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Full Length Research Paper

Extrapolation of long-term risk-free interest rates: A case study for the Taiwan insurance market

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This study constructed a risk free term structure based on the Taiwan government bond market, with maturities of up to 120 years. In Taiwan, only government bonds with maturities of up to 30 years could be observed. Additionally, the short-term interest rate also has had spurious volatility and caused the GARCH volatility models to be difficult to converge in the estimation of long-term volatility levels. This paper suggested a threshold GARCH model to infer the equilibrium volatility term structure. Furthermore, this paper used the Vasicek equilibrium interest rate model to extrapolate the long-term interest rate to the Unconditional Forward Rates (UFR) suggested by Quantitative Impact Study 5 (QIS5). The proposed method avoided the arbitrage determination of parameters in QIS5. The numerical analysis showed that the proposed method produced liability values for long-term annuities that were less than that of QIS5.

Key words: Extrapolation, fair valuation, threshold GARCH, unconditional forward rates, vasicek model.

INTRODUCTION

The term structure of the interest rate plays an important role in the financial industry, especially in the life insurance industry, which sells long-term contracts. To reveal the economic value of insurance policies, and to increase the transparency of financial reports, modern accounting principles and risk management disciplines such as International Financial Reporting Standards (IFRS) and Solvency II require fair valuation, which is supposed to perform with a risk free term structure. Traditionally, the cash flows or duration in the insurance field last for several decades, and can even last more than one hundred years. Therefore, it is necessary to extrapolate a risk-free term structure for long-term policy valuation, which can have a great impact on both insurance policies and the value of insurance companies.

However, there is not enough market trading data for the long-term bond market, especially for maturities that are greater than 30 years. There has been some research on extrapolating the yield curve beyond the last point with available market data. QIS5 Technical Specifications uses the Smith-Wilson (2001) method to extrapolate the forward rate by inputting the zero coupon bond price into the matrix arithmetic. EIOPA suggested the forward rate curve should reach the unconditional ultra forward rate (UFR) at a maturity of between 70 and 120 years. Thomas (2007) set a convergence parameter arbitrarily. Liu (2008) constructed a volatility term structure based on the GARCH and EWMA methods for the liquid available market data. Liu (2008) then determined the speed of reversion to UFR via fitting the volatility term structure with Vasicek (1977) models. Similarly to Thomas (2007), in EWMA, a memory parameter for the weighs of past volatilities should be determined subjectively, and it is sensitive to the pattern of volatility term structure.

In Taiwan, only government bonds with maturities of up to 30 years can be observed. There is insufficient information about the yield curve for valuing long-term claims and assessing risk. Additionally, liquidity data for Taiwan government bonds suggests that instruments are insufficiently traded beyond 10 years. Even when data on

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Abbreviations: UFR, Unconditional forward rates; **QIS5,** quantitative Impact study 5; **IFRS,** international financial reporting standards; **UFR,** ultra forward rate; **ADF,** augmented dickey fuller; **MLE,** maximum likelihood estimation.

longer term nominal forward rates is available, the information that can be obtained from the market prices may be spuriously volatile, due to the methods used by practitioners (and central banks) to construct the forward curve. Not only is there little observed data for the long term interest rate market, short term interest rates are spurious due to recent governmental financial or monetary policies, especially after the financial crisis during 2009. Gospodinov (2005) indicated that the short rate directly affects the slope of the yield curve, and therefore the inflation expectation and aggregate demand in the economy. A thorough understanding of the time-series properties of the short rate is of ultimate importance for policy makers and economic agents. The main aim of this paper was to extrapolate the long-term forward rate curve of the Taiwan economy. While it is inherently difficult to estimate long-term interest rate volatility and understand how long it takes to reach such levels, this paper aimed to use a more objective approach that was easy to understand. First, this paper suggested the GARCH model to determine the equilibrium volatility term structure without a subjective setting of parameters.

It was observed that, in the Taiwan market, the decrease of short-term forward rates during 2009 caused the GARCH model to be difficult to converge for parameter estimation. To overcome the non-linearity evolution of short-term rates, this study used Threshold GARCH to obtain the volatility term structure of short-term interest rates. Next, adopting the method put forward by Liu (2008)0, the theoretical volatility term structure was derived, based on the equilibrium Vasicek interest rate models. Using the extrapolated term structure, the liability values were examined under the principles of stability and consistency for a long-term annuity. The rest of this work is organized as follows: First is a summary of the collected market data used for extrapolation. This is followed by an introduction of the GARCH and Threshold GARCH model to obtain the volatility term structure. The relevant parameters are then derived by which a long-term risk free term structure is extrapolated. Subsequently, a comparison of the difference of liability values for the deferred annuities between the term structures by the proposed method and QIS5 is considered. Finally, conclusions and further research directions are then outlined.

DATA AND METHODOLOGY

The data used for empirical analysis was the annualized yield to maturity of government bond price at monthly frequency, taken from GreTai Securities Market, which is the over-the-counter market in Taiwan, for the period of January 2006 to December 2010. For the published term structures, some parametric models (Nelson and Siegel, 1987) and the extension by Svensson (1994) were used to construct forward rate curves from available bond prices. These models could induce spurious volatility for longer maturity forward rates, because the estimated terminal values (which are closely linked to long forward rates) often vary to give a better fit to the data. Due to the consideration of liquidity in the Taiwan bond market, this paper adopted yields with maturities between 1 to10 years as a proxy for the volatility term structure. According to Carriere (1999), there are correlations between the spot yields of consequent maturities that can be eliminated by using the forward rate, which is derived from taking the difference on spot rates. Let y_t^n represent the spot yield with a maturity of *n* years at time *t*. The forward rate between year *n*-1 to *n*, denoted by f_t^n, can be obtained as follows.

$$f_t^n = n \cdot \log(1+y_t^n) - (n-1) \cdot \log(1+y_t^{n-1}) \quad (1)$$

The augmented Dickey Fuller (ADF) test demonstrates that there is a significant unit-root relationship between two observed forward rates. After taking the log and difference transformation to the forward rate data, the unit-root phenomena were not significant for the change in the log-forward rate after year three. Throughout this paper, the focus remained on the year-to-year change of log forward rates.

$$u_t^n = \log f_t^n - \log f_{t-1}^n \quad (2)$$

This paper performed extrapolations with the volatility term structure, rather than the term structure from market bond prices. As shown previously, the volatility term structure depicts the apparent decay of variability for changes in the long-term forward rates based on available liquid market data, which may be helpful for determining the speed of convergence to the Unconditional Forward Rate (UFR). Furthermore, some literature, such as Andersen (1997) calibrated the market price data with many parameters, implying a numerical difficulty, such as the local optimization relevant to the initial parameter settings. Another way to gain the volatility term structure is to use the market implied volatility from interest rate derivatives such as caps or floors. However, many empirical studies have suggested that market implied volatility derived from the option market is not an efficient and unbiased predictor of realized volatility (Amin and Ng, 1997), Canina and Figlewski (1993) and Christensen and Prabhala (1998).

DERIVATIONS

Estimation of equilibrium volatility term structure

To infer the real equilibrium level of volatility term structure more subjectively, it makes sense to understand the dynamics of the short-term volatility of changes in the forward rates. Intuitively, the GARCH (1.1) model may be used to measure the equilibrium or long-term average volatilities of the change in log forward rates u_t^n for the forward periods $n=1$ to 10. The GARCH (1.1) model can be expressed as follows:

$$\sigma_t^2 = a_0 + a_1\varepsilon_{t-1}^2 + b_1\sigma_{t-1}^2 \quad (3)$$

Another advantage of the GARCH (1.1) models is the recognition that over time, the variance tends to get

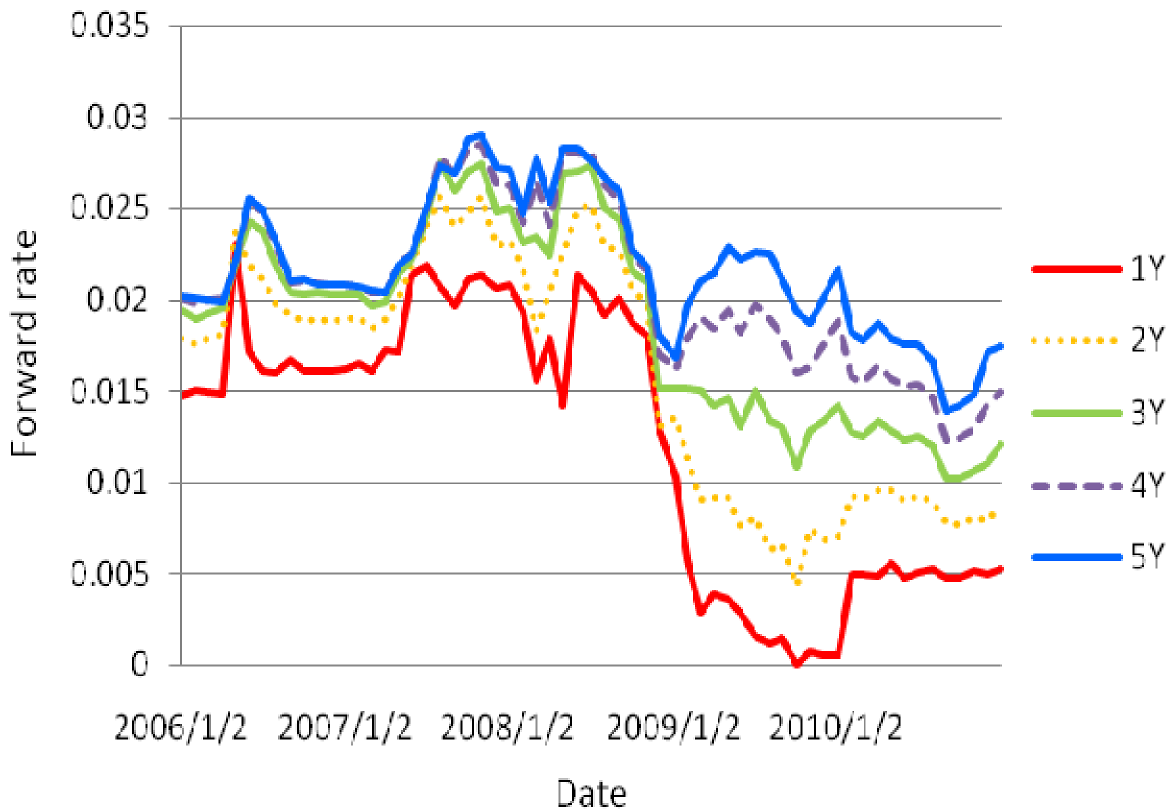

Figure 1. Time series for the forward rate.

pulled back to a long-run average level of $a_0/(1-a_1-b_1)$, implying the mean reversion, which is suitable for the determination of equilibrium volatility level. It is remarkable that when the Maximum Likelihood Estimation (MLE) method is applied to the GARCH models to estimate the long-run average volatility $a_0/(1-a_1-b_1)$, the convergence fails for short-term volatilities such as maturity years one and three. Figure 1 indicates there were significant drops in the forward rate during 2009, due to the Taiwan government's monetary and financial policies during the financial crisis. According to Gospodinov (1995), the presence of possible nonlinearities in the conditional moments of the short rate may have important implications for the dynamics of the long rates. Although EWMA can avoid the problem, its parameter π plays a key role in the term structure model of the interest rate but is exposed to ambiguity due to the arbitrage determination. To explain the dropping short-term rates during 2009 and overcome the divergence of the long term average volatility in GARCH (1.1) models, this study adopted the Threshold GARCH (TGARCH) model proposed by Zakoian (1994), which is characterized by a leverage effect for the downward interest rate scenarios and has the following form:

$$\sigma_t^2 = a_0 + a_1\varepsilon_{t-1}^2 + \gamma_1 S_{t-1}\varepsilon_{t-1}^2 + b_1\sigma_{t-1}^2 \quad (4)$$

Where

$$S_{t-1} = \begin{cases} 1 \ , \ if \ \ \varepsilon_{t-1} < 0 \\ 0 \ , \ if \ \ \varepsilon_{t-1} \geq 0 \end{cases} \quad (5)$$

That is, depending on whether ε_{t-1} is above or below the threshold value of zero, ε_{t-1}^2 has different effects on the conditional variance s_t^2: when ε_{t-1} is positive, the total effects are given by $a_1\varepsilon_{t-1}^2$; when ε_{t-1} is negative, the total effects are given by $(a_1+\gamma_1)\varepsilon_{t-1}^2$. The leverage effect can be used to explain the high volatility in short-term interest rates due to the dramatic drops in 2009, as depicted in Figure 1. The model is also known as the GJR model, because Glosten, Jagannathan and Runkle (1993) all proposed essentially the same model. Appendix C shows the long-term average volatility, estimated parameters and model diagnosis under both TGARCH and GARCH.

As previously described, for the short-term forward

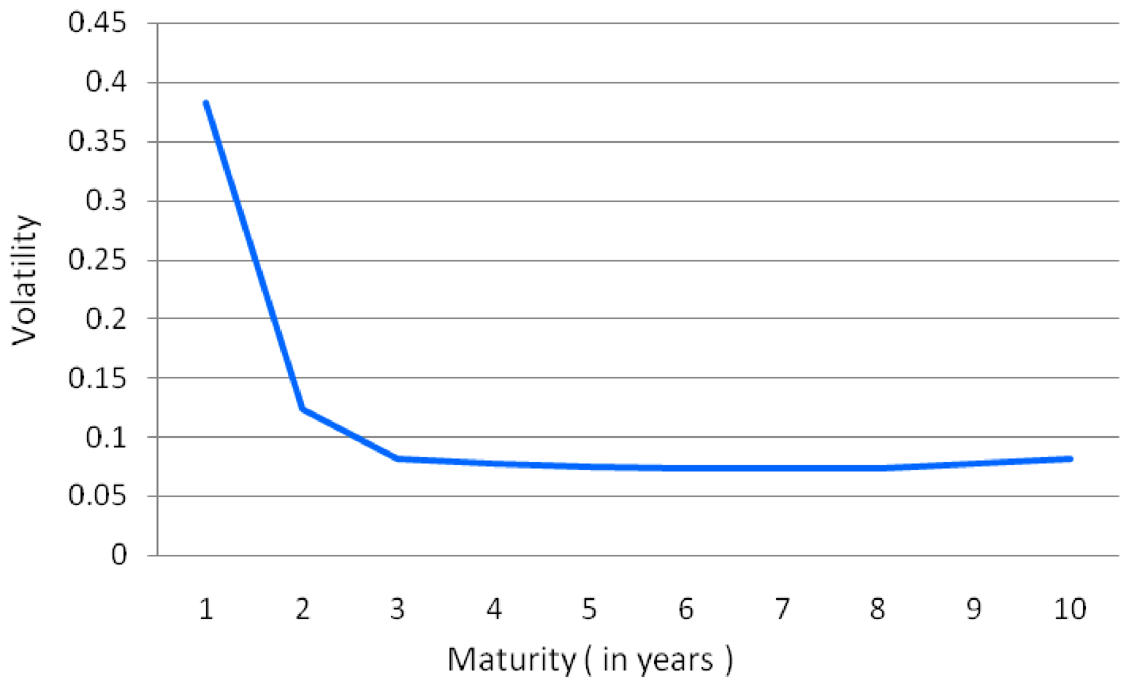

Figure 2. Estimated volatility term structure for maturity year 1 to 10 using GARCH and TGARCH.

rates with the forward period *n*=1 and *n*=3, the GARCH model failed to derive the asymptotic volatility whereas TGARCH successfully obtained the asymptotic volatility. According to the significance of leverage effect parameter γ_1 or the gamma statistics (significant if larger than two), the leverage effects were significant within the forward period *n*=3 and became insignificant thereafter, implying lower non-linearity for higher maturity, which coincided with Figure 2. This paper adopted TGARCH for the forward years *n*=1 to 3, and GARCH for the maturity years four to 10. Figure 2 shows the estimated volatility term-structure. The model diagnosis, shown in the right panel of APPENDIX , verified that the TGARCH model achieved acceptable fitness for the data of the forward years one to three in terms of the Ljung-Box statistics. The residual ARCH and Correlation effects were not significant. Notably, the normality was not accepted for the case *n*=1. Similarly, the GARCH model achieved acceptable fitness for the data of the forward years four to ten, in terms of the Ljung-Box statistics. The residual ARCH and Correlation effects were not significant.

Extrapolation by volatility term structure

This paper adopted the Vasicek (1977) model to fit the proposed volatility term structure. The mean version inherent in the Vasicek model guarantees the existence of the long-term average short rate or the equilibrium term structure. Theoretically, the Vasicek model also verifies the decay of variability for the change of long-term forward rates, which coincides with the empirical results described previously. Particularly, this paper employed the optimization with constraints, which produced the long term UFR consistent with the *QIS5 technical specifications* (2010). Assume that the short rate follows the Vasicek model:

$$r_t = (1-\varphi)\theta + \varphi r_{t-1} + \sigma\varepsilon_t \tag{6}$$

Where σ represents the volatility of the short rate, ϕ is the autocorrelation of the short rate, θ represents the short rate in equilibrium and ε is white noise. The forward rate during the *n*-1 to *n* year ahead at time *t* can be derived as follows:

$$f_t^n = (1-\varphi^{n-1})\theta - \frac{1}{2}\sigma^2\left(\frac{1-\varphi^{n-1}}{1-\varphi}\right)^2 + \phi^{n-1} r_t \tag{7}$$

In equilibrium, the forward rate during the *n*-1 to *n* year ahead can be derived by letting observed time *t* tend to infinity. That is,

$$\bar{f}^n = (1-\varphi^n)\theta - \frac{1}{2}\sigma^2\left(1+\varphi+\varphi^2+\cdots+\varphi^{n-1}\right)^2 + \varphi^{n-1}\theta \tag{8}$$

The volatility of change in the log forward rate is given by:

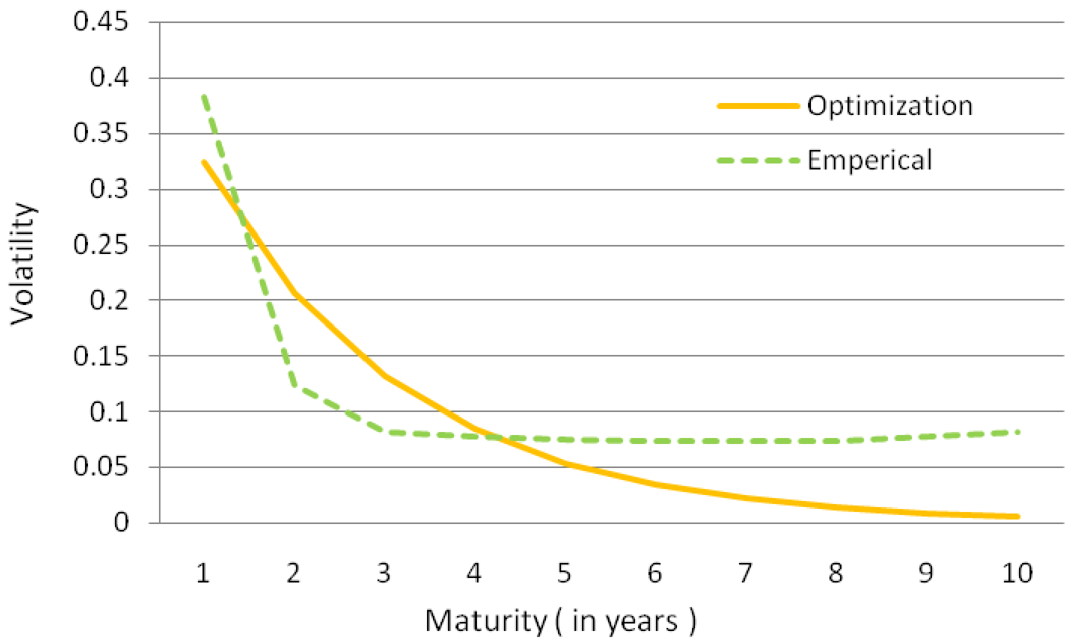

Figure 3. The in-sample fitting for the volatility term structure.

$$s_n \equiv \lim_{t\to\infty}\sqrt{Var\left(\log f_t^n - \log f_{t-1}^n\right)} = \sqrt{2(1-\varphi)\left(\frac{1}{\bar{f}^n}\right)^2 \frac{\sigma^2\varphi^{2(n-1)}}{1-\varphi^2}} \tag{9}$$

The details are described in APPENDIX . Notably, as seen in Equation 9, it was found that the volatility of the forward rate would go to zero as long as $0<\phi<1$, that is, the time series of the short rate was stationary. The UFR is defined as the forward rate (or yield rate) with infinity maturity, that is:

$$\bar{f} = \lim_{n\to\infty}\bar{f}^n = \theta - \frac{1}{2}\left(\frac{\sigma}{1-\varphi}\right)^2 \tag{10}$$

Liu (2008) derived the theoretical volatility of changes of the log forward rate by replacing the forward rate during the *n*-1 to *n* year ahead $\bar{f}^n$ with the UFR $\bar{f}$ in Equation (9) and then deriving a Nelson-Siegel (1987) form for the volatility of change in the log forward rate, where the decay parameter would be chosen arbitrarily. Different from Liu (2008), this paper directly fit the previous estimated volatility term structure from Equation (9) for parameters θ, φ and σ, by which the forward rate could be extrapolated to the UFR. The sum of the squared error between the estimated $\hat{s}_n$ and theoretical volatility term structures s_n for maturity *n* from 1 to 10 was chosen as the fitting criteria. In practice, to sustain the consistency of the term structure, QIS5 will suggest an exogenous UFR for each economy area by comparing many economical factors such as inflation, regional differences and long-term expectation. This paper adopted this information by performing the optimization procedure with the UFR constraint. Equation (9) shows the optimization for the Vasicek parameters to extrapolate the forward rate.

$$\min_{\varphi,\theta,\sigma}\sum_{n=1}^{10}\left|\hat{s}_n - s_n\right|^2 \quad \text{s.t.} \quad \bar{f} = 4.2\% \tag{11}$$

The estimated parameters were $\hat{\varphi} = 0.636966$, $\hat{\theta} = 4.2589\%$ and $\hat{\sigma} = 1.2476\%$, with an average error of 0.315% for each maturity. The optimal value for the AR (1) parameter value $\hat{\varphi}$ was between 0 and 1, implying a stationary short rate process and convergence for the equilibrium of the volatility term structure. Figure 3 demonstrates the results of the optimization in Equation (11). The short-term volatilities were lower than the ones from the TGARCH model. After year five, the fitted volatilities decayed to zero due to Equation (9). Figure 4 compares the risk-free nominal term structures at the end of year 2010 by the proposed GARCH method and the Smith-Wilson approach used by QIS5. The term structure by the proposed extrapolating method based on the GARCH model converged to UFR more rapidly than that based on QIS5. A steep term structure implies a high speed of mean reversion. Using Equations 7 and 8,

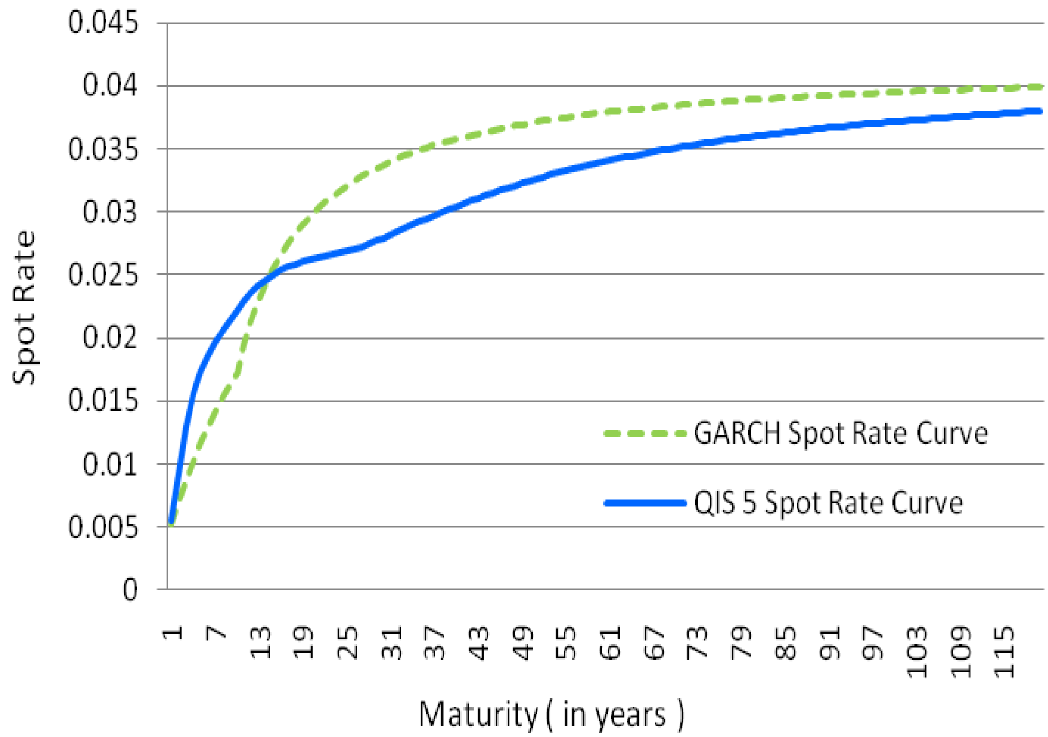

Figure 4. The spot rate curves for GARCH and QIS 5.

this paper inferred the equilibrium forward rate by letting the time horizon tend to infinity. However, the Smith-Wilson method fit the observed market data directly, which may reflect low interest rate levels in the current government bond market.

CASE COMPARISONS

To compare the term structures of the proposed GARCH method and the Smith-Wilson method suggested by QIS5, this paper used these two term structures to value a deferred annuity with a deferred period of 30 years and a payment period of 25 years, and a benefit payment of $1 per year. To consider mortality and gender effects, this paper valued a life annuity by referring to the TSO 2002 Life Table and a certain annuity. Due to a high speed of convergence, the annuity values by the proposed method were less than that of QIS5. Table 1 points out that the term structure by the proposed GARCH method produced lower liability values than that of QIS5. For the male case with $4.7 QIS5 liability value, this method produced a liability value that was less than that of QIS5 by $1. For the female case with $5.7 QIS5 liability value, this method produced a liability value that was less than that of QIS5 by $1.2. For annuity-certain with $7.2 QIS5 liability value, the proposed method had a lower liability value by $1.4.

Conclusions

This paper addressed the problems encountered during the extrapolation of nominal risk-free term structures based on the Taiwan government bond market. The scarcity of liquid trading data makes it necessary to extrapolate long-term interest rates using mark-to-model methods. This paper worked on the GARCH models implying mean reversion and equilibrium property, with the advantages of simplicity and avoiding the arbitrary determination of model parameters. An empirical study showed that the dramatic decreases for short-term yields in the Taiwan government bond market during 2009 caused the GARCH models to fail to infer the volatility of short-term interest rates. This paper proposed the Threshold GARCH model to determine the volatility term structure through which to extrapolate the long-term risk-free nominal term structure under the Vasicek interest rate model. The empirical experiments indicated that the proposed Threshold GARCH method successfully deduced the equilibrium volatility level of short-term interest rates.

This paper suggests using the Threshold GARCH model for maturities of less than three years and the GARCH model for maturities from four to ten years. The GARCH type models also achieved statistically acceptable inference in goodness of fit. The proposed optimization process extrapolated the long-term rate to the

Table 1. Valuation of deferred annuity liability with GARCH and QIS 5 term-structures.

Case	GARCH	QIS 5
Male	$3.89691	$4.76344
Female	$4.68426	$5.73535
Annuity-certain	$5.88527	$7.22349

the UFR level suggested by QIS5. Compared with QIS5for deferred annuities, the proposed term structure produced lower liability values for the insurers, due to the higher speed of mean reversion. Future studies are still required regarding the extrapolation of long-term interest rates. Extrapolation must be undertaken for maturities over 10 years and available market data shorter than 10 years should be adopted. The smoothness should be taken into account due to the stability consideration required by QIS5. Furthermore, the term structures in Taiwan were only available after year 2006, and are interpolated using the Nielsen and Siegel methods, which may cause the term structure to be more volatile and create bias between the published term structure and market trading prices. The consistency property needs to be investigated, and should be performed with more liquid market data.

REFERENCES

Amin K, Ng V (1997). Inferring future volatility from information in implied volatility in Eurodollar options: A new approach. Rev. Finan. Stud., 10: 333-367.

Andersen TG, Lund J (1997). Estimating continuous-time stochastic volatility models of the short-term interest rate. J. Econ., 77: 343-377.

Canina L, Figlewski S (1993). The informational content implied volatility. Rev. Financ. Stud., 6: 659-681.

Carriere JF (1999). Long-term yield rates for actuarial valuations. N. Am. Actuarial J., 3: 13-24.

Christensen BJ, Prabhala NR (1998). The relation between implied and realized volatility. J. Financ. Econ., 50: 125-150.

European Insurance and Occupational Pensions Authority (2010). QIS 5 risk-free interest rates-extrapolation method.

European Insurance and Occupational Pensions Authority (2010). QIS 5 technical specifications.

Glosten LR, Jagannathan R, Runkle DE (1993). On the relation between the expected value and the volatility of nominal excess return on stocks. J. Financ., 48: 1779-1801.

Gospodinov N (2005). Testing for threshold nonlinearity in short-term interest rates. J. Financ. Econ., 3: 344-371.

Liu Z (2008). How to construct a volatility term-structure of interest rate in the absence of market price. Finan. Econ. Res., 1.0. Barrie Hibbert. Working Paper.

Nelson CR, Siegel AF (1987). Parsimonious modeling of yield curves. J. Bus., 60: 473-489.

Smith A, Wilson T (2001). Fitting yield curves with long term constraints. Res. Notes, Bacon and Woodrow. Working Paper.

Svensson LEO (1994). Estimating and interpreting forward interest rates: Sweden, 1992-1994. National Bureau of Economic Research, Working Paper Series 4871.

Thomas M, Maré E (2007). Long term forecasting and hedging of the South African yield curve. Working Paper.

Vasicek O (1977). An equilibrium characterization of the term structure. J. Finan. Econ., 5(2): 177-188.

Zakoian J (1994). Threshold heteroskedastic model. J. Econ. Dyn. Control, 18: 931-955.

APPENDIX A

Descriptive statistics

Table 3 shows the descriptive statistics for the NSS monthly data of changes in the log forward rates.

Table 1. Descriptive statistics of changes in the log forward rates.

Maturity	Mean	Median	Min	Max	Std	Skew	Kurt	ACF(1)	ACF(2)	ACF(12)
1	-0.0171	-0.0088	-3.6869	3.0471	0.7187	-0.4336	17.7712	-0.4264	0.0773	-0.0212
2	-0.0117	-0.0036	-0.4300	0.5553	0.1430	0.3539	5.1102	-0.2653	0.2582	0.1492
3	-0.0080	0.0008	-0.3228	0.1836	0.0856	-0.6326	2.4782	-0.0488	0.1915	0.2010
4	-0.0049	-0.0016	-0.2359	0.1534	0.0782	-0.4103	0.5791	0.0722	0.0254	0.0713
5	-0.0024	-0.0048	-0.1835	0.1515	0.0752	-0.1871	0.3004	0.1405	-0.0098	-0.0107
6	-0.0003	-0.0075	-0.1830	0.1815	0.0735	0.0620	0.7183	0.1700	0.0136	-0.0571
7	0.0014	-0.0025	-0.1848	0.1975	0.0739	0.3600	1.4639	0.1523	0.0308	-0.0865
8	0.0026	0.0000	-0.1848	0.2311	0.0757	0.5905	2.0585	0.1069	0.0195	-0.1059
9	0.0035	0.0014	-0.1827	0.2591	0.0780	0.7164	2.3906	0.0567	-0.0068	-0.1176
10	0.0040	0.0016	-0.1782	0.2744	0.0803	0.7378	2.4011	0.0156	-0.0384	-0.1188
11	0.0042	-0.0014	-0.1742	0.2781	0.0824	0.7065	2.1774	-0.0107	-0.0671	-0.1059
12	0.0041	-0.0040	-0.1821	0.2706	0.0845	0.6402	1.7453	-0.0270	-0.0930	-0.0840
13	0.0037	0.0018	-0.1908	0.2551	0.0872	0.5553	1.2473	-0.0360	-0.1205	-0.0554
14	0.0032	-0.0018	-0.1974	0.2399	0.0900	0.4422	0.8502	-0.0413	-0.1500	-0.0249
15	0.0026	-0.0068	-0.2224	0.2496	0.0935	0.3004	0.6987	-0.0486	-0.1880	-0.0029
16	0.0018	-0.0072	-0.2698	0.2602	0.0974	0.1531	0.8278	-0.0581	-0.2323	0.0110
17	0.0010	-0.0107	-0.3155	0.2701	0.1018	0.0345	1.2255	-0.0695	-0.2834	0.0178
18	0.0001	-0.0096	-0.3573	0.2818	0.1073	-0.0344	1.7476	-0.0841	-0.3313	0.0169
19	-0.0009	-0.0074	-0.3964	0.2916	0.1140	-0.0316	2.2331	-0.1000	-0.3748	0.0107
20	-0.0018	-0.0096	-0.4322	0.3007	0.1228	0.0156	2.4529	-0.1153	-0.4030	0.0059
21	-0.0029	-0.0164	-0.4635	0.3446	0.1335	0.0740	2.3602	-0.1313	-0.4125	0.0027
22	-0.0039	-0.0212	-0.4923	0.3862	0.1466	0.1038	2.0421	-0.1480	-0.4009	0.0034
23	-0.0049	-0.0206	-0.5173	0.4275	0.1625	0.1058	1.6283	-0.1639	-0.3726	0.0077
24	-0.0061	-0.0155	-0.5367	0.4672	0.1812	0.0856	1.1807	-0.1791	-0.3288	0.0153
25	-0.0072	-0.0120	-0.5560	0.5033	0.2034	0.0404	0.7804	-0.1919	-0.2756	0.0237
26	-0.0084	-0.0106	-0.6006	0.5396	0.2303	0.0111	0.4485	-0.2025	-0.2108	0.0342
27	-0.0097	-0.0059	-0.6906	0.5729	0.2641	0.0052	0.2796	-0.2085	-0.1366	0.0434
28	-0.0110	-0.0083	-0.7789	0.8055	0.3103	0.0284	0.5468	-0.2071	-0.0457	0.0503
29	-0.0125	-0.0190	-1.1902	1.2272	0.3862	0.0491	1.8517	-0.1964	0.0817	0.0488
30	-0.0165	-0.0231	-2.4383	2.1238	0.5586	-0.4655	8.4662	-0.1222	0.0032	-0.0750

ACF (k) is the k-th order autocorrelation function.

APPENDIX B

Augmented Dickey-Fuller Test

Table 2 shows the Augmented Dickey-Fuller Test for the forward rate data. The second column represents the statistics (with the p-value in parentheses) of the original forward rate data, and the third column represents the statistics (with the p-value in parentheses) of changes in the log-forward rate data.

Table 2. Augmented dickey-fuller test

Maturity (in years)	Original forward rate	Change in log-forward rate
1	-2.2423 (0.4768)	-3.3069 (0.0791)
2	-2.4236 (0.4035)	-2.9607 (0.1865)
3	-2.2500 (0.4737)	-3.3156 (0.0777)
4	-2.3280 (0.4422)	-3.7554 (0.0277)
5	-2.5220 (0.3637)	-4.1369 (0.0100)
6	-2.7261 (0.2812)	-4.3982 (0.0100)
7	-2.8486 (0.2316)	-4.5286 (0.0100)
8	-2.8953 (0.2127)	-4.5193 (0.0100)
9	-2.8910 (0.2145)	-4.4296 (0.0100)
10	-2.8490 (0.2315)	-4.3332 (0.0100)

The numbers in parentheses denote the p-value.

APPENDIX C

Estimation and diagnosis for GARCH and TGARCH

Table 3 contains the parameters and statistics related to the GARCH and TGARCH volatility models.

Table 3. GARCH and TGARCH table for the volatility of changes in the log forward rates.

Maturity	Estimated parameter						Model diagnosis			
n=1	Asym.Stdev.	a_0	a_1	b_1	γ_1	Gamma statistics	Normality (JB test)	Ljung-Box statistics	Residual ARCH effect	Residual correlation effect
GARCH	na	-0.0001469 (0.8246)	4.6092087 (0.0000)	0.2509544 (0.0000)			220.4 (0.0000)	6.199 (0.9057)	3.9026 (0.9851)	6.1987 (0.9057)
TGARCH	0.3829292	0.0052641 (0.009627)	-0.3926156 (0.0000)	0.7271505 (0.0000)	1.2591307 (0.0000)	13.76372	581.4 (0.0000)	12.78 (0.385)	1.4062 (0.9999)	12.7828 (0.3850)

Table 3. Contd.

n=2	Asym.Stdev.	a_0	a_1	b_1	γ_1	Gamma statistics	Normality (JB test)	Ljung-Box statistics	Residual ARCH effect	Residual correlation effect
GARCH	0.14364983	0.005365 (0.18509)	0.240428 (0.08047)	0.09133 (0.499567)			27.09 (0.0000)	13.05 (0.3657)	6.9038 (0.8639	13.0467 (0.3657)
TGARCH	0.1226665	0.0009037 (0.00488)	-0.2086650 (0.0000)	0.9532635 (0.0000)	0.3906833 (0.0000)	7.875236	4.444 (0.1084)	12.06 (0.4408)	2.997 (0.9956)	12.0612 (0.4408)
n=3	Asym. Stdev.	a_0	a_1	b_1	γ_1	Gamma statistics	Normality (JB test)	Ljung-Box statistics	Residual ARCH effect	Residual correlation effect
GARCH	na	-0.0005485 (0.0000)	-0.0232482 (0.2084)	1.0891589 (0.0000)			5.406 (0.06699)	21.49 (0.04359)	11.7708 (0.4643)	21.4946 (0.0436)
TGARCH	0.08160859	0.001560 (0.03741)	-0.282826 (0.01423)	0.878132 (0.0000)	0.341060 (0.01999)	2.396209	8.32 (0.01561)	17.75 (0.1234)	6.3901 (0.8952)	17.7531 (0.1234)
n=4	Asym. Stdev.	a_0	a_1	b_1	γ_1	Gamma statistics	Normality (JB test)	Ljung-Box statistics	Residual ARCH effect	Residual Correlation Effect
GARCH	0.07665592	0.008068 (0.3928)	-0.095961 (0.3401)	-0.277066 (0.8733)			0.911 (0.6341)	7.149 (0.8476)	15.0864 (0.2367)	7.1494 (0.8476)
TGARCH	0.07692194	0.001672 (0.1325)	-0.284555 (0.01028)	0.897621 (0.0000)	0.208776 (0.0144)	2.52729	1.442 (0.4862)	10.58 (0.5653)	14.6316 (0.2622)	10.5789 (0.5653)
n=5	Asym. Stdev.	a_0	a_1	b_1	γ_1	Gamma statistics	Normality (JB test)	Ljung-Box statistics	Residual ARCH effect	Residual correlation effect
GARCH	0.07462323	0.007285 (0.1510)	-0.117117 (0.3005)	-0.191169 (0.8377)			0.1879 (0.9103)	6.457 (0.8913)	15.2531 (0.2279)	6.4570 (0.8913)
TGARCH	0.07715432	0.0006958 (0.2196)	-0.2911573 (0.000082)	1.0537704 (0.0000)	0.2410047 (0.001007)	3.473995	1.225 (0.542)	7.262 (0.8398)	10.4816 (0.5738)	7.2621 (0.8398)
n=6	Asym. Stdev.	a_0	a_1	b_1	γ_1	Gamma statistics	Normality (JB test)	Ljung-Box statistics	Residual ARCH effect	Residual correlation effect
GARCH	0.07327679	0.006820 (0.04597)	-0.156541 (0.06955)	-0.113649 (0.85187)			0.08367 (0.959)	9.881 (0.6264)	15.4069 (0.2199)	9.8814 (0.6264)
TGARCH	0.08261096	0.0003467 (0.2212)	-0.2311733 (0.0000)	1.0662396 (0.0000)	0.2282704 (0.000442)	3.739251	0.4912 (0.7822)	9.86 (0.6282)	10.8077 (0.5455)	9.8604 (0.6282)
n=7	Asym. Stdev.	a_0	a_1	b_1	γ_1	Gamma statistics	Normality (JB test)	Ljung-Box statistics	Residual ARCH effect	Residual correlation effect
GARCH	0.07305755	0.008209 (0.000592)	-0.145751 (0.00048)	-0.392204 (0.128858)			2.278 (0.3201)	8.671 (0.7308)	15.5597 (0.2122)	8.6708 (0.7308)
TGARCH	na	0.000116 (0.7484)	-0.144687 (0.002603)	1.054933 (0.0000)	0.190457 (0.006184)	2.847485	0.04408 (0.9782)	10.36 (0.5841)	13.6624 (0.3228)	10.3632 (0.5841)

Table 3. Contd.

n=8	Asym.Stdev.	a_0	a_1	b_1	γ_1	Gamma statistics	Normality (JB test)	Ljung-Box statistics	Residual ARCH effect	Residual correlation effect
GARCH	0.07314084	0.0071379 (0.039219)	-0.1097516 (0.004216)	-0.2245342 (0.706775)			5.784 (0.05547)	7.051 (0.8542)	14.2016 (0.2880)	7.0510 (0.8542)
TGARCH	0.1440554	0.0002531 (0.2527)	-0.1438501 (0.0000)	1.0658850 (0.0000)	0.1315341 (0.09963)	1.674906	0.1862 (0.9111)	8.882 (0.713)	12.0566 (0.4411)	8.8823 (0.7130)
n=9	Asym.Stdev.	a_0	a_1	b_1	γ_1	Gamma statistics	Normality (JB test)	Ljung-Box statistics	Residual ARCH effect	Residual correlation effect
GARCH	0.07732394	0.002108 (0.8943)	0.029026 (0.8436)	0.618460 (0.8236)			14.29 (0.000788)	5.784 (0.9266)	13.6843 (0.3213)	5.7835 (0.9266)
TGARCH	na	0.0001552 (0.4422)	-0.1165725 (0.0000)	1.0706573 (0.0000)	0.1267125 (0.02954)	2.234411	0.6952 (0.7064)	8.651 (0.7324)	12.4478 (0.4104)	8.6512 (0.7324)
n=10	Asym. Stdev.	a_0	a_1	b_1	γ_1	Gamma statistics	Normality (JB test)	Ljung-Box statistics	Residual ARCH effect	Residual correlation effect
GARCH	0.08100820	0.002298 (0.5230)	0.157581 (0.5429)	0.492178 (0.5052)			18.47 (0.000097)	5.075 (0.9554)	10.4485 (0.5767)	5.0749 (0.9554)
TGARCH	na	0.0001181 (0.4326)	-0.1050579 (0.0000)	1.0694472 (0.0000)	0.1131277 (0.04023)	2.101094	0.4457 (0.8002)	9.143 (0.6907)	12.6533 (0.3947)	9.1428 (0.6907)

"na" represents divergence for the optimization. The numbers in parentheses denote the p-value. "Asym. Stdev" represents the long-term average level of volatility.

APPENDIX D

Volatility term structure of changes in the log forward rates

Using the Vasicek model, the short rate follows an AR (1) process, that is,

$$r_t = (1-\varphi)\theta + \varphi r_{t-1} + \sigma\varepsilon_t \ , \qquad \text{(D.1)}$$

Where ε_t is an independent Gaussian process with a mean of zero and a variance of one. The unconditional second moments of the short rates are:

$$\mathrm{Var}(r_t) = \sigma^2 \frac{1-\varphi^{2t}}{1-\varphi^2} \ , \qquad \text{(D.2)}$$

And

$$\mathrm{Cov}\left(r_{t-1}, r_t\right) = \mathrm{Cov}\left(r_{t-1}, (1-\varphi)\theta + \varphi r_{t-1} + \sigma\varepsilon_t\right) = \varphi * \mathrm{Var}\left(r_{t-1}\right) = \varphi\frac{1-\varphi^{2(t-1)}}{1-\varphi^2}. \quad \text{(D.3)}$$

With the stationary condition 0<φ<1, the equilibrium short rate can be inferred by letting time *t* tend to infinity, that is:

$$\bar{r} = \frac{(1-\varphi)\theta}{1-\varphi} = \theta \ . \quad \text{(D.4)}$$

Let b_t^n denote the discount factor for cash flow at *n* years ahead. Under the Vasicek model, the affine term structure is related to the short rate r_t and can be transformed as follows:

$$-\log b_t^n = A_n + B_n r_t \ . \quad \text{(D.5)}$$

Given the no-arbitrage condition:

$$b_t^{n+1} = E_t\left(b_t^1 \cdot b_{t+1}^n\right). \quad \text{(D.6)}$$

The recursive relationships for the sequences A_n and B_n can be induced as follows:

$$A_{n+1} = A_n + B_n(1-\varphi)\theta - \frac{1}{2}\sigma^2 B_n^2 \ , \quad \text{(D.7)}$$

and

$$B_n = \frac{1-\varphi^n}{1-\varphi} \ . \quad \text{(D.8)}$$

From Equations D.7 and D.8, the *n*-period ahead one-period forward rate, denoted by f_t^n, can be derived as follows:

$$f_t^n = \left(A_n - A_{n-1}\right) + \left(B_n - B_{n-1}\right) r_t = \left(1-\varphi^{n-1}\right)\theta - \frac{1}{2}\sigma^2\left(\frac{1-\varphi^{n-1}}{1-\varphi}\right)^2 + \varphi^{n-1} r_t \ . \quad \text{(D.9)}$$

Using Equations D.2 and D.3 the second moments of the log forward rates can be expressed as follows:

$$\mathrm{Var}\left(f_t^n\right) = \left(\frac{\varphi^{n-1} - \varphi^n}{1-\varphi}\right)^2 \sigma^2 \frac{1-\varphi^{2t}}{1-\varphi^2} \ , \quad \text{(D.10)}$$

and

$$\text{Cov}\left(\log f_t^n - \log f_{t-1}^n\right) = \left(\frac{\varphi^{n-1} - \varphi^n}{1-\varphi}\right)^2 \sigma^2 \varphi \frac{1-\varphi^{2t}}{1-\varphi^2} \ . \qquad \text{(D.11)}$$

Equation D.12 derives the volatility of the change in the log forward rates by the Delta method with the transformation $g\left(f_t^n, f_{t-1}^n\right) \triangleq \log f_t^n - \log f_{t-1}^n$.

$$\text{Var}\left(\log f_t^n - \log f_{t-1}^n\right)$$

$$= \left(\frac{1}{f_t^n}, -\frac{1}{f_{t-1}^n}\right) \begin{pmatrix} \text{Var}\left(\log f_t^n\right) & \text{Cov}\left(\log f_t^n, \log f_{t-1}^n\right) \\ \text{Cov}\left(\log f_t^n, \log f_{t-1}^n\right) & \text{Var}\left(\log f_{t-1}^n\right) \end{pmatrix} \begin{pmatrix} \dfrac{1}{f_t^n} \\ -\dfrac{1}{f_{t-1}^n} \end{pmatrix} . \qquad \text{(D.12)}$$

Substituting Equations D.9, D.10, D.11 and into D.12 and letting time *t* tend to infinity, the equilibrium volatility term structure can be derived for the change in the log forward rates as follows:

$$\lim_{t\to\infty} \sqrt{\text{Var}\left(\log f_t^n - \log f_{t-1}^n\right)} = \sqrt{2(1-\varphi)\left(\frac{1}{\overline{f}^n}\right)^2 \frac{\sigma^2 \varphi^{2(n-1)}}{1-\varphi^2}} \ . \qquad \text{(D.13)}$$

Meanwhile,

$$\overline{f}^n = \left(1-\varphi^n\right)\theta - \frac{1}{2}\sigma^2\left(1+\varphi+\varphi^2+\cdots+\varphi^{n-1}\right)^2 + \varphi^{n-1}\theta \ . \qquad \text{(D.13)}$$