The SABR model

Keywords: SABR, volatiltiy, smile, risk

In the SABR model, the volatility is a function of the underlying asset and a stochastic process. The SABR model is a popular model for pricing vanilla options and can be extended to include more complex derivatives.

Formula: 
\[ \sigma_S(t) = \sigma_0 + \alpha \langle \Delta \rangle^{\beta} \] 

where
- \( \sigma_S(t) \) is the implied volatility at time \( t \)
- \( \sigma_0 \) is the initial volatility
- \( \alpha \) is the volatility parameter
- \( \beta \) is the skew parameter
- \( \langle \Delta \rangle \) is the instantaneous variance

The SABR model is widely used in quantitative finance and is an important tool for understanding and pricing interest rate derivatives.
\[
\mathbf{W} = \sum_{j=1}^{N} \mathbf{W}_j
\]

In the context of our model, we have the expression for the weight matrix as a sum of individual weight matrices. This representation allows us to analyze the contribution of each individual component to the overall model. We can further simplify the expression by considering the properties of these weight matrices and their interactions.
2. Modelling using the explicit formulas for the SABR

\[
\frac{\partial}{\partial \tau} \mathbb{E} \left[ e^{\alpha \sigma} \right] = \frac{\partial}{\partial \sigma} \mathbb{E} \left[ e^{\alpha \sigma} \right] = 0
\]

The derivative of the expectation with respect to \( \sigma \) gives us the information about the sensitivity of the SABR model to changes in volatility. This is known as the 'vega' of the SABR model.

\[
\mathbb{E} \left[ e^{\alpha \sigma} \right] = \int_0^\infty e^{\alpha \sigma} f(t, \sigma) \, dt
\]

where \( f(t, \sigma) \) is the density function of the forward rate process in the SABR model.

\[
\frac{\partial}{\partial \sigma} \mathbb{E} \left[ e^{\alpha \sigma} \right] = \alpha \mathbb{E} \left[ e^{\alpha \sigma - \delta} \right]
\]

This derivative is used in the 'volga' formula to hedge the position in an SABR model.

\[
\mathbb{E} \left[ e^{\alpha \sigma} \right] = \int_0^\infty e^{\alpha \sigma} f(t, \sigma) \, dt = \int_0^\infty e^{\alpha \sigma} \left( e^{\alpha \sigma} + \alpha \right) f(t, \sigma) \, dt
\]

The second moment of the SABR density function is used in the 'gamma' formula to measure the convexity of the option price with respect to changes in volatility.

\[
\frac{\partial^2}{\partial \sigma^2} \mathbb{E} \left[ e^{\alpha \sigma} \right] = \alpha^2 \mathbb{E} \left[ e^{2\alpha \sigma} \right]
\]

This derivative is used in the 'volga' formula to hedge the position in an SABR model.

\[
\mathbb{E} \left[ e^{\alpha \sigma} \right] = \int_0^\infty e^{\alpha \sigma} f(t, \sigma) \, dt = \int_0^\infty e^{\alpha \sigma} \left( e^{\alpha \sigma} + \alpha \right) f(t, \sigma) \, dt
\]

The expected value of the SABR model is used to price options in the SABR framework.

\[
\frac{\partial}{\partial \sigma} \mathbb{E} \left[ e^{\alpha \sigma} \right] = \alpha \mathbb{E} \left[ e^{\alpha \sigma - \delta} \right]
\]

This derivative is used in the 'volga' formula to hedge the position in an SABR model.
\[ a = \frac{\partial}{\partial x} \]
4. Results

\[ Y = f \left( (Y - \text{mean})^{m} \right) \]

The function above was derived to study the relationship between two variables. The mean was calculated using the formula:

\[ \text{mean} = \frac{\sum x}{n} \]

where \( x \) represents the data points and \( n \) is the number of data points. The function was then applied to the data to study the relationship between the variables.
The integration of the proposed direction is dependent on the fact that

\[ \int_{\Omega} \sqrt{g} |\nabla v| \, d\Omega = \oint \nabla v \cdot n \, ds \]

(6)

And the resulting

\[ \int_{\Omega} \sqrt{g} |\nabla v| \, d\Omega = \oint \nabla v \cdot n \, ds \]

(7)

By the variational

\[ \int_{\Omega} \sqrt{g} |\nabla v| \, d\Omega = \oint \nabla v \cdot n \, ds \]

(8)

One can then

\[ \int_{\Omega} \sqrt{g} |\nabla v| \, d\Omega = \oint \nabla v \cdot n \, ds \]

(9)

The variation of the proposed

\[ \int_{\Omega} \sqrt{g} |\nabla v| \, d\Omega = \oint \nabla v \cdot n \, ds \]

(10)

Appendices

A Condition of the Effective Formulation

Here we present the effective model:

\[ \frac{d^2 \theta}{dr^2} + \frac{1}{r} \frac{d \theta}{dr} = 0 \]

(11)

\[ \frac{d^2 \theta}{dr^2} + \frac{1}{r} \frac{d \theta}{dr} = 0 \]

(12)

\[ \frac{d^2 \theta}{dr^2} + \frac{1}{r} \frac{d \theta}{dr} = 0 \]

(13)

\[ \frac{d^2 \theta}{dr^2} + \frac{1}{r} \frac{d \theta}{dr} = 0 \]

(14)

The effective model is

\[ \frac{d^2 \theta}{dr^2} + \frac{1}{r} \frac{d \theta}{dr} = 0 \]

(15)

\[ \frac{d^2 \theta}{dr^2} + \frac{1}{r} \frac{d \theta}{dr} = 0 \]

(16)

\[ \frac{d^2 \theta}{dr^2} + \frac{1}{r} \frac{d \theta}{dr} = 0 \]

(17)

\[ \frac{d^2 \theta}{dr^2} + \frac{1}{r} \frac{d \theta}{dr} = 0 \]

(18)

\[ \frac{d^2 \theta}{dr^2} + \frac{1}{r} \frac{d \theta}{dr} = 0 \]

(19)

\[ \frac{d^2 \theta}{dr^2} + \frac{1}{r} \frac{d \theta}{dr} = 0 \]

(20)

\[ \frac{d^2 \theta}{dr^2} + \frac{1}{r} \frac{d \theta}{dr} = 0 \]

(21)
\[ (\mathbf{v} \cdot \mathbf{v})_{\mathbf{a} \cdot \mathbf{a}} = (\mathbf{v} \cdot \mathbf{v})_{\mathbf{a} \cdot \mathbf{a}} \]

Differentiating both sides with respect to \( t \), we get:

\[ \frac{d}{dt} (\mathbf{v} \cdot \mathbf{v})_{\mathbf{a} \cdot \mathbf{a}} = \frac{d}{dt} (\mathbf{v} \cdot \mathbf{v})_{\mathbf{a} \cdot \mathbf{a}} \]

Using the product rule, we have:

\[ \frac{d}{dt} (\mathbf{v} \cdot \mathbf{v})_{\mathbf{a} \cdot \mathbf{a}} = 2 (\mathbf{v} \cdot \mathbf{a}) \frac{d\mathbf{a}}{dt} + 2 (\mathbf{a} \cdot \mathbf{v}) \frac{d\mathbf{v}}{dt} \]

Substituting the derivative of \( \mathbf{a} \) and \( \mathbf{v} \), we get:

\[ \frac{d}{dt} (\mathbf{v} \cdot \mathbf{v})_{\mathbf{a} \cdot \mathbf{a}} = 2 (\mathbf{v} \cdot \mathbf{a}) \frac{d\mathbf{a}}{dt} + 2 (\mathbf{a} \cdot \mathbf{v}) \frac{d\mathbf{v}}{dt} \]

Solving for \( \frac{d\mathbf{a}}{dt} \), we get:

\[ \frac{d\mathbf{a}}{dt} = \frac{1}{2} \left( \frac{d}{dt} (\mathbf{v} \cdot \mathbf{v})_{\mathbf{a} \cdot \mathbf{a}} - 2 (\mathbf{a} \cdot \mathbf{v}) \frac{d\mathbf{v}}{dt} \right) \]

Similarly, for \( \frac{d\mathbf{v}}{dt} \), we have:

\[ \frac{d\mathbf{v}}{dt} = \frac{1}{2} \left( \frac{d}{dt} (\mathbf{v} \cdot \mathbf{v})_{\mathbf{a} \cdot \mathbf{a}} - 2 (\mathbf{v} \cdot \mathbf{a}) \frac{d\mathbf{a}}{dt} \right) \]

These equations can be solved by numerical methods to determine the motion of \( \mathbf{a} \) and \( \mathbf{v} \).
\[ f = \int_{\Omega} g \, d\mu \]
\((91)\quad \frac{\partial}{\partial t} \frac{\partial^2}{\partial x^2} (D \alpha(x,t)) = \frac{\partial}{\partial x} \left( \frac{1}{D} \frac{\partial D}{\partial x} \right) \frac{\partial^2}{\partial x^2} (D \alpha(x,t)) = \frac{1}{D} \frac{\partial D}{\partial x} \frac{\partial^2}{\partial x^2} (D \alpha(x,t))
\]

\((92)\quad \frac{\partial}{\partial t} \frac{\partial^2}{\partial x^2} (D \alpha(x,t)) = \frac{1}{D} \frac{\partial D}{\partial x} \frac{\partial^2}{\partial x^2} (D \alpha(x,t))
\]

\[(93)\quad \frac{\partial}{\partial t} \frac{\partial^2}{\partial x^2} (D \alpha(x,t)) = \frac{1}{D} \frac{\partial D}{\partial x} \frac{\partial^2}{\partial x^2} (D \alpha(x,t))
\]

\[(94)\quad \frac{\partial}{\partial t} \frac{\partial^2}{\partial x^2} (D \alpha(x,t)) = \frac{1}{D} \frac{\partial D}{\partial x} \frac{\partial^2}{\partial x^2} (D \alpha(x,t))
\]

\[(95)\quad \frac{\partial}{\partial t} \frac{\partial^2}{\partial x^2} (D \alpha(x,t)) = \frac{1}{D} \frac{\partial D}{\partial x} \frac{\partial^2}{\partial x^2} (D \alpha(x,t))
\]

\[(96)\quad \frac{\partial}{\partial t} \frac{\partial^2}{\partial x^2} (D \alpha(x,t)) = \frac{1}{D} \frac{\partial D}{\partial x} \frac{\partial^2}{\partial x^2} (D \alpha(x,t))
\]

\[(97)\quad \frac{\partial}{\partial t} \frac{\partial^2}{\partial x^2} (D \alpha(x,t)) = \frac{1}{D} \frac{\partial D}{\partial x} \frac{\partial^2}{\partial x^2} (D \alpha(x,t))
\]

\[(98)\quad \frac{\partial}{\partial t} \frac{\partial^2}{\partial x^2} (D \alpha(x,t)) = \frac{1}{D} \frac{\partial D}{\partial x} \frac{\partial^2}{\partial x^2} (D \alpha(x,t))
\]
\[
\begin{align*}
\omega_1(t) &= C(t) + \phi_1(t) \\
\omega_2(t) &= C(t) + \phi_2(t) \\
\omega_3(t) &= C(t) + \phi_3(t) \\
\end{align*}
\]

where \( C(t) \) is the constant term and \( \phi_1(t), \phi_2(t), \phi_3(t) \) are the variable parts.
\textbf{C1. Generation}

We also consider the case of the multidimensional case where the problem is split into two problems on the constraint manifold: \( \mathcal{D}_U = \mathcal{D}_L \times \mathcal{D}_R \) and \( \mathcal{D}_W = \mathcal{D}_L \), and the constraints are the two-dimensional constraint manifold: \( \mathcal{D}_L = \mathcal{D}_L \times \mathcal{D}_R \).

\( f_i = f_i^L \times f_i^R \)

\( f_i^L = f_i^L \times f_i^R \)

\( f_i^R = f_i^L \times f_i^R \)

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\( f_i^R = f_i^L \times f_i^R \)
(7.57) \[
\frac{\partial^2 f}{\partial x^2} = 0.0 + k^2 \frac{\partial f}{\partial x} \tag{5.07}
\]

(7.58) \[
\frac{\partial f}{\partial x} = 0.0 + \lambda^2 \frac{\partial f}{\partial x} \tag{5.08}
\]

(7.59) \[
\frac{\partial^2 f}{\partial x^2} = 0.0 + \lambda^4 \frac{\partial f}{\partial x} \tag{5.09}
\]

(7.60) \[
\frac{\partial^2 f}{\partial x^2} = 0.0 + \lambda^6 \frac{\partial f}{\partial x} \tag{5.10}
\]

(7.61) \[
\frac{\partial^2 f}{\partial x^2} = 0.0 + \lambda^8 \frac{\partial f}{\partial x} \tag{5.11}
\]

(7.62) \[
\frac{\partial^2 f}{\partial x^2} = 0.0 + \lambda^{10} \frac{\partial f}{\partial x} \tag{5.12}
\]

(7.63) \[
\frac{\partial^2 f}{\partial x^2} = 0.0 + \lambda^{12} \frac{\partial f}{\partial x} \tag{5.13}
\]

(7.64) \[
\frac{\partial^2 f}{\partial x^2} = 0.0 + \lambda^{14} \frac{\partial f}{\partial x} \tag{5.14}
\]

(7.65) \[
\frac{\partial^2 f}{\partial x^2} = 0.0 + \lambda^{16} \frac{\partial f}{\partial x} \tag{5.15}
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(7.66) \[
\frac{\partial^2 f}{\partial x^2} = 0.0 + \lambda^{18} \frac{\partial f}{\partial x} \tag{5.16}
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(7.67) \[
\frac{\partial^2 f}{\partial x^2} = 0.0 + \lambda^{20} \frac{\partial f}{\partial x} \tag{5.17}
\]

(7.68) \[
\frac{\partial^2 f}{\partial x^2} = 0.0 + \lambda^{22} \frac{\partial f}{\partial x} \tag{5.18}
\]

(7.69) \[
\frac{\partial^2 f}{\partial x^2} = 0.0 + \lambda^{24} \frac{\partial f}{\partial x} \tag{5.19}
\]

(7.70) \[
\frac{\partial^2 f}{\partial x^2} = 0.0 + \lambda^{26} \frac{\partial f}{\partial x} \tag{5.20}
\]

(7.71) \[
\frac{\partial^2 f}{\partial x^2} = 0.0 + \lambda^{28} \frac{\partial f}{\partial x} \tag{5.21}
\]

(7.72) \[
\frac{\partial^2 f}{\partial x^2} = 0.0 + \lambda^{30} \frac{\partial f}{\partial x} \tag{5.22}
\]

The corrected formula is as follows:

\[
\frac{\partial^2 f}{\partial x^2} = 0.0 + \lambda^{2} \frac{\partial f}{\partial x} \tag{5.23}
\]
\[ f = \frac{\left(1 + \frac{d-1}{\alpha + \frac{d-1}{\alpha} + 1}\right)}{\left(1 + \frac{\alpha}{\alpha + \frac{d-1}{\alpha} + 1}\right)} \]

where

\[ \alpha = \frac{\gamma}{\bar{\gamma}} \]

The Shreve SABR model is described below. This is a local volatility model that allows for changes in the volatility surface over time. The model is based on the SABR model and is used to model the price of an option in the presence of a stochastic volatility process.