QIS 5
Risk-free interest rates – Extrapolation method

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1 Introduction

For liabilities expressed in any of the EEA currencies, Japanese yen, Swiss franc, Turkish lira or USA dollar, QIS5 provides to participants risk-free interest rate term structures. Additionally, risk-free interest rate term structures are provided for a secondary list of currencies.

The appropriate risk-free interest rate term structure is in practice constructed from a finite number of data points. Therefore, both interpolation between these data points and extrapolation beyond the last available data point of sufficient liquidity is required.

The purpose of this paper is to describe how the extrapolated part of the interest rate term structures for currencies where the relevant risk-free interest rate term structures are provided in the spreadsheet included in QIS5 package was set up.

2 Liquid points of risk-free interest rate curve

The non-extrapolated part of the risk-free interest rate curve for QIS5 purposes was delivered by the industry. The aspects of the risk-free interest rate term structure that had to be considered were the selection of the basic risk-free interest rate term structure, the method for adjusting inter-bank swaps for credit risk, the assessment of the last liquid point to enter the yield curve extrapolation and the derivation of the liquidity premium. Details on the determination of these can be found in the documentation given by CFO and CRO Forum.

This data basis consists of continuously compounded and already smoothed spot rates. The rates have been derived from market data by fitting a smoothed regression spline to market swap rates using Barrie & Hibbert's yield curve fitting methodology.

The extrapolation tool delivers spot rates with annual compounding for all other maturities up to 135 years as output. Furthermore, the Smith-Wilson approach achieves both interpolation (for maturities in the liquid end of the term structure where risk-free zero coupon rates are missing) and extrapolation. A consistent approach for inter- and extrapolation would be preferred by CEIOPS and we would thus recommend not to use already smoothed and interpolated market data as a starting point.

3 Extrapolation method

For QIS5, macroeconomic extrapolation techniques are used to derive the extrapolation beyond the last available data point. The overall aim is to construct a stable and robust extrapolated yield curve which reflects current market conditions and at the same time embodies economical views on how unobservable long term rates are expected to behave. Macroeconomic extrapolation techniques assume a long-term equilibrium interest rate. A transition of observed interest rates of short-term maturities to the assessed equilibrium interest rate of long-term maturities takes place within a certain maturity spectrum.

Valuation of technical provisions and the solvency position of an insurer or reinsurer shall not be heavily distorted by strong fluctuations in the short-term interest rate. This is particularly important for currencies where liquid reference rates are only available for short term maturities and simple extrapolation of these short term interest rates may cause excessive volatility. A macro-economic model meets the demands on a model that ensures relatively stable results in the long term.

There are some considerations that have to be faced when specifying the macro-economic extrapolation method for QIS5 purposes. These are examined further in the following sections.

3.1 Determination of ultimate forward rate

A central feature is the definition of an unconditional ultimate long-term forward rate (UFR) for infinite maturity and for all practical purposes for very long maturities. The UFR has to be determined for each currency. While being subject to regular revision, the ultimate long term forward rate should be stable over time and only change due to fundamental changes in long term expectations. The unconditional ultimate long-term forward rate is determined for each currency by macro-economic methods.

Common principles governing the methods of calculation should ensure a level playing field between the different currencies. For all currencies interest rates beyond the last observable maturity - where no market prices exist - are needed.

The most important economic factors explaining long term forward rates are long-term expected inflation and expected real interest rates. From a theoretical point of view it can be argued that there are at least two more components: the expected long-term nominal term premium and the long-term nominal convexity effect.

The term premium represents the additional return an investor may expect on risk-free long dated bonds relative to short dated bonds, as compensation for the longer term investment. This factor can have both a positive and a negative value, as it depends on liquidity considerations and on preferred investor habitats. As no empirical data on the term premium for ultra-long maturities exists, a practical estimation of the term premium is not undertaken for QIS5 purposes.

The convexity effect arises due to the non-linear (convex) relationship between interest rates and the bond prices used to estimate the interest rates. This is a purely technical effect and always results in a negative component.

In order to have a robust and credible estimate for the UFR the assessment is based on the estimates of the expected inflation and the expected short term real rate only.

Making assumptions about expectations this far in the future for each economy is difficult. However, in practice a high degree of convergence in forward rates can be expected when extrapolating at these long-term horizons. From a macro economical point of view it seems consistent to expect broadly the same value for the UFR around the world in 100 years. Nevertheless, where the analysis of expected long term inflation or real rate for a currency indicates significant deviations, an adjustment to the long term expectation and thus the UFR has to be applied. Therefore, three categories are established capturing the medium UFR as well as deviations up or down.

Thus, the macro economically assessed UFR for use in the QIS5 is set to 4.2 per cent (+/-1 percentage points) per anno. This value is assessed as the sum of the expected inflation rate of annually 2 per cent (+/- 1 percentage points) and of an expected short term return on risk free bonds of 2.2 per cent per anno. Further details on the estimation of expected inflation rate and expected real rate can be found in Appendix A.

For QIS5 the following UFR are used:

Category	Currencies	Macro economically assessed UFR
1	JPY, CHF	3.2
2	Euro, SEK, NOK, DKK, GBP, USD, CZK, BGN, LVL, LTL, EEK, PLN, RON, HUF, ISK, CAD, AUD, SGD, MYR, KRW, THB, HKD, TWD, CNY	4.2
3	TRY, ZAR, MXN, INR, BRL	5.2

3.2 Transition to the equilibrium rate

This paragraph considers the issue of how to extrapolate between the estimated forward rates and the unconditional ultimate forward rate.

3.2.1 Technique for transition

For QIS5 the Smith-Wilson method will be used. If applied to observed zero coupon bond prices from the liquid market, this method ensures that the term structure is fitted exactly to all observed zero coupon bond prices, i.e. all liquid market data points are used without smoothing. If applied to the already smoothed market data that the industry has delivered for the liquid part of the term structure, the extrapolated term structure will pass through all zero coupon market rates that are given as input.

Furthermore, with the Smith-Wilson approach both interpolation (for maturities in the liquid end of the term structure, if risk-free zero coupon rates are missing) and extrapolation can be achieved. The industry has already delivered risk-free rates for all maturities from one year up to the last liquid maturity (given in whole years) for the currencies in question. Thus, for these currencies, the Smith-Wilson approach will be used only for interpolation in cases of non-integer maturities and for the extrapolation beyond the last liquid data point.

It is a sophisticated approach that is still easy to use, and gives both a relative smooth forward rate and a smooth spot rate curve in the extrapolated part. Further details on the Smith-Wilson technique can be found in Appendix C.

Nevertheless, the linear method has been also run in order to provide a kind of cross-checking, avoiding a full reliance in a single method and enhancing the robustness of results provided by the Smith-Wilson approach.

3.2.2 Speed of transition

The speed of transition towards the UFR can be specified by the maturity T2 at which the forward rate curve "reaches" the UFR. A range for T2 between 70 and 120 years is considered appropriate. The forward rate curve is deemed to reach the UFR at T2, if the spread between the UFR and the annual forward rate at T2 - in absolute values - lies within predefined limits. These limits are chosen as a threshold of 3 BP for QIS 5 purposes.

The choice for the maturity at which the ultimate forward rate will be "reached" has an impact on the stability of the yield curve over time. On the one hand the yield curve follows a quite flat course beyond the maturity the UFR is "reached". Therefore, the earlier the UFR is "reached" the more stable is the yield curve for long maturities. On the other hand, it has to be considered that the earlier the UFR is "reached", the more sensitive will the yield curve be towards changes in the choice of the UFR.

In light of this, CEIOPS provides a set of extrapolated interest rate curves for T2 set to 90 years.

Nevertheless, there is no fixed, predefined maturity where the UFR is deemed to be arrived at in the Smith-Wilson approach, but the speed of convergence to the unconditional ultimate forward rates has to be set. Therefore, in the Smith-Wilson technique the speed of transition as defined above has to be translated into a convergence parameter α (alpha). Thomas[1], who was looking at Australian term structures, fitted this parameter empirically to α = 0.1 as it ensured sensible results and economically appropriate curves in most cases.

For QIS5, in a first step the default value for alpha is 0.1. Only if the extrapolated rates deviate from the UFR at the predefined time T2 by more than 0.03% (i.e. 3 BP), then alpha is recalibrated in a second step, such that the spread lies within this threshold. This ensures that the extrapolated curve is sufficiently close to the chosen UFR at T2.

3.3 Allowance for liquidity premium

For every currency, the liquidity premium is allowed for in the risk-free interest rate curve up to a cut-off point. Past that cut-off point a phasing-out period of 5 years for the LP is applied.

There are two alternatives for the application of the liquidity premium to the basic risk-free interest rate curve. The first alternative is to interpret the LP as a spot rate and adjust the risk-free basic term structure (i.e. the basic zero coupon rates) with this premium. The second alternative would be to interpret the LP as a forward rate and adjust the forward rates with the LP.

The main difference between adjusting the spot rates and adjusting the forward rates is the impact on discounting cash flows beyond maturities for which a liquidity premium in assets is deemed to exist and is deemed to be measurable. In the first alternative, the LP is restricted to the discounting of cash flows of maturities for which a liquidity premium for assets can be assessed. In the second alternative the LP has an impact on the discounting of cash flows of all maturities even beyond maturities for which a liquidity premium in assets is deemed to exist. This is due to the fact that spot rates are a kind of average of forward rates, and thus spot rates implicitly contain the liquidity adjustments on the forward rates that enter the average.

After having discussed the pros and cons of the issue, the extrapolation team decided to attach the utmost weight to the condition that no liquidity premium should be included in the extrapolated part of the interest rate curve and hence to implement the LP as an adjustment of the spot rate for QIS5.

[1] Michael Thomas, Eben Maré: Long Term Forecasting and Hedging of the South African Yield Curve, Presentation at the 2007 Convention of the Actuarial Society of South Africa

4 Appendix A – Estimation of UFR

4.1 Estimation of expected long term inflation rate

The expected inflation should not solely be based on historical averages of observed data, as the high inflation rates of the past century do not seem to be relevant for the future. The fact is that in the last 15-20 years many central banks have set an inflation target or a range of inflation target levels and have been extremely successful in controlling inflation, compared to previous periods.

Barrie Hibbert[2] propose to assess the inflation rate as 80 per cent of the globally prevailing inflation target of 2 per cent per anno and 20 per cent of an exponentially weighted average of historical CPI inflations when modelling the term structure in their Economic Scenario Generator. When they assess the historical inflation average of the main economies they still compute a high level as of December 2007 (they assess an expected global inflation rate of 2.4 per cent per anno) but with a strong downward trend over the sample of data they considered.

In order to have a robust and credible estimate for the UFR, the standard expected long term inflation rate is set to 2 per cent per anno, consistently to the explicit target for inflation most central banks operate with[3].

Nevertheless, based on historical data for the last 10-15 years and current inflation, two additional categories are introduced to capture significant deviations either up or down in the expected long term inflation rate for certain countries. Table 1 shows inflation data for the OECD-countries in the period 1994 – 2009.

Table 1: Inflation 1994 – 2009 OECD Countries and some Non-OECD members

Price indices (MEI) : Consumer prices - Annual inflation

Data extracted on 22 Apr 2010 13:13 UTC (GMT) from OECD.Stat

Frequency	Annual															
Time	1994	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
Country																
Australia	1.9	4.6	2.6	0.3	0.9	1.5	4.5	4.4	3.0	2.8	2.3	2.7	3.5	2.3	4.4	1.8
Austria	3.0	2.2	1.9	1.3	0.9	0.6	2.3	2.7	1.8	1.4	2.1	2.3	1.4	2.2	3.2	0.5
Belgium	2.4	1.5	2.1	1.6	0.9	1.1	2.5	2.5	1.6	1.6	2.1	2.8	1.8	1.8	4.5	-0.1
Canada	0.2	2.1	1.6	1.6	1.0	1.7	2.7	2.5	2.3	2.8	1.9	2.2	2.0	2.1	2.4	0.3
Czech Republic	10.0	9.1	8.8	8.5	10.7	2.1	3.9	4.7	1.8	0.1	2.8	1.9	2.6	3.0	6.3	1.0
Denmark	2.0	2.1	2.1	2.2	1.8	2.5	2.9	2.4	2.4	2.1	1.2	1.8	1.9	1.7	3.4	1.3
Finland	1.1	0.8	0.6	1.2	1.4	1.2	3.0	2.6	1.6	0.9	0.2	0.6	1.6	2.5	4.1	0.0
France	1.7	1.8	2.0	1.2	0.6	0.5	1.7	1.6	1.9	2.1	2.1	1.7	1.7	1.5	2.8	0.1

[2] Steffen Sørensen, Interest rate calibration – How to set long-term interest rates in the absence of market prices, Barrie+Hibbert Financial Economic Research, September 2008.
[3] Also the European Central bank aims at an annual inflation just below 2 per cent.

Germany		2.8	1.8	1.4	1.9	1.0	0.6	1.4	1.9	1.5	1.0	1.7	1.5	1.6	2.3	2.6	0.4
Greece		10.9	8.9	8.2	5.5	4.8	2.6	3.2	3.4	3.6	3.6	2.9	3.6	3.2	2.9	4.2	1.2
Hungary		18.9	28.3	23.5	18.3	14.2	10.0	9.8	9.1	5.3	4.7	6.7	3.6	3.9	8.0	6.0	4.2
Iceland		1.6	1.7	2.3	1.8	1.7	3.2	5.1	6.4	5.2	2.1	3.2	4.0	6.7	5.1	12.7	12.0
Ireland		2.4	2.5	1.7	1.4	2.4	1.6	5.6	4.9	4.6	3.5	2.2	2.4	3.9	4.9	4.1	-4.5
Italy		4.1	5.2	4.0	2.0	2.0	1.7	2.5	2.8	2.5	2.7	2.2	2.0	2.1	1.8	3.3	0.8
Japan		0.7	-0.1	0.1	1.8	0.7	-0.3	-0.7	-0.8	-0.9	-0.2	0.0	-0.3	0.2	0.1	1.4	-1.4
Korea		6.3	4.5	4.9	4.4	7.5	0.8	2.3	4.1	2.7	3.6	3.6	2.8	2.2	2.5	4.7	2.8
Luxembourg		2.2	1.9	1.2	1.4	1.0	1.0	3.2	2.7	2.1	2.0	2.2	2.5	2.7	2.3	3.4	0.4
Mexico		7.0	35.0	34.4	20.6	15.9	16.6	9.5	6.4	5.0	4.5	4.7	4.0	3.6	4.0	5.1	5.3
Netherlands		2.8	1.9	2.0	2.2	2.0	2.2	2.3	4.2	3.3	2.1	1.2	1.7	1.2	1.6	2.5	1.2
New Zealand		1.7	3.8	2.3	1.2	1.3	-0.1	2.6	2.6	2.7	1.8	2.3	3.0	3.4	2.4	4.0	2.1
Norway		1.4	2.4	1.2	2.6	2.3	2.3	3.1	3.0	1.3	2.5	0.5	1.5	2.3	0.7	3.8	2.2
Poland		33.0	28.0	19.8	14.9	11.6	7.2	9.9	5.4	1.9	0.7	3.4	2.2	1.3	2.5	4.2	3.8
Portugal		5.4	4.2	3.1	2.3	2.8	2.3	2.9	4.4	3.6	3.3	2.4	2.3	3.1	2.5	2.6	-0.8
Slovak Republic		13.4	9.8	5.8	6.1	6.7	10.6	12.0	7.3	3.1	8.6	7.5	2.7	4.5	2.8	4.6	1.6
Spain		4.7	4.7	3.6	2.0	1.8	2.3	3.4	3.6	3.1	3.0	3.0	3.4	3.5	2.8	4.1	-0.3
Sweden		2.2	2.5	0.5	0.7	-0.3	0.5	0.9	2.4	2.2	1.9	0.4	0.5	1.4	2.2	3.4	-0.3
Switzerland		0.9	1.8	0.8	0.5	0.0	0.8	1.6	1.0	0.6	0.6	0.8	1.2	1.1	0.7	2.4	-0.5
Turkey		105.2	89.1	80.4	85.7	84.6	64.9	54.9	54.4	45.0	21.6	8.6	8.2	9.6	8.8	10.4	6.3
United Kingdom		2.0	2.7	2.5	1.8	1.6	1.3	0.8	1.2	1.3	1.4	1.3	2.0	2.3	2.3	3.6	2.2
United States		2.6	2.8	2.9	2.3	1.6	2.2	3.4	2.8	1.6	2.3	2.7	3.4	3.2	2.9	3.8	-0.4
G7		2.2	2.3	2.3	2.0	1.3	1.4	2.2	2.0	1.3	1.8	2.0	2.4	2.4	2.2	3.2	-0.1
OECD - Europe		8.6	8.7	7.6	7.2	7.0	5.4	5.7	5.6	4.9	3.0	2.4	2.4	2.6	2.7	3.9	1.2
OECD - Europe excluding high inflation countries		2.8	2.9	2.6	2.0	1.6	1.3	2.3	2.6	2.0	1.9	2.0	2.0	2.1	2.2	3.4	0.8
OECD - Total		4.8	6.1	5.7	4.8	4.2	3.6	4.0	3.7	2.8	2.4	2.4	2.6	2.6	2.5	3.7	0.5
OECD - Total excluding high inflation countries		2.4	2.5	2.4	2.1	1.6	1.5	2.6	2.5	1.7	2.0	2.2	2.5	2.5	2.3	3.5	0.4
Non-OECD Member Economies	Brazil	2 075.9	66.0	15.8	6.9	3.2	4.9	7.0	6.8	8.5	14.7	6.6	6.9	4.2	3.6	5.7	4.9
	Chile	0.7	0.7	0.5	0.5	0.4	0.2	0.4	0.2	0.2	0.1	0.2	0.3	0.2	0.6	0.6	-0.2
	China	24.3	16.8	8.3	2.8	-0.8	-1.4	0.3	0.7	-0.7	1.1	3.8	1.8	1.6	4.8	5.9	-1.0
	Estonia	..	..	..	..	..	3.3	4.0	5.7	3.6	1.3	3.0	4.1	4.4	6.6	10.4	-0.1
	India	10.2	10.2	9.0	7.2	13.2	4.7	4.0	3.8	4.3	3.8	3.8	4.2	5.8	6.4	8.3	10.9
	Indonesia	8.5	9.4	8.0	6.2	58.4	20.5	3.7	11.5	11.9	6.8	6.1	10.5	13.1	6.4	10.2	4.4
	Israel	12.4	10.0	11.3	9.0	5.4	5.2	1.1	1.1	5.7	0.7	-0.4	1.3	2.1	0.5	4.6	3.3
	Russian Federation	307.5	197.5	47.9	14.7	27.8	85.7	20.8	21.5	15.8	13.7	10.9	12.7	9.7	9.0	14.1	11.7
	Slovenia	21.0	13.5	9.9	8.4	7.9	6.2	8.9	8.4	7.5	5.6	3.6	2.5	2.5	3.6	5.7	0.9
	South Africa	8.9	8.7	7.4	8.6	6.9	5.2	5.3	5.7	9.5	5.7	-0.7	2.1	3.2	6.2	10.1	7.2

Singapore, Malaysia, Thailand, Hong Kong and Taiwan are not included in the list from the OECD database. The data for these currencies can be found in Table 2 and are taken from EcoWin (Reuters) database.

Table 2: Inflation 1994 - 2009 Certain Asian Countries

Country		Year															
	Consumer Prices,	1994	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
Hong Kong,	Composite CPI, Total, Index, 2004-2005=100	9,6 %	7,0 %	6,7 %	5,2 %	-1,6 %	-4,0 %	-2,1 %	-3,6 %	-1,5 %	-1,9 %	0,3 %	1,4 %	2,3 %	3,8 %	2,0 %	1,3 %
Malaysia,	Total, Index, 2005=100 [c.o.p 1 year]	3,5 %	3,2 %	3,3 %	2,9 %	5,3 %	2,5 %	1,2 %	1,2 %	1,7 %	1,2 %	2,1 %	3,2 %	3,1 %	2,4 %	4,4 %	1,1 %
Singapore,	All items, Index, 2009=100 [c.o.p 1 year]	2,9 %	0,8 %	2,0 %	2,0 %	-1,4 %	0,7 %	2,1 %	-0,6 %	0,4 %	0,7 %	1,3 %	1,3 %	0,8 %	3,7 %	5,5 %	-0,5 %
Thailand,	Total, Index, 2007=100 [c.o.p 1 year]	4,7 %	7,5 %	4,7 %	7,7 %	4,3 %	0,6 %	1,5 %	0,7 %	1,7 %	1,7 %	3,0 %	5,8 %	3,5 %	3,2 %	0,4 %	3,5 %
Taiwan,	Total, Index, 2006=100 [c.o.p 1 year]	2,7 %	4,6 %	2,5 %	0,3 %	2,1 %	0,1 %	1,6 %	-1,7 %	0,8 %	-0,1 %	1,6 %	2,2 %	0,7 %	3,3 %	1,3 %	-0,2 %

Table 1 and Table 2 show that three OECD-countries and three non-OECD members we are looking at had inflation above 4.5 percent in 2009: Iceland (12 percent), Mexico (5.3 percent), Turkey (6.3 percent), Brazil (4.9 percent), India (10.9 percent) and South Africa (7.2 percent).

During the last 15 years, Turkey has been categorised by OECD as a high inflation country[4]. Turkey's inflation target is also higher (5-7.5% for the period 2009 - 2012) than in other countries. Mexico, Brazil, and India have had persistent high inflation rates in the last 15 years. South Africa has had high inflation rates during the decade from 1994 to 2003, a drop to negative inflation in 2004 and rising inflation rates up to 2008. Moreover, Mexico's inflation target for 2010 is 3 percent, Brazil's national monetary council has set the inflation target at 4.5 percent plus or minus two percentage points for this year and 2011, South African's central bank has set the upper end of its inflation target at 3%-6% and in India the central bank does not follow a policy of targeting inflation.

Based on this data basis, all five currencies are ranked in the high inflation group.

Hungary and Iceland are also possible candidates for the high inflation group. However, deviations to the average inflation rate are far more moderate than those for the other high inflation countries. Furthermore, Hungary and Iceland are expected to join the Euro sooner or later (and thus have to fulfil the convergence criteria). Therefore, Hungary and Iceland are classified in the standard inflation category.

Japan, having deflation in the period since 1994, is an obvious candidate for the "low inflation"-group. Switzerland can also be seen as an outlier. This is due to the fact that historically relatively low inflation rates can be observed and that Switzerland is particular attractive in the international financial markets (exchange rate conditions, liquidity, "save haven"[5]...). For these reasons, lower inflation assumptions are applied for the Swiss francs.

The estimate covers one-year inflation rate 70 - 100 years from now. It is arbitrary to say whether the inflation differences we see today and have seen the last 15 years will persist 100 years into the future. However, historical evidence and current long term interest rates indicate that it is reasonable to have three groups of currencies with different inflation assumptions. The standard inflation rate is set to 2 per cent per anno. To allow for deviations up and down to the standard inflation rate, an adjustment to the estimate of +/- 1 percentage point is applied for the high inflation group and the low inflation group respectively. This adjustment of 1 percentage point will be applied to the estimated inflation rate for outliers based on differences in current long term interest rates (30Y), observed historical differences between the average interest rate and differences in short term inflation expectations.

The following grouping is used for the estimated expected long term inflation rate:

[4] http://stats.oecd.org/index.aspx

[5] http://www.cepr.org/pubs/dps/DP5181.asp "Why are Returns on Swiss Francs so low? Rare events may solve the puzzle." Peter Kugler, Weder di Mauro

Standard inflation rate set to 2%:	Euro-zone, UK, Norway, Sweden, Denmark, USA, Poland, Hungary, Iceland, Czech Republic, Bulgaria, Latvia, Lithuania, Estonia, Romania, Canada, Australia, Korea, China, Singapore, Malaysia, Thailand, Hong Kong and Taiwan
High inflation rate set to 3%:	Turkey, Brazil, Mexico, South Africa and India
Low inflation rate set to 1%:	Japan, Switzerland*

** combined effects*

4.2 Estimation of the expected real rate of interest

We expect that the real rates should not differ substantially across economies as far out as 100 years from now. Elroy Dimson, Paul Marsh and Mike Staunton provide a global comparison of annualized bond returns over the last 110 years (1900 to 2009) for the following 19 economies: Belgium, Italy, Germany, Finland, France, Spain, Ireland, Norway, Japan, Switzerland, Denmark, Netherlands, New Zealand, UK, Canada, US, South Africa, Sweden and Australia[6].

Figure 1: Real return on bonds 1900 – 2009

Source: Dimson, Marsh and Staunton – Credit Suisse Global Investment Returns Yearbook

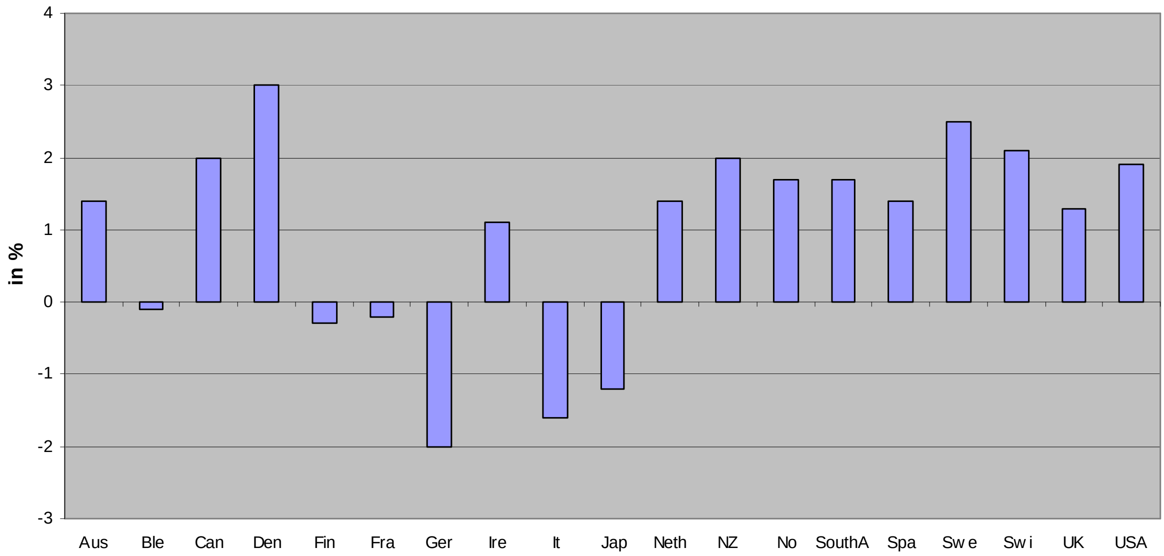


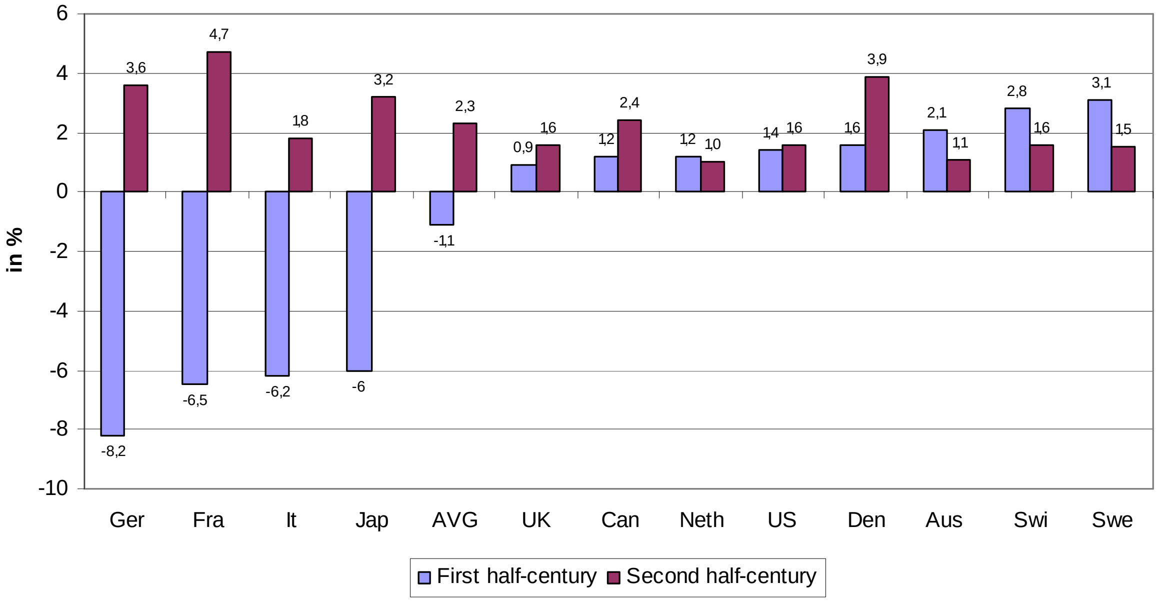
Figure 1 shows that, while in most countries bonds gave a positive real return, six countries experienced negative returns. Mostly the poor performance dates back to the first half of the 20th century and can be explained with times of high or hyperinflation[7]. Aggregating the real returns on bonds for each currency[8] to an annual rate of real return on globally diversified bonds gives a rate of 1.7 per cent.

[6] Credit Suisse Global Investment Returns Yearbook 2010, To be found at www.tinyurl.com/DMS2010
[7] German hyperinflation in 1922/1923, in Italy an inflation of 344% in 1944, in France 74% in 1946 and in Japan 317% in 1946.
[8] Average where each return is weighted by its country's GDP.

In an earlier publication, the same authors compared the real bond returns from the second versus the first half of the 20th century for the following 12 economies: Italy, Germany, France, Japan, Switzerland, Denmark, Netherlands, UK, Canada, US, Sweden and Australia[9]. The average real bond return over the second half of the 20th century was computed as annually 2.3 per cent (compared to -1.1 percent for the first half of the 20th century).

*Figure 2: Real bond returns: first versus second half of 20th century**

Source: Dimson, Marsh and Staunton (ABN- Ambro/LBS)



* Data for Germany excludes 1922-23. AVG = Average

In light of the above data, 2.2 per cent is an adequate estimate for the expected real interest rate.

5 Appendix B - Relation between spot and forward rates

The risk-free spot rate for a given maturity T can be interpreted as the yield of a risk-free zero coupon bond with maturity T. Forward rates are the rates of interest implied by spot rates for periods of time in the future. The relation between spot and forward term structures can best be illustrated by the following formulae, the first for annually compounded spot rates and the second for continuously compounded spot rates:

Annual compounding:

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[9] Elroy Dimson, Paul Marsh and Mike Staunton: Risk and return in the 20th and 21th, Business Strategy Review, 2000, Volume 11 issue 2, pp 1-18. See Figure 4 on page 5. The article can be downloaded at: http://docs.google.com/viewer?a=v&q=cache:07V7vM0gu5oJ:citeseerx.ist.psu.edu/viewdoc/download%3Fdoi%3D10.1.1.11.7613%26rep%3Drep1%26type%3Dpdf+Risk+and+return+in+the+20th+and+21th+Centuries&hl=no&gl=no&sig=AHIEtbQbxwuXZNO6ViVlqkV0KZ63LKhB0g

$$(1+R_T)^T = (1+R_{T-1})^{T-1}\cdot\left(1+FR(T-1,T)\right) = (1+R_{T-2})^{T-2}\cdot\left(1+FR(T-2,T-1)\right)\cdot\left(1+FR(T-1,T)\right)=$$
$$= \ldots = \left(1+FR(0,1)\right)\cdot\left(1+FR(1,2)\right)\cdot\left(1+FR(2,3)\right)\cdot\ldots\cdot\left(1+FR(T-2,T-1)\right)\cdot\left(1+FR(T-1,T)\right),$$

where R_T , (R_{T-1}) denotes the spot rate for maturity T, (T-1), while FR(i,i+1) denotes the annual forward rate for the period from year end i to year end i+1, for i=0, 1, 2,…. T.

Continuous compounding:

$$e^{T\cdot\tilde{R}_T} = e^{(T-1)\cdot\tilde{R}_{T-1}}\cdot e^{FRc(T-1,T)} = e^{(T-2)\cdot\tilde{R}_{T-2}}\cdot e^{FRc(T-2,T-1)}\cdot e^{FRc(T-1,T)} = ..$$
$$= e^{FRc(0,1)}\cdot e^{FRc(1,2)}....e^{FRc(T-2,T-1)}\cdot e^{FRc(T-1,T)},$$

where $\tilde{R}_T$ ($\tilde{R}_{T-1}$) is the continuously compounded basic spot rate for maturity T, (T-1), while FRc(i,i+1) denotes the annual continuous forward rate for the period from year end i to year end i+1, for i=0, 1, 2,…. T.

6 Appendix C - Smith-Wilson technique

6.1 Introduction

The Smith-Wilson technique is a macroeconomic approach: a spot (i.e. zero coupon) rate curve is fitted to observed prices of financial instruments, with the macroeconomic ultimate long term forward rate as input parameter.[10]

The output from the Smith-Wilson calculation is the discount factor *P(t), t>0*. *P(t)*is the market price at valuing time for a zero coupon bond paying 1 at some future date *t* (the maturity).

Depending on whether we need the spot rates as continuously compounded rates $\tilde{R}_t$ or as rates R_t with annual compounding, the following relation between the discount factor and the spot rate can be used to assess the spot rates: $P(t)=\exp(-t\cdot\tilde{R}_t)$ for continuously compounded rates, and $P(t)=(1+R_t)^{-t}$ for annual compounding.

The relation between the two rates is $\tilde{R}_t = \ln(1+R_t)$.

The aim is to assess the price function *P(t)* for <u>all maturities</u> *t, t > 0.* From the relations referred to above it can be seen that therewith the whole risk-free term structure at valuing date is defined.

In its most general form the input data for the Smith-Wilson approach can consist of different financial instruments that relate to interest rates. We will first present the

[10] The mathematical background and a further discussion of the method can be found in the original paper by Andrew Smith and Tim Wilson, see Smith A. & Wilson, T. – “Fitting Yield curves with long Term Constraints” (2001), Research Notes, Bacon and Woodrow. (Remark: We will refer later on to an actualised version of the paper.)

formulae in the case where the inputs are zero coupon bond prices. The formulae in this simple case are quite easy to understand and straightforward to implement. Then we will present the formulae for the general case, where a large set of arbitrary financial instruments can be the input.

All financial instruments specified through

- their market price at valuation date,
- the cash payment dates up to maturity, and
- the size of the cash flows at these dates,

can be input instruments for the Smith-Wilson method.

In the last part of this note we will look at the input for fitting to zero coupon bond rates, to coupon bond rates and to par swap rates.

We will proceed as follows: After some general remarks on extrapolation techniques in section 2 we list the advantages and disadvantages of the Smith-Wilson technique in section 3, give the formulae in section 4, apply these formulae to different input instruments in section 5 and illustrate the method through two worked examples for par swap rates in section 6.

6.2 Some general remarks

Most extrapolation methods start from the price function, and assume that the price function is known for a fixed number of *N* maturities. In order to get the price function for all maturities, more assumptions are needed.

The most common procedure is to impose – in a first step - a functional form with *K* parameters on the price function, on the spot rate curve or on the forward rate curve[11]. These functional forms could be polynomials, splines, exponential functions, or a combination of these or different other functions[12].

In some of the methods, in a second step, the *K* parameters are estimated by minimizing the sum of the squares of the differences between estimated data and market data at each point in time where market data is given. In other methods K equations are set up from which the K parameters are calculated.

The equations are – as a rule - set up in a manner that guarantees that *P* has (most of) the features desired for a price function. The desired features are:

- *P* is a positive function,
- strictly decreasing,
- with value 1 at time t=0,
- passing through all given data points,
- to a certain degree smooth, and
- with values converging to 0 for large t.

In some of the methods the term structure is estimated by using one approach for all maturities, in others different methods are used depending on whether spot rates are assessed in the liquid part or in the extrapolated part of the term structure. The most

[11] Svensson imposes a parametric form with 6, Nelson-Siegel one with 4 parameters.
[12] BarrieHibbert use cubic splines for the liquid part and Nelson-Siegel for the extrapolated part.

prominent examples of the first procedure are the Svensson method and the Nelson-Siegel[13] method, where the same parametric form is used throughout the whole term structure. BarrieHibbert on the other hand apply splines for the liquid part and Nelson-Siegel for the extrapolated part of the term structure.

In the Smith-Wilson method the pricing function *P(t)*, for all *t>0*, is set up as the sum of a term $e^{-UFR \cdot t}$ for the asymptotical long term behavior of the discount factor and a linear combination of *N* kernel functions[14] $K_i(t)$, $i=1,2,\dots,N$ (the number *N* of kernel functions being equal to the number of input instruments).

The kernel functions are appropriately defined functions of the input market data and two input parameters: the ultimate forward rate (*UFR*) and a parameter (*alpha*) that determines how fast the estimated forward rates converge to *UFR*.

If *N* input instruments are given, we know *N* market prices and can thus set up *N* linear equations. In most of the cases the resulting system of linear equations (SLE) can be solved automatically, i.e. without interfering from the outside. By plugging the solution of the SLE (solution assessed for the maturities of the *N* input instruments) into the Smith-Wilson pricing function at any given time *t* we receive the discount function for maturity *t*. With the discount function, the spot rate curve is known.

6.3 Advantages and disadvantages of the Smith-Wilson (S-W) approach

Compared to the other extrapolation methods, the main advantages can be summed up as follows:

- S-W is a method in the open domain. Both the formulae and a computing tool can be published on CEIOPS homepage. Thus, the method is wholly transparent and fully accessible to all companies, at all times.
- S-W is very flexible concerning the input, and at the same time it is very easy to implement. The risk-free term structure can be assessed from a choice of bonds (with or without coupons) or from swap rates, all by using one simple[15] excel-spread sheet.
- S-W can be used as a widely mechanized approach. However, even so in most cases the assessment of the extrapolated rates will consist in automatically applying the formulas to the input data, in some situations - where the input data is biased, or where the linear equations that have to be solved are linearly dependent or nearly linearly dependent[16] - judgment may still be needed.

[13] Method used by the ECB and many other central banks, when assessing the published zero coupon rates.
[14] The idea behind the choice of the kernel functions can be found in Smith A. & Wilson, T. – "Fitting Yield curves with long Term Constraints" (2001), Research Notes, Bacon and Woodrow.
[15] Especially, VBA code will not be needed, as in many companies the opening of macro code from sources outside the company is considered a breach of security and will not be allowed.
[16] The system of linear equations that have to be solved can become linearly dependent or nearly linearly dependent for certain input data. This will require that the user of the method has to decide to remove some of the data from the input in order to compute a valid solution. The function W(t,u) can be interpreted as the covariance function of an Integrated Ornstein Uhlenbeck yield curve model. From this follows that linear dependency can only occur in cases where two or more of the input instruments have the same maturity; these are cases in which also the other extrapolation methods will have a problem. For details see Frankland, Smith,

- S-W provides a perfect fit of the estimated term structure to the liquid market data. In many other methods the term structure is assessed as a smoothed curve that is only reasonably close to the market data[17]. A trade-off is often made between the goodness of fit and the smoothness of the term structure. In S-W all relevant data from the liquid market is taken as input, no smoothing is performed.
- S-W is based on solving a linear system of equations analytically. This is an advantage compared to methods that are based on e.g. minimizing sums of least square deviations, as these are susceptible to catastrophic jumps when the least-squares fit jumps from one set of parameters to another set of quite different values[18]. This problem is due to the non-linearity in the least squares formula which can give rise to more than one local minimum.
- S-W can be applied directly to the raw data from financial markets. No bootstrapping or other methods are needed to transform market par swap rates into zero coupon bond rates, as the case is in for example the linear extrapolation method, where the input has to be first converted into zero coupon bond rates.
- S-W is a uniform approach, both interpolation between the liquid market data points and extrapolation beyond the last data point are performed. For many other methods interpolation and extrapolation are done separately, often based on different principles and mostly using different kinds of functions for assessing the different parts of the curve. This can lead to inconsistencies between the interpolated and extrapolated part of the same curve and also to inconsistencies over time for each part of the curve. (If e.g. due to higher liquidity at the long end, the entry point for the extrapolation changes significantly from one period to the next, the rates for maturities between these two points in time will be assessed with quite different methods from one period to the next.)
- In S-W the ultimate forward rate will be reached asymptotically[19]. How fast the extrapolated forward rates converge to the *UFR* will depend on how the rates in the liquid part of the term structure behave and on an exogenic parameter *alpha*. For higher *alpha* the extrapolated forward rates converge faster to the *UFR*, i.e. the market data from the liquid part of the curve are of less impact for the extrapolated rates.

Some of the disadvantages of the Smith-Wilson approach:

Wilkins, Varnell, Holtham, Biffis, Eshun, Dullaway – "Modelling Extreme Market Values – A Report of the Benchmarking Stochastic Models Working Party" (2008). The paper can be downloaded at: http://www.actuaries.org.uk/__data/assets/pdf_file/0007/140110/sm20081103.pdf

[17] The Svensson and Nelson-Siegel method can be used as macroeconomic methods if the parameter defining the flat component of the curve is taken as UFR. The other 5 (3) parameters will be determined through a least square optimisation. For a market with a large set of market data the estimated term structure will not fit the market data exactly. Another example is given by the method CRO-Forum used to assess the risk-free rates from par swap rates when they proposed the input for QIS5 for CEIOPS. They use a "regression spline with smoothing constraints" method, the "Barrie&Hibbert standard yield curve fitting methodology". They clarify on page 8 of their note "QIS5 Technical Specification Risk –free interest rates" the following: "This method produces rates that are very close to but not exactly equal to market rates. The average absolute error is generally less than 1 basis point." It is not very clear whether this means that the error is assessed by first netting out positive and negative deviations for each currency, and so taking the average of the absolute value of these netted errors over all currencies. Should this be implied by what CRO-Forum writes, the fit of the term structures to the market data could be much worse than the 1 bps suggest.

[18] For a thorough discussion of these problems see Andrew J. G. Cairns – "Descriptive Bond-Yield and Forward-rate models for the British Government Securities' Market" (1997).
The paper can be found at http://www.ma.hw.ac.uk/~andrewc/papers/ajgc11.pdf

[19] Introducing the maturity T2 as the maturity where the UFR is reached literally can be avoided if the S-W outcome is no longer compared to the linear extrapolation outcome.

- The parameter *alpha* has to be chosen outside the model. Thus, in general, expert judgment would be needed to assess this input parameter for each currency and each point in time separately. In order to have a harmonized approach over all currencies in Solvency II we will for all currencies use the Smith-Wilson approach with the parameter *alpha* starting at 0.1[20]. If this alpha is not appropriate for the currency it is applied to, we will increase it iteratively, until it is deemed – based on given criteria - to be appropriate. A lot more work needs to be done here to develop objective criteria for setting the alpha, in order to avoid that expert judgment is needed in all these cases.
- There is no constraint forcing the discount function *P(t)* to decrease. In the liquid part of the assessed term structure we could have cases were *P(t)* is a decreasing function on the given liquid market data points, but were two neighboring data points have values that are quite near. As an example *P(0)*=1, P(1)=0.95001, *P(2)*=0.95000, *P(3)*=0.9, and so on. When we fit a smooth curve through this points we will for large *alpha* get a curve between *P(1)* and *P(2)* that will bend down (i.e. $P(t) < P(2)=0.95$ for some $1<t<2$) because of the enforced smooth continuation of the fit between *P(0)* and *P(1)*. Many other methods would have the same problem here.
- Beyond the liquid part of the curve, *P(t)* may become negative. This situation can arise when the last forward rate in the liquid part of the curve is high compared to the sum of *UFR* and *alpha*. This is a disadvantage of S-W compared e.g. to parametric methods, as parametric methods often are based on formulas for the spot rate which per definition can not produce negative discount functions. If for certain sets of input market data *P(t)* will become negative, one has to take higher *alpha*s. This procedure will have to be based on expert judgment.

[20] Larger values of alpha give greater weight to the ultimate forward rate, while smaller values of alpha give more weight to the liquid market data. More work has to be done in order to see if a lower value of alpha than 0.1 could be more appropriate as starting value, as the resulting curves could be deemed to be more objective and market consistent.

6.4 Smith - Wilson technique

We will now explain how the term structure can be assessed by using the S-W technique.

Smith-Wilson for zero coupon bond prices as input

We start by assuming that in the liquid part of the term structure the price function is known for a fixed number of *N* maturities: u_1, u_2, u_3, up to u_N. This is the same as saying that the risk-free zero coupon rates for these *N* liquid maturities are given beforehand.

Depending on whether the market data spot rates are given as continuously compounded rates $\tilde{R}_{u_i}$ or as rates R_{u_i} with annual compounding, the input zero bond prices at maturities u_j can be expressed as:

$m_i = P(u_i) = \exp(-u_i \cdot \tilde{R}_{u_i})$ for continuously compounded rates, and

$m_i = P(u_i) = (1 + R_{u_i})^{-u_i}$ for annual compounding.

In this case, where zero coupon bond prices are the input, the task consists in assessing the price function, i.e. the spot rates for the remaining maturities. These can be both maturities in the liquid end of the term structure where risk-free zero coupon rates are missing (interpolation) and maturities beyond the last observable maturity (extrapolation).

The pricing function proposed by Smith and Wilson[21] reduces in this simple case to:

$$P(t) = e^{-UFR \cdot t} + \sum_{j=1}^{N} \zeta_j \cdot W(t, u_j), \quad t \geq 0 \tag{1}$$

With the symmetric Wilson $W(t, u_j)$ functions defined as:

$$W(t, u_j) = e^{-UFR \cdot (t + u_j)} \cdot \left\{ \alpha \cdot \min(t, u_j) - 0.5 \cdot e^{-\alpha \cdot \max(t, u_j)} \cdot \left(e^{\alpha \cdot \min(t, u_j)} - e^{-\alpha \cdot \min(t, u_j)} \right) \right\} \tag{2}$$

The following notation holds:

- *N*, the number of zero coupon bonds with known price function
- m_i, $i = 1, 2, \ldots N$, the market prices of the zero coupon bonds
- u_i, $i = 1, 2, \ldots N$, the maturities of the zero coupon bonds with known prices
- *t*, the term to maturity in the price function
- *UFR*, the ultimate unconditional forward rate, continuously compounded
- α, mean reversion, a measure for the speed of convergence to the UFR
- ζ_i, $i = 1, 2, \ldots N$, parameters to fit the actual yield curve

The so called kernel functions $K_j(t)$ are defined as functions of the maturity t:

[21] Smith A. & Wilson, T. – "Fitting Yield curves with long Term Constraints" (2001), Research Notes, Bacon and Woodrow. Referred to in Michael Thomas, Eben Maré: "Long Term Forecasting and Hedging of the South African Yield Curve", Presentation at the 2007 Convention of the Actuarial Society of South Africa.
Andrew Smith: Pricing Beyond the Curve – derivatives and the Long Term (2001), presentation to be found at http://www.cfr.statslab.cam.ac.uk/events/content/20001/asmith2001.pdf

$$K_j(t) = W(t, u_j),\ \mathrm{t} > 0 \quad \text{and } j=1, 2, 3 \ldots N \tag{3}$$

They depend only on the input parameters and on data from the input zero coupon bonds. For each input bond a particular kernel function is computed from this definition. The intuition behind the model is to assess the function *P(t)* as the linear combination of all the kernel functions. This is similar to the Nelson-Siegel method, where the forward rate function is assessed as the sum of a flat curve, a sloped curve and a humped curve, and the Svensson method, where a second humped curve is added to the three curves from Nelson-Siegel.

The unknown parameters needed to compute the linear combination of the kernel functions, ζ_j, $j=1, 2, 3 \ldots N$, are given as solutions of the following linear system of equations:

$$
\begin{aligned}
m_1 &= P(u_1) = e^{-UFR \cdot u_1} + \sum_{j=1}^{N} \zeta_j \cdot W(u_1, u_j) \\
m_2 &= P(u_2) = e^{-UFR \cdot u_2} + \sum_{j=1}^{N} \zeta_j \cdot W(u_2, u_j) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
m_N &= P(u_N) = e^{-UFR \cdot u_N} + \sum_{j=1}^{N} \zeta_j \cdot W(u_N, u_j)
\end{aligned} \tag{4}
$$

In vector space notation this becomes:

$$\mathbf{m} = \mathbf{p} = \boldsymbol{\mu} + \mathbf{W}\boldsymbol{\zeta}\ ,$$

with:

$$\mathbf{m} = (m_1, m_2, \ldots\ldots, m_N)^T,$$ [22]

$$\mathbf{p} = (P(u_1), P(u_2), \ldots\ldots P(u_N))^T,$$

$$\boldsymbol{\mu} = (e^{-UFR \cdot u_1}, e^{-UFR \cdot u_2}, \ldots. e^{-UFR \cdot u_N})^T,$$

$$\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \ldots \zeta_N)^{\mathrm{T}},$$

and

$\mathbf{W} = (W(u_i, u_j))_{i=1,2,\ldots N;\, j=1,2,\ldots N}$ a *NxN*-matrix of certain Wilson functions

From this notation it is clear that the solution (ζ_1, ζ_2, ζ_3, … ζ_N) is calculated by inverting the *NxN*-matrix *($W(u_i$, $u_j)$)* and multiplying it with the difference of the **p**-vector and the **μ**-vector, i.e. the difference of the market prices of the zero coupon bonds and the asymptotical term.

$$\boldsymbol{\zeta} = \mathbf{W}^{-1}(\mathbf{p} - \boldsymbol{\mu}) = \mathbf{W}^{-1}(\mathbf{m} - \boldsymbol{\mu}), \tag{5}$$

[22] The superscript T denotes the transposed vector

We can now plug these parameters ζ_1, ζ_2, ζ_3, … ζ_N into the pricing function and get the value of the zero coupon bond price for all maturities t for which no zero bonds were given to begin with:

$$P(t) = e^{-UFR \cdot t} + \sum_{j=1}^{N} \zeta_j \cdot W(t, u_j), \ t>0 \tag{6}$$

From this value it is straightforward to calculate the spot rates by using the definition of the zero coupon bond price. The spot rates are calculated as $\tilde{R}_t = \frac{1}{t} \cdot \ln(\frac{1}{P(t)})$ for continuous compounded rates and $R_t = (\frac{1}{P(t)})^{\frac{1}{t}} - 1$ if annual compounding is used.

Smith-Wilson for a set of general input

We now assume that we have *N* interest related financial instruments as input from the liquid part of the term structure and that *J* is the number of different dates at which a cash payment has to be made on behalf of at least one of these instruments. The following input shall be given:

- The market prices m_i of the instruments i at valuation date, for $i=1,2,3,\ldots,N$.
- All cash payment dates $u_1, u_2, u_3, \ldots, u_J$, for the instruments, and
- The cash flows $c_{i,1}, c_{i,2}, c_{i,3}, \ldots, c_{iJ}$ that are due for instrument i at time $u_1, u_2, \ldots u_J$, for all i. (If no cash payment is due at time $t = u_j$ on instrument i, then $c_{i,j}$ is set to nil).

The general pricing function at valuing time proposed by Smith and Wilson[23] is:

$$P(t) = e^{-UFR \cdot t} + \sum_{i=1}^{N} \zeta_i \cdot (\sum_{j=1}^{J} c_{i,j} \cdot W(t, u_j)), \quad t \geq 0 \tag{7}$$

with the symmetric Wilson-functions $W(t, u_j)$ defined as in (2) above and the same notation for t, UFR, α and ζ_i as was given for the zero coupon case.

The function defined by the inner parenthesis in (7) is called the kernel functions $K_i(t)$:

$$K_i(t) = \sum_{j=1}^{J} c_{i,j} \cdot W(t, u_j), \ t > 0, \mathrm{i} = 1,2,3,\ldots\mathrm{N} \tag{8}$$

For each input instrument a particular kernel function is computed. The intuition here is to assess the function *P(t)* as the linear combination of all the kernel functions.

In the simple case, where the zero coupon prices $P(u_i)$ for certain maturities are given as market price input m_i, i.e. where m_i equaled $P(u_i)$ for $i=1,2,3,\ldots,N$, the left side of the linear system of equations (LSE) in (1) was known and it was straightforward to compute the ζ_i from this LSE. In the general case we have the market prices m_i of the instruments, but the zero coupon prices $P(u_i)$ are not known.

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23 Smith A. & Wilson, T. – "Fitting Yield curves with long Term Constraints" (2001), Research Notes, Bacon and Woodrow. Referred to in Michael Thomas, Eben Maré: "Long Term Forecasting and Hedging of the South African Yield Curve", Presentation at the 2007 Convention of the Actuarial Society of South Africa

We do know how to assess the market price of an instrument i if all cash payment dates $u_1, u_2, u_3, \ldots, u_J$ for the instrument, the cash flows $c_{i,1}, c_{i,2}, c_{i,3}, \ldots, c_{iJ}$ at times $u_1, u_2, \ldots u_J$, and the discount factors $P(u_j)$, $j=1,2,3,\ldots,J$, are known. Then we have to discount the cash flows $c_{i,j}$ to the valuation date (i.e. multiply $c_{i,j}$ with $P(u_j)$ and sum over all cash flow dates.

$$m_i = \sum_{j=1}^{J} c_{i,j} \cdot P(u_j),\ \mathrm{i}=1,2,3,\ldots.\mathrm{N} \tag{9}$$

In the above relation, we know the market prices m_i and the cash flows $c_{i,j}$.

We set the definition of the price function for $P(u_j)$ (7) into relation (9) and get the LSE:

$$m_1 = \sum_{j=1}^{J} c_{1,j} \cdot P(u_j) = \sum_{j=1}^{J} c_{1,j} \cdot (e^{-UFR \cdot u_j} + \sum_{l=1}^{N} \zeta_l \cdot \sum_{k=1}^{J} c_{l,k} \cdot W(u_j, u_k)) \tag{10}$$

$$m_2 = \sum_{j=1}^{J} c_{2,j} \cdot P(u_j) = \sum_{j=1}^{J} c_{2,j} \cdot (e^{-UFR \cdot u_j} + \sum_{l=1}^{N} \zeta_l \cdot \sum_{k=1}^{J} c_{l,k} \cdot W(u_j, u_k))$$

....

$$m_N = \sum_{j=1}^{J} c_{N,j} \cdot P(u_j) = \sum_{j=1}^{J} c_{N,j} \cdot (e^{-UFR \cdot u_j} + \sum_{l=1}^{N} \zeta_l \cdot \sum_{k=1}^{J} c_{l,k} \cdot W(u_j, u_k))$$

We can rearrange the above expressions to get:

$$\sum_{j=1}^{J} c_{1,j} \cdot P(u_j) = \sum_{j=1}^{J} c_{1,j} \cdot e^{-UFR \cdot u_j} + \sum_{l=1}^{N} (\sum_{k=1}^{J} (\sum_{j=1}^{J} c_{1,j} \cdot W(u_j, u_k)) \cdot c_{l,k}) \cdot \zeta_l \tag{11}$$

$$\sum_{j=1}^{J} c_{2,j} \cdot P(u_j) = \sum_{j=1}^{J} c_{2,j} \cdot e^{-UFR \cdot u_j} + \sum_{l=1}^{N} (\sum_{k=1}^{J} (\sum_{j=1}^{J} c_{2,j} \cdot W(u_j, u_k)) \cdot c_{l,k}) \cdot \zeta_l$$

....

$$\sum_{j=1}^{J} c_{N,j} \cdot P(u_j) = \sum_{j=1}^{J} c_{N,j} \cdot e^{-UFR \cdot u_j} + \sum_{l=1}^{N} (\sum_{k=1}^{J} (\sum_{j=1}^{J} c_{N,j} \cdot W(u_j, u_k)) \cdot c_{l,k}) \cdot \zeta_l$$

In vector space notation we write the left side of (10) as:

$$\mathbf{m} = \mathbf{Cp}, \tag{12}$$

and (11) as:

$$\mathbf{Cp} = \mathbf{C\mu} + (\mathbf{CWC}^{\mathrm{T}})\boldsymbol{\zeta}, \tag{13}$$

with:

$$\mathbf{m} = (m_1, m_2, \ldots\ldots, m_N)^T,$$
$$\mathbf{p} = (P(u_1), P(u_2), \ldots\ldots P(u_J))^T,$$

$$\mathbf{C} = \begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,3} & \cdots & \cdots & c_{1,j} & \cdots & c_{1,J} \\ c_{2,1} & c_{2,2} & c_{2,3} & \cdots & \cdots & c_{2,j} & \cdots & c_{2,J} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{i,1} & c_{i,2} & c_{i,3} & \cdots & \cdots & c_{i,j} & \cdots & c_{i,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{N,1} & c_{N,2} & \cdots & \cdots & \cdots & c_{N,j} & \cdots & c_{N,J} \end{bmatrix}. \qquad \text{the NxJ cash flow matrix}$$

$$\boldsymbol{\mu} = (e^{-UFR \cdot u_1}, e^{-UFR \cdot u_2}, \ldots . e^{-UFR \cdot u_J})^T,$$

$$\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \ldots \zeta_N)^{\mathrm{T}},$$

and

$$\mathbf{W} = \begin{bmatrix} w(u_1,u_1) & w(u_1,u_2) & \cdots & \cdots & w(u_1,u_i) & \cdots & \cdots & w(u_1,u_J) \\ w(u_2,u_1) & w(u_2,u_1) & \cdots & \cdots & w(u_2,u_i) & \cdots & \cdots & w(u_2,u_J) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ w(u_i,u_1) & w(u_i,u_2) & \cdots & \cdots & w(u_i,u_i) & \cdots & \cdots & w(u_i,u_J) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ w(u_J,u_1) & \cdots & \cdots & \cdots & w(u_J,u_i) & \cdots & \cdots & w(u_J,u_J) \end{bmatrix}$$

the *JxJ* matrix of certain Wilson functions.

Combining (12) and (13) leads to:

$$\mathbf{m} = \mathbf{C}\boldsymbol{\mu} + (\mathbf{C}\mathbf{W}\mathbf{C}^{\mathrm{T}})\boldsymbol{\zeta}\ , \qquad (14)$$

and we see at once that the solution ζ_1, ζ_2, ζ_3, … ζ_N is calculated by inverting the NxN-matrix ***CWC^T*** and multiplying it with the difference of the market value vector and the vector assessed as product of matrix **C** with the **μ**-vector, the asymptotical term:

$$\boldsymbol{\zeta} \ = (\mathbf{C}\mathbf{W}\mathbf{C}^{\mathrm{T}})^{-1}(\mathbf{m} - \mathbf{C}\boldsymbol{\mu}) \qquad (15)$$

Now we can plug these parameters ζ_1, ζ_2, ζ_3, … ζ_N for *t= 1,2,3,…* into the pricing function *P(t)* and get the value of the discount function for all maturities, and thus the term structure for the spot rates.

Remark: When using swap rates to fit the risk-free term structure, an adjustment to allow for the credit risk in swaps has to be made. Assuming that the adjustment can be expressed as a delta credit risk spread of *Δcr* basis points of swaps above basic risk-free rates, it seems most adequate to adjust the continuously compounded spot rates with *Δcr* basis points. This means that *Δcr* basis points are subtracted from the continuously compounded spot rates, which were assessed with the S-W technique from the unadjusted swaps. This is equivalent to multiplying the discount factors *P(t)* (assessed from swaps), with an adjustment factor $e^{(\Delta cr/10000)\cdot t}$.

6.5 Fitting the spot rate term structure to bond prices and swap rates

With the Smith-Wilson technique the term structure can be fitted to all the different financial instruments that may be eligible as basis for assessing the risk-free interest rate curve.

Each set of instruments that is taken as input is defined by

- the vector of the market prices (of *N* instruments) at valuation date,
- the vector of the cash payment dates (*J* different dates) up to the last maturity, and
- the *NxJ*-matrix of the cash flows on the instruments in these dates.

We will now look at this input when the spot rate curve is fitted to zero coupon bond rates, to coupon bond rates and to par swap rates. We will furthermore give some simple computed examples for par swap rates as input.

Instruments	Market prices	Cash payment dates	Cash flow matrix
Fitting the term structure to zero coupon bonds	• Market prices of the *N* input instruments, given as the per cent amount of the notional amount • The market prices of the zero coupon input bonds translate at once into spot rates for input maturities	• The cash payment dates are the maturity dates of the *N* zero coupon input bonds (i.e. *J=N*)	• An *NxN* matrix with entries: - $c_{i,j} = 1$ for $i=j$, and - $c_{i,j} = 0$ else. • Remark: The **C** matrix reduces to the unity matrix. It can easily be seen that all the complex formulae given in (7,8,10,15) reduce to the simpler ones in (1,3,4,5).
Fitting the term structure to coupon bonds	• Market prices of the *N* coupon input bonds, given as the per cent amount of the notional amount of the bond.	• The cash payment dates are, in addition to the maturity dates of the input bonds all coupon dates. • Order these *J* cash payment dates in increasing order, i.e. $u_1 < u_2 < \ldots < u_J$ • Order the bonds depending on their time to maturity, such that if the time of maturity of the i^{th} bond is denominated with $u_{t(i)}$, the following holds: $u_{t(1)} < u_{t(2)} \ldots < u_{t(N)} = u_J$	• An NxJ matrix with the following entries (all i): - $c_{i,j} = r_c(i)/s$, $j<t(i)$ - $c_{i,t(i)} = 1+r_c(i)/s$, - $c_{i,j} = 0$, $j>t(i)$, where $r_c(i)$ is the coupon rate of i^{th} bond, and s is the settlement frequency. • Remark: We propose to take the simple approach, and to not allow for day count details

<table>
<tr><td></td><td></td><td></td><td></td></tr>
<tr>
<td>Fitting the term structure to par swap rates</td>
<td>

- The market prices of the *N* par swap input instruments are taken as unit (i.e. 1).
- To receive the swap rate, a floating rate has to be earned, that can be swapped against the fixed rate. To earn the variable rate a notional amount has to be invested. At maturity, the notional amount is de-invested.

</td>
<td>

- The cash payment dates are, in addition to the maturity dates of the swap agreements all swap rate payment dates.
- Order these *J* cash payment dates in increasing order, i.e. $u_1 < u_2 < \ldots < u_J$
- Order the swaps depending on their time to maturity, such that if the time to maturity of the i^th swap is denominated with $u_{i,t(i)}$, the following holds: $u_{1,t(1)} < u_{2,t(2)} \ldots < u_{N,t(N)} = u_J$

</td>
<td>

- An *NxJ* matrix with the following entries (all i):
- $c_{i,j} = r_c(i)/s,\ j<t(i)$
- $c_{i,t(i)} = 1+r_c(i)/s$,
- $c_{i,j} = 0,\ j>t(i)$,
where $r_c(i)$ is the swap rate of agreement i, and s is the settlement frequency.
- Remark: We propose to take the simple approach, and to not allow for day count details.

</td>
</tr>
</table>

6.6 Worked examples

When fitting the spot rate term structure to the input data from the following examples, we set the long term forward rate to 4.2% (for annual compounding; i.e. ln(1+4.2%) = 4.114% for continuous compounding), and the alpha parameter to 0.1.

Example 1.

<u>Market data input for example 1:</u>

<table>
<tr><td>Par swap rates</td><td>Market prices</td><td>Cash payment dates</td><td>Transposed Cash flow matrix C^T (i.e. cash flows of instrument i in column i)</td></tr>
<tr>
<td>Maturities:
1,2,3,5 years

s=1

coupon rates
<table>
<tr><td>$r_c(1)$</td><td>1%</td></tr>
<tr><td>$r_c(2)$</td><td>2%</td></tr>
<tr><td>$r_c(3)$</td><td>2.6%</td></tr>
<tr><td>$r_c(5)$</td><td>3.4%</td></tr>
</table>
</td>
<td>
<table>
<tr><td>i</td><td>m_i</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>1</td></tr>
<tr><td>3</td><td>1</td></tr>
<tr><td>4</td><td>1</td></tr>
</table>
</td>
<td>

- $u_1 = 1$, $u_2 = 2$, $u_3 = 3$, $u_4 = 4$, $u_5 = 5$
- $u_{1,t(1)} = 1$, $u_{2,t(2)} = 2$, $u_{3,t(3)} = 3$, $u_{4,t(4)} = 5$

</td>
<td>
<table>
<tr><td></td><td>1</td><td>2</td><td>3</td><td>4</td></tr>
<tr><td>u_1</td><td>1.01</td><td>0.2</td><td>0.026</td><td>0.034</td></tr>
<tr><td>u_2</td><td>0</td><td>1.02</td><td>0.026</td><td>0.034</td></tr>
<tr><td>u_3</td><td>0</td><td>0</td><td>1.026</td><td>0.034</td></tr>
<tr><td>u_4</td><td>0</td><td>0</td><td>0</td><td>0.034</td></tr>
<tr><td>u_5</td><td>0</td><td>0</td><td>0</td><td>1.034</td></tr>
</table>
</td>
</tr>
</table>

The steps in the S-W technique:

The 5x5 matrix of Wilson functions is computed straightforward from formula (2):

$$\mathbf{W}=\begin{bmatrix}0.009 & 0.016 & 0.022 & 0.027 & 0.031\\ 0.016 & 0.030 & 0.041 & 0.051 & 0.058\\ 0.022 & 0.041 & 0.058 & 0.072 & 0.083\\ 0.027 & 0.051 & 0.072 & 0.090 & 0.104\\ 0.031 & 0.058 & 0.083 & 0.104 & 0.122\end{bmatrix}$$

If multiplied with **C** from the right and $\mathbf{C}^{\mathbf{T}}$ from the left, the resulting 4x4 matrix is:

$$\mathbf{CWC^T}=\begin{bmatrix}0.009 & 0.017 & 0.023 & 0.035\\ 0.017 & 0.032 & 0.045 & 0.067\\ 0.023 & 0.045 & 0.065 & 0.097\\ 0.035 & 0.067 & 0.097 & 0.150\end{bmatrix}$$

The inverse of this matrix $\mathbf{CWC^T}$ is computed as:

$$\left(\mathbf{CWC^T}\right)^{-1}=\begin{bmatrix}10658.6 & -10190.4 & 3653.0 & -268.0\\ -10190.4 & 14337.6 & -7987.5 & 1116.2\\ 3653.0 & -7987.5 & 6252.3 & -1323.2\\ -268.0 & 1116.2 & -1323.2 & 426.6\end{bmatrix}$$

We first multiply the cash flows in **C** with the vector $\boldsymbol{\mu}$ of the asymptotic terms, and then subtract this vector from the vector of the market values:

$$\mathbf{m}-\mathbf{C}\boldsymbol{\mu}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}-\begin{bmatrix}0.969\\0.959\\0.956\\0.965\end{bmatrix}=\begin{bmatrix}0.031\\0.041\\0.044\\0.035\end{bmatrix}$$

Multiply $\mathbf{(CWC^T)^{-1}}$ with $\boldsymbol{m}\text{-}\boldsymbol{\mu}$. The resulting vector represents the solution of the LSE that was set up in (14):

$$\zeta=\begin{bmatrix}57.79\\ -33.5\\ 111.40\\ -5.47\end{bmatrix}$$

To assess the discount function *P(t)* in arbitrary *t, t>0*, the Wilson functions *W(t,uj)*, *j=1,2,3…J* have to be assessed and multiplied with **C**, as defined in (7). We want to compute the discount factor for t=4, and calculate therefore $\mathbf{w^T}=(W(4,u_j))_{j=1,2,3,4,5}$

$$\mathbf{w}^T = \left[0.27 \quad 0.051 \quad 0.072 \quad 0.090 \quad 0.104\right]$$

This vector multiplied with $\boldsymbol{C}^T$ gives the values of the kernel functions in t=4, i.e.:

$$(\mathrm{K}_i(4))_{i=1,2,3,4} = \mathbf{w}^\mathbf{T}\mathbf{C}^\mathbf{T} = \left[0.027 \quad 0.052 \quad 0.076 \quad 0.116\right]$$

From the linear combination of these kernel functions we get:

$$(\mathbf{w}^\mathbf{T}\mathbf{C}^\mathbf{T})\zeta = 0.037$$

and adding the asymptotical factor $e^{-0.04114\times 4} = 0.8483$, the discount function at maturity 4 years has the value *P(4)*= 0.848+0.037=0.885. This gives a spot rate (with annual compounding) of 3.10%.

We can table the Wilson functions for all maturities (years, month, days) for which risk-free spot rates will be needed, perform the above calculation for each maturity, and thus assess the risk-free interest rate term structure.

Example 2:

Market data input for example 2:

<table>
<tr><th>Par swap rates</th><th>Market prices</th><th>Cash payment dates</th><th>Transposed Cash flow matrix C^T (i.e. cash flows of instrument i in column i)</th></tr>
<tr>
<td>
Maturities: 1,2,3,5 years

s=1

coupon rates
<table>
<tr><td>$r_c(1)$</td><td>1%</td></tr>
<tr><td>$r_c(2)$</td><td>2%</td></tr>
<tr><td>$r_c(3)$</td><td>2.6%</td></tr>
<tr><td>$r_c(5)$</td><td>3.4%</td></tr>
</table>
</td>
<td>
<table>
<tr><td>i</td><td>m_i</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>1</td></tr>
<tr><td>3</td><td>1</td></tr>
<tr><td>4</td><td>1</td></tr>
</table>
</td>
<td>

u_1=0.25, u_2=0.5, u_3=0.75, u_4=1, u_5=1.25, ……. u_{19}=4.75, u_{20}=5,
$u_{1,t(1)}$ =1, $u_{2,t(2)}$ =2, $u_{3,t(3)}$ =3, $u_{4,t(4)}$ =5

</td>
<td>
<table>
<tr><td></td><td>1</td><td>2</td><td>3</td><td>4</td></tr>
<tr><td>u_1</td><td>0.0025</td><td>0.005</td><td>0.0065</td><td>0.0085</td></tr>
<tr><td>u_2</td><td>0.0025</td><td>0.005</td><td>0.0065</td><td>0.0085</td></tr>
<tr><td>u_3</td><td>0.0025</td><td>0.005</td><td>0.0065</td><td>0.0085</td></tr>
<tr><td>u_4</td><td>1.0025</td><td>0.005</td><td>0.0065</td><td>0.0085</td></tr>
<tr><td>u_5</td><td>0</td><td>0.005</td><td>0.0065</td><td>0.0085</td></tr>
<tr><td>u_7</td><td>0</td><td>0.005</td><td>0.0065</td><td>0.0085</td></tr>
<tr><td>u_8</td><td>0</td><td>1.005</td><td>0.0065</td><td>0.0085</td></tr>
<tr><td>u_9</td><td>0</td><td>0</td><td>0.0065</td><td>0.0085</td></tr>
<tr><td>..</td><td>0</td><td>0</td><td>…</td><td>…</td></tr>
<tr><td>u_{19}</td><td>0</td><td>0</td><td>0</td><td>0.0085</td></tr>
<tr><td>u_{20}</td><td>0</td><td>0</td><td>0</td><td>1.0085</td></tr>
</table>
</td>
</tr>
</table>

The steps in the S-W technique:

The 20x20 matrix of Wilson functions is computed straightforward from formula (2):

$$\mathbf{W} = \begin{bmatrix}
0.001 & 0.001 & 0.002 & 0.002 & 0.003 & 0.003 & 0.004 & 0.004 & 0.005 & 0.005 & 0.005 & 0.006 & 0.006 & 0.006 & 0.007 & 0.007 & 0.007 & 0.007 & 0.008 & 0.008 \\
0.001 & 0.002 & 0.003 & 0.004 & 0.005 & 0.006 & 0.007 & 0.008 & 0.009 & 0.010 & 0.011 & 0.011 & 0.012 & 0.013 & 0.013 & 0.014 & 0.014 & 0.015 & 0.015 & 0.016 \\
0.002 & 0.003 & 0.005 & 0.007 & 0.008 & 0.009 & 0.011 & 0.012 & 0.013 & 0.014 & 0.016 & 0.017 & 0.018 & 0.019 & 0.019 & 0.020 & 0.021 & 0.022 & 0.023 & 0.023 \\
0.002 & 0.004 & 0.007 & 0.009 & 0.011 & 0.012 & 0.014 & 0.016 & 0.018 & 0.019 & 0.020 & 0.022 & 0.023 & 0.024 & 0.026 & 0.027 & 0.028 & 0.029 & 0.030 & 0.031 \\
0.003 & 0.005 & 0.008 & 0.011 & 0.013 & 0.015 & 0.017 & 0.020 & 0.022 & 0.023 & 0.025 & 0.027 & 0.029 & 0.030 & 0.032 & 0.033 & 0.034 & 0.036 & 0.037 & 0.038 \\
0.003 & 0.006 & 0.009 & 0.012 & 0.015 & 0.018 & 0.021 & 0.023 & 0.026 & 0.028 & 0.030 & 0.032 & 0.034 & 0.036 & 0.037 & 0.039 & 0.041 & 0.042 & 0.044 & 0.045 \\
0.004 & 0.007 & 0.011 & 0.014 & 0.017 & 0.021 & 0.024 & 0.027 & 0.029 & 0.032 & 0.034 & 0.037 & 0.039 & 0.041 & 0.043 & 0.045 & 0.047 & 0.049 & 0.050 & 0.052 \\
0.004 & 0.008 & 0.012 & 0.016 & 0.020 & 0.023 & 0.027 & 0.030 & 0.033 & 0.036 & 0.039 & 0.041 & 0.044 & 0.046 & 0.049 & 0.051 & 0.053 & 0.055 & 0.057 & 0.058 \\
0.005 & 0.009 & 0.013 & 0.018 & 0.022 & 0.026 & 0.029 & 0.033 & 0.036 & 0.040 & 0.043 & 0.046 & 0.049 & 0.051 & 0.054 & 0.056 & 0.059 & 0.061 & 0.063 & 0.065 \\
0.005 & 0.010 & 0.014 & 0.019 & 0.023 & 0.028 & 0.032 & 0.036 & 0.040 & 0.043 & 0.047 & 0.050 & 0.053 & 0.056 & 0.059 & 0.062 & 0.064 & 0.067 & 0.069 & 0.071 \\
0.005 & 0.011 & 0.016 & 0.020 & 0.025 & 0.030 & 0.034 & 0.039 & 0.043 & 0.047 & 0.051 & 0.054 & 0.058 & 0.061 & 0.064 & 0.067 & 0.070 & 0.072 & 0.075 & 0.077 \\
0.006 & 0.011 & 0.017 & 0.022 & 0.027 & 0.032 & 0.037 & 0.041 & 0.046 & 0.050 & 0.054 & 0.058 & 0.062 & 0.065 & 0.069 & 0.072 & 0.075 & 0.078 & 0.080 & 0.083 \\
0.006 & 0.012 & 0.018 & 0.023 & 0.029 & 0.034 & 0.039 & 0.044 & 0.049 & 0.053 & 0.058 & 0.062 & 0.066 & 0.070 & 0.073 & 0.077 & 0.080 & 0.083 & 0.086 & 0.089 \\
0.006 & 0.013 & 0.019 & 0.024 & 0.030 & 0.036 & 0.041 & 0.046 & 0.051 & 0.056 & 0.061 & 0.065 & 0.070 & 0.074 & 0.078 & 0.081 & 0.085 & 0.088 & 0.091 & 0.094 \\
0.007 & 0.013 & 0.019 & 0.026 & 0.032 & 0.037 & 0.043 & 0.049 & 0.054 & 0.059 & 0.064 & 0.069 & 0.073 & 0.078 & 0.082 & 0.086 & 0.089 & 0.093 & 0.096 & 0.099 \\
0.007 & 0.014 & 0.020 & 0.027 & 0.033 & 0.039 & 0.045 & 0.051 & 0.056 & 0.062 & 0.067 & 0.072 & 0.077 & 0.081 & 0.086 & 0.090 & 0.094 & 0.097 & 0.101 & 0.104 \\
0.007 & 0.014 & 0.021 & 0.028 & 0.034 & 0.041 & 0.047 & 0.053 & 0.059 & 0.064 & 0.070 & 0.075 & 0.080 & 0.085 & 0.089 & 0.094 & 0.098 & 0.102 & 0.105 & 0.109 \\
0.007 & 0.015 & 0.022 & 0.029 & 0.036 & 0.042 & 0.049 & 0.055 & 0.061 & 0.067 & 0.072 & 0.078 & 0.083 & 0.088 & 0.093 & 0.097 & 0.102 & 0.106 & 0.110 & 0.113 \\
0.008 & 0.015 & 0.023 & 0.030 & 0.037 & 0.044 & 0.050 & 0.057 & 0.063 & 0.069 & 0.075 & 0.080 & 0.086 & 0.091 & 0.096 & 0.101 & 0.105 & 0.110 & 0.114 & 0.118 \\
0.008 & 0.016 & 0.023 & 0.031 & 0.038 & 0.045 & 0.052 & 0.058 & 0.065 & 0.071 & 0.077 & 0.083 & 0.089 & 0.094 & 0.099 & 0.104 & 0.109 & 0.113 & 0.118 & 0.122
\end{bmatrix}$$

If multiplied with **C** from the right and $\mathbf{C}^\mathbf{T}$ from the left, the resulting 4x4 matrix is:

$$\mathbf{CWC^T} = \begin{bmatrix} 0.009 & 0.016 & 0.023 & 0.034 \\ 0.016 & 0.031 & 0.044 & 0.066 \\ 0.023 & 0.044 & 0.063 & 0.095 \\ 0.034 & 0.066 & 0.095 & 0.147 \end{bmatrix}$$

The inverse of this matrix $\mathbf{CWC^T}$ is computed as:

$$\left(\mathbf{CWC^T}\right)^{-1} = \begin{bmatrix} 10765.8 & -10328.0 & 3708.9 & -272.7 \\ -10328.0 & 14579.0 & -8136.8 & 1139.6 \\ 3708.9 & -8136.8 & 6381.5 & -1353.7 \\ -272.7 & 1139.6 & -1353.7 & 437.4 \end{bmatrix}$$

We first multiply the cash flows in **C** with the vector mu of the asymptotic terms, and then subtract the vector from the vector of the market values:

$$\mathbf{m - C\mu} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \mathbf{C} \cdot \begin{bmatrix} 0.9898 \\ 0.9796 \\ 0.9696 \\ 0.9597 \\ 0.9499 \\ 0.9402 \\ 0.9305 \\ 0.9210 \\ 0.9116 \\ 0.9023 \\ 0.8930 \\ 0.8839 \\ 0.8748 \\ 0.8659 \\ 0.8570 \\ 0.8483 \\ 0.8396 \\ 0.8310 \\ 0.8225 \\ 0.8141 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0.969 \\ 0.959 \\ 0.957 \\ 0.967 \end{bmatrix} = \begin{bmatrix} 0.031 \\ 0.041 \\ 0.043 \\ 0.033 \end{bmatrix}$$

Multiply $\mathbf{(CWC^T)^{-1}}$ with **m-μ**. The resulting vector represents the solution of the LSE that was set up in (14):

$$\zeta = \begin{bmatrix} 58.6 \\ -34.1 \\ 11.8 \\ -5.7 \end{bmatrix}$$

To assess the discount function *P(t)* in arbitrary t, t>0, the Wilson functions $W(t,u_j)$, $j=1,2,3...20$ have to be assessed and multiplied with $\boldsymbol{C}$, as defined in (7). We want to compute the discount factor for t=4. We calculate $\mathbf{w^T}=(W(4,u_j))_{j=1,2,3,..20}$, multiply it with $\boldsymbol{C^T}$ and get the values of the kernel functions in t=4, i.e.:

$$(\mathrm{K_i}(4))_{\mathrm{i}=1,2,3,4} = \mathbf{w^T C^T} = [0{,}027 \quad 0{,}052 \quad 0{,}075 \quad 0{,}115]$$

From the linear combination of these kernel functions we get

$$(\mathbf{w}^{\mathbf{T}}\mathbf{C}^{\mathbf{T}})\boldsymbol{\zeta} = 0.0353$$

and adding the asymptotical factor $e^{-0.04114\times 4} = 0.8483$, the discount function at maturity 4 years has the value *P(4)* = 0.0353 +0.8483=0.8836. This gives a spot rate (with annual compounding) of 3.141%.