Aggregating credit risk in a corporate bond portfolio

Srichander Ramaswamy
Head of Investment Analysis

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Summary of presentation

- Portfolio credit risk
- Loss correlation
- Default correlation
- Asset return correlation
- Computing loss correlation under default mode
- Computing loss correlation under migration mode
- Numerical examples

Portfolio credit risk

- We have seen so far that for a single corporate bond, the credit risk can be quantified in terms of its expected loss EL and unexpected loss UL
- One can think of extending this credit risk quantification frame-work to a portfolio of corporate bonds
- Under such a framework, the credit risk of the corporate bond portfolio will be quantified in terms of expected portfolio loss EL_P and unexpected portfolio loss UL_P
- To compute EL_P and UL_P we need to work with the random loss variable $\tilde{\ell}_i$ of ith bond in the portfolio
- This random variable has mean EL_i and standard deviation UL_i

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Two-bond portfolio

- Let us take a simple 2-bond portfolio example and try to compute the portfolio credit risk
- An investor holding a 2-bond portfolio will be faced with a loss distribution given by,

$$\tilde{\ell}_P = \tilde{\ell}_1 + \tilde{\ell}_2$$

- The expected portfolio loss is given by,

$$EL_P = E(\tilde{\ell}_1 + \tilde{\ell}_2) = EL_1 + EL_2$$

- The variance of the portfolio loss is given by,

$$\begin{aligned} Var(\tilde{\ell}_P) &= E[(\tilde{\ell}_1 + \tilde{\ell}_2)^2] - [E(\tilde{\ell}_1 + \tilde{\ell}_2)]^2 \\ &= E(\tilde{\ell}_1^2) + E(\tilde{\ell}_2^2) + 2E(\tilde{\ell}_1\tilde{\ell}_2) - EL_1^2 - EL_2^2 - 2EL_1 \times EL_2 \\ &= UL_1^2 + UL_2^2 + 2E(\tilde{\ell}_1\tilde{\ell}_2) - 2EL_1 \times EL_2 \end{aligned}$$

Loss correlation

- Let us denote the correlation between the two random loss variables $\tilde{\ell}_1$ and $\tilde{\ell}_2$ as ρ^{ℓ}_{12}
- We will denote this the loss correlation and is given by,

$$\rho^{\ell}_{12} = \frac{E(\tilde{\ell}_1 \tilde{\ell}_2) - EL_1 \times EL_2}{UL_1 \times UL_2}$$

- From probability theory we know that if x and y are two random variables, then the correlation coefficient between these random variables can be estimated using observed data as given below

$$\rho_{xy} = \frac{\frac{1}{n}\sum_{i=1}^{n} x_i y_i - \bar{x} \cdot \bar{y}}{\sigma_x \sigma_y}$$

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Putting it together

- We have seen that for a 2-bond portfolio, the portfolio credit risk can be quantified as follows:

$$EL_P = EL_1 + EL_2$$

$$UL_P = \sqrt{UL_1^2 + UL_2^2 + 2\rho^{\ell}_{12} UL_1 UL_2}$$

- Computing UL_P will require knowledge of the loss correlation between the two loss variables
- Depending on whether the loss variable $\tilde{\ell}_i$ models the credit risk under default mode or migration mode, we can distinguish between two loss correlation numbers
- However, it is not possible to compute loss correlation directly!

Quantifying portfolio credit risk

- When an investor is holding bonds issued by n obligors in the portfolio, the expected and unexpected losses are given by,

$$EL_P = \sum_{i=1}^{n} EL_i$$

$$UL_P = \sqrt{\sum_{i=1}^{n}\sum_{k=1}^{n} \rho_{ik}^{\ell} \times UL_i \times UL_k} \qquad where,\ \rho_{ik}^{\ell} = 1\ if\ i = k$$

- The above equations generalizes to any portfolio that could potentially include more than one bond issued by same obligor

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But wait …

- What happened to the famous default correlation that is discussed in every credit risk text book?
- Default correlation between two obligors is defined as the correlation between the default indicators for these two obligors over some specified interval of time (1-year typically)
- From the standard definition of correlation between two random variables, we have the following relation for default correlation:

$$\rho_{ik}^{\delta} = \frac{E(I_{[\delta_i=1]} \cdot I_{[\delta_k=1]}) - E(I_{[\delta_i=1]})E(I_{[\delta_k=1]})}{\sqrt{Var(I_{[\delta_i=1]}) \times Var(I_{[\delta_k=1]})}}$$

$$\rho_{ik}^{\delta} = \frac{prob(\delta_i = 1, \delta_k = 1) - PD_i \times PD_k}{\sqrt{PD_i \times (1 - PD_i) \times PD_k \times (1 - PD_k)}}$$

(since default indicator is a Bernoulli random variable)

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Relationship between loss and default correlation

- Loss correlation we introduced is applicable to both the default mode as well as the migration mode
- Default correlation, on the other hand, is only relevant under the default mode
- After some mathematical manipulations, one can show that default and loss correlation are related as follows when credit risk is aggregated under the default mode

$$\rho_{ik}^{\ell} = \frac{NE_i \times NE_k \times \sigma_{PD_i} \times \sigma_{PD_k} \times LD_i \times LD_k}{UL_i \times UL_k} \times \rho_{ik}^{\delta}$$

- In deriving the above relation, we assume that the recovery rates between two obligors are independent

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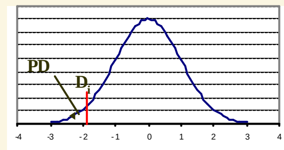
Computing loss correlation under default mode

- Due to lack observations on joint credit events, we mentioned earlier that loss correlation cannot be directly estimated
- This statement is also true for default correlation
- Of course, if we know the default correlation, we can determine the loss correlation using the earlier relationship
- To make an estimate of default correlation or loss correlation, we need a theoretical framework
- The theoretical framework looks for variables that are drivers of credit events
- In Merton's framework, credit events are driven by asset returns

Computing loss correlation under default mode

- Therefore, correlation between asset returns of two firms can be used for indirect estimation of the default correlation between the two firms
- Under Merton's theoretical framework, an obligor default is triggered if the asset returns of the firm exceed a certain default threshold, denoted D_i
- The default threshold can be determined if we know probability of default for the issuer and the distribution of asset returns

$$PD_i = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{D_i} \exp(-\tfrac{1}{2} z^2)\, dz$$



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Computing loss correlation under default mode

- If we make the simplifying assumption that asset returns are jointly normal with asset return correlation being ρ_{ik}^{α} the joint probability of default between the two obligors is given by,

$$prob(\delta_i = 1, \delta_k = 1) = \frac{1}{2\pi\sqrt{1-(\rho_{ik}^{\alpha})^2}} \int_{-\infty}^{D_i}\int_{-\infty}^{D_k} \exp\left(-\frac{(x^2 - 2\rho_{ik}^{\alpha} xy + y^2)}{2[1-(\rho_{ik}^{\alpha})^2]}\right) dxdy$$

- Once the joint probability of default is determined, the default correlation between two obligors can be computed
- From default correlation, we can compute the loss correlation under the default mode between two obligors

Numerical example

Bond level details of example considered		
Description	**Bond 1**	**Bond 2**
Bond issuer	Oracle Corp	Alliance Capital
Issuer rating grade	A3	A2
Settlement date	24 Apr 2002	24 Apr 2002
Bond maturity date	15 Feb 2007	15 Aug 2006
Coupon rate	6.91%	5.625%
Dirty price for $1 nominal	1.0533	1.0029
Nominal exposure	$1,000,000	$1,000,000
PD (historical)	10 bp	8 bp
KMV's EDF	58 bp	158 bp
Mean recovery rate	47%	47%
Volatility of RR	25%	25%

Numerical example

- ***Using historical PD and asset return correlation = 30%***
- Joint default probability = 1.2505e-05
- Default correlation = 0.01301
- Loss correlation when recovery rates between issuers are independent = 0.0109
- Recovery rate correlation when assumption that loss correlation and default correlation are equal is made = 0.9401
- Expected portfolio loss = $1,010
- Unexpected portfolio loss using loss correlation = $26,205
- Unexpected portfolio loss using default correlation = $26,233

But how do we compute asset return correlation?

- Equity of a firm is traded and not the firms assets
- The firm's assets can be represented as

  $$A_t = S_t + B_t$$

- In this case equity return correlation can be used to infer the asset return correlation
- In some cases, even equity is not traded (Ford Motor Credit is not traded, only parent company is traded)
- If mergers or acquisitions take place or the a company's core business changes, past equity returns are not representative
- Computing asset return correlation directly is also not an option
- We are basically running around in circles here!

Factor models can help here

- Some of the above difficulties in estimating asset return correlation can be addressed through the use of factor models
- A factor model relates the systematic or non-diversifiable components of the firm's asset returns to various factors that drive the firm's asset returns
- Knowledge of the sensitivities to the common factors and the correlation between common factors will then allow us to estimate the asset return correlation between obligors
- KMV Corporation makes use of a factor model and will provide asset return correlation between firms (for a price!)
- It is possible to derive approximate asset return correlations

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Portfolio credit risk under migration mode

- While computing default correlation between two obligors, we ignored the rating transitions to non-default states completely
- Hence, loss correlation computed under default mode also ignores transitions to non-default states for the obligors
- In a more general setting, loss correlation between obligors can also result from rating migrations to different grades
- To compute the loss correlation, we will have to work with the joint probability distribution of credit losses for obligor pairs
- Let us first recall the equation for loss correlation stated earlier

$$\rho_{12}^{\ell} = \frac{E(\tilde{\ell}_1 \tilde{\ell}_2) - EL_1 \times EL_2}{UL_1 \times UL_2}$$

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Computing loss correlation under migration mode

- From this relationship, we can compute loss correlation between two obligors if we can compute the quantity $E(\tilde{\ell}_1 \tilde{\ell}_2)$
- Since we are dealing with a discrete distribution with 18 possible states, we can enumerate all the states of the joint loss distribution of a 2-obligor portfolio, which is 18x18=324 states
- To compute the quantity $E(\tilde{\ell}_1 \tilde{\ell}_2)$ we need to estimate the credit loss associated with each of the 324 states and the probability of occupying these states
- If p_i denotes the probability of occupying each state and L_i is the corresponding credit loss in the state, we have

$$E(\tilde{\ell}_1 \tilde{\ell}_2) = \sum_{i=1}^{324} p_i L_i$$

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Let us first get a feel for this

- Consider a 2-obligor portfolio with 3 possible states for each obligor numbered as 1, 2 and 3
- The total number of states in which this 2-obligor portfolio could be in 1-year from now will be 9 as given below
- (1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)
- If the current ratings of obligor A is 1 and that of obligor B is 2, the state (1,2) will denote that both obligors at the end of one year remain in their current ratings
- The state (2,1), on the other hand, represents a rating downgrade for obligor A and a rating upgrade for obligor B
- There is a probability 1 of being in one of the 9 states

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Probability of occupying different states

- But how do we determine the probability of occupying any one of the 324 states we enumerated in the joint distribution?
- We first have to extend Merton's framework to include transitions to non-default credit states
- Under the extended Merton's framework, we have to compute thresholds for transitioning to different credit states
- These thresholds will be a function of the rating migration probabilities, which in turn will depend on the current credit rating
- For a given rating transition matrix, we will have a corresponding threshold matrix which we will call the z-threshold matrix

Z-thresholds for A-rated obligor

Rating	Transition probability	Z-threshold
AAA	0.07%	infinity
AA	2.25%	3.1947
A	91.75%	1.9917
BBB	5.19%	-1.5607
BB	0.49%	-2.4372
B	0.20%	-2.8071
CCC	0.01%	-3.2905
Default	0.04%	-3.3527

Computing joint migration probabilities

- We saw how to compute the various z-thresholds for credit migration of any obligor
- What we are really interested in is the joint migration probabilities to different states (324 of them) for a 2-obligor portfolio
- We will once again make the assumption that credit events are driven by asset returns and correlation between asset returns drive joint credit migrations
- Under this framework, we can compute the joint migration probabilities if we know the joint distribution of asset returns
- Let us make the assumption that this is bivariate normal

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Computing joint migration probabilities

- If ρ denotes the asset return correlation between two obligors, the joint migration probabilities are given by,

$$h_{ik} = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{z_{u,i}}^{z_{u,i+1}} \int_{z_{v,k}}^{z_{v,k+1}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)} \right) dxdy$$

- The above represents the joint probability that in 1-year time obligor 1 has a credit rating i and obligor 2 credit rating k
- Following this approach, it is easy to calculate the joint migration probabilities to any of the 324 states for a given obligor pair
- We will have to do this calculation for all obligor pairs in the portfolio

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Computing joint credit loss

- The next stet is to compute the credit loss associated with each of the 324 states of the joint probability distribution
- The joint credit loss as a result of the rating migration of obligor 1 to state i from state u and obligor 2 to state k from state v is,

$$g_{ik} = NE_1 \times NE_2 \times \Delta P_{ui,1} \times \Delta P_{vk,2}$$

- The credit loss of each obligor in the above equation can be computed if we know the yield spreads between rating grades
- The expected value of joint losses can now be computed

$$E(\tilde{\ell}_1 \tilde{\ell}_2) = \sum_{i=1}^{18} \sum_{k=1}^{18} h_{ik} \times g_{ik}$$

Numerical example

- We will consider the earlier 2-bond portfolio example and assume asset return correlation between the obligors is 30%
- Under the migration mode the various portfolio credit risk quantities of interest are given below

Description	Migration mode	Default mode
Loss correlation	0.0633	0.0109
Expected loss	$4,740	$1,010
Unexpected loss	$31,610	$26,205

23-bond portfolio example

Portfolio credit risk under migration mode				
Description	**EL_P (mn)**	**UL_P (mn)**	**$\%EL_P$**	**$\%UL_P$**
Under migration mode and 30% asset return correlation	$ 1.622	$ 4.603	34.0 bp	96.6 bp
Under default mode and 30% asset return correlation	$ 0.660	$ 3.268	13.8 bp	68.6 bp