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Advanced value-at-risk

Cash-flow mapping

Advanced Risk Management

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Agenda

- Value-at-risk
- Cash-flow mapping description
- Cash-flow mapping properties and comparisons

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Value-at-risk

Fixed income portfolio

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Risk factors

The change of value of a portfolio can be explained by a finite number of risk factors. The type of risk factors we are interested in in this presentation are interest rate risk factors.

The interest rate risk factors are composed of a limited number of zero-coupon bonds of standard tenors (1m, 3m, 6m, 1y, 2y,…).

One needs to associate a fixed income portfolio to a vector in the space of the risk factors.

The goal of this presentation is to describe several ways to do the association (mapping) between cash-flows at non-standard dates and the risk factors.

Linear dependence

We are interested by cash-flow mapping. It implicitly means that we consider that fixed income instruments can be represented by fixed cash-flows and exclude contingent instruments.

The standard way to treat options in this framework is to replace the option by there delta equivalent in the underlying.

Normal distribution

In standard RiskMetrics type of value-at-risk computation, one supposes that the change of the risk factors is (joint-) normally distributed with mean 0.

This is not required for our presentation on cash-flow mapping. Nevertheless we will use this hypothesis for the geometrical representation of the VaR and in some examples.

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Definition

The value-at-risk of a portfolio is the forecast of a given (high) quintile of the distribution of the returns of the portfolio over a given horizon.

We give the definition of value-at-risk for the delta-normal version of VaR.

Suppose that the portfolio is associated to a vector $p=(p_1,\dots,p_N)$ describing its decomposition in the risk factors. If we denote V the covariance matrix of the changes of value of the risk factors over one period, the VaR associated to the portfolio is

$$VaR(p) = F^{-1}(?)\ (p^TVp)^{1/2}$$

where F^{-1} is the inverse of the cumulative normal distribution and ? is the probability level.

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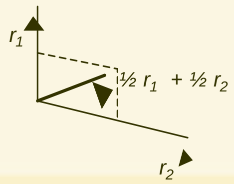
Geometric interpretation

As the matrix V is positive definite, the number

$$p^TVq$$

Is a scalar product. The length of the vector p associated to a portfolio is the VaR of the portfolio. The correlation between two positions is the cosine of the angle between the vectors.



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Cash-flow mapping

Description

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Introduction

For the description of the maps we always consider a cash-flow with present value 1. We need to allocate the cash-flow to positions with present value X_1 and X_2 on the surrounding vertices. If r_1 and r_2 and the surrounding risk factor, the cash-flow will be represented by $r = X_1 r_1 + X_2 r_2$.

Elementary map

- Construction

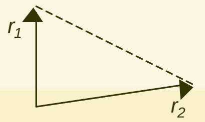
The elementatry mapping, also called *duration* mapping, preserves the present value and the duration.

Using those two conditions, we obtain

$$X_1 = \frac{t_2 - t}{t_2 - t_1} \quad \text{and } X_2 = \frac{t - t_1}{t_2 - t_1}\ .$$

- Geometric description

The vector r is a convex combination of the two vectors r_1 and r_2.



Elementary map (2)

- Forward rate

Another way to describe the mapping is to suppose that the *forward rate* (continuously compounded) are constant between standard tenors. We have a rate for a tenor *t* equal to

$$r_t = ?\frac{t_1}{t} r_1 + (1-?)\frac{t_1}{t} r_2 \qquad ? = \frac{t - t_1}{t_2 - t_1}\ .$$

By using $P_t = \exp(-r_t\, t)$ and the first order approximation, we have

$$\frac{\bar{P}_t - P_t}{P_t} \sim ?\frac{\bar{P}_1 - P_1}{P_1} + (1-?)\frac{\bar{P}_2 - P_2}{P_2}.$$

Rate map

Suppose that the (continuously compounded) rate is linearly interpolated, i.e.

$$r_t = ?\ r_1 + (1-?)\ r_2 \qquad ? = \frac{t_2 - t}{t_2 - t_1}\ .$$

By using $P_t = \exp(-r_t\ t)$ and the first order approximation, we have

$$\frac{\overline{P_t} - P_t}{P_t} \sim ?\ \frac{t}{t_1} \frac{\overline{P_1} - P_1}{P_1} + (1-?)\ \frac{t}{t_2} \frac{\overline{P_2} - P_2}{P_2}\ .$$

which means that investing 1 in t generates the same profit as investing $X_1 = ?\ \frac{t}{t_1}$ in t_1 and $X_2 = (1-?)\ \frac{t}{t_2}$ in t_2.

This approach is coherent with the way yield curves are often constructed.

RiskMetric map

- Name

We call *RiskMetrics* map the one descibed in the RiskMetrics Technical Document (‘96). It is the one originally used for the computation of the value-at-risk. In Return to RiskMetrics (‘01) this method is not used anymore and the rate map is suggested.

- Construction

This map preserves the present value, the sign of the present value and the volatility obtain by a linear interpolation. We then have

$$X_1 + X_2 = 1 \quad \text{and } ? = ?_1 + \frac{t - t_1}{t_2 - t_1}(?_2 - ?_1).$$

RiskMetric map (2)

After some computation we obtain

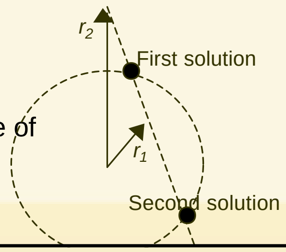
$$X1 = (-b \pm (b^2 - ac)^{½}) / a$$

$$A = ?_1^2\text{-}2?\ ?_1?_2+?_2^2 \quad b = ?\ ?_1?_2\text{-}\ ?_2^2 \quad \text{and} \quad c = ?_2^2\text{-}\ ?_1^2$$

Between the two possible solutions (±) one has to chose the one such that X_1 and X_2 are between 0 and 1.

- Geometric interpretation

The solution is on the line between the vectors r_1 and r_2. The length is interpolated linearly between the one of r_1 and r_2.



Other mapping

Other types of mappings have been described in the financial literature. Some to overcome problems with the *RiskMetrics* mapping, some to show that there was some freedom in the choice of the mapping. But none of them have been constructed with a financial reasoning in mind.

- Schaller (‘96)
- Polar coordinate (’00)
- Three dimensional map(’00)

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Cash-flow mapping

Properties and comparison

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RiskMetric map: discontinuity

A problem with the *RiskMetrics* map is that the map may be non-continuous in the sense that one does not necessarily have X_1 -> 1 and X_2 -> 0 when t-> t_1. This happens when

$$(r_2-r_1)^T V r_1 < 0 \quad \text{and} \quad ?_1 < ?_2.$$

i.e. when r_2 is in the half plane with boundary perpendicular and passing through r_1. In that case there are two solutions that satisfy the mapping description for t= t_1.

For a given pair of risk factors, this is not appearing very often. But if we consider all the possible consecutive risk factors for one day, then this phenomenon is not rare at all. Some test (done in 99) reveals that 5 days out of 6 this appeared at least once. Also it is not restricted to exotic currencies as we have occurrences in GBP, JPY, FRF, CAD,…

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Interesting property for multi-curve portfolios

For the mapping described, once the present value if the cash-flows is computed, the actual rates are not used in the different mapping formulas.

This is an interesting property if different products are prices from similar curves that differs by a spread but only one curve is part of the risk factors.

An algorithm to compute VaR suggested by the above property is the following.

1. Decompose the portfolio into equivalent cash-flows

2. Discount them with the correct curve (even if the curve is not used in the risk factors)

3. Do the mapping to risk factors using the present value.

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Interesting property for multi-curve portfolios (2)

To see the impact of this approach we take the example of a one year liability at a rate of *r-s* hedged with a one year asset at *r*. The cash-flow are such that the sensitivities are hedged

$$- c_2 / (1+r-s)^2 = - c_1/(1+r)^2.$$

Using the usual approach we have a present value of the cash-flow used for the VaR computation of

$$(c_2 - c_1) / (1+r) = c_1 /(1+r) \left(s^2 / (1+r)^2 - 2 s / (1+r)\right)$$

In the second case we have

$$c_2 /(1+r-s) - c_1/(1+r) = - c_1 /(1+r) \; s/(1+r)$$

which is approximatively an improvement by a factor of two.

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Mapping of mapping

The elementary and rate maps enjoy the property that “the mapping of a mapping is a mapping”.

Suppose that you have five times $t_1 < {}_{t2} < t < t_3 < t_4$. If we map the cash-flow in t to t_2 and t_3 and then the resulting cash-flows to t_1 and t4, we obtain the same results that the direct mapping of the initial cash-flow to t_1 and t_4.

This means that one can do first a very precise cash-flow mapping, that will summarise a full portfolio by a limited number of positions in risk factors. This is for example done to compute the sensitivity of a short-term book with respect to monthly buckets. Those results can be reused and map to the tenor of the VaR computation. In this case one does not need to use the full description of the book, the first monthly mapping can be used as source of the second mapping.

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Numerical complexity

The elementary and rate maps, computed from present value, uses only elementary arithmetic operations. As such they are very fast to compute. The *RiskMetrics* map required the computation of a square root.

Moreover the *RiskMetrics* map uses, on the top of the rate, the volatilities and correlations of the risk factors. So this approach can not be used to do some precise bucketing of the cash-flows if we don’t have the full variance-covariance matrix associated to it.

Comparison

We compare the different mappings using the following technique. We hedge a cash-flow of present value 1 in one standard tenor by mapping it to the preceding and following tenors. We compute the VaR of the hedged position.

The VaR computed is the residual risk created by the mapping technique, or the VaR (risk) of the error.

Comparison (2)

	RiskMetrics	Rates
15-Mar-99	38	62
15-Apr-99	38	61
14-May-99	41	64
15-Jun-99	37	72
14-Jul-99	44	64
10-Aug-99	42	70

Number of improvements for the different mapping with respect ot the elementary mapping. Number of risk factors: 119.

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Comparison (3)

	Elementary	*RiskMetrics*	Rates
Govt 3Y	112	122	6
Govt 4Y	160	173	91
Govt 5Y	383	375	419
Govt 7Y	312	318	250
Govt 9Y	188	196	159
Govt 10Y	486	483	489
Govt 15Y	572	558	136
Govt 20Y	1639	1778	460

Risk induced by a position of 1m in the risk factor hedged by the same amount mapped to the adjacent factors. Figures for the USD with the covariance matrix of 14 July 99.

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Comparison (4)

If I have to give a (subjective) ranking, it would be

1. Rate mapping
2. Elementary mapping
3. *RiskMetrics mapping*

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References

- Henrard, M. Comparison of cash flow maps for the value-at-risk. *The Journal of Risk,* 3(1):57-71, 2000.
- Henrard, M. *Value-at-Risk: The Delta-normal Approach*.
- Mina, J. Improved cash-flow map. Working paper, RiskMetrics Group, April 1999.
- Morgan Guaranty Trust Company. *RiskMetrics Technical Document*, 4th edition, 1996.
- Schaller, P. On cash-flow mapping in VaR estimation. Technical report, Credietanstalt-Bankverein, 1996.