Factors at Risk

G. Studer
Prof. H.–J. Lüthi

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Abstract:

The identification of scenarios which have a particularly low or high P&L helps to get a better understanding of the portfolio's risk exposure. Therefore, the notions of safe (resp. dangerous) regions are introduced, which represent sets where the P&L is greater (resp. less) than a given critical level. In order to describe such sets in an easily interpretable way, one–dimensional intervals are used. Such intervals can be determined by solving a sequence of restricted maximum loss problems.

Keywords: Risk Management — Maximum Loss Optimization

1 Introduction

Maximum Loss (ML) was introduced as a method for measuring market risks of nonlinear portfolios (cf. [Studer]). The basic idea of ML is to determine the worst case out of a specific set A of scenarios, called "trust region". Maximum Loss is a *coherent* risk measure (cf. [Artzner et al.]) and it is always more conservative than the corresponding VAR (for a more detailed discussion of VAR refer to [Beckström and Campbell] and [RiskMetrics]).
Mathematically, the ML problem can be formulated as follows: the risk factors $\omega = (\omega_1, \ldots, \omega_M)$ represent shifted market rates (e.g. commodity prices, foreign exchange rates, equity indices, interest rates), such that $\omega_i = 0$ corresponds to the actual value of market rate i. The profit and loss (P&L) function $v : \mathbb{R}^M \to \mathbb{R}; \omega \mapsto v(\omega)$ gives the *change* in portfolio value (satisfying $v(0) = 0$). If $A \subset \mathbb{R}^M$ denotes the trust region, then ML is defined as:

$$\begin{aligned} \mathrm{ML} \;=\; & \min \quad v(\omega) \\ & \text{s.t.} \quad \omega \in A. \end{aligned} \tag{1}$$

To get a univocal definition of ML, the trust region A has to be defined more precisely. If the risk factors ω are multinormally distributed (i.e., $\omega \sim \mathcal{N}(0, \Sigma_t)$, where Σ_t is the covariance matrix for a holding period of length t), then our standard choice will be

$$A = \{\omega \mid \omega^T \Sigma_t^{-1} \omega \leq c_\alpha\}, \tag{2}$$

where c_α is the α–quantile of a chi–square distribution with M degrees of freedom. This choice assures that exactly α percent of all possible outcomes are covered by the set A. For quadratic P&L functions $v(\omega) = \frac{1}{2}\omega^T G\omega + g^T\omega$, where G is a symmetric $M \times M$ matrix and g an M–dimensional vector, the resulting problem

$$\begin{aligned} \text{ML} \;=\; \min \quad & \frac{1}{2}\omega^T G\omega + g^T\omega \\ \text{s.t.} \quad & \omega^T \Sigma_t^{-1} \omega \leq c_\alpha, \end{aligned} \tag{3}$$

can be solved *efficiently* (i.e., in polynomial time up to d digits). Solving this problem repetitively for an increasing sequence of confidence levels $0\% = \alpha^{(0)} < \alpha^{(1)} < \ldots < \alpha^{(N)} \leq 100\%$ leads to the path of ML scenarios (for a detailed discussion see [Studer and Lüthi]). At this point, the question arises how to extend the analysis in order to pass from one–dimensional ML paths to multidimensional *regions*.

2 Definition of Safe and Dangerous Regions

For investigating higher dimensional objects, the notions of *safe* and *dangerous scenarios* will be used in the sequel. The set $\mathcal{S}(c)$ of safe scenarios for a critical level c is defined as

$$\mathcal{S}(c) = \left\{\omega \in \mathrm{I\!R}^M : v(\omega) > c\right\}. \tag{4}$$

Then, a given subset $S \subset \mathrm{I\!R}^M$ is said to be a *safe region* with respect to the trust region A and for the level c if

$$S \cap A \subseteq \mathcal{S}(c) \cap A, \tag{5}$$

which claims that the P&L of all scenarios lying inside the intersection of S and the trust region A exceeds the level c:

$$v(\omega) > c; \quad \omega \in A \cap S. \tag{6}$$

Similarly, the set of dangerous scenarios for a critical level c is

$$\mathcal{T}(c) = \left\{\omega \in \mathrm{I\!R}^M : v(\omega) < c\right\}, \tag{7}$$

and $T \subset \mathbb{R}^M$ is a *dangerous region* with respect to the trust region A and for the level c if

$$T \cap A \subseteq \mathcal{T}(c) \cap A, \tag{8}$$

which means that the P&L of all scenarios lying inside the intersection of T and the trust region A has to be less than c:

$$v(\omega) < c; \quad \omega \in A \cap T. \tag{9}$$

Since $\mathcal{S} \cap \mathcal{T} = \emptyset$, an arbitrary set $U \subset \mathbb{R}^M$ may be characterized as being either safe, dangerous or none of both, but never simultaneously safe *and* dangerous. For quadratic P&L functions $v(\omega) = \frac{1}{2}\omega^T G\omega + g^T\omega$, the level set

$$L(c) = \{\omega : \frac{1}{2}\omega^T G\omega + g^T\omega = c\}, \tag{10}$$

is a second order surface (e.g. ellipsoid, cylinder, paraboloids, hyperboloids, cones, planes) and can be described parametrically. In principle, it would be possible to use such equations to characterize safe and dangerous regions completely. In practice, however, this approach is not promising since such equations are very difficult to explicate and handle in higher dimensional spaces.

3 Calculation of Safe and Dangerous Intervals

A practicable way to describe safe and dangerous regions is to use a *set* of one–dimensional intervals I_j having the form

$$I_j = \{\omega \mid a \leq \omega_j \leq b\}; \quad a < b \in \mathbb{R}, \tag{11}$$

for each risk factor $\omega_j; j = 1, \ldots, M$. By equation (6), I_j is safe if and only if

$$v(\omega) > c; \quad \omega \in A \cap I_j. \tag{12}$$

Hence, a one point interval $I_j = \{y\}$ is safe if and only if

$$v(\omega) > c; \quad \omega \in A; \omega_j = y. \tag{13}$$

To decide whether $I_j = \{y\}$ is safe, it is possible to refer to the restricted maximum loss problem

$$\begin{aligned} \mathrm{ML}_j(y) = \quad & \min \quad v(\omega) \\ & \text{s.t.} \quad \omega^T \Sigma_t^{-1} \omega \leq c_\alpha \\ & \qquad\;\; \omega_j = y. \end{aligned} \tag{14}$$

Obviously, $I_j = \{y\}$ is safe if and only if

$$\mathrm{ML}_j(y) > c. \tag{15}$$

This way, the decision problem (13) has been transformed to the maximum loss computation $\mathrm{ML}_j(y)$. Correspondingly, $I_j = \{y\}$ is dangerous if and only if

$$\mathrm{MP}_j(y) < c, \tag{16}$$

where

$$\begin{aligned} \mathrm{MP}_j(y) = \quad & \max \quad v(\omega) \\ & \text{s.t.} \quad \omega^T \Sigma_t^{-1} \omega \le c_\alpha \\ & \qquad\;\; \omega_j = y. \end{aligned} \tag{17}$$

To calculate $\mathrm{ML}_j(y)$ and $\mathrm{MP}_j(y)$ for quadratic P&L functions $v(\omega) = \frac{1}{2}\omega^T G\omega + g^T\omega$, these problems are converted into the standard form of (3); this process is described in appendix A. Then, the determination of the maximal set of safe intervals (in the sense of inclusion) for risk factor ω_j requires a discretization of axis j. Considering the trust region $A = \{\omega \mid \omega^T \Sigma_t^{-1}\omega \le c_\alpha\}$, lower and upper bounds of ω_j can be obtained by solving the problem

$$\begin{aligned} \min \quad & \pm\omega_j \\ \text{s.t.} \quad & \omega^T \Sigma_t^{-1} \omega \le c_\alpha, \end{aligned} \tag{18}$$

and the results are (cf. [Studer]):

$$\begin{aligned} \omega_j^l &= -\sqrt{c_\alpha}\sqrt{[\Sigma_t]_{j,j}} \\ \omega_j^u &= \sqrt{c_\alpha}\sqrt{[\Sigma_t]_{j,j}}, \end{aligned} \tag{19}$$

which means that $\omega^T \Sigma_t^{-1}\omega > c_\alpha$ if $\omega \notin [\omega_j^l, \omega_j^u]$. Then, $(n+1)$ equally spaced points

$$\omega_j^{(i)} = \omega_j^l + \frac{i}{n}(\omega_j^u - \omega_j^l); \quad i = 0, \ldots, n, \tag{20}$$

are chosen and $\mathrm{ML}_j(\omega_j^{(i)})$ and $\mathrm{MP}_j(\omega_j^{(i)})$ for $i = 0, \ldots, n$ are calculated. Once these calculations have been performed, safe and dangerous intervals for various levels of c can be determined immediately by formulas (15) and (16).
The dashed lines in figure 1 represent the functions $\mathrm{ML}_j(\omega_j)$ and $\mathrm{MP}_j(\omega_j)$ of a quadratic portfolio. The safe intervals for level c_1 are the segments lying below the ML curve, whereas the dangerous intervals for c_2 correspond to the segments lying above the MP curve.

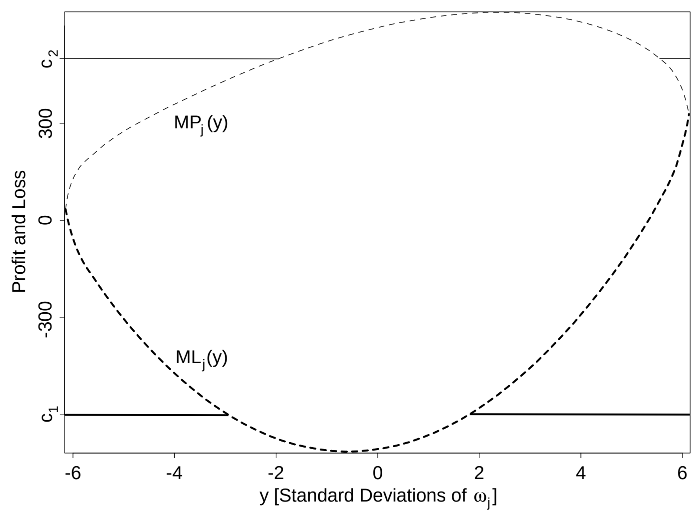

Figure 1: Determining safe and dangerous intervals of risk factor ω_j

4 Interpretations

By formula (13) a scenario is safe as soon as *one* of its components lies inside a safe interval (cf. figure 2).

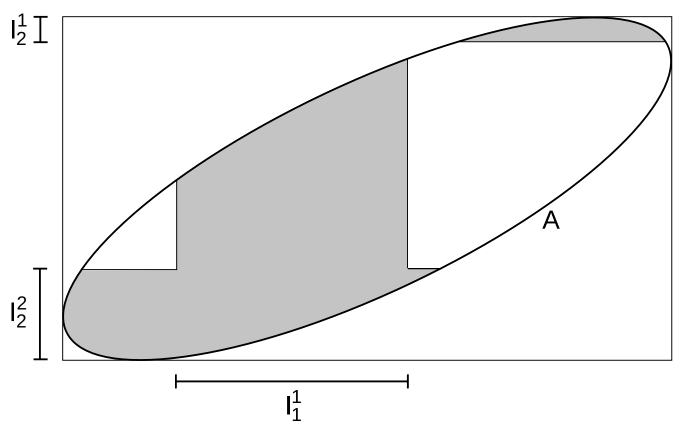

Figure 2: Additivity of safe intervals

However, this statement applies only to those scenarios which lie inside the trust region A with confidence level α. Therefore, it is possible to assign an error probability of $(1-\alpha)$ to each set of intervals. Since higher confidence levels enlarge the feasible domain in problem (14) it follows that lower error probabilities result

in smaller intervals (i.e., there is a tradeoff between accuracy and the size of identifiable intervals).
Moreover, the graphs of $\mathrm{ML}_j(y)$ and $\mathrm{MP}_j(y)$ reflect information about the risk sensitivity of the individual factors (cf. figure 3). The value ω_j^{ML} where $\mathrm{ML}_j(y)$ attains its minimum represents the j^{th} component of the global worst case scenario (i.e., the solution to (1)). On the other hand, $\mathrm{ML}_j(0)$ is equivalent to the maximum loss of a portfolio where all open positions in risk factor j have been closed. Thus, the difference

$$\Delta\mathrm{ML}_j = \mathrm{ML}_j(\omega_j^{ML}) - \mathrm{ML}_j(0), \tag{21}$$

is the amount by which ML is reduced if there are no longer positions in risk factor j. Similarly, this analysis can also be applied to the maximum profit:

$$\Delta\mathrm{MP}_j = \mathrm{MP}_j(\omega_j^{MP}) - \mathrm{MP}_j(0). \tag{22}$$

Appendix B explains how to calculate the expected P&L of the scenarios which lie inside the trust region. This way, closing all positions in risk factor j results in a reduction of

$$\Delta\mathrm{EV}_j = \mathrm{E}(v(\omega) \mid \omega \in A) - \mathrm{E}(v(\omega) \mid \omega \in A, \omega_j = 0). \tag{23}$$

Contrasting the values of $\Delta\mathrm{ML}_j$, $\Delta\mathrm{MP}_j$ and $\Delta\mathrm{EV}_j$ for all risk factors $j = 1, \ldots, M$ helps to decide which positions should be closed in order to reduce the total risk of a portfolio.

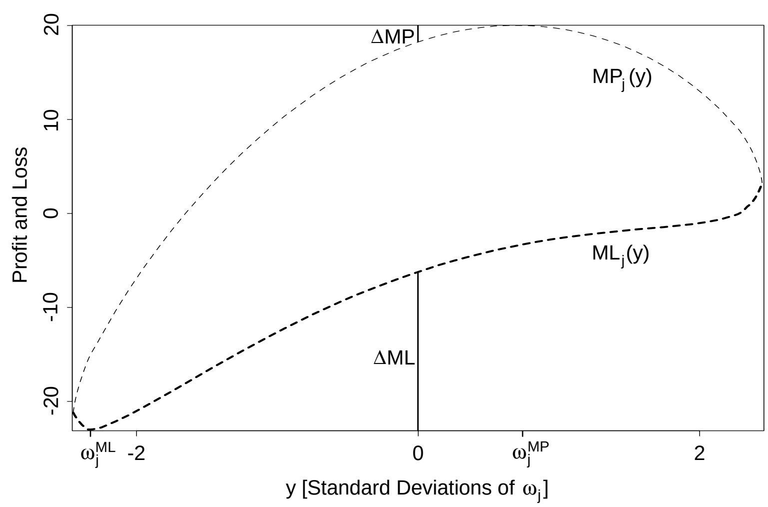

Figure 3: Effect of closing all positions in risk factor ω_j

5 Conclusion

The identification of safe and dangerous regions helps to get a better feel of the portfolio's risk exposure. One way to describe such regions in an easily interpretable manner is by one–dimensional intervals: a scenario is safe as soon as one component belongs to a safe interval. However, the fact of working with simple representations carries the cost of incomplete information: there may exist other scenarios whose P&L is above (resp. below) the critical level c, but which cannot be described by means of one–dimensional intervals. Nevertheless, the spread of P&L among all scenarios with $\omega_j = y$ can be obtained easily: it is the difference $\mathrm{MP}_j(y) - \mathrm{ML}_j(y)$.
Furthermore, a comparison of the global maximum loss and $\mathrm{ML}_j(0)$ (resp. the global MP and $\mathrm{MP}_j(0)$) for all risk factors $j = 1, \dots, M$, gives insight into which positions should be closed in order to reduce the total risk of a portfolio.

A Transformation of Restricted ML Problem into Standard Form

This chapter shows how to convert the restricted problem (14) with quadratic P&L function $v(\omega) = \frac{1}{2}\omega^T G\omega + g^T\omega$ into the standard form (3). For ease of notation, the procedure for risk factor $j = 1$ is presented. In this case, the input parameters can be rewritten as

$$G = \begin{bmatrix} G_{1,1} & G_1^T \\ G_1 & \tilde{G} \end{bmatrix}, \quad \Sigma_t^{-1} = \begin{bmatrix} \Sigma_{1,1}^{-1} & \Sigma_1^{-T} \\ \Sigma_1^{-1} & \tilde{\Sigma}^{-1} \end{bmatrix}, \quad g = \begin{bmatrix} g_1 \\ \hat{g} \end{bmatrix}, \quad \omega = \begin{bmatrix} \omega_1 \\ \hat{\omega} \end{bmatrix}, \tag{24}$$

which represents a partition into the first column/row and the remaining $(M-1)$ ones. Then, new variables are defined as follows:

$$\begin{aligned} \tilde{\omega} &= \hat{\omega} + \omega_1 \tilde{\Sigma}\Sigma_1^{-1} \\ \tilde{g} &= \hat{g} + \omega_1 \left[G_1 - \tilde{G}\tilde{\Sigma}(\Sigma_1^{-1})\right] \\ \tilde{k} &= \omega_1 \left[g_1 - \hat{g}^T\tilde{\Sigma}(\Sigma_1^{-1})\right] + \frac{1}{2}\omega_1^2 \left[G_{1,1} + (\Sigma_1^{-1})^T\tilde{\Sigma}\tilde{G}\tilde{\Sigma}(\Sigma_1^{-1})\right] \\ \tilde{c}_\alpha &= c_\alpha + \omega_1^2 \left[(\Sigma_1^{-1})^T\tilde{\Sigma}(\Sigma_1^{-1}) - \Sigma_{1,1}^{-1}\right], \end{aligned} \tag{25}$$

where $\tilde{\Sigma}$ is the inverse of $\tilde{\Sigma}^{-1}$. This way, the variable ω_j is eliminated and a new, $(M-1)$–dimensional problem is obtained:

$$\begin{aligned} \mathrm{ML}_j(y) = \quad & \min \quad \frac{1}{2}\tilde{\omega}^T\tilde{G}\tilde{\omega} + \tilde{g}^T\tilde{\omega} + \tilde{k} \\ & \text{s.t.} \quad \tilde{\omega}^T\tilde{\Sigma}_t^{-1}\tilde{\omega} \le \tilde{c}_\alpha. \end{aligned} \tag{26}$$

The geometric meaning of this procedure is shown in figure 4: the M–dimensional trust region A is cut by a plane, which lies orthogonal to ω_1. The result is an $(M-1)$–dimensional ellipsoid. To solve the minimization problem, this new ellipsoid has to be recentered at the origin; the transformation (25) keeps the objectif function quadratic (introducing a supplementary constant $\tilde{k}$).

B Conditional Expectation of Profit and Loss Inside an Ellipsoid

According to [Studer and Lüthi] the conditional expectation of the profit and loss function $v(\omega)$ *on the surface* of an ellipsoid is

$$\mathrm{E}(v(\omega) \mid \omega^T\Sigma_t^{-1}\omega = c_\alpha) = \frac{c_\alpha}{2}\frac{\mathrm{Tr}(\hat{G})}{M}, \tag{27}$$

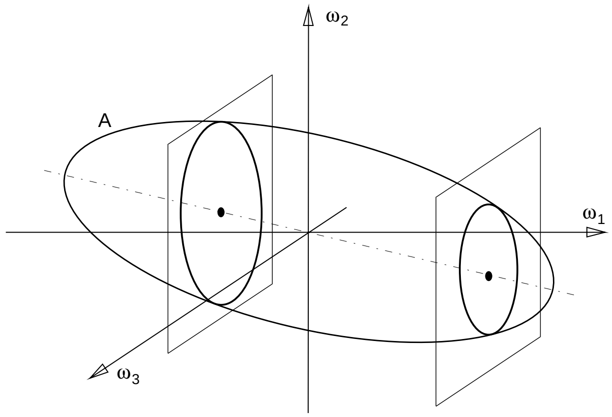

Figure 4: Transformation of trust region

where $\hat{G} = UGU^T$ is the transformed matrix of the quadratic function $v(\omega) = \frac{1}{2}\omega^T G\omega + g^T\omega$, where U is the Cholesky decomposition of the covariance matrix

$$\Sigma_t = U^T U. \tag{28}$$

Since c_α is the α quantile of a chi–square distribution (cf. [Studer]), c_α can be seen as random variate with density $f(x) = \frac{x^{\frac{M}{2}-1}}{\Gamma(\frac{M}{2})2^{\frac{M}{2}}}\exp(-\frac{x}{2})$. Thus, the conditional expectation of $v(\omega)$ *in the interior* of the trust region is:

$$\begin{aligned} \mathrm{E}(v(\omega) \mid \omega^T\Sigma_t^{-1}\omega \le c_\alpha) &= \int_0^{c_\alpha} \frac{x}{2}\frac{\mathrm{Tr}(\hat{G})}{M}\frac{1}{\alpha}\frac{x^{\frac{M}{2}-1}}{\Gamma(\frac{M}{2})2^{\frac{M}{2}}}\exp(-\frac{x}{2})dx \\ &= \frac{\mathrm{Tr}(\hat{G})}{2\alpha M}\frac{1}{\Gamma(\frac{M}{2})}\int_0^{c_\alpha}\frac{x^{\frac{M}{2}}}{2^{\frac{M}{2}}}\exp(-\frac{x}{2})dx \\ &= \frac{\mathrm{Tr}(\hat{G})}{\alpha M}\frac{1}{\Gamma(\frac{M}{2})}\int_0^{\frac{c_\alpha}{2}} y^{\frac{M}{2}}\exp(-y)dy \\ &= \frac{\mathrm{Tr}(\hat{G})}{\alpha M}\frac{\Gamma_{\frac{c_\alpha}{2}}(\frac{M}{2}+1)}{\Gamma(\frac{M}{2})}, \end{aligned} \tag{29}$$

where $\Gamma_t(x)$ denotes the incomplete Gamma function $\Gamma_t(x) = \int_0^t y^{x-1}\exp(-y)dy$.

References

[Artzner et al.] Ph. Artzner, F. Delbaen, J.–M. Eber, D. Heath. A Characterization of Measures of Risk. Working paper, February 1996.

[Beckström and Campbell] R. Beckström and A. Campbell. An Introduction to VAR. C·ATS Software Inc., Palo Alto, 1995.

[RiskMetrics] J.P. Morgan & Co. RiskMetricsTM - Technical Document. New York, May 1995.

[Studer] G. Studer. Value At Risk and Maximum Loss Optimization. RiskLab: Technical Report, Zurich, December 1995.

[Studer and Lüthi] G. Studer and H.–J. Lüthi. Quadratic Maximum Loss for Risk Measurement of Portfolios. RiskLab: Technical Report, Zurich, September 1996.