.8	4.5	-0.1
Canada	i 0.2	2.1	1.6	1.6	1.0	1.7	2.7	2.5	2.3	2.8	1.9	2.2	2.0	2.1	2.4	0.3
Czech Republic i	i 10.0	9.1	8.8	8.5	10.7	2.1	3.9	4.7	1.8	0.1	2.8	1.9	2.6	3.0	6.3	1.0
Denmark	i 2.0	2.1	2.1	2.2	1.8	2.5	2.9	2.4	2.4	2.1	1.2	1.8	1.9	1.7	3.4	1.3
Finland	i 1.1	0.8	0.6	1.2	1.4	1.2	3.0	2.6	1.6	0.9	0.2	0.6	1.6	2.5	4.1	0.0
France	i 1.7	1.8	2.0	1.2	0.6	0.5	1.7	1.6	1.9	2.1	2.1	1.7	1.7	1.5	2.8	0.1
Germany i	i 2.8	1.8	1.4	1.9	1.0	0.6	1.4	1.9	1.5	1.0	1.7	1.5	1.6	2.3	2.6	0.4
Greece	i 10.9	8.9	8.2	5.5	4.8	2.6	3.2	3.4	3.6	3.6	2.9	3.6	3.2	2.9	4.2	1.2
Hungary	i **18.9**	**28.3**	**23.5**	**18.3**	**14.2**	**10.0**	**9.8**	**9.1**	**5.3**	**4.7**	**6.7**	**3.6**	**3.9**	**8.0**	**6.0**	**4.2**
Iceland	i **1.6**	**1.7**	**2.3**	**1.8**	**1.7**	**3.2**	**5.1**	**6.4**	**5.2**	**2.1**	**3.2**	**4.0**	**6.7**	**5.1**	**12.7**	**12.0**

[2] Steffen Sørensen, Interest rate calibration – How to set long-term interest rates in the absence of market prices, Barrie+Hibbert Financial Economic Research, September 2008.
[3] Also the European Central bank aims at an annual inflation just below 2 per cent.

Ireland	2.4	2.5	1.7	1.4	2.4	1.6	5.6	4.9	4.6	3.5	2.2	2.4	3.9	4.9	4.1	-4.5
Italy	4.1	5.2	4.0	2.0	2.0	1.7	2.5	2.8	2.5	2.7	2.2	2.0	2.1	1.8	3.3	0.8
Japan	**0.7**	**-0.1**	**0.1**	**1.8**	**0.7**	**-0.3**	**-0.7**	**-0.8**	**-0.9**	**-0.2**	**0.0**	**-0.3**	**0.2**	**0.1**	**1.4**	**-1.4**
Korea	6.3	4.5	4.9	4.4	7.5	0.8	2.3	4.1	2.7	3.6	3.6	2.8	2.2	2.5	4.7	2.8
Luxembourg	2.2	1.9	1.2	1.4	1.0	1.0	3.2	2.7	2.1	2.0	2.2	2.5	2.7	2.3	3.4	0.4
Mexico	7.0	35.0	34.4	20.6	15.9	16.6	9.5	6.4	5.0	4.5	4.7	4.0	3.6	4.0	5.1	5.3
Netherlands	2.8	1.9	2.0	2.2	2.0	2.2	2.3	4.2	3.3	2.1	1.2	1.7	1.2	1.6	2.5	1.2
New Zealand	1.7	3.8	2.3	1.2	1.3	-0.1	2.6	2.6	2.7	1.8	2.3	3.0	3.4	2.4	4.0	2.1
Norway	1.4	2.4	1.2	2.6	2.3	2.3	3.1	3.0	1.3	2.5	0.5	1.5	2.3	0.7	3.8	2.2
Poland	33.0	28.0	19.8	14.9	11.6	7.2	9.9	5.4	1.9	0.7	3.4	2.2	1.3	2.5	4.2	3.8
Portugal	5.4	4.2	3.1	2.3	2.8	2.3	2.9	4.4	3.6	3.3	2.4	2.3	3.1	2.5	2.6	-0.8
Slovak Republic	13.4	9.8	5.8	6.1	6.7	10.6	12.0	7.3	3.1	8.6	7.5	2.7	4.5	2.8	4.6	1.6
Spain	4.7	4.7	3.6	2.0	1.8	2.3	3.4	3.6	3.1	3.0	3.0	3.4	3.5	2.8	4.1	-0.3
Sweden	2.2	2.5	0.5	0.7	-0.3	0.5	0.9	2.4	2.2	1.9	0.4	0.5	1.4	2.2	3.4	-0.3
Switzerland	**0.9**	**1.8**	**0.8**	**0.5**	**0.0**	**0.8**	**1.6**	**1.0**	**0.6**	**0.6**	**0.8**	**1.2**	**1.1**	**0.7**	**2.4**	**-0.5**
Turkey	**105.2**	**89.1**	**80.4**	**85.7**	**84.6**	**64.9**	**54.9**	**54.4**	**45.0**	**21.6**	**8.6**	**8.2**	**9.6**	**8.8**	**10.4**	**6.3**
United Kingdom	2.0	2.7	2.5	1.8	1.6	1.3	0.8	1.2	1.3	1.4	1.3	2.0	2.3	2.3	3.6	2.2
United States	2.6	2.8	2.9	2.3	1.6	2.2	3.4	2.8	1.6	2.3	2.7	3.4	3.2	2.9	3.8	-0.4
G7	2.2	2.3	2.3	2.0	1.3	1.4	2.2	2.0	1.3	1.8	2.0	2.4	2.4	2.2	3.2	-0.1
OECD - Europe	8.6	8.7	7.6	7.2	7.0	5.4	5.7	5.6	4.9	3.0	2.4	2.4	2.6	2.7	3.9	1.2
OECD - Europe excluding high inflation countries	2.8	2.9	2.6	2.0	1.6	1.3	2.3	2.6	2.0	1.9	2.0	2.0	2.1	2.2	3.4	0.8
OECD - Total	4.8	6.1	5.7	4.8	4.2	3.6	4.0	3.7	2.8	2.4	2.4	2.6	2.6	2.5	3.7	0.5
OECD - Total excluding high inflation countries	2.4	2.5	2.4	2.1	1.6	1.5	2.6	2.5	1.7	2.0	2.2	2.5	2.5	2.3	3.5	0.4

Table 1 shows that two OECD-countries had inflation above 5 percent in 2009: Iceland (12 percent) and Turkey (6.3 percent). During the last 15 years, Turkey has been categorised by OECD as a high inflation country[4]. Turkey's inflation target is also higher (5-7.5% for the period 2009 - 2012) than in other countries.

Based on this data basis, Hungary and Iceland are possible candidates for the high inflation group. However, deviations to the average inflation rate are far more moderate than those for Turkey. Furthermore, these countries are expected to join the Euro sooner or later (and thus have to fulfil the convergence criteria). Therefore, Hungary and Iceland are classified in the standard inflation category.

Japan, having deflation in the period since 1994, is an obvious candidate for the "low inflation"-group. Switzerland can also be seen as an outlier. This is due to the fact that historically relatively low inflation rates can be observed and that Switzerland is particular attractive in the international financial markets (exchange rate conditions, liquidity, "save haven"[5]...). For these reasons, lower inflation assumptions are applied for the Swiss francs.

The estimate covers one-year inflation rate 70 - 100 years from now. It is arbitrary to say whether the inflation differences we see today and have seen the last 15 years will persist 100 years into the future. However, historical evidence and current long term interest rates indicate that it is reasonable to have three groups of currencies with different inflation assumptions. The standard inflation rate is set to 2 per cent per anno. To allow for deviations up and down to the standard inflation rate, an adjustment to the estimate of +/- 1 percentage point is applied for the high inflation group and the low inflation group respectively. This adjustment of 1 percentage point will be applied to the

[4] http://stats.oecd.org/index.aspx
[5] http://www.cepr.org/pubs/dps/DP5181.asp "Why are Returns on Swiss Francs so low? Rare events may solve the puzzle." Peter Kugler, Weder di Mauro

estimated inflation rate for outliers based on differences in current long term interest rates (30Y), observed historical differences between the average interest rate and differences in short term inflation expectations.

The following grouping is used for the estimated expected long term inflation rate:

Standard inflation rate set to 2%:	Euro-zone, UK, Norway, Sweden, Denmark, USA, Poland, Hungary, Iceland, Czech Republic, Bulgaria, Latvia, Lithuania, Estonia and Romania
High inflation rate set to 3%:	Turkey
Low inflation rate set to 1%:	Japan, Switzerland*

** combined effects*

Estimation of the expected real rate of interest

We expect that the real rates should not differ substantially across economies as far out as 100 years from now. Elroy Dimson, Paul Marsh and Mike Staunton provide a global comparison of annualized bond returns over the last 110 years (1900 to 2009) for the following 19 economies: Belgium, Italy, Germany, Finland, France, Spain, Ireland, Norway, Japan, Switzerland, Denmark, Netherlands, New Zealand, UK, Canada, US, South Africa, Sweden and Australia[6].

[6] Credit Suisse Global Investment Returns Yearbook 2010, To be found at www.tinyurl.com/DMS2010

Figure 1: Real return on bonds 1900 – 2009

Source: Dimson, Marsh and Staunton – Credit Suisse Global Investment Returns Yearbook

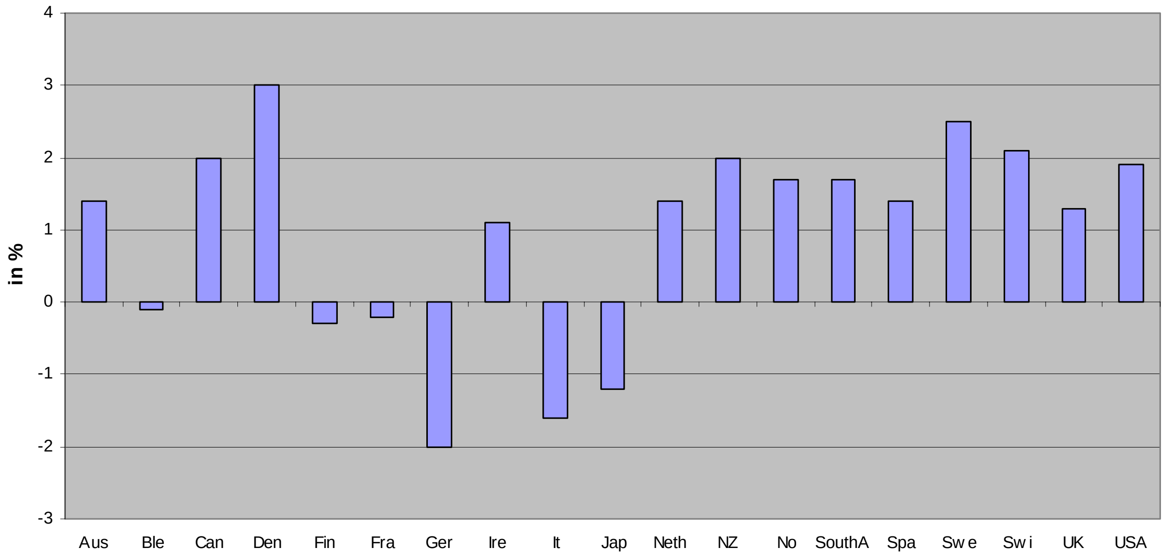


Figure 1 shows that, while in most countries bonds gave a positive real return, six countries experienced negative returns. Mostly the poor performance dates back to the first half of the 20th century and can be explained with times of high or hyperinflation[7]. Aggregating the real returns on bonds for each currency[8] to an annual rate of real return on globally diversified bonds gives a rate of 1.7 per cent.

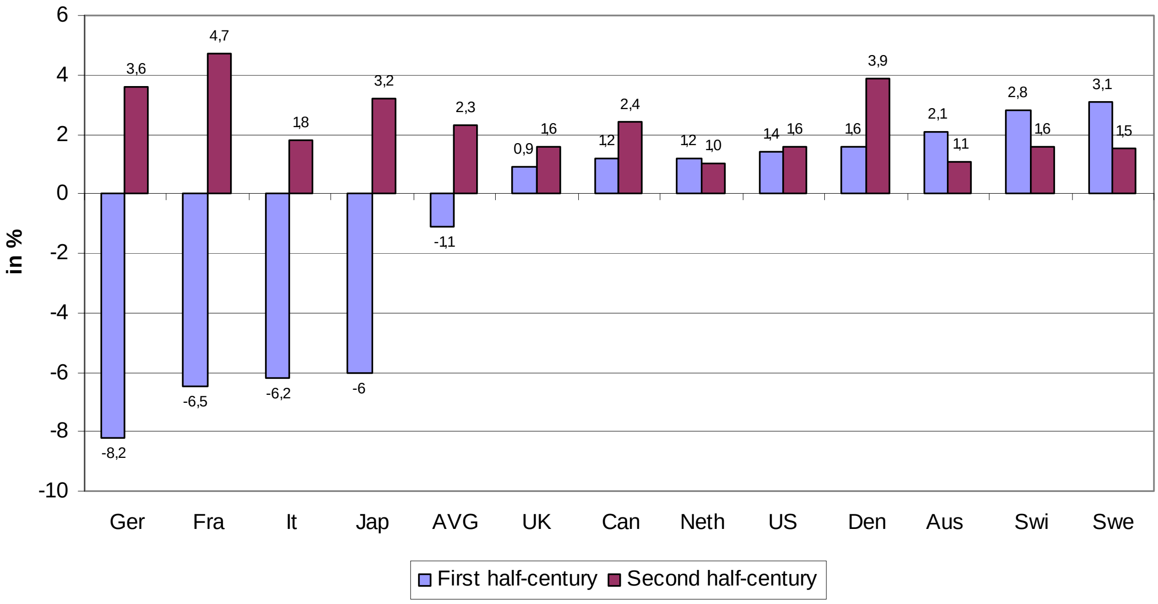
In an earlier publication, the same authors compared the real bond returns from the second versus the first half of the 20th century for the following 12 economies: Italy, Germany, France, Japan, Switzerland, Denmark, Netherlands, UK, Canada, US, Sweden and Australia[9]. The average real bond return over the second half of the 20th century was computed as annually 2.3 per cent (compared to -1.1 percent for the first half of the 20th century).

[7] German hyperinflation in 1922/1923, in Italy an inflation of 344% in 1944, in France 74% in 1946 and in Japan 317% in 1946.

[8] Average where each return is weighted by its country's GDP.

[9] Elroy Dimson, Paul Marsh and Mike Staunton: Risk and return in the 20th and 21th, Business Strategy Review, 2000, Volume 11 issue 2, pp 1-18. See Figure 4 on page 5. The article can be downloaded at: http://docs.google.com/viewer?a=v&q=cache:07V7vM0gu5oJ:citeseerx.ist.psu.edu/viewdoc/download%3Fdoi%3D10.1.1.11.7613%26rep%3Drep1%26type%3Dpdf+Risk+and+return+in+the+20th+and+21th+Centuries&hl=no&gl=no&sig=AHIEtbQbxwuXZNO6ViVlqkV0KZ63LKhB0g

*Figure 2: Real bond returns: first versus second half of 20th century**
Source: Dimson, Marsh and Staunton (ABN- Ambro/LBS)



* Data for Germany excludes 1922-23. AVG = Average

In light of the above data, 2.2 per cent is an adequate estimate for the expected real interest rate.

Appendix B

Smith-Wilson technique

The Smith-Wilson approach is a macroeconomic method: a spot (i.e. zero coupon) rate curve is fitted to observed bond prices with the macroeconomic ultimate long term forward rate as input parameter.

In its most general form, the input data for the Smith-Wilson approach can consist of a large set of different financial instruments relating to interest rates. We will limit the input to zero coupon bond prices, and will only put down the formulae for this simple case.

In other words: we assume that in the liquid part of the term structure the risk-free zero coupon rates for all liquid maturities are given beforehand. Our task is to assess the spot rate for the remaining maturities. These are both maturities in the liquid end of the term structure where risk-free zero coupon rates are missing (interpolation) and maturities beyond the last observable maturity (extrapolation).

Let's assume that we have market zero coupon rates for J different maturities: u_1, u_2, u_3, and so on. The last maturity for which market data is given is u_J.

The market price P(t) for a zero coupon bond of maturity t is the price, at valuing time

t_0 = 0, of a bond paying 1 at some future date t. Depending on whether the market data spot rates are given as continuously compounded rates $\tilde{R}_{u_j}$ or as rates R_{u_j} with annual compounding, the input zero bond prices at maturities u_j are:

$$P(u_j) = \exp(-u_j * \tilde{R}_{u_j}) \quad \text{for continuously compounded rates, and}$$

$$P(u_j) = (1 + R_{u_j})^{-u_j} \quad \text{for annual compounding.}$$

The relation between the two rates is given through $R_{u_j} = \ln(1 + \tilde{R}_{u_j})$.

Our aim is to assess the function P(t) for <u>all maturities</u> t, t > 0. From the definition of the price function $P(t) = \exp(-t * \tilde{R}_t)$ for continuously compounded rates and $P(t) = (1 + R_t)^{-t}$ for annual compounding, we then can assess the whole risk-free term structure at valuing date t_0 = 0.

<u>General on extrapolation technique</u>

Most extrapolation methods start from the price function, and assume that the price function is known for a fixed number of say J maturities. In order to get the price function for all maturities, some <u>more assumptions are needed</u>.

The most common procedure is to impose – in a first step - a functional form with K parameters on the price function P[10]. These functional forms could be polynomials, splines, exponential functions, or a combination of these or different other functions[11]. In some of the methods, in a second step, the K parameters are estimated via least squares at each point in time. In other methods K equations are set up from which the K parameters are calculated. The equations are set up in a manner that guarantees that P has the features desired for a price function: A positive function, with value 1 at time t=0, passing through all given data points, to a certain degree smooth, and with values converging to 0 for large t.

<u>Smith-Wilson approach</u>

Smith and Wilson[12][13] proposed a pricing function (here reproduced in a restricted form, only for valuing at point t_0 =0) of the following form:

$$P(\tau) = e^{-UFR*\tau} + \sum_{i=1}^{J} \varsigma_i * W(\tau, u_i), \qquad \tau \geq 0$$

with the symmetric Wilson - functions $W(\tau, u_i)$ defined as :

[10] In their respective models, Svensson for instance imposes a parametric form with 6 parameters and Nelson-Siegel one with 4 parameters.
[11] BarrieHibbert use cubic splines in the liquid part of the term structure and Nelson-Siegel for the extrapolation part.
[12] Smith A. & Wilson, T. – "Fitting Yield curves with long Term Constraints" (2001), Research Notes, Bacon and Wodrow. Referred to in Michael Thomas, Eben Maré: "Long Term Forecasting and Hedging of the South African Yield Curve", Presentation at the 2007 Convention of the Actuarial Society of South Africa
[13] Andrew Smith: Pricing Beyond the Curve – derivatives and the Long Term (2001), presentation to be found at http://www.cfr.statslab.cam.ac.uk/events/content/20001/asmith2001.pdf

$$W(\tau,u_i) = e^{-UFR*(\tau+u_i)} * \left\{\alpha * \min(\tau,u_i) - 0.5 * e^{-\alpha*\max(\tau,u_i)} * (e^{\alpha*\min(\tau,u_i)} - e^{-\alpha*\min(\tau,u_i)})\right\}$$

The following notation holds:

- J = the number of zero coupon bonds with known price function
- u_i, i=1, 2, … J, the maturities of the zero coupon bonds with known prices
- τ = term to maturity in the price function
- UFR = the ultimate unconditional forward rate,
- α = mean reversion, a measure for the speed of convergence to the UFR
- ζ_i = parameters to fit the actual yield curve

The so called kernel functions $K_i(\tau)$ are defined as functions of τ:

$$K_i(\tau) = W(\tau,u_i),\ \tau > 0 \quad \text{and i=1,2,...J}$$

They depend only on the input parameters and on data from the input zero coupon bonds. For each input bond a particular kernel function is computed from this definition. The intuition behind the model is to assess the function P(t), from which we aim to calculate the term structure, as the linear combination of all the kernel functions. This reminds of the Nelson-Siegel method, where the forward rate function is assessed as the sum of a flat curve, a sloped curve and a humped curve, and the Svensson method, where a second humped curve is added to the three curves from Nelson-Siegel.

The unknown parameters needed to compute the linear combination of the kernel functions, ζ_i, i= 1, 2, 3 … J are given as solutions of the following linear system of equations:

$$P(u_1) = e^{-UFR*u_1} + \sum_{i=1}^{J} \varsigma_i * W(u_1,u_i)$$

$$P(u_2) = e^{-UFR*u_2} + \sum_{i=1}^{J} \varsigma_i * W(u_2,u_i)$$

....

$$P(u_J) = e^{-UFR*u_J} + \sum_{i=1}^{J} \varsigma_i * W(u_J,u_i)$$

In vector space notation this becomes:

$$\vec{\mathrm{P}} = \vec{\mathrm{E}} + W * \vec{\zeta} \ ,$$

with:

$$\vec{\mathrm{P}} = (P(u_1), P(u_2), ...P(u_J))^T \quad \text{(The superscript T denoting the transposed vector)}$$

$$\vec{\mathrm{E}} = (e^{-UFR*u_1}, e^{-UFR*u_2}, ...e^{-UFR*u_J})^T,$$

$$\vec{\varsigma} = (\varsigma_1, \varsigma_2, ...\varsigma_i,)^{\mathrm{T}},$$

and

$W = (W(u_i, u_j))_{i=1,\ldots J,\ j=1,\ldots J}$, a JxJ-matrix of certain Wilson functions

From this notation we see at once that the solution (ζ_1, ζ_2, ζ_3, … ζ_J) is easily calculated by inverting the JxJ-matrix ($W(u_i, u_j)$) and multiplying it with the difference of the P-vector and the E-vector, i.e.

$$\vec{\zeta} = W^{-1} * (\vec{P} - \vec{E}),$$

We can now plug these parameters ζ_1, ζ_2, ζ_3, … ζ_J into the pricing function and get the value of the zero coupon bond price for all maturities τ, for which no zero bonds were given to begin with:

$$P(\tau) = e^{-UFR*\tau} + \sum_{i=1}^{J} \varsigma_i * W(\tau, u_i),\ \tau>0$$

From this value it is straightforward to calculate the spot rates by using the definition of the zero coupon bond price. The spot rates are calculated as $\tilde{R}_\tau = \frac{1}{\tau} * \ln\left(\frac{1}{P(\tau)}\right)$ for continuous compounded rates and $R_\tau = \left(\frac{1}{P(\tau)}\right)^{\frac{1}{\tau}} - 1$ if annual compounding is used.

Appendix C

Relation between spot and forward rates

The risk-free spot rate for a given maturity T can be interpreted as the yield of a risk-free zero coupon bond with maturity T. Forward rates are the rates of interest implied by spot rates for periods of time in the future. The relation between spot and forward term structures can best be illustrated by the following formulae, the first for annually compounded spot rates and the second for continuously compounded spot rates:

Annual compounding:

$$\begin{aligned}(1+R_T)^T &= (1+R_{T-1})^{T-1} \cdot \left(1+FR(T-1,T)\right) = (1+R_{T-2})^{T-2} \cdot \left(1+FR(T-2,T-1)\right) \cdot \left(1+FR(T-1,T)\right) = \\ &= \ldots = \left(1+FR(0,1)\right) \cdot \left(1+FR(1,2)\right) \cdot \left(1+FR(2,3)\right) \cdot \ldots \cdot \left(1+FR(T-2,T-1)\right) \cdot \left(1+FR(T-1,T)\right),\end{aligned}$$

where R_T , (R_{T-1}) denotes the spot rate for maturity T, (T-1), while FR(i,i+1) denotes the annual forward rate for the period from year end i to year end i+1, for i=0, 1, 2,…. T.

Continuous compounding:

$$\begin{aligned}&e^{T*\tilde{R}_T} = e^{(T-1)*\tilde{R}_{T-1}} \cdot e^{FRc(T-1,T)} = e^{(T-2)*\tilde{R}_{T-2}} \cdot e^{FRc(T-2,T-1)} \cdot e^{FRc(T-1,T)} = .. \\ &= e^{FRc(0,1)} \cdot e^{FRc(1,2)} \ldots e^{FRc(T-2,T-1)} \cdot e^{FRc(T-1,T)},\end{aligned}$$

where $\tilde{R}_T$ ($\tilde{R}_{T-1}$) is the continuously compounded basic spot rate for maturity T, (T-1), while FRc(i,i+1) denotes the annual continuous forward rate for the period from year end i to year end i+1, for i=0, 1, 2,…. T.

The Liquidity Premium - An Example

In Annex A of the Task Force report a possible proxy for the LP observable in financial markets is given. Applying the simplified formula given a LP in per annum bps relative to swap was computed for EUR, GBP and USD at given dates. The most recent estimated LP for Euro was 59 bps as per End September 2009.

In accordance with principle #3 of point I–4 Task Force report (page 13), the addition of a liquidity premium shall be limited to maturities where an additional return can be earned. Let us assume that for the Euro-zone appropriate instruments with maturities up to 30 years are available and that the portion of the LP observed in financial markets corresponding to (re-)insurance obligations is 100%.

Following the instructions in the Task Force report we compute the LP given in Table 1 below.

If the LP from this example is applied as spot rate adjustment (first alternative in 3.3), then the rates in Table 1 can be added to the spot rates from the basic risk-free term structure. A cash flow with maturity 26 years will thus be discounted with the basic risk-free rate for maturity 26 years increased by 47 bps. A cash flow with maturity 33 years will accordingly be discounted with the basic risk-free rate for maturity 33 increased by o bps. As can be seen, the LP will have no impact on the discounting of the cash flows with maturities of 30 years and from 30 years onwards.

Table 1: Liquidity premium 100%, EUR, as of 30.09.2009

Maturity in years	LP in bps	Maturity in years	LP in bps	Maturity in years	LP in bps	Maturity in years	LP in bps	Maturity in years	LP in bps	Maturity in years	LP in bps
1	59	21	59	41	0	61	0	81	0	101	0
2	59	22	59	42	0	62	0	82	0	102	0
3	59	23	59	43	0	63	0	83	0	103	0
4	59	24	59	44	0	64	0	84	0	104	0
5	59	25	59	45	0	65	0	85	0	105	0
6	59	26	47	46	0	66	0	86	0	106	0
7	59	27	35	47	0	67	0	87	0	107	0
8	59	28	24	48	0	68	0	88	0	108	0
9	59	29	12	49	0	69	0	89	0	109	0
10	59	30	0	50	0	70	0	90	0	110	0
11	59	31	0	51	0	71	0	91	0	111	0
12	59	32	0	52	0	72	0	92	0	112	0
13	59	33	0	53	0	73	0	93	0	113	0
14	59	34	0	54	0	74	0	94	0	114	0
15	59	35	0	55	0	75	0	95	0	115	0
16	59	36	0	56	0	76	0	96	0	116	0
17	59	37	0	57	0	77	0	97	0	117	0
18	59	38	0	58	0	78	0	98	0	118	0
19	59	39	0	59	0	79	0	99	0	119	0
20	59	40	0	60	0	80	0	100	0	120	0

Source: Task Force on the Illiquidity Premium, Report. Ceiops-SEC-34/10, 1 March 2010

If, on the other hand, the LP in Table 1 is applied as forward rate adjustment (second alternative in 3.3), then we have to add the LPs to the forward rates implied by the risk-free basic spot rate term structure. With these (adjusted) forward rates we can compute the adjusted spot rate curve.

We are interested in the effect of the LP on the adjusted spot rate curve, if this method (second alternative in 3.3) is chosen. The best way to see the difference is to calculate the spreads between the adjusted and the unadjusted spot rates, and compare them to the values given in table 1.

We want to keep our example simple. Due to the following remark we can avoid assuming an explicit basic spot rate term structure, as it is possible to assess the spread directly by applying the method given below in point 2.

For <u>continuously compounded</u> spot rates, we can see at once from the formula given in 6.1 that method 1 and method 2 described below give the same adjusted spot rate term structure:

1. Add the LPs to the basic risk-free forward rates and apply the formula from 6.1
2. Compute spot adjustment rates from the LPs (using formula in 6.1 with the LP as forward rates) and add these to the basic spot rate term structure.

The spot rate adjustments computed from the rates in Table 1 (using method 2), are presented in Table 2. Cash flows with maturity 26 years will have to be discounted with the basic risk-free rate for maturity 26 years increased by 59 bps, cash flows with maturity 33 years with the basic risk-free rate for maturity 33 increased by 48 bps, and so on. Even in the discounting of cash flows as far out as 120 years from now, a 13 bps adjustment due to the liquidity premium has to be taken into account.

Table 2: Liquidity premium spot rates adjustment derived from the LP rates in Table1

Maturity in years	LP in bps	Maturity in years	LP in bps	Maturity in years	LP in bps	Maturity in years	LP in bps	Maturity in years	LP in bps	Maturity in years	LP in bps
1	59	21	59	41	39	61	26	81	20	101	16
2	59	22	59	42	38	62	26	82	19	102	16
3	59	23	59	43	37	63	25	83	19	103	15
4	59	24	59	44	36	64	25	84	19	104	15
5	59	25	59	45	35	65	24	85	19	105	15
6	59	26	59	46	35	66	24	86	18	106	15
7	59	27	58	47	34	67	24	87	18	107	15
8	59	28	56	48	33	68	23	88	18	108	15
9	59	29	55	49	32	69	23	89	18	109	15
10	59	30	53	50	32	70	23	90	18	110	14
11	59	31	51	51	31	71	22	91	17	111	14
12	59	32	50	52	31	72	22	92	17	112	14
13	59	33	48	53	30	73	22	93	17	113	14
14	59	34	47	54	29	74	21	94	17	114	14
15	59	35	45	55	29	75	21	95	17	115	14
16	59	36	44	56	28	76	21	96	17	116	14
17	59	37	43	57	28	77	21	97	16	117	14
18	59	38	42	58	27	78	20	98	16	118	13
19	59	39	41	59	27	79	20	99	16	119	13
20	59	40	40	60	27	80	20	100	16	120	13