Local time for the SABR model
Connection with the "complex" Black Scholes
And application to CMS and Spread Options

E. Benhamou[1],[3] O. Croissant[2],[3]
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Abstract: It is well known that the cost of a call and put option is equal to its intrinsic value plus the cost of a stop loss strategy. This stop loss strategy can be re-expressed in terms of the local time. It provides easily closed forms solution for model like Black Scholes [8] or [3]. This paper examines the theory of local time for stochastic volatility models and in particular the SABR model [5]. It gives an approximated formula for the local time in SABR and shows that this model can be valued using a Black Scholes formula but where all the terms are complex number. This formula turns out to be more robust for low and high strikes. This solves in particular the problem of valuing the whole smile in SABR as required in the replication method for CMS and the copula integration for CMS spread options.

1. Expected local time and vanilla option prices

The strength of the local time theory lies in its general framework. Let the forward asset F_t follow a stochastic volatility diffusion:

$$dF_t = \sigma(t, F_t)C(F_t)dW_t^F \qquad F_0 = f \tag{1.1}$$

and $$d\sigma(t, F_t) = \sigma_t M(t,\sigma)dt + \sigma_t S(t,\sigma)dW_t^\sigma \qquad \sigma_0 = \alpha \tag{1.2}$$

where W_t^F and W_t^σ are two standard Brownian motions potentially correlated $\langle dW_t^F, dW_t^\sigma \rangle = \rho_t dt$ and $C(F_t)$ is a mapping function (usually a CEV $C(F_t) = F_t^\beta$ or a displaced diffusion function $C(F_t) = F_t + \mu_t$) and $M(t,\sigma)$, $S(t,\sigma)$ are some smooth functions. The Meyer Tanaka formula, extension of the Ito formula to convex payoff ([6] or [9]) provides an interesting framework to compute the price of a call with strike k given by ([3]):

$$(F_T - k)^+ = (f-k)^+ + \int_0^T 1_{F_u^- > k} dF_u + \frac{1}{2}\int_0^T 1_{F_u = k} \sigma^2(F_u, u) C^2(F_u) du \tag{1.3}$$

Taking the expected value of the above equation leads to a forward price of a call option given by

$$E\left[(F_T - k)^+\right] = (f-k)^+ + E\left[\frac{1}{2}\int_0^T 1_{F_u = k} \sigma^2(F_u, u) C^2(F_u) du\right] \tag{1.4}$$

The above equation summarizes that the forward price of a call is the sum of its intrinsic and its time value represented by its expected continuous[4] local time at strike k for a maturity time T. But more than a new formulation for the value of a vanilla option, it shows that this value lies in the computation of the expected local time for any model. In particular, in a Black Scholes model with volatility α whose diffusion equation is given by $dF_t = \alpha F_t dW_t^F$, the expected local time at strike k for a maturity time T is easy to compute and given by ([3]):

$$E\left[\frac{1}{2}\int_0^T 1_{F_u = k}\alpha^2 du\right] = \frac{1}{2}\alpha^2 k^2 \int_0^T \frac{1}{\alpha k\sqrt{u}} \frac{e^{-\frac{1}{2}\left(\frac{Log(f/k)}{\alpha\sqrt{u}} - \frac{1}{2}\alpha\sqrt{u}\right)^2}}{\sqrt{2\pi}} du \tag{1.5}$$

[1] Eric.benhamou@pricingpartners.com, Head of Quantitative Research, PRICING PARTNERS, 204 Rue de Crimée, 75019 Paris, France.
[2], OCroissant@ixis-cib.com, Interest Rates Derivatives Research, IXIS CIB, 47 Quai d'Austerlitz, 75648 Paris Cedex 13, France.
[3] All ideas and opinion expressed herein are the ones of the authors and do not necessarily reflect those of their respective companies.
[4] We call it continuous in opposition to discrete local time. Discrete local time terms would appear in the case of a jump-diffusion. In the rest of the paper, we will drop continuous as we only deal with continuous diffusion.

which implies a closed form solution given by:

$$BS\_Call(f,k,\alpha,T) = (f-k)^+ + \frac{1}{2}\alpha^2 k^2 \int_0^T \frac{1}{\alpha k\sqrt{u}} \frac{e^{-\frac{1}{2}\left(\frac{Log(f/k)}{\alpha\sqrt{u}} - \frac{1}{2}\alpha\sqrt{u}\right)^2}}{\sqrt{2\pi}} du \quad (1.6).$$

The above result can be derived either from a direct computation of the local time for geometric Brownian motion or by relating the vega to the local time (see appendix section 7.1).

2. Expected Local time and probability density

General framework

Define the probability density $p(t, f, \alpha; T, F, A)$ by:

$$p(t, f, \alpha; T, F, A)dFdA = \Pr ob\{F < F_T < F + dF, A < \sigma_T < A + dA | F_t = f, \sigma_t = \alpha\} \quad (2.1),$$

we then have the following connection between the local time and the probability density:

Proposition 2.1

The expected local time at strike is related to the density probability as follows:

$$E\left[\frac{1}{2}\int_0^T 1_{F_u=k}\sigma^2(F_u,u)C^2(F_u)du\right] = \frac{1}{2}C(k)^2 \int_{u=0}^{T}\int_{\sigma=0}^{+\infty} \sigma^2 p(t, f, \alpha, u, k, \sigma)d\sigma du \quad (2.2),$$

which implies that the forward price of a call option $V(f,k,\alpha,T)$ with forward worth f today, with strike k, with volatility with boundary condition equal to α and with maturity T is equal to :

$$V(f,k,\alpha,\mathrm{T}) = (f-k)^+ + \frac{1}{2}C(k)^2 \int_{u=0}^{T}\int_{\sigma=0}^{+\infty} \sigma^2 p(t, f, \alpha, u, k, \sigma)d\sigma du \quad (2.3).$$

Proof: see 7.2. □

The equation (2.2) shows that any expected local time problem can be reformulated as a probability density problem.

3. Explicit computation of the local time for stochastic models

Computing the local time in stochastic volatility models is not an easy task in general. The above theory assumes either a closed form for the expected local time or for the probability density. But this is far from being the case for general stochastic volatility models. However, we will see that we can derive very good approximation of the solution using perturbation theory for a stochastic volatility model given by equations (1.2) and (1.3). Clearly, equation (1.2) and (1.3) encompasses most of the common models, namely:

- The seminal SABR model [5], $dF_t = \sigma_t F_t^{\beta} dW_t^F$, $d\sigma_t = \nu\sigma_t dW_t$ (3.1)
  with $C(F_t) = F_t^{\beta}$, $M(\sigma_t) = 0$, $S(\sigma_t) = \nu$ (CEV process on the underlying with a lognormal stochastic volatility).
- The Heston model: $dF_t = \sigma_t F_t dW_t^F$, $d\sigma_t^2 = (\kappa - \lambda\sigma_t^2)dt + \nu\sigma_t dW_t$) (3.2),

which can be reformulated into the framework of (1.3) by $d\sigma_t = \left(\frac{\kappa - \nu^2/4}{2\sigma_t} - \lambda\sigma_t\right)dt + \frac{\nu}{2}dW_t$ (3.3),

with $C(F_t) = F_t$, $M(\sigma_t) = \left(\frac{\kappa - \nu^2/4}{2\sigma_t^2} - \lambda\right)$, $E(\sigma_t) = \nu / 2\sigma_t$ (lognormal diffusion on the underlying with a mean reverting square root stochastic square volatility).

- Some common extensions of Heston[5]:
  - The shifted log Heston:

$$dF_t = \sigma_t(\mu F_t + (1-\mu)F_0)dW_t^F, \; d\sigma_t = \left(\frac{\kappa - \nu^2/4}{2\sigma_t} - \lambda\sigma_t\right)dt + \frac{\nu}{2}dW_t \quad (3.4).$$

  - The CEV Heston:

[5] These extensions of the Heston models have flourished in various banks as proprietary version of the Heston model, especially on foreign exchange option desks.

$$dF_t = \sigma_t F_t^{\beta} dW_t^F \ , d\sigma_t = \left(\frac{\kappa - v^2/4}{2\sigma_t} - \lambda\sigma_t \right) dt + \frac{v}{2} dW_t \quad (3.5).$$

- The Beta volatility Heston:

$$dF_t = \sigma_t F_t dW_t^F \ , \ d\sigma_t = \left(\frac{\kappa - v^2/4}{2\sigma_t} - \lambda\sigma_t \right) dt + \frac{v}{2} dW_t \quad (3.6).$$

- Combinations of the above, namely the beta volatility shifted log, the beta volatility CEV Heston.

* Older stochastic volatility models such as Stein and Stein, Hull and White.

The main result of the paper is the following. We can obtain a very good approximation of the expected local time of our general stochastic volatility model.

Proposition 3.1 The expected local time of the stochastic volatility model (3.1) can be approximated at the second order by the following expression.

$$E\left[\frac{1}{2} \int_0^T 1_{F_u = k} \sigma^2(F_u, u) C^2(F_u) du \right] = \frac{1}{2} \alpha \sqrt{C(0) C(z_0)} I^{1/2}(v z_0) e^{\frac{1}{4}\rho v \alpha b_1 z^2} \int_0^T \frac{1}{\sqrt{2\pi\tau}} e^{-x^2/\tau} e^{\kappa\tau} d\tau \quad (3.7)$$

where $I(\zeta) = \sqrt{1 - 2\rho\zeta + \zeta^2}$ and $z_0 = \frac{1}{\alpha} \int_k^f \frac{df'}{C(f')}$ (3.8),

and $\kappa = v^2 \left(\frac{1}{4} I''(\varepsilon v z_0) I(\varepsilon v z_0) - \frac{1}{8} [I'(\varepsilon v z_0)]^2 + \alpha^2 \left(\frac{1}{4} b_2 - \frac{3}{8} b_1^2 \right) + \frac{3}{4} \rho v \alpha b_1 \right)$ (3.9).

Proof: see 7.3. □

In the particular case of a SABR model, this leads to the following approximation

Proposition 3.2 The forward value of a call option is given in a SABR model by

$$SABR\_Call(f, k, T, \alpha, \beta, \rho, v) = (f - k)^+ + \frac{f - k}{2x\sqrt{2\pi}} e^{\theta} \int_0^T \frac{e^{-\frac{x^2}{2u} + \kappa u}}{\sqrt{u}} du \quad (3.10),$$

with

$$\theta = \frac{1}{4} \rho v \alpha b_1 z^2 + Log\left(\frac{\alpha (fk)^{\frac{\beta}{2}} z}{f - k} \right) + Log\left(\frac{x}{z} \left(1 - 2v\rho z + v^2 z^2\right)^{\frac{1}{4}} \right) \quad (3.11),$$

$$\kappa = \frac{1}{8} \left(\frac{\alpha^2 (\beta - 2) \beta k^{2\beta}}{\left(k - \alpha(\beta - 1) k^{\beta} z_0\right)^2} + \frac{6\alpha\beta k^{\beta} v \rho}{k - \alpha(\beta - 1) k^{\beta} z_0} + \frac{v^2 \left(2 - 3\rho^2 + 2v\rho z_0 - v^2 z_0^{\ 2}\right)}{1 - 2v\rho z_0 + v^2 z_0^{\ 2}} \right) \quad (3.12),$$

$$x = \frac{1}{v} Log\left(\frac{-\rho + vz + \sqrt{1 - 2v\rho z + v^2 z^2}}{1 - \rho} \right) \quad (3.13).$$

$$z = \frac{f^{1-\beta} - k^{1-\beta}}{\alpha(1 - \beta)} \qquad z_0 = \frac{f_0^{\ 1-\beta} - k^{1-\beta}}{\alpha(1 - \beta)} \quad (3.14)$$

$$b_1 = \frac{\beta}{\alpha(1 - \beta) z_0 + k^{1-\beta}} \quad (3.15)$$

where f_0 is the effective projected forward. It can safely be put to the strike up to the order of approximation we are looking at. An explanation of its role is given in the appendix section.
Proof: see 7.4. □

Compared to the final result of [5] (2.17a), the solution is more accurate as it does not involve the last approximation on the integral term $\int_0^T \frac{e^{-\frac{x^2}{2u}+\kappa u}}{\sqrt{u}} du$ as in (A.48c) where it is approximated as $\int_0^T \frac{e^{-\frac{x^2}{2u}}}{\sqrt{u}\left(1-\frac{2}{3}\kappa u\right)^{3/2}} du$. This approximation is clearly very off in general. We will see now how to efficiently compute this integral.

4. Fast computation of the SABR Stochastic integral and connection with Black Scholes

The SABR standard integral given by $\int_0^T \frac{e^{-\frac{x^2}{2u}+\kappa u}}{\sqrt{u}} du$ can be computed as follows:

$$\int_0^T \frac{e^{-\frac{x^2}{2u}+\kappa u}}{\sqrt{u}} du = \frac{\sqrt{\pi}}{2i\sqrt{\kappa}}\left[e^{i\sqrt{2}x\sqrt{\kappa}}\left(erf\left(\frac{x}{\sqrt{2T}}+i\sqrt{\kappa T}\right)-1\right)+e^{-i\sqrt{2}x\sqrt{\kappa}}\left(erf\left(\frac{-x}{\sqrt{2T}}+i\sqrt{\kappa T}\right)+1\right)\right] \quad (4.1)$$

with the additional formula for the complex error function

$$\begin{aligned} erf(x+iy) &= erf(x) + \frac{e^{-x^2}}{2\pi x}\left[1-\cos(2xy)+i\sin(2xy)\right] \\ &+ \frac{2}{\pi}e^{-x^2}\sum_{n=1}^{\infty}\frac{e^{-\frac{1}{4}n^2}}{n^2+4x^2}\left\{2x-2x\cosh(ny)\cos(2xy)+n\sinh(ny)\sin(2xy)+i\left(2x\cosh(ny)\sin(2xy)+n\sinh(ny)\cos(2xy)\right)\right\} \end{aligned} \quad (4.2)$$

Proof: see 7.5. □

The last series converges very quickly (5 terms is in general enough for a good precision).

Connection with Black Scholes

Identifying the local time component both in Black Scholes and SABR, we can find a relationship between SABR and Black Scholes. Namely, we can prove that the SABR model can be seen as an extension of the Black Scholes formula in the complex filed as proven by proposition 3.2

Proposition 3.2

The SABR model is equivalent to a Black Scholes models with the following parameters (complex numbers potentially, but the price remains real):

$$\tilde{f} = \frac{x\sqrt{-2\kappa}e^{\pm x\left(\sqrt{-\frac{\kappa}{2}}+\sqrt{-2\kappa}\right)-\theta}}{\left(e^{\pm x\sqrt{-2\kappa}}-1\right)} x\sqrt{-2\kappa}e^{x\sqrt{-\frac{\kappa}{2}}-\theta} \quad (4.3)$$

$$\tilde{k} = \frac{x\sqrt{-2\kappa}e^{\pm x\sqrt{-\frac{\kappa}{2}}-\theta}}{\left(e^{\pm x\sqrt{-2\kappa}}-1\right)} x\sqrt{-2\kappa}e^{x\sqrt{-\frac{\kappa}{2}}-\theta} \quad (4.4)$$

$$\tilde{\alpha} = \sqrt{-2\kappa} \quad (4.5)$$

$$SABR\_Call(f,k,T,\alpha,\beta,\rho,v) = BS\_Call\left(\tilde{f},\tilde{k},\tilde{\alpha},T\right) \quad (4.6)$$

Proof: see 7.6. □

Compared to the Hagan et al. formula, described in [5], the above computation of the SABR model requires to compute the Black Scholes formula with the complex error function $erf(x+iy)$ using the remark (4.1) and (4.2).

Quality of our approximation and numerical examples

Our formula is addressing one of the key issues of the original Hagan formula, namely, its poor quality for long approximation, like for instance 20 years. More precisely, for long term maturities, the original Hagan formula breaks downs for low strike as we are far away from the moneyness and hence cannot apply the approximation. In contrast, our formula is more robust for long term maturities. This is because the integration does not need the last approximation given by the formula (A.48c) in the original article of Hagan, where the integral term $\int_0^T \frac{e^{-\frac{x^2}{2u}}}{\sqrt{u}\left(1-\frac{2}{3}\kappa u\right)^{3/2}} du$ is approximated by $\int_0^T \frac{e^{-\frac{x^2}{2u}+\kappa u}}{\sqrt{u}} du$. This approximation is clearly very off in general.

To illustrate our point, we took the following numerical example where the spot value of the Libor is 0.05 and other parameters are beta=0.7, rho= -0.5, nu=0.2 and we display the graphics of the digital call given by our formula and the one of Hagan. We know that the price of the digital call should converge to one for low strike. This is obviously the case for the Black Scholes formula in the complex field but not for the original Hagan formula as shown by figure 1.

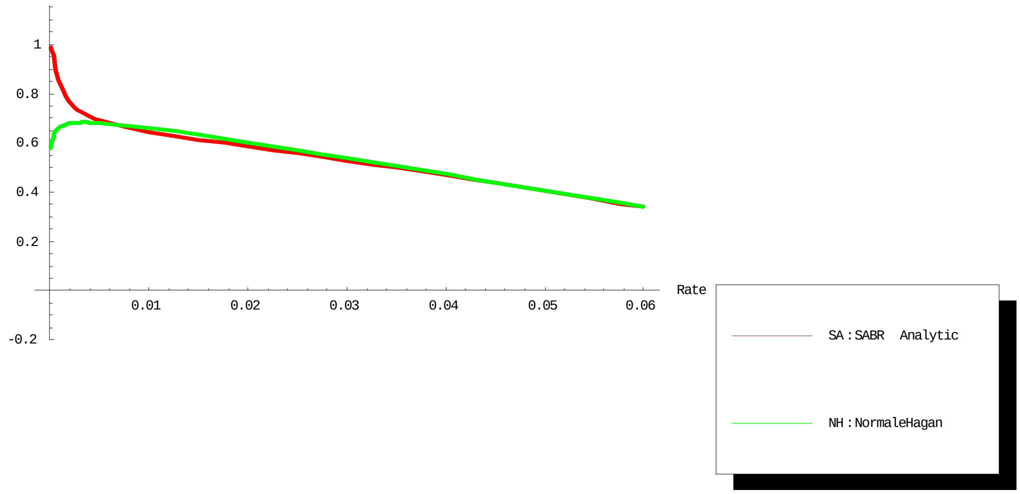


FIGURE 1 : Price of a digital call the original Hagan formula and in original Hagan and our SABR analytical formula for alpha= 0.11

And not surprisingly, higher at-the-money volatility makes the things worse as shown by figure 2

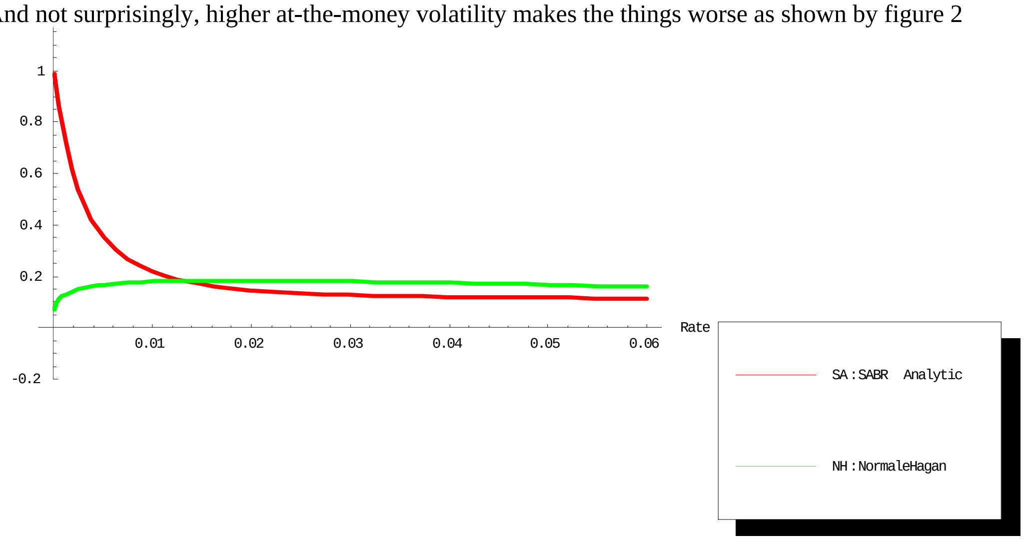


FIGURE 2 : Price of a digital call the original Hagan formula and in original Hagan and our SABR analytical formula for alpha= 0.3

An even better insight is given by the implicit probability density function. This implicit probability density function is computed from the digital price as simply the opposite of its derivatives with respect to strike. With the preceding example parameters the density associated with Normal Hagan's formula is negative for strike below 0.5%, which is clearly a non desirable feature.

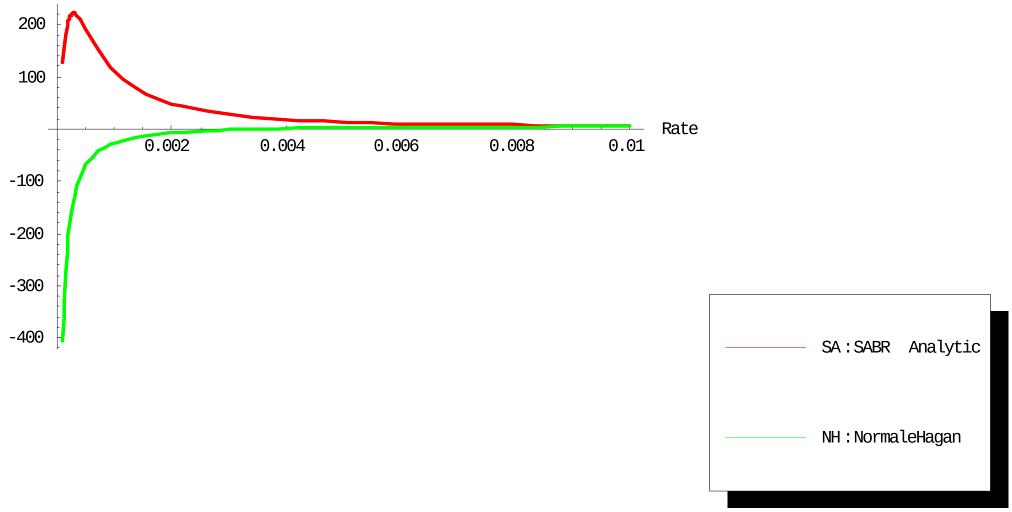


FIGURE 3: Implicit density in the original Hagan formula and our SABR analytical formula for alpha= 0.1

5. Application to CMS replication pricing

CMS can be replicated as explained in [3]. Namely, a payoff which is a function of an underlying market on which options are available can be replicated with these options. If F_T is the value of the underlying market at maturity T, and if $f(F_T)$ is the payoff to replicate, the replication formula says that the price of the European derivative is simply an (infinite) linear combination of call and puts payoff:

$$f(F_T) = f(\kappa) + f'(\kappa)((F_T - \kappa)^+ - (\kappa - F_T)^+) + \int_0^\kappa f''(K)(\kappa - F_T)^+ dK + \int_\kappa^\infty f''(K)(F_T - \kappa)^+ dK \quad \textbf{(5.1)}$$

And if we select a value of κ such that $f'(\kappa) = 0$ and take the expectation of both members of this stochastic equation, we get the weightings of these calls and puts as follows:

$$E\left[f(F_T)\right] = f(\kappa) + \int_0^\kappa f''(K)Put(K,T)dK + \int_\kappa^\infty f''(K)Call(K,T)dK \quad \textbf{(5.2)}$$

This formula tells us that from the price of vanilla options (call and put option), one can compute the arbitrage-free price of any European payoff that delivers $f(F_T)$ at maturity.

We can apply this result to a security that pays a swap rate of maturity M at the date T. By means of the change of numeraire formula, we know that the forward price of such contract can be easily computed under its natural numeraire measure associated with the annuity. Recall that the annuity is given by:

$$A_T(T) = \sum_{i=0}^{n-1} \delta_i B_{T_i}(T) \quad \textbf{(5.3)}$$

So the forward price is given by:

$E^{Q_T}\left[s_{CMS}\right] = E^{Q_A}\left[s_{CMS}\frac{dQ_T}{dQ_A}\right] = E^{Q_A}\left[s_{CMS}\frac{B_T(t)/B_T(0)}{A_T(t)/A_T(0)}\right]$, and if we designate by $r_T(t)$ the zero coupon rate of maturity T at the date t, we can reformulate the expected price as follows :

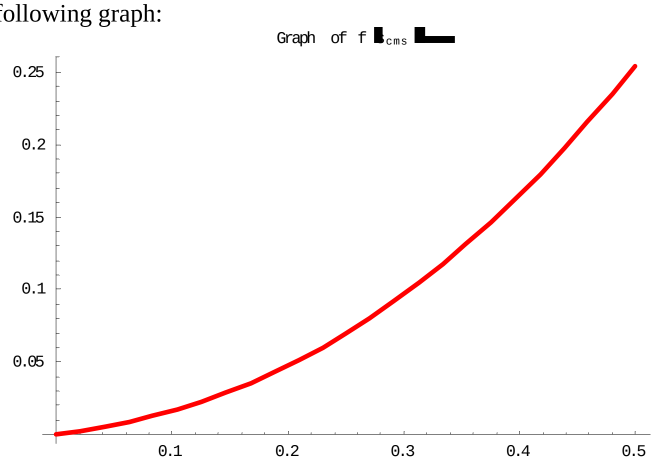
$$E^{Q_T}\left[s_{CMS}\right] = \frac{A_T(0)}{B_T(0)} E^{Q_A}\left[\frac{s_{CMS}(t)}{\sum_{i=0}^{n-1} \delta_i (1 + r_{T-T_i}(t))^{T_i - T}}\right] \quad \textbf{(5.4)}$$

Obviously, we need a model able to prices all forward rates and the swap rate. But, as a first cut, we can appreciate the influence of the volatility on the pricing of such non linear functional and say with a good approximation that all rates have a correlation close to 1. So we introduce the following common approximation:

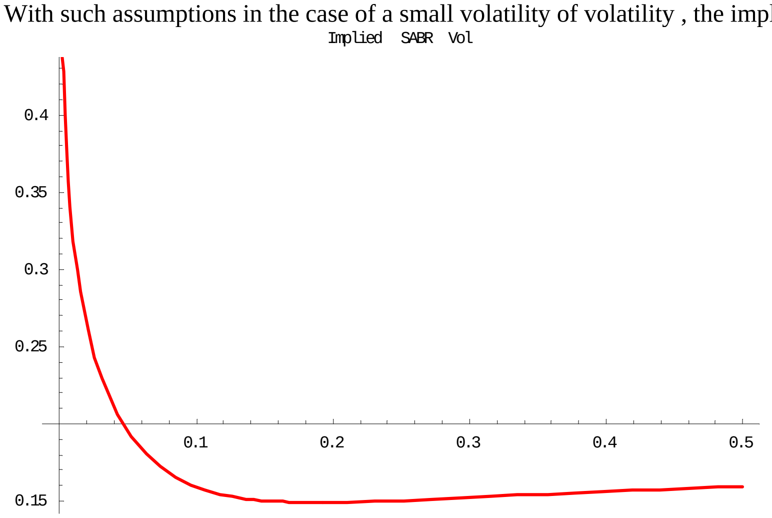
$$E^{Q_T}\left[s_{CMS}\right] \approx \frac{A_T(0)}{B_T(0)} E^{Q_A}\left[\frac{s_{CMS}(t)}{\sum_{i=0}^{n-1} \delta_i (1+s_{CMS}(t))^{T_i - T}}\right] \quad \textbf{(5.5)}$$

We see now how we are going to apply the replication algorithm described in the beginning of the paragraph.

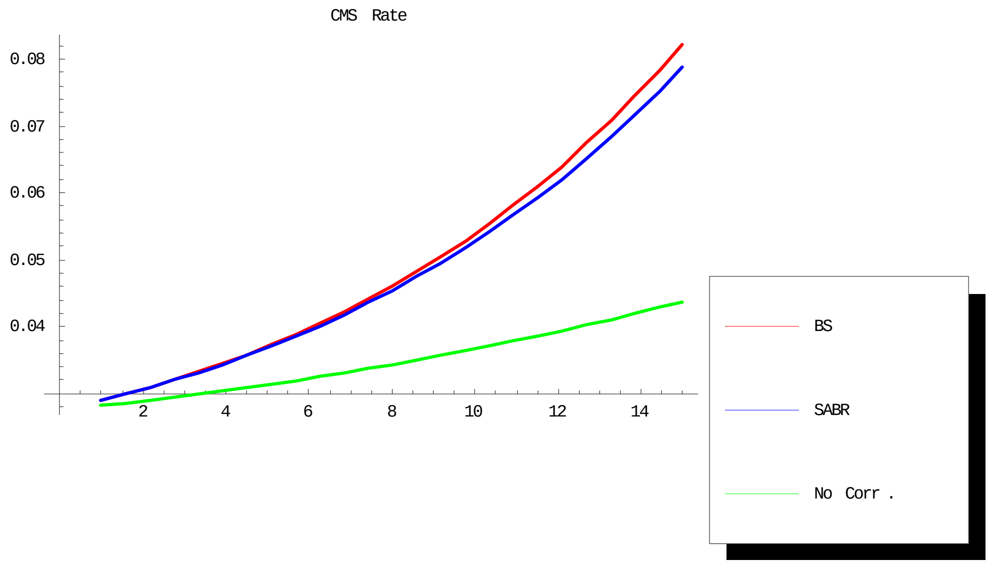
We replicate the function $f(s_{CMS}) = \dfrac{s_{CMS}(t)}{\sum_{i=0}^{n-1} \delta_i (1+s_{CMS}(t))^{T_i - T}}$ which has a positive convexity as we can see it on the following graph:



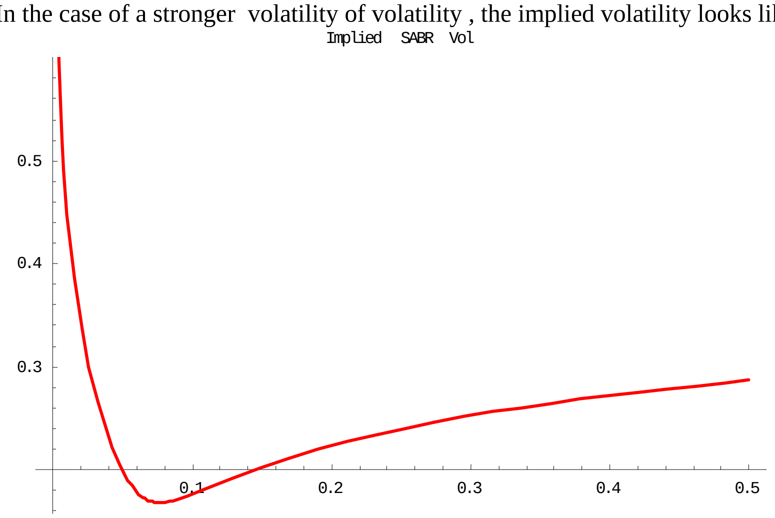
We want to compare the result of such replication for the SABR model and the Black and Scholes model. It is obvious that the parameters of the SABR model do depend on the maturity. This is checked by practitioners and comes from the fact that the SABR models assume a lognormal process for the volatility and the evolution of the market prices for the smile is more consistent with a stationary character of the volatility process. Therefore we use a black and sholes curve for the BS model calibrated at the money with the SABR model.
With such assumptions in the case of a small volatility of volatility , the implied volatility looks like



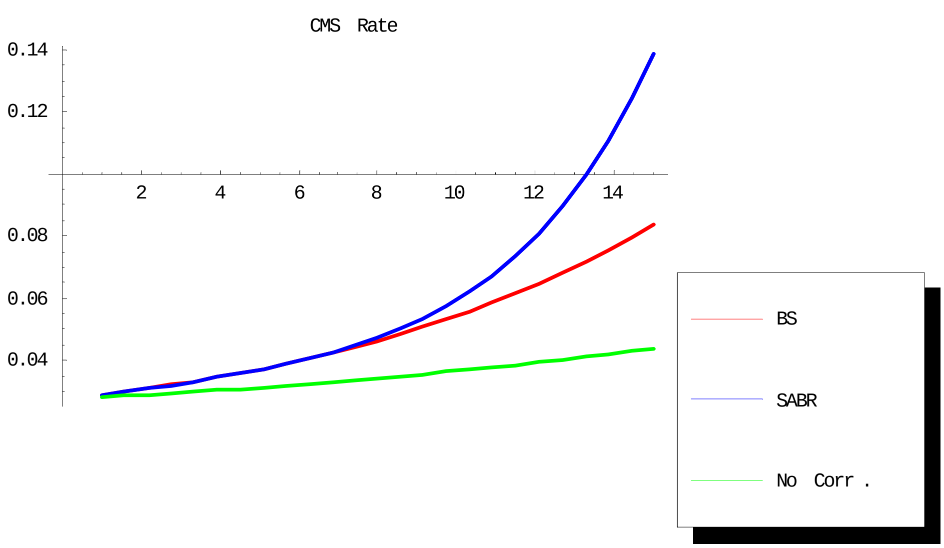
the prices look like:



Where we observe that the influence of the smile on the CMS price is weak.
In the case of a stronger volatility of volatility , the implied volatility looks like



the prices look like:



Where we see that the influence of the volatility of volatility is important after 7 years. After 15 years, the SABR computations become inexact and the explosion of the difference that we observe is inaccurate.

6. Spread option with Copula

We assume that X and Y are to random variables with a SABR distribution (at a maturity T and different parameters px and py).
The codependence structure can be defined using historical values or using risk neutral considerations.
Let be respectively D_X and D_Y the cumulative distributions of respectively X and Y.
Let be N the cumulative distribution of a standard Gaussian random variable.
We know that $D_X(X)$ and $D_Y(Y)$ are uniform density law on [0,1], and if we compose by N^{-1} we get :
$N^{-1}\circ D_X(X)$ and $N^{-1}\circ D_Y(Y)$ are Gaussian random variables which may have very complicated codependence structure. We mean by this statement that as a bidimensional variable , (X,Y) it is very likely distributed as a non Gaussian variable but with Gaussian marginals.
We are making here a drastic hypothesis: we model the codependence by postulating that the distribution of (X,Y) is Gaussian, with a correlation ρ . With this hypothesis, we can compute the price of any cashflow f(X,Y) by :

$$Call = E\left[f(X,Y)\right] = \iint dxdy\cdot f(D_X{}^{-1}\circ N(x), D_Y{}^{-1}\circ N(y))l(x,y,\rho) \tag{6.1}$$

where

$$l(x,y,\rho) = \frac{1}{2\pi t\sqrt{1-\rho^2}} e^{-\frac{1}{2t(1-\rho^2)}\left\{x^2+y^2-2\rho xy\right\}} \tag{6.2}$$

is the bidimensional standard correlated Gaussian density
An inconvenience of this above formula is the need of a wide range of integration points. To perform the numerical integration, we need indeed to account for points that can be far from the origin, depending on the correlation. When rho is close to one , this can be quite far away points. This is in particular the case of underlyings like LIBOR or CMS rate. To get rid of this problem and to make the handling of x and y uniform, we can make the following change of variable:

$$\begin{matrix} x' = \dfrac{x-\rho y}{\sqrt{1-\rho^2}} \\ y' = y' \end{matrix} \Leftrightarrow \begin{matrix} x = x'\sqrt{1-\rho^2} + y'\rho \\ y = y' \end{matrix} \tag{6.3}$$

We then get

$$l(x,y,\rho)dxdy = \frac{1}{2\pi t} e^{-\frac{1}{2t}\left\{x'^2+y'^2\right\}} dx'dy' = l(x',y')dx'dy' \tag{6.4}$$

What is interesting in the last formula is that we get rid of the dependency on rho. So by renormalizing the y axis, we handle the correlation and we get :

$$Call = E\left[f(X,Y)\right] = \iint dxdy\cdot f(D_X{}^{-1}\circ N(x\sqrt{1-\rho^2}+y\rho), D_Y{}^{-1}\circ N(y))l(x,y) \tag{6.5}$$

We see that the important thing here is to be able to compute $D_X{}^{-1}$ and $D_Y{}^{-1}$, and the range of values on which we should be able to get a value is more explicit on (3.20). It just requires being able to compute $D_X{}^{-1}$ for arguments close to 0 and 1 but not too close. The trick here is to use some kind of importance sampling through a optimal system of roots and the corresponding weights. We compute numerically the values of the digitals by
$D_X(f) \approx \frac{1}{2\varepsilon}(SABR(K+\varepsilon) - SABR(K-\varepsilon))$.Because a numeric derivative is simpler to obtain than a full analytical derivative. Then by using a numerical inversion algorithm (Brent or Newton-Raphson) we can obtain an algorithm able to compute $D_X{}^{-1}$. The final numerical integral is to be done with a Legendre Gaussian integral method. 50 points are usually enough to get prices with 4 digits accuracy.

In order to perform the integration in good conditions we need to be able to compute good prices for very low and very large prices. For certain set of parameters we know that our approximate formula is breaking, it is therefore

important to have an extrapolation framework that do not introduce to much error in the integration process. A good idea is to perform this extrapolation in the Gaussian space, that means to extrapolate the function $D_X^{-1} \circ N$.
To have a good feeling about how to perform such thing, it is useful to realize that if X is Gaussian , then $D_X^{-1} \circ N$ should be a linear function and if X is lognormal then $D_X^{-1} \circ N$ is an exponential function.

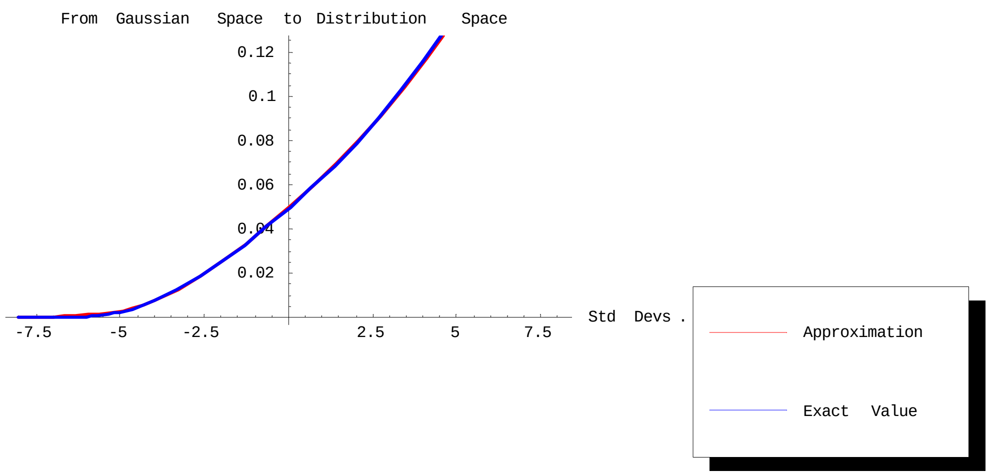
For the extrapolation in the direction of the low strike, we have no choice but to use a schema that guarantee the variable to stay positive. So we decide to use:

$$f(x) = f(x_0) e^{\frac{f'(x_0)}{f(x_0)}(x - x_0)} \qquad (6.6)$$

For the extrapolation in the direction of high strike we have the choice to use the same schema or to use a linear continuation like:

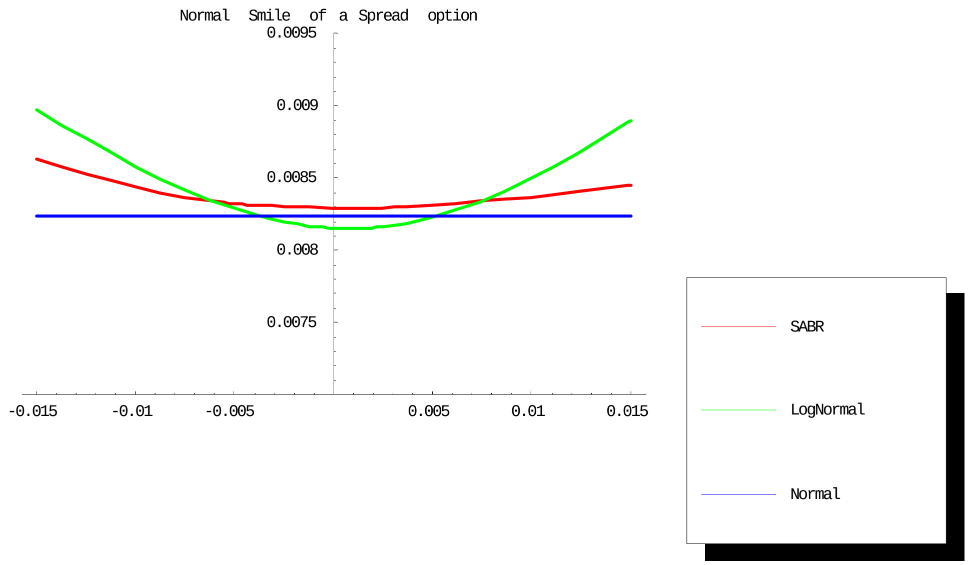
$$f(x) = f(x_0) + f'(x_0)(x - x_0) \qquad (6.7)$$

We assume that the strikes that interest us are far from these transition values (usually, 2-4 standard deviations, depending on the searched accuracy of the computations). To get a good sense how good these approximations are, we can compare approximated values computed from a knowledge of the function on a limited interval (here 4 standard deviations) to the true function value.

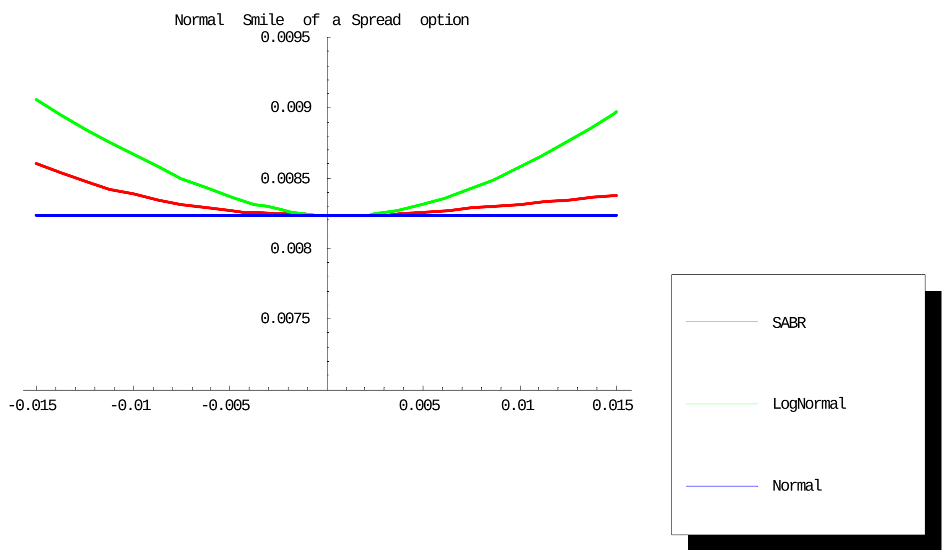


Parameters: beta=0.7,rho=-0.5,nu=0.2,T=1 year

We will also compute the price of a spread options with a model assuming for each underlying a lognormal diffusion and a correlation between the Brownian motions. In order to compare the prices, we need to have a kind of “neutral” framework . If we use a normal approximation for the spread, we can compute implicit normal volatilities for the lognormal model and the SABR+ Copula model. Doing so, we visualize the normal smile:



We see that even at the money prices do not coincide. This is understandable because the correlation assumed to be the same in the 3 models does not play the same role in those 3 models. But it makes the interpretation of such graph difficult.
A better idea is to fit separately the 3 models at the money and find an implicit correlation for each of the 3 models separately. We get the following graph



We see that the spread option smile for the SABR model lies between lognormal and normal. This makes sense as our beta, equals to 0.7 lies between the lognormal case (beta=1) and the normal one (beta=0). This implies in particular that the copula approach has entitled us to transfer the smile from the underlying to the spread option in a consistent way.

7. Conclusion

In this paper, we show that the local time formulation of the vanilla option problem enables us to relate the SABR model to the Black Scholes one. After providing a closed form approximation for the local time problem in SABR, we explain how to numerically compute the complex error function $erf(x+iy)$ using fast convergent series. This "complex Black Scholes" formula turns out to be more robust for low and high strike. This is very useful for CMS replication and spread option pricing using copula.

8. References

[1] Abramowitz, M. & Stegun, I.A. “Handbook of Mathematical Functions. Dover”, New York, 1964.

[2] Amblard G., Lebuchoux J. (2000) “Models for CMS caps”, Euro Derivatives, Risk publication.

[3] Carr P, Jarrow R.A., “The Stop-Loss Start-Gain Paradox and Option Valuation - A New Decomposition into Intrinsic and Time Value”, Cornell University, Review Of Financial Studies , 3, 469-492, 1990.

[4] Dybvig P, “Inefficient Dynamic Portfolio Strategies or How to Throw Away a Million Dollars in the Stock Market” Review of Financial Studies, 1(1), 1988.

[5] Hagan, P., D. Kumar, A. Lesniewski, and D. Woodward. “Managing Smile Risk”. Wilmott magazine, 84–108, September 2002.

[6] Meyer, P. “Un Cours sur les Intégrales stochastiques”, Séminaire de Probabilités X, Lecture Notes inMathematics, 511, Springer-Verlag, Berlin, 1976.

[7] W. H. Press, S. A. Teukolsky, W. T. Vetterling, B. P. Flannery, “Numerical Recipes in C++” Second edition, Cambridge University Press 2002 Printing

[8] Omberg, E., “Binary Trading Strategies, Brownian Local Time, and the Tanaka Formulas,”
working paper, Santa Clara University, 1988.

[9] Yor M. “Rappels et Préliminaires Généraux” in: Temps Locaux, Société Mathématique de France, Astérisque, 52-53, 17-22, 1978

9. Appendix

Proof 7.1: rather than computing the expected local time for a geometric Brownian motion, one can notice the following vega decomposition of a call option:

$$BS\_Call(f,k,\alpha,T)=\int_0^{\alpha}\frac{\partial}{\partial\sigma}BS\_Call(f,k,\sigma,T)d\sigma+(f-k)^{+} \qquad (7.1.1)$$

which can be re-expressed when plugging the vega formula for call option as

$$BS\_Call(f,k,\alpha,T)=\int_0^{\alpha}k\sqrt{T}N'(d_2(\sigma))d\sigma+(f-k)^{+} \qquad (7.1.2)$$

where $N'(x)=\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$ is the standard normal density and $d_2(\sigma)=\frac{\log(f/k)}{\sigma\sqrt{T}}-\frac{1}{2}\sigma\sqrt{T}$. The result (1.6) is then trivially obtained by doing a change of variable that interchanges time u and volatility σ:

$$u=\frac{\sigma^2T}{\alpha^2} \qquad (7.1.3).\square$$

Proof 7.2: the result (2.2) is immediate when using conditional expectation and the definition of the transition probability:

$$E\left[\frac{1}{2}\int_0^T 1_{F_u=k}\sigma^2(F_u,u)C^2(F_u)du\right]=E\left[E\left[\frac{1}{2}\int_0^T 1_{F_u=k}\sigma^2C^2(F_u)du\middle|\sigma\right]\right] \qquad (7.2.1).\square$$

Proof 7.3: Let us denote by $P(t,f,\alpha,u,k)$ the expected average squared volatility at time u for a conditional level of the underlying k given by

$$P(t,f,\alpha,u,k)=\int_{\sigma=0}^{+\infty}\sigma^2p(t,f,\alpha,u,k,\sigma)d\sigma \qquad (7.3.1).$$

If we solve the problem for the expected average square volatility, we get as well the one for the expected local time since the expected local time is just the time average of the expected average square volatility over the option's life:

$$E\left[\frac{1}{2}\int_0^T 1_{F_u=k}\sigma^2(F_u,u)C^2(F_u)du\right]=\frac{1}{2}C(k)^2\int_{u=0}^{T}P(t,f,\alpha,u,k)du \qquad (7.3.2).$$

As a function of time t,f,α, the expected average square volatility satisfies the backward Kolmogorov equation given by:

$$P_t+\frac{1}{2}\alpha^2C(f)^2P_{ff}+\rho\alpha^2S(\alpha)C(f)P_{f\alpha}+\frac{1}{2}\alpha^2S(\alpha)^2P_{\alpha\alpha}+\alpha M(\alpha)P_\alpha=0, \qquad (7.3.3),$$

with the boundary condition given by $P=\alpha^2\delta(f-k)$ for $t=u$ (7.3.4).

Equation (7.3.3) with boundary (7.4.4) cannot be solved in general. However, using singular perturbation technique, we will show how to compute an approximated solution. The proof goes along the same line as in [5]. Namely, we use singular perturbation techniques around the non stochastic volatility models. Compared to [5], the problem has additional terms coming from both the mean reverting drift of the stochastic volatility and the more general form for the volatility of volatility. Our perturbed diffusion will therefore be written as

$$\begin{aligned} dF_t &= \varepsilon\sigma_tC(F_t)dW_t^F \quad \text{with } F_0=f \\ d\sigma_t &= \varepsilon\sigma_t\left(M(\sigma_t)dt+S(\sigma_t)dW_t^\sigma\right) \quad \text{with } \sigma_0=\alpha \end{aligned} \qquad (7.3.5),$$

where ε is supposed to be small. The backward Kolmogorov equation (7.3.3) is then changed into

$$P_t + \frac{1}{2}\varepsilon^2\alpha^2 C(f)^2 P_{ff} + \rho\varepsilon^2\alpha^2 S(\alpha)C(f)P_{f\alpha} + \frac{1}{2}\varepsilon^2\alpha^2 S(\alpha)^2 P_{\alpha\alpha} + \varepsilon\alpha M(\alpha)P_\alpha = 0, \quad (7.3.6),$$

with the boundary condition given by $P = \alpha^2\delta(f-k)$ for $t = u$ (7.3.7).

At the first order, the solution for (7.3.6) with boundary condition (7.3.7) should be given by a Gaussian process:

$$P = \frac{\alpha}{\sqrt{2\pi\varepsilon^2 C^2(k)\tau}}\exp\left\{\frac{-(f-k)^2}{2\alpha^2\varepsilon^2 C^2(k)\tau}(1+\ldots)\right\} \quad (7.3.8),$$

with $\tau = u - t$. We will hence expand around this solution. We will first change the terminal condition into an initial condition with the change of time variable given by $\tau = u - t$. Then, a second simple transformation is to renormalize by the volatility. It is given by $z = \frac{1}{\varepsilon\alpha}\int_k^f \frac{df'}{C(f')}$ (7.3.9),

with the additional convention that $C(f) = B(\varepsilon\alpha f)$. Then, we have $\frac{\partial}{\partial f} = \frac{\partial z}{\partial f}\frac{\partial}{\partial z} = \frac{1}{\varepsilon\alpha B(\varepsilon\alpha z)}\frac{\partial}{\partial z}$ while

$$\begin{aligned}
&\frac{\partial}{\partial\alpha} = \frac{\partial}{\partial\alpha} - \frac{z}{\alpha}\frac{\partial}{\partial z}, \qquad \frac{\partial^2}{\partial f^2} \to \frac{1}{\varepsilon^2\alpha^2 B^2(\varepsilon\alpha z)}\left\{\frac{\partial^2}{\partial z^2} - \varepsilon a\frac{\alpha B'(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}\frac{\partial}{\partial z}\right\}, \\
&\frac{\partial^2}{\partial f\partial\alpha} \to \frac{1}{\varepsilon\alpha B(\varepsilon\alpha z)}\left\{\frac{\partial^2}{\partial z\partial\alpha} - \frac{z}{\alpha}\frac{\partial^2}{\partial z^2} - \frac{1}{\alpha}\frac{\partial}{\partial z}\right\}, \qquad \frac{\partial^2}{\partial\alpha^2} \to \frac{\partial^2}{\partial\alpha^2} - \frac{2z}{\alpha}\frac{\partial^2}{\partial z\partial\alpha} + \frac{z^2}{\alpha^2}\frac{\partial^2}{\partial z^2} + \frac{2z}{\alpha^2}\frac{\partial}{\partial z}
\end{aligned} \quad (7.3.10),$$

with the additional boundary condition given by $\delta(f-k) = \delta(\varepsilon\alpha z C(k)) = \frac{1}{\varepsilon\alpha C(k)}\delta(z)$. (7.3.6) then becomes

$$\begin{aligned}
P_\tau = &\frac{1}{2}\left(1 - 2\varepsilon\rho S z + \varepsilon^2 S^2 z^2\right)P_{zz} - \frac{1}{2}\varepsilon\alpha\frac{B'}{B}P_z + \left(\varepsilon\rho S - \varepsilon^2 S^2 z\right)\left(\alpha P_{z\alpha} - P_z\right) \\
&+ \frac{1}{2}\varepsilon^2\alpha^2 S^2 P_{\alpha\alpha} + \varepsilon M\left(\alpha P_\alpha - zP_z\right)
\end{aligned}, \text{ for } \tau \ge 0 \quad (7.3.11),$$

with the boundary solution given by $P = \frac{\alpha}{\varepsilon C(k)}\delta(z)$ (7.3.12).

Like in [5], we can renormalize our target function and define $P = \frac{\alpha}{\varepsilon C(k)}\hat{P}(z)$ (7.3.13).

This change of function enables to get a real Dirac initial condition. In terms of $\hat{P}$, we get

$$\begin{aligned}
&P_z = \frac{\alpha}{\varepsilon C(k)}\hat{P}_z, \qquad P_{zz} = \frac{\alpha}{\varepsilon C(k)}\hat{P}_{zz}, \qquad P_\alpha = \frac{\alpha}{\varepsilon C(k)}\left(\hat{P}_\alpha + \frac{\hat{P}}{\alpha}\right), \qquad P_{\alpha z} = \frac{\alpha}{\varepsilon C(k)}\left(\hat{P}_{\alpha z} + \frac{\hat{P}_z}{\alpha}\right), \\
&P_{\alpha\alpha} = \frac{\alpha}{\varepsilon C(k)}\left(\hat{P}_{\alpha\alpha} + \frac{2\hat{P}_a}{\alpha}\right)
\end{aligned} \quad (7.3.14).$$

(7.3.11) then becomes

$$\begin{aligned}
\hat{P}_\tau = &\frac{1}{2}\left(1 - 2\varepsilon\rho S z + \varepsilon^2 S^2 z^2\right)\hat{P}_{zz} - \frac{1}{2}\varepsilon\alpha\frac{B'}{B}\hat{P}_z + \left(\varepsilon\rho S - \varepsilon^2 S^2 z\right)\alpha\hat{P}_{z\alpha} + \varepsilon M\left(\alpha\hat{P}_\alpha + \hat{P} - z\hat{P}_z\right) \\
&+ \frac{1}{2}\varepsilon^2\alpha^2 S^2\left(\hat{P}_{\alpha\alpha} + \frac{2\hat{P}_\alpha}{\alpha}\right)
\end{aligned} \quad (7.3.15).$$

To the leading order, (7.3.15) is solution of the standard Heat equation $\hat{P}_\tau = \frac{1}{2}\hat{P}_{zz}$, with an initial Dirac condition. Hence, the solution does only depend in α at the order the order $O(\varepsilon)$. The same applies for its derivatives. At the order $O(\varepsilon^3)$, the equation (7.3.15) is equivalent to the simpler one:

$$\hat{P}_\tau = \frac{1}{2}\left(1 - 2\varepsilon\rho S z + \varepsilon^2 S^2 z^2\right)\hat{P}_{zz} - \frac{1}{2}\varepsilon\alpha\frac{B'}{B}\hat{P}_z + \varepsilon\rho S\alpha\hat{P}_{z\alpha} + \varepsilon M\left(\alpha\hat{P}_\alpha + \hat{P} - z\hat{P}_z\right) \quad (7.3.16),$$

with the initial condition $\hat{P} = \delta(z)$ at $\tau = 0$ (7.3.17).

Let us now make this equation only one dimensional. Let us define $H(\tau, z, \alpha)$ exactly like in [5]:

$$\hat{P} = \sqrt{C(f)/C(k)}H = \sqrt{B(\varepsilon\alpha z)/B(0)}H \qquad (7.3.18).$$

Then

$$\hat{P}_z = \sqrt{B(\varepsilon\alpha z)/B(0)}\left\{H_z + \frac{1}{2}\varepsilon\alpha\frac{B'}{B}H\right\}, \qquad \hat{P}_\alpha = \sqrt{B(\varepsilon\alpha z)/B(0)}\left\{H_\alpha + \frac{1}{2}\varepsilon z\frac{B'}{B}H\right\}$$

$$\hat{P}_{zz} = \sqrt{B(\varepsilon\alpha z)/B(0)}\left\{H_{zz} + \varepsilon\alpha\frac{B'}{B}H_z + \varepsilon^2\alpha^2\left[\frac{B''}{2B} - \frac{B'^2}{4B^2}\right]H\right\}, \qquad (7.3.19). \text{ And}$$

$$\hat{P}_{z\alpha} = \sqrt{B(\varepsilon\alpha z)/B(0)}\left\{H_{z\alpha} + \frac{1}{2}\varepsilon z\frac{B'}{B}H_z + \frac{1}{2}\varepsilon\alpha\frac{B'}{B}H_\alpha + \frac{1}{2}\varepsilon\frac{B'}{B}H + O(\varepsilon^2)\right\}$$

(7.3.16) is modified into

$$\begin{aligned} H_\tau = {} & \frac{1}{2}\left(1 - 2\varepsilon\rho Sz + \varepsilon^2 S^2 z^2\right)H_{zz} - \left\{\frac{1}{2}\varepsilon^2\rho Sz\alpha\frac{B'}{B} + \varepsilon Mz\right\}H_z \\ & + \left(\varepsilon^2\alpha^2\left(\frac{B''}{4B} - \frac{3}{8}\frac{B'^2}{B^2}\right) + \frac{1}{2}\varepsilon^2\rho S\alpha\frac{B'}{B} + \varepsilon M\right)H + \varepsilon\rho S\alpha H_{z\alpha} + \varepsilon\left\{\rho S\alpha\left(\frac{1}{2}\varepsilon\alpha\frac{B'}{B}\right) + \alpha M\right\}H_\alpha \end{aligned} \qquad (7.3.20).$$

At the leading order, equation (7.3.20) does not depend in α. Hence the derivatives of the function $H(\tau, z, \alpha)$ are no larger than $O(\varepsilon)$ and the equation (7.3.20) can be simplified at the order $O(\varepsilon^3)$ into

$$\begin{aligned} H_\tau = {} & \frac{1}{2}\left(1 - 2\varepsilon\rho Sz + \varepsilon^2 S^2 z^2\right)H_{zz} - \left\{\frac{1}{2}\varepsilon^2\rho Sz\alpha\frac{B'}{B} + \varepsilon Mz\right\}H_z \\ & + \left(\varepsilon^2\alpha^2\left(\frac{B''}{4B} - \frac{3}{8}\frac{B'^2}{B^2}\right) + \frac{1}{2}\varepsilon^2\rho S\alpha\frac{B'}{B} + \varepsilon M\right)H + \varepsilon\rho S\alpha H_{z\alpha} + \varepsilon\alpha M H_\alpha \end{aligned} \qquad (7.3.21).$$

At order $O(\varepsilon^2)$, equation (7.3.20) is equal to $$H_\tau = \frac{1}{2}(1 - 2\varepsilon\rho Sz)H_{zz} - \varepsilon Mz H_z \qquad (7.3.22).$$

To the leading order, $M(\sigma_t)$ and $S(\sigma_t)$ can be taken as constant since $\sigma_t = \sigma_0 + \varepsilon\int_0^t d\sigma_t$ and we can make the following approximation:

$$M(\sigma_t) = M(\sigma_0) \text{ and } S(\sigma_t) = S(\sigma_0) \qquad (7.3.23).$$

Hence to the order $O(\varepsilon^2)$, the equation (7.3.21) is independent from α. This implies in particular that the derivatives of $H(\tau, z, \alpha)$ with respect to α are also of order $O(\varepsilon^2)$. Consequently (7.3.21) can be simplified even further into

$$H_\tau = \frac{1}{2}\left(1 - 2\varepsilon\rho Sz + \varepsilon^2 S^2 z^2\right)H_{zz} - \left\{\frac{1}{2}\varepsilon^2\rho S\alpha\frac{B'}{B} + \varepsilon M\right\}(zH_z - H) + \varepsilon^2\alpha^2\left(\frac{B''}{4B} - \frac{3}{8}\frac{B'^2}{B^2}\right)H \qquad (7.3.24).$$

(7.3.24) shows that the original two-dimensional equation can be in fact reduced to a one-dimensional one. At this stage, the problem is closed to be solved as shown in [5]. Taking the same approach as in [5], we can first notice that the functions $\frac{B'}{B}$ and $\frac{B''}{B}$ can be considered to constant the order $O(\varepsilon^2)$. We will denote as follows $b_1 = \frac{B'}{B}$ and $b_2 = \frac{B''}{B}$. Like in [5], we can do the following change of variables to obtain a closed form for the equation (7.3.24) to the order $O(\varepsilon^2)$.

Defining the new function Q by $$H = e^{\varepsilon^2\rho\nu b_1 z^2/4}\left(1 - 2\varepsilon\rho\nu z + \varepsilon^2\nu^2 z^2\right)^{1/4}Q \qquad (7.3.25)$$

And looking at the new variable $x=\frac{1}{\varepsilon\nu}\log\left(\frac{\sqrt{1-2\varepsilon\rho\nu z+\varepsilon^2\nu^2 z^2}-\rho+\varepsilon\nu z}{1-\rho}\right)$ (7.3.26),

We can show that the new function Q is the solution of the following PDE:

$$for\,\tau>0,\qquad Q_\tau=\frac{1}{2}Q_{xx}+\varepsilon^2\nu^2\left(\frac{1}{4}I''I-\frac{1}{8}I'^2\right)Q+\varepsilon^2\alpha^2\left(\frac{1}{4}b_2-\frac{3}{8}b_1^2\right)Q+\frac{3}{4}\varepsilon^2\rho\nu\alpha b_1 Q \quad (7.3.27),$$

with the boundary condition given by the Dirac function

$$for\,\tau=0,\qquad Q=\delta(x). \quad (7.3.28),$$

and with the function $I(\zeta)$ defined by $I(\zeta)=\sqrt{1-2\rho\zeta+\zeta^2}$ (7.3.29).

The equation (7.3.27) can be summarized into a standard Heat equation at the second order as follows. Define the constant

$$\kappa=\nu^2\left(\frac{1}{4}I''(\varepsilon\nu z_0)I(\varepsilon\nu z_0)-\frac{1}{8}[I'(\varepsilon\nu z_0)]^2+\alpha^2\left(\frac{1}{4}b_2-\frac{3}{8}b_1^2\right)+\frac{3}{4}\rho\nu\alpha b_1\right) \quad (7.3.30),$$

we have to solve the following Heat equation:

$$for\,\tau>0,\qquad Q_\tau=\frac{1}{2}Q_{xx}+\varepsilon^2\kappa Q \quad (7.3.31),$$

whose solution is given by

$$Q=\frac{1}{\sqrt{2\pi\tau}}e^{-x^2/\tau}e^{\varepsilon^2\kappa\tau} \quad (7.3.32),$$

Remember that

$$E\left[\frac{1}{2}\int_0^T 1_{F_u=k}\sigma^2(F_u,u)C^2(F_u)du\right]=\frac{1}{2}\varepsilon\alpha\sqrt{B(0)B(\varepsilon\alpha z)}I^{1/2}(\varepsilon\nu z)e^{\frac{1}{4}\varepsilon^2\rho\nu\alpha b_1 z^2}\int_0^T Q(\tau,x)d\tau \quad (7.3.33)$$

Combining the results (7.3.25), (7.3.32) and (7.3.33) leads to the following solution for our initial problem at the second order precision $O(\varepsilon^2)$.

$$E\left[\frac{1}{2}\int_0^T 1_{F_u=k}\sigma^2(F_u,u)C^2(F_u)du\right]=\frac{1}{2}\varepsilon\alpha\sqrt{B(0)B(\varepsilon\alpha z)}I^{1/2}(\varepsilon\nu z)e^{\frac{1}{4}\varepsilon^2\rho\nu\alpha b_1 z^2}\int_0^T\frac{1}{\sqrt{2\pi\tau}}e^{-x^2/\tau}e^{\varepsilon^2\kappa\tau}d\tau \quad (7.3.34).$$

Taking the limit for ε equal to one leads to the result. □

Proof 7.4: It is shown in [5] that using $z_0=z_0(z)$, κ is constant at the second order when $\nu\to 0$. It is the diffusion coefficient of a heat equation with only one space dimension as a projection of a true two space dimensions equation exact up to the second order when $\nu\to 0$.This diffusion coefficient, and through it, the value z_0 can be used to calibrate the model at have it match exactly the true solution of the two dimensional space equation of one option. z_0 is the value of a point in the space of z which is just a renormalization of the prices's dimension. It is therefore natural to associate the corresponding f_0 which the same point but in the ordinary price coordinate. It explains the name of this parameter: effective projected forward. This projection business can locally be made rigorous with the use of normal coordinates (given with geodesics) associated with the partial differential equation. But this goes beyond the scope of this paper. □

Proof 7.5: in Abramowitz et al. [1] formula 7.4.33, we have

$$\int_0^t e^{-b^2x^2-\frac{a^2}{x^2}}dx=\frac{\sqrt{\pi}}{4a}\left[e^{2ab}\left(erf\left(bx+\frac{a}{x}\right)+e^{-2ab}\left(erf\left(bx-\frac{a}{x}\right)\right)\right)\right] \quad (7.4.1)$$

But a change of variable: $u=x^2$, we have

$$\int_0^t\frac{e^{-b^2u-\frac{a^2}{u}}}{\sqrt{u}}du=\frac{\sqrt{\pi}}{2b}\left[e^{2ab}\left(erf\left(\frac{a}{\sqrt{t}}+b\sqrt{t}\right)-1\right)+e^{-2ab}\left(erf\left(\frac{-a}{\sqrt{t}}+b\sqrt{t}\right)+1\right)\right] \quad (7.4.2)$$

By analyticity, with $b=i\bar{b}$, we get

$$\int_0^t \frac{e^{-\frac{a^2}{u}+b^2u}}{\sqrt{u}}du = \frac{\sqrt{\pi}}{2ib}\left[e^{2iab}\left(erf\left(\frac{a}{\sqrt{t}}+ib\sqrt{t}\right)-1\right)+e^{-2iab}\left(erf\left(\frac{-a}{\sqrt{t}}+ib\sqrt{t}\right)+1\right)\right] \quad (7.4.3),$$

which is exactly the formula (4.1). (4.2) is a standard expansion for the complex erf function given in Abramowitz et al. [1] in 7.1.29.□

Proof 7.6:
Immediate, when comparing the local time formulation for the Black Scholes and the SABR model given by (1.4), (3.3):

$$BS\_Call(f,k,\alpha,T) = (f-k)^+ + \frac{1}{2}\alpha^2k^2\int_0^T \frac{1}{\alpha k\sqrt{u}}\frac{e^{-\frac{1}{2}\left(\frac{Log(f/k)}{\alpha\sqrt{u}}-\frac{1}{2}\alpha\sqrt{u}\right)^2}}{\sqrt{2\pi}}du \quad (7.5.1)$$

$$SABR\_Call(f,k,T,\alpha,\beta,\rho,v) = (f-k)^+ + \frac{f-k}{2x\sqrt{2\pi}}e^{\theta}\int_0^T \frac{e^{-\frac{x^2}{2u}+\kappa u}}{\sqrt{u}}du \quad (7.5.2).\ □$$