Using the SABR Model

Jason Vinar

Ameriprise

Workshop 2012

Overview

The Black-76 model has been the standard model for European options on currency, interest rates, and stock indices with it's main drawback being the constant volatility assumption. The SABR (stochastic, α, β, and ρ) model is a stochastic model which attempts to capture the volatility smile. This project will consist of

- Calibrating the SABR model
- Simulating the forward
- Pricing a vanilla and barrier option
- Creating dynamic hedges for the barrier option

Option Prices with Black-76

The Black model for European gives the forward price of the option, V as

$$V = wF\Phi(wd_1) - wK\Phi(wd_2)$$

$$d_1 = \frac{\ln\frac{F}{K} + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

where $w = 1$ for call options and $w = -1$ for put options. Here the volatility, σ, is constant. With the SABR model you can derive a value for σ that depends on the strike K.

Definition of the SABR Model

The SABR Model: stochastic-$\alpha\beta\rho$

$$dF = \alpha F^{\beta} dW_1$$

$$d\alpha = \nu \alpha dW_2$$

$$dW_1 dW_2 = \rho dt$$

The model has 4 parameters: α, β, ρ, and ν.

- In terms of the model dynamics, ρ and β determine the degree of the smile.
- The ν parameter, volatility of volatility, determines the skew.
- Lastly, α determines the at-the-money forward (ATM) volatility. It is common to use the ATM volatility[1] rather than α since it can be directly observed in the market.

[1] West showed that the ATM volatility can be computed by solving a cubic root.

SABR Implied Volatility

The SABR model has a unique feature that allows you to compute the implied volatility directly for a given strike.

$$\sigma_B(F,K) = \left[\frac{\alpha}{(FK)^{(1-\beta)/2}\left(1+\frac{(1-\beta)^2}{24}(\ln\frac{F}{K})^2+\frac{(1-\beta)^4}{1920}(\ln\frac{F}{K})^4\right)}\frac{z}{\chi(z)}\right]$$
$$\cdot \left[1+\left(\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(FK)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta\nu\alpha}{(FK)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\nu^2\right)\right]T$$

where

$$z = \frac{\nu}{\alpha}(FK)^{(1-\beta)/2}k$$

$$\chi(z) = \ln\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right)$$

Calibration

The SABR model is calibrated to a set of option prices (volatilities) for a given expiration. The calibration begins with chosing β either by empirical analysis of the asset price and the σ_{ATM} or by setting $\beta = 0$ for a normal process or $\beta = 1$ for a lognormal process. Next, with β chosen, you have two choices in calibrating the other 3 parameters.

- Let your minimization determine α and in turn the σ_{ATM}.

$$(\hat{\alpha}, \hat{\rho}, \hat{\nu}) = \underset{\alpha,\rho,\nu}{\text{argmin}} \sum_i [\sigma_i^M - \sigma_B(F, K_i; \alpha, \rho, \nu)]^2$$

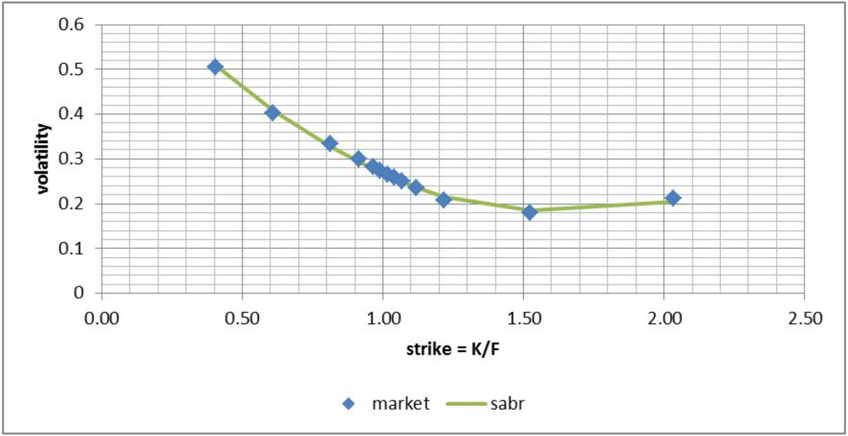
  This approach is typically faster.

- Infer the α parameter from the ATM σ.

$$(\hat{\alpha}, \hat{\rho}, \hat{\nu}) = \underset{\alpha,\rho,\nu}{\text{argmin}} \sum_i [\sigma_i^M - \sigma_B(F, K_i; \alpha(\rho, \nu), \rho, \nu)]^2$$

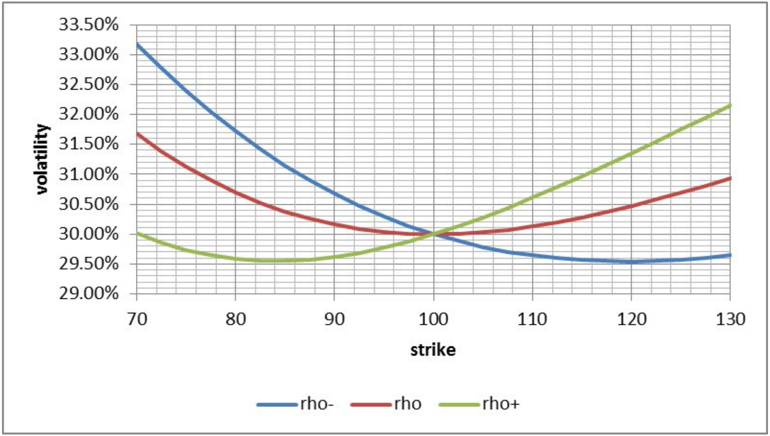
  This approach allows for exact matching of the σ_{ATM}.

Calibration Fit



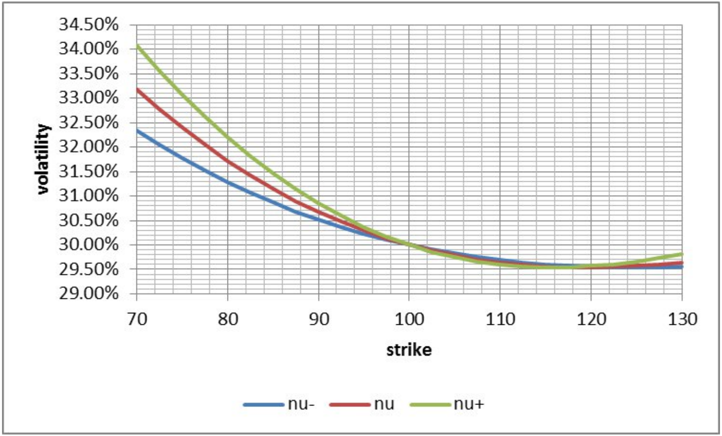
ρ Dynamics

The smile is determined with ρ. Changing ρ pivots the curve on the point $K = F$ and $\sigma = \sigma_{ATM}$.



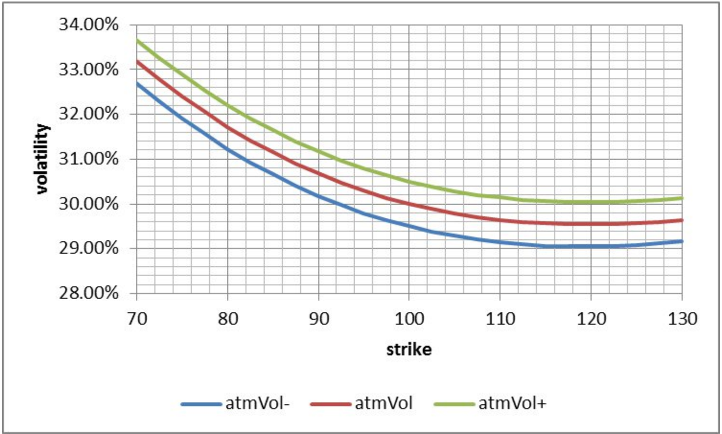
ν dynamics

The volatility of volatility parameter ν determines the degree of the skew. As ν increases both the put and call prices increase.



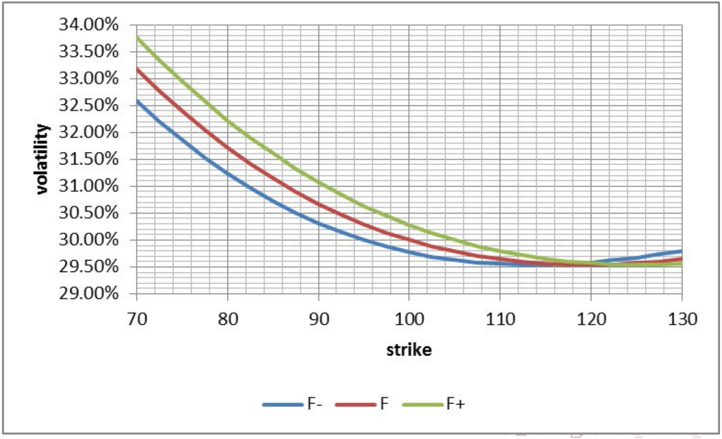
ATM σ Dynamics

The σ_{ATM} gives the volatility for $K = F$. Notice this is not a parallel shift across the strike dimension. Here, for example where $\rho < 0$, $\delta\sigma_P < \delta\sigma_{ATM} < \delta\sigma_C$.



F Dynamics

The volatility follows a *sticky-delta* relationship for changes in the forward, i.e. the volatility is a function of the relationship K/F, $\sigma_B(F,K) = \sigma_{ATM} - b(K/F - 1)F_0$.



Monte Carlo Simulation

Direct simulation of the SDE's, e.g. with an Euler scheme

$$F_{t+\Delta} = F_t + \sigma_t F_t^{\beta} Z_1 \sqrt{\Delta}$$

$$\sigma_{t+\Delta} = \sigma_t \exp\left(-\frac{1}{2}\nu^2 \Delta + \nu Z_2 \sqrt{\Delta} \right)$$

where $\sigma_0 = \alpha$, $F_0 = F$, and Z_1 and Z_2 have correlation ρ.

Monte Carlo Simulation

Simulation using the local volatility function

$$S_{t+\Delta} = S_t \frac{F^M_{t+\Delta}}{F^M_t} \exp\left(-\frac{1}{2}\sigma^2_{loc}(t, S_t)\Delta + \sigma_{loc}(t, S_t) Z \sqrt{\Delta} \right)$$

$$\sigma_{loc}(t, S_t) = \sqrt{\frac{\frac{\partial w}{\partial T}}{1 - \frac{y}{w} + \frac{1}{4}(-\frac{1}{4} - \frac{1}{w} + \frac{y^2}{w^2})(\frac{\partial w}{\partial y})^2 + \frac{1}{2}\frac{\partial^2 w}{\partial y^2}}}$$

where $w = \sigma^2_B(F^M_t, S_t)T$, $S_0 = S$, and $y = \log \frac{S_t}{F^M_t}$.

Monte Carlo Simulation

The two simulation methods presented are quite different. As such, they are simulating two slightly different quantities. The first method is simulating the value of the forward and has no drift. The second method is simulating the spot level while matching the market forward through simulation time.

While the SABR model is not often used for equity derivatives, recently in the literature it has been paired with several short rate models to price long maturity equity derivatives, particularly exotic options.

For equity derivatives, it is most useful to have a simulation of the spot process. This project will primarily consist of building a simulation engine for the spot level, further it must

- Be able to use time dependent SABR parameters;
- Match the implied volatility surface; and
- Replicate the $T = 0$ forward levels throughout simulation time.

Project Plan

1. Calibrate
   - Black-76 price and implied volatility functions
   - Calculate the SABR implied volatility, $\sigma_B(F, K)$
2. Volatility Surface
   - Time interpolation
   - Convert to local volatility
3. Simulate
   - Generate correlated standard normal random variables
   - Simulate directly with the Euler scheme
   - Simulate using local volatility
   - Compare simulation approaches
4. Simulation with time dependent SABR parameters
5. Price and hedge a barrier option

References

Chen, B., C. W. Oosterlee, H. van der Weide (2011): "A low-bias simulation scheme for the SABR stochastic volatility model," *TDB,* TBA.

Hagan, P.S., D. Kumar, A. S. Lesniewski, and D. E. Woodward (2002): "Managing Smile Risk," *Wilmott Magazine,* September, 1(1).

Haug, E. G. (2007): *The Complete Guide to Option Pricing Formulas, 2nd ed.* (New York, NY: McGraw-Hill).

Rouah, F. D.: "The SABR Model," *http://www.volopta.com*