Hedging under SABR Model

Bruce Bartlett, Gorilla Science, BruceTheQuant@yahoo.com

Abstract

In this note we take a fresh look at the delta and vega risks within the SABR stochastic volatility model Hagan et al. (2002). These risks can be hedged more precisely by adding new terms to the formulas contained in the original SABR paper. The effect of these new terms is minimized when one hedges *both* vega risks and delta risks, but are substantial when only delta is hedged. In the SABR model, one usually specifies the CEV exponent β and then selects the correlation parameter ρ to match the volatility skew. The delta risk (as specified in the original SABR paper) then depends on the β chosen. With the new term, the delta risk is much less sensitive to the particular value of β, and depends mainly on the slope of the implied volatility curve.

1 Introduction

The SABR model Hagan et al. (2002) is given by the system of stochastic differential equations:

$$\begin{aligned} df_t &= \alpha_t f_t^\beta dW_t^1, \\ d\alpha_t &= \nu \alpha_t dW_t^2, \end{aligned} \tag{1}$$

with the initial condition:

$$\begin{aligned} f_0 &= f, \\ \alpha_0 &= \alpha. \end{aligned} \tag{2}$$

The state variable f_t represents a forward rate (say, a LIBOR forward, a forward swap rate, or a forward yield on a bond), and α_t, the "SABR vol" is a volatility parameter. The movements in the underlying forward rate are correlated with the movements in the underlying volatility:

$$\mathbf{E}\left[dW_t^1 dW_t^2\right] = \rho dt. \tag{3}$$

A martingale measure appropriate for the problem at hand is implicitly assumed.

Consider an option on the forward rate f struck at K and expiring T years from now. It was shown in Hagan (2002) that the value of this option under SABR is given by the modified Black formula:

$$V = \mathrm{B}\left(f, K, \sigma\left(K; f, \alpha, T\right), T\right). \tag{4}$$

Here, $\mathrm{B}\left(f, K, \sigma, T\right)$ is the usual Black valuation formula for vanilla options. The implied volatility $\sigma_B\left(K; f, \alpha, T\right)$ is given by the SABR formula derived in Hagan (2002). Note that the implied volatility depends not only on the strike K, but also on the forward f, the "SABR vol" α, and the time-to-expiry T.

Now in the SABR model, the underlying forward rate f rate is correlated with the SABR vol α. Thus, whenever the forward rate f changes, the vol α also changes, at least on average. Accounting for this average change in α caused by movements in the forward f leads to a new term in the formula for the delta risk, a term not contained in the original SABR paper. One can hedge delta more precisely by adding this term into the delta risk.

In this note we derive the new delta risk and show that:

A) The effect of the new term is minimal when one hedges *both* vega risks and delta risks, but can be substantial when only delta is hedged.

B) Suppose one specifies β and then selects ρ to match the observed volatility skew. (This is the usual method for fitting the SABR model to market data). The delta risk (as specified in the original SABR paper) then depends on the β chosen. With the new term, the delta risk is less sensitive to the β chosen, depending mainly on the slope of the implied vol curve.

2 Original Risk Formulas

In the original SABR paper, the delta hedge was calculated by shifting the current value of the underlying while keeping the current value of α fixed:

$$\begin{aligned} f &\to f + \Delta f, \\ \alpha &\to \alpha. \end{aligned} \tag{5}$$

This scenario leads to the change in the option value

$$\Delta V = \left\{ \frac{\partial B}{\partial f} + \frac{\partial B}{\partial \sigma}\frac{\partial \sigma}{\partial f} \right\} \Delta f,$$

and thus the option delta is given by:

$$\Delta = \frac{\partial B}{\partial f} + \frac{\partial B}{\partial \sigma}\frac{\partial \sigma}{\partial f}. \tag{6}$$

The first term on the right hand side in the formula above is the original Black delta, and the second accounts for the systematic change in the implied volatility as the underlying changes.

Similarly, the vega risk was calculated from

$$\begin{aligned} f &\to f, \\ \alpha &\to \alpha + \Delta\alpha, \end{aligned} \tag{7}$$

to be

$$\Lambda = \frac{\partial B}{\partial \sigma}\frac{\partial \sigma}{\partial \alpha}. \tag{8}$$

Formulas (6) and (8) are the classic SABR greeks. In the next section we derive modified SABR greeks which make a better use of the model dynamics.

3 New Risk Formulas

Since α and f are correlated, whenever f changes, on average α changes as well. A delta scenario which is more realistic than (5) is thus

$$\begin{aligned} f &\to f + \Delta f, \\ \alpha &\to \alpha + \delta_f \alpha. \end{aligned} \tag{9}$$

Here $\delta_f \alpha$ is the average change in α caused by the change in the underlying forward. In order to calculate $\delta_f \alpha$, we write the SABR dynamics in terms of *independent* Brownian motions W_t and Z_t[1]:

$$\begin{aligned} df_t &= \alpha_t f_t^{\beta} dW_t, \\ d\alpha_t &= \nu \alpha_t \left(\rho dW_t + \sqrt{1-\rho^2} dZ_t \right). \end{aligned} \tag{10}$$

This implies that

$$d\alpha_t = \frac{\rho \nu}{f_t^{\beta}} df_t + \nu \alpha_t \sqrt{1-\rho^2} dZ_t. \tag{11}$$

In other words, the time evolution of α_t can be decomposed into two independent components: one due to the change of f_t and one due to the idiosyncratic change in α_t. The average change in α due to a change in the forward is therefore given by

$$\delta_f \alpha = \frac{\rho \nu}{f^{\beta}} \Delta f. \tag{12}$$

The change in the option value is

$$\Delta V = \left[\frac{\partial B}{\partial f} + \frac{\partial B}{\partial \sigma} \left(\frac{\partial \sigma}{\partial f} + \frac{\partial \sigma}{\partial \alpha} \frac{\rho \nu}{f^{\beta}} \right) \right] \Delta f, \tag{13}$$

and so the new delta risk is given by

$$\Delta = \frac{\partial B}{\partial f} + \frac{\partial B}{\partial \sigma} \left(\frac{\partial \sigma}{\partial f} + \frac{\partial \sigma}{\partial \alpha} \frac{\rho \nu}{f^{\beta}} \right). \tag{14}$$

This risk incorporates the average change in volatility caused by changes in the underlying. Hedging this risk should be more effective than hedging the classic SABR delta risk.

The new term in the delta risk,

$$\frac{\partial B}{\partial \sigma} \frac{\partial \sigma}{\partial \alpha} \frac{\rho \nu}{f^{\beta}}, \tag{15}$$

is just $\rho\nu/f^{\beta}$ times the classic SABR vega risk. In a vega-hedged portfolio this term is zero, so if the (classic) vega risk *and* delta risk are *both* hedged, then the new delta risk is also hedged.

Let us now turn to the vega risk. We argue, as we did in the case of delta, that the vega risk should be calculated from the scenario:

$$\begin{aligned} f &\to f + \delta_\alpha f, \\ \alpha &\to \alpha + \Delta\alpha, \end{aligned} \tag{16}$$

where $\delta_\alpha f$ is the average change in f caused by the change in SABR vol. We find readily that

$$\delta_\alpha f = \frac{\rho f^{\beta}}{\nu} \Delta\alpha. \tag{!7}$$

and thus the change in the option value is

$$\Delta V = \frac{\partial B}{\partial \sigma} \left(\frac{\partial \sigma}{\partial \alpha} + \frac{\partial \sigma}{\partial f} \frac{\rho f^{\beta}}{\nu} \right) \Delta\alpha. \tag{18}$$

Hedging out the vega risk means constructing portfolios where the net

$$\Lambda = \frac{\partial B}{\partial \sigma} \left(\frac{\partial \sigma}{\partial \alpha} + \frac{\partial \sigma}{\partial f} \frac{\rho f^{\beta}}{\nu} \right) \tag{19}$$

summed over all deals is zero. This is the new vega hedge which replaces the classic SABR vega.

4 Example

Figures 1-4 analyze the market data for the 3M into 10Y Euro swaptions on January 23, 2006. The relatively short tenor (3M) has been chosen so that the implied volatilities exhibit a substantial smile. The 10Y tenor has been chosen due to its liquidity. To obtain these figures, we first selected β and then fitted the remaining SABR parameters (α, ρ, and ν) to the market's implied volatility curve. Shown are the results for $\beta = 0.0$, 0.25, 0.50, 0.75, and 1.0.

Figure 1 shows the theoretical volatilities $\sigma(K)$ obtained from the market fit. All five curves fit the observed implied volatility reasonably well. Figure 2 graphs the time value of the swaptions as a function of K for these five curves; the coincidence of these time-values demonstrates that the differences between the volatility curves $\sigma(K)$ for different β are immaterial. We conclude that it is difficult to determine which β should be used based solely from fitting the market smile, as reported in the original SABR paper Hagan (2002).

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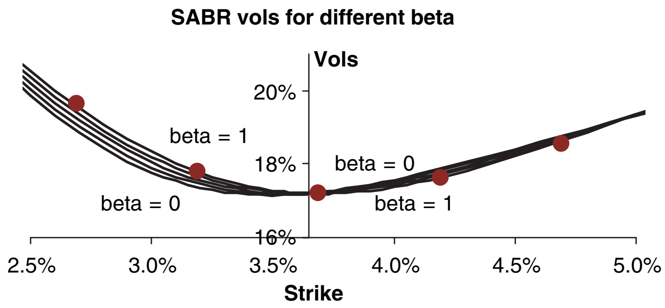

Figure 1: SABR implied volatility smiles fitted to market data (solid dots) for $\beta = 0, 0.25, 0.50, 0.75$, and 1.

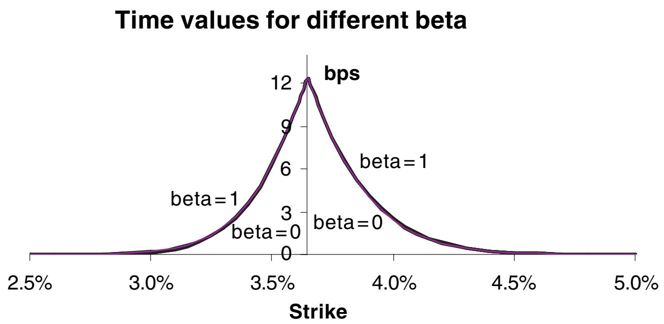

Figure 2: Option time value as function of strike. The curves for $\beta = 0, 0.25, 0.50, 0.75, 1.0$ coincide to within the width of the line.

Figure 3 shows the $\partial\sigma/\partial f$ term from the original SABR article. This term represents the systematic change in the implied vol curve $\sigma_B(K; f, \alpha, T)$ due to changes in the forward f, and leads to the delta risk proposed in the original SABR paper. This term, and thus the delta risks of the original SABR paper, varies substantially depending on which value is chosen for β.

Figure 4 shows our proposed replacement,

$$d\sigma/df = \partial\sigma/\partial f + \left(\rho\nu/f^{\beta}\right)\partial\sigma/\partial\alpha,$$

for calculating delta risks. The new term accounts for the average change in α that occurs when f changes. Clearly this new term is about the same size as the original $\partial\sigma/\partial f$, and should not be neglected. In addition, comparing Figure 3 to Figure 4 shows that this term also makes the delta hedge relatively insensitive to the particular value of β chosen, especially near the ATM point.

Since $\alpha f^{\beta} dW = \left(\alpha/f^{1-\beta}\right) f dW$, we can interpret $\alpha/f^{1-\beta}$ as the "effective local volatility." If $\beta < 1$, this local volatility changes immediately with any changes in the forward f. Similarly, if $\rho \neq 0$ then on average, the volatility α changes when the f changes. Thus we can interpret figure 4 as showing that the delta hedges are not overly sensitive to which fraction of the skew/smile is arises from deterministic changes in the local volatility, and which arise from average changes in the local volatility.

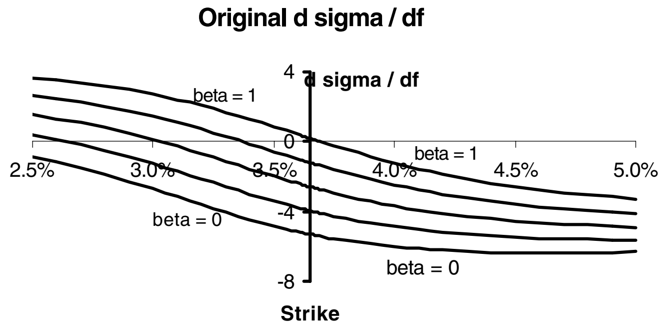

Figure 3: Systematic change in implied vol, $\partial\sigma/\partial f$, specified in the original paper for all five β's.

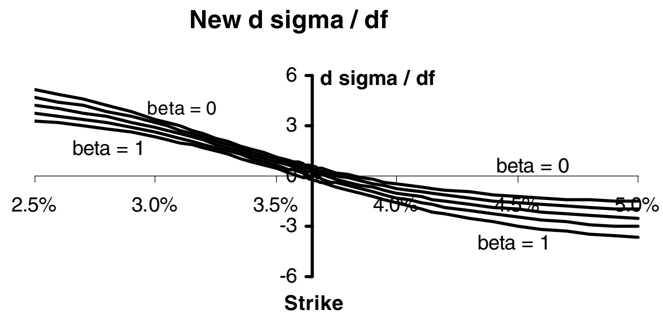

Figure 4: Proposed replacement, $\partial\sigma/\partial f + (\rho\nu/f^{\beta})\,\partial\sigma/\partial\alpha$ for the systematic change in implied vol for all five β's.

FOOTNOTE & REFERENCE

1. Explicitly, $W^1_t = W_t$, $W^2_t = \rho W_t + \sqrt{1-\rho^2}\, Z_t$.

■ Hagan, P., Kumar, D., Lesniewski, A., and Woodward, D.: Managing smile risk, *Wilmott Magazine*, **September**, 84–108 (2002).