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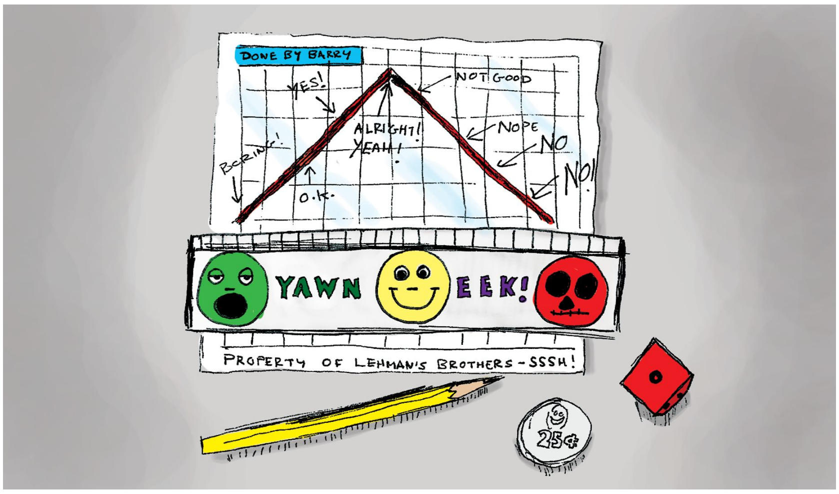
A Primer on Risk Measures

In this column,[1] we present a survey of key results and references to papers on risk measures

The concepts of utility and risk attitude, such as decreasing risk aversion, are general and qualitative. But risk is basic to decision-making models and requires a precise definition. The essential characteristics of risk are the chance of a potential loss and the size of the potential loss. If financial decisions are to be considered in the framework of risk management, it is clear that a quantitative measure of risk, which incorporates the essential characteristics, is needed.

Originally, risk was associated with variance. Markowitz (1952) provided a quantitative framework for measuring portfolio risk and return using mean and variance. At the same time, Roy (1952) stated that an investor will prefer safety of principal first and will set some minimum acceptable return that will conserve the principal. This focused risk on the lower partial moment (LPM). These initial measures were intuitive, but they lacked a theoretical foundation.

The returns on investment are uncertain, so the financial status is given by a random variable X with a distribution function F. A *risk measure* is a functional $\rho(X)$ which depends only on the distribution function F, so that two random variables with the same distribution have the same risk value. This property is called *law invariance*. Random variables can be ordered with risk preferences. Typically risk preferences have been formulated through utility and, equivalently, stochastic dominance. Risk aversion is identified with concave utilities, and the stochastic dominance defined by the family of concave utilities is second-order stochastic dominance (SSD). It is reasonable that the ordering of random variables with



a risk measure should be *consistent* with the ordering from stochastic dominance. See Ogryczak and Ruscynski (1999).

The financial position X depends on the uncertain returns on assets and the investment decision which determines the amount of capital allocated to various assets. Assume that the unit returns on assets are functions on a probability space (Ω, B, P), and that the set of possible investment strategies determines a class of financial positions χ, with $X \in \chi$. So the risk measure is a functional $\rho : \chi \to \mathbb{R}$.

From a financial perspective, it is natural to associate risk with losses (Kusy and Ziemba, 1986), and to view the financial risk of X as the capital requirement $\rho(X)$ to make the position X acceptable (Artzner et al., 1999; Föllmer and Knispel, 2013). *The acceptance set* of ρ is

$$A_\rho : \{X \in \chi | \rho(X) \le 0\} .$$

The risk measure can be defined from the acceptance set

$$\rho(X) = inf\left\{m \in \mathbb{R} | X + m \in A_\rho\right\} .$$

Rockafellar and Ziemba (2000) established the following result.

Equivalence theorem

There is a one-to-one correspondence between acceptance sets A_ρ and the risk measures ρ.

The concept of capital requirement to cover the losses from investment captures the financial risk idea, but the probability of loss is not taken into account. Various measures, which use the distribution of the financial status X, have been proposed and used in risk management practice. Details on the measures are provided by Krokhmal et al. (2011) and Föllmer and Knispel (2013) (see also Föllmer and Schied, 2004).

[1] Variance

Markowitz (1952, 1959, 1987) and Markowitz and van Dijk (2006) used the variance of the position X to measure risk:

$$\rho(X) = E(X - EX)^2,$$

where $E(X)$ is the mean of X. Variability or uncertainty does capture aspects of risk, but the gains are treated the same as losses. In addition, variance is not consistent with SSD.

[2] Semi-variance
A useful estimate of risk exposure or downside risk was proposed by Markowitz (1959), Porter (1974), and Markowitz and van Dijk (2006), with the semi-variance:

$$\rho(X) = E(X - EX)_-^2,$$

where $(.)\_$ denotes negative part: $y\_= 0$ if $y \geq 0$, $y\_=y$ if $y<0$.

[3] Deviation risk measures
Rockafellar et al. (2006) generalized the variance-type measure for any square integrable deviation measure $D: \chi \to \mathbb{R}^+$. Then, the risk measure is:

$$\rho(X) = D(X) - EX.$$

There is a one-to-one relationship between averse risk measures and deviation risk measures through the relationship $D(X) = \rho(X - EX)$. (see Krokhmal et al., 2011). The aversion property follows from $\rho(X) > \rho(EX)$ for nonconstant X.

[4] *VaR*
The focus on downside risk measures started with the development of the LPM risk measure by Bawa (1975) and Fishburn (1977). This very popular risk measure was further developed by Jorion (2006). It is defined by the α^{th} quantile of the distribution F for the financial position X:

$$\rho(X) = VaR_\alpha(X) = F^{-1}(\alpha) = inf\{x \in \mathbb{R} | F(x) \geq \alpha\}.$$

In terms of losses, VaR_α is the maximum possible loss at the confidence level $1 - \alpha$. This measure accounts for losses and probabilities, in the sense that it has acceptance sets

$$A_\rho = \{X \in \chi | VaR_\alpha(X) \leq 0\}$$

An interesting takeoff on *VaR* is its application in governance for endowments, trusts, and pension plans. Worldwide adoption of the Basel II Accord established this measure as a standard.

[5] *AVaR*
The VaR measure does not account for the losses below it, as all losses below the cutoff are considered the same. A variation is to consider the average of the values at risk for $\lambda \in (0, \alpha]$

$$\rho(X) = AVaR_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha VaR_\alpha(X)\, d\alpha.$$

Föllmer and Knispel (2013) showed how *AVaR* is a building block for law-invariant risk measures.

[6] *CVar*
A related measure, proposed by Rockafellar and Uryasev (2000) and others (e.g., Acerbi and Tasche, (2002), is:

$$CVaR_\alpha(X) = E(X | X < F^{-1}(\alpha)).$$

CVaR is the conditional expectation of losses exceeding the $VaR_\alpha(X)$ level. Obviously, $CVaR_\alpha(X) = AVaR_\alpha(X)$ when the distribution F for X is continuous. However, for discontinuous distributions the measures may differ.

As a functional, there exist mathematical properties which may reasonably be expected to be satisfied by risk measures ρ.Those properties can be considered as properties of either the functional or the associated acceptance set.

(A1) Law invariance: $\rho(X) = \rho(Y)$ *for all* $X, Y \in \chi$ *such that* $F_X = F_Y$
(A2) Monotonicity: $F_X \geq F_Y \Rightarrow \rho(X) \geq \rho(Y)$
(A3) Translation invariance: $\rho(X + m) = \rho(X) - m$ *for all* $X \in \chi$, $m \in \mathbb{R}$
(A4) Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$ *for all* $X, Y \in \chi$
(A5) Positive homogeneity: $\rho(\lambda X) = \lambda\rho(X)$ *for all* $X \in \chi, \lambda > 0$
(A6) Consistency: X *SSD* $Y \Rightarrow \rho(X) \geq \rho(Y)$
(A7) Risk aversion: $\rho(X) = c$ *for constant c, and* $\rho(X) > -EX$ *for nonconstant X.*

The properties do not constitute a definition of a risk measure, and they are not sufficient to build a risk measure. For a proposed risk measure, these properties can be verified. Law invariance is clear. Monotonicity requires that the risk measure is ordered as implied by the distribution (i.e., if one density is to the left of the other, it implies greater risk). Translation invariance states that adding cash to a financial position reduces the risk by the same amount.

The subadditivity property is significant as it implies that diversification typically reduces risk. There are severe times when correlated assets X and Y are such that the risk of $(X + Y)$ is more than the risk of X plus the risk of Y, because of price pressure. The Long-Term Capital demise of 1998 was one such example (see Ziemba and Ziemba, 2013).

As a set of properties, A2–A5 have been used by Artzner et al. (1999) to characterize the class of *coherent risk measures*.

Positive homogeneity and subadditivity together imply convexity:

$$\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y).$$

Convexity does not imply both positive homogeneity and subadditivity hold. As convexity is the desired property, it can replace subadditivity and homogeneity. The resulting class of *convex risk measures* is considered by Föllmer and Knispel (2013), following Rockafellar and Ziemba (2000) and Föllmer and Schied (2002, 2004, 2011). Convexity of the risk measure ρ or the corresponding acceptance set A_ρ is important for decision problems where the risk measure is used to formulate constraints. *Averse risk measures*, which were introduced by Rockafellar et al. (2006), satisfy A4, A5, and A7.

If the commonly used risk measures defined above are checked against the properties, the following results hold:

- Standard deviation: satisfies A1, A5, and A4 if $cov(X, Y) \geq 0$.
- *VaR*: satisfies A1, A2, and A3.
- *AVaR*: satisfies A1, A2, A3, A4, A5, A6, and A7.

So, *AVaR* (or *CVaR*, in the case of continuous distributions) has all the properties of a reasonable risk measure.

A general framework for defining a risk measure, based on the distribution function F for a financial position X, is with a Choquet integral. A distortion function g is defined such that $g: [0, 1] \to [0, 1]$, $g(0) = 0, g(1) = 1$:

Then risk measures are of the form:

$$\rho_g(X) = \int_0^\infty g(1 - F(t))\,dt + \int_{-\infty}^0 [g(1 - F(t)) - 1]\,dt.$$

The elegance of this formulation is the association of the risk measure with the distortion/weighting of the probability distribution F to capture risk perception. There is a family of risk measures defined by the distortion functionals. Föllmer and Knispel (2013) discuss various such risk measures. It is shown that the risk measure is convex if, and only if, the distortion functional is concave. The distortion functional for *VaR* is not concave, so it is neither convex nor consistent with SSD.

The distortion functional operates on the distribution function F or, alternatively, on the fixed measure P in the probability space $(\Omega, \mathrm{B}, \mathrm{P})$. There are shortcomings of the distortion functional approach. In particular, there is excessive reliance on a single probabilistic measure P. More generally, it raises the issue of model uncertainty or model ambiguity, often called Knightian uncertainty. Föllmer and Knispel (2013) discuss a robustification where the probability measure P is a member of a class $\wp$. The class could be the set of probability measures within a specified distance from the reference measure P. On the class, the risk measure could be:

$$\rho_{\wp}(X) := \sup_{\mathrm{P}\in\wp} \rho_{\mathrm{P}}(X) .Z$$

This set-of-measures approach could deal with the issue of estimation error. Calculations in MacLean et al. (2007) show that the inflation of risk, as measured by *CVaR*, can be increased as much as five times from estimation error. That is, an investment strategy chosen to have a *CVaR* requirement violated 5 percent of the time, in practice can have a 25 percent violation. As the known empirical distribution $\hat{F}$ is an estimate of the true distribution F, the risk measure $\hat{\rho}$ from $\hat{F}$ is an estimate. If F is parametric, then a confidence interval of distributions could define the class over which a risk measure is defined.

In the framework of financial decision making, the management of risk can be viewed in the style of Markowitz's mean-variance analysis. Krokhmal et al. (2011) present the decision problem as a tradeoff between risk and reward. Given a payoff (profit) function $X = X(x, \omega)$ that is dependent on the decision vector x and random element $\omega \in \Omega$, the risk measure is $\rho(X) = \rho(X(x, \omega))$ and the reward function is $\pi(X) = \pi(X(x,\omega))$. The problem is to select the decision x that maximizes the reward $\pi(X)$ while assuring that the risk does not exceed ρ_0:

$$\max_x \left\{\pi(X) \,|\rho(X) \le \rho_0\right\} .$$

Alternatively, a weighted combination of risk and reward is optimized:

$$\max_x \{\pi(X) - \lambda\rho(X)|\lambda \ge 0,\} .$$

In this problem, λ is a risk aversion parameter. This penalty parameter may incorporate the potential effects of both model uncertainty and estimation error.

An application with the use of the penalty parameter approach is provided by the financial planning model InnoALM for the Austrian pension fund of the electronics firm Siemens (Geyer and Ziemba, 2008). The model uses a multiperiod stochastic linear programming framework, where uncertainty is modeled using multiperiod discrete probability scenarios for random returns and other model parameters. The concave risk-averse preference function is to maximize the expected present value of terminal wealth at the specified horizon net of the expected discounted convex (piecewise-linear) penalty costs for wealth and benchmark targets shortfalls in each decision period. Earlier applications of the convex penalty approach are found in Kallberg et al. (1982), Kuzy and Ziemba (1986), Cariño and Ziemba (1998), and Cariño et al. (1998). Ziemba (2013) presents the case for the use of convex risk measures. See also MacLean and Ziemba (2013).

The implementation of a scenario-based asset allocation model leads to more flexible allocation restraints, which allows for more risk tolerance and ultimately results in better long-term investment performance.

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ENDNOTE

1. This column has been modified from an introduction in our *Handbook of Financial Decision Making, Part II* (World Scientific, 2013), where readers can find many of the papers cited here, plus other papers and discussion relevant to financial decision making.