BANK FOR INTERNATIONAL SETTLEMENTS

A framework for valuing fixed income securities

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Presentation outline

1. The need for valuation models
2. Factors affecting value
3. Valuing bullet bonds
4. Valuing callable bonds
5. Valuing mortgage-backed securities
6. Questions and answers
7. Appendix

In this presentation we take a step-by-step approach in developing a robust framework that can be used to value interest rate-dependent securities. This framework is needed to value securities with embedded options whose payoff is dependent on the path followed by interest rates.

We start by considering a simple bullet bond. The payments associated with this bond are know in advance. We show how this bond can be valued by using the zero coupon yield curve to discount each of the payments that is made by the bond. We then show that this is equivalent to successively using the forward short rate to discount the payments that get made by the bond.

We then consider callable bonds, these bonds can be called by the issuer on pre-defined dates at a pre-defined price. The call option of the issuer means that there is some uncertainty in the bond’s cash flow stream since the issuer will call the bond when it is in his financial interest to do so but will not call if it is not in his interest. The value of the call option depends on the evolution of interest rates: in order to value the callable bond it necessary to model the evolution of interest rates.

We finally consider MBS. The borrowers in a pool of mortgages backing an MBS have the right to prepay the outstanding balance of their loan. This prepayment option is, in many respects, similar to the call option of a callable bond. It is however much more difficult to predict prepayment behaviour of MBS than it is to predict the exercise of the call option on a callable bond.

As we move from bullet bond to callable bond to MBS we show how a simple extension of the valuation methodology enables it to accommodate the more complex nature of each instrument.

The step-by-step approach that has been adopted is intended to be intuitively simple. The presentation concentrates on clarifying the approach of the valuation framework. More rigorous mathematical support of the approach is given in the appendix (which will not be covered during the presentation).

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The need for valuation models

- Valuation models attempt to determine fair value of securities
- Reconciling model price with market price
  - Risk premia
- Relative vs absolute value
  - Models are often much better at determining relative value
- Risk/return analysis
  - We need to understand how their value is affected by change in the underlying variables
  - Valuation models are needed for scenario analysis

Valuation models are obviously necessary to value over-the-counter (OTC) securities for which a price is not observable in the market.

Reconciling model price with market price of traded securities may be used to determine the premium that the market places on certain types of risk. For example, the price of off-the-run Treasury instruments will be below that of on-the-run Treasury instruments, this means that the yield of off-the-run bonds is above that of on-the-run bonds, this yield pick-up is compensation for assuming liquidity risk.

Valuation models that are not very good at determining the absolute value of a security may, nevertheless, be very good at determining relative value between similar securities. This means that valuation models can be used to help us decide which security to buy when we have several alternatives.

Valuation models may also be used to help us quantify the risks inherent in holding a security. As a first step in quantifying risk investors often look at historic price changes of the security. Historic analysis will not identify risks that may be specific to the current investment horizon it is for this reason that historic analysis should be complemented with forward-looking analysis of possible security return. In order to carry out this forward-looking return analysis we need a valuation model to determine security return under various possible future states of the world and we need to assign probabilities to each outcome.

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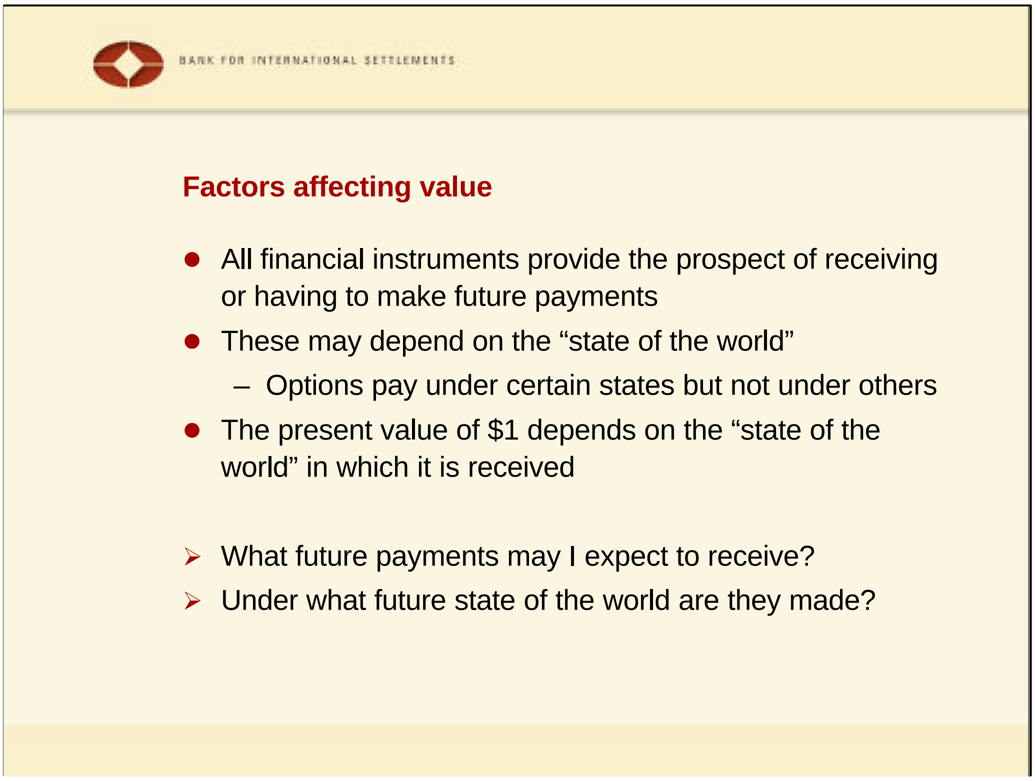


The one thing that all financial instruments have in common is the prospect of receiving (or having to make) future payments.

Bullet bonds are among the few types of security for which there is no uncertainty about what payments that will be made. The payments that get made by more complex securities depend on the "state of the world", payments may be made in some states but not in other states. Consider a world that has two possible future states: the "up" state and the "down" state. In the down state borrowing is cheap (rates are low), in this state issuers will call callable bonds they may have issued and refinance their borrowing at a cheaper rate. In the up state borrowing is expensive, in this state of the world issuers will not call callable bonds they may have issued. This example shows that the payments that get made by financial instruments may depend on the state of the world.

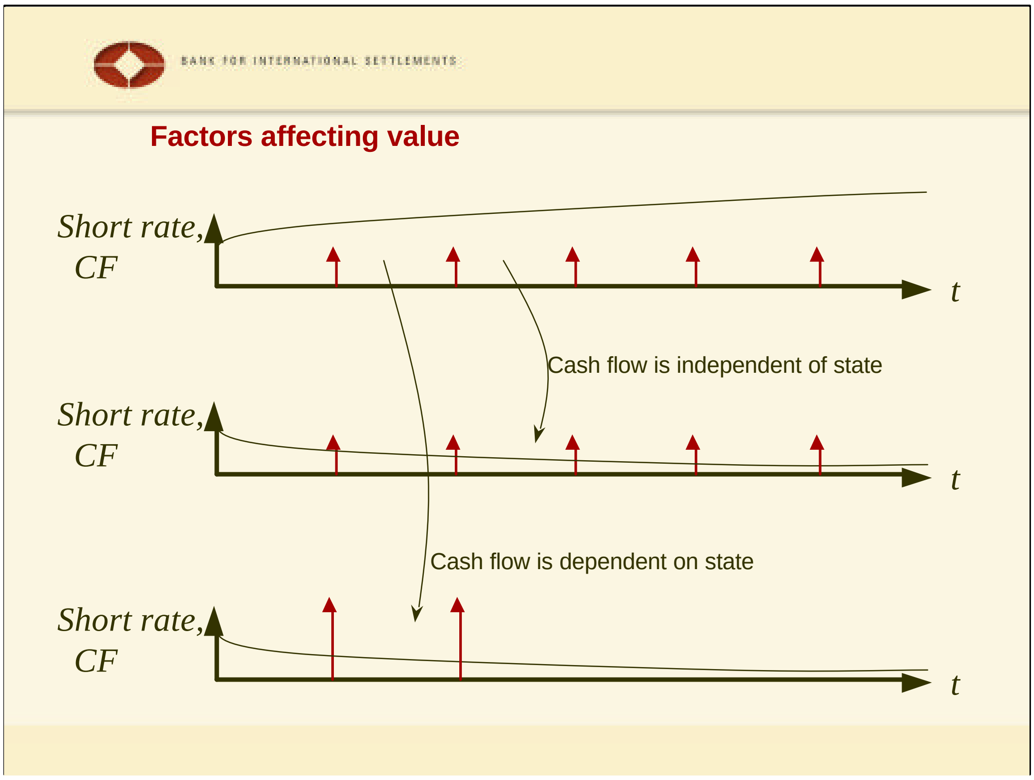
In general, the occurrence of future payments is dependent on the state of the world. Furthermore, the present value of a $1 payment also depends in which future state of the world it is made. A $1 payment made in one state may be worth more today than a $1 payment that gets made in a different state (even if the two states correspond to the same instance in the future). Consider a world that has got two possible future states, and there is equal probability that either state is realized. Let us assume that state 1 emerges after a period of high inflation and that state 2 emerges after a period of low inflation. Let us also assume that there are two securities: security 1 makes a payment of $1 only in state 1, security 2 makes makes a $1 payment only in state 2.

Which security has greater value?

Which one would you pay more for?

I would pay more for the prospect of getting $1 after a period of low inflation than for an equal prospect of receiving $1 after a period of high inflation.

Security 2 is worth more than security 1.



This slide shows different possible (hypothetical) evolutions of short rate as well as the payments that get made by some hypothetical security.

The short rate is the rate that should be used to discount payments that get made at the end of one (a "short" period that in practice often varies in length between one day and one year) period of time.

The slide provides a graphical representation of the concepts that were introduced in the previous slide. The top graph shows the world evolving across states of progressively higher short rates whereas the two lower graphs show the world evolving across states of progressively lower short rates.

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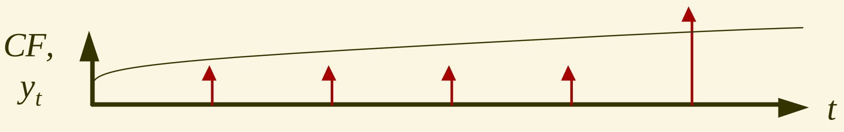
Valuing bullet bonds (the standard way)

- Bullet bonds provide a known future cash flow stream
  - The PV of each payment is equal to the payment discounted using the zero coupon yield
  - The value of the bond is the sum of the present value of each payment



$$PV = \frac{CF_1}{(1+y_1)^1} + \frac{CF_2}{(1+y_2)^2} + \frac{CF_3}{(1+y_3)^3} + \ldots \frac{CF_N}{(1+y_N)^N} = \sum_{t=1}^{N} \frac{CF_t}{(1+y_t)^t}$$

A bullet bond has got a certain (known) cash flow stream associated with it. The standard way to value such a security is to discount each of the bond's payments using the appropriate zero coupon yield. The present value (PV) of the bond is then equal to the sum of the present values of the individual payments.

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Valuing bullet bonds: an example

- Consider a $10-coupon bond maturing in three years
  - This bond will be used in examples throughout the presentation

Year	Zero coupon yield	Discount factor	Cash	PV
1	10%	0.909	$10.00	$9.09
2	11%	0.812	$10.00	$8.12
3	12%	0.712	$110.00	$78.30
				$95.50

$$= \frac{1}{(1+0.12)^3}$$

This example shows how to value a bond that pays $10 coupons once a year and that matures in three years.

This bond makes a $10 (coupon) payment in one year, a $10 (coupon) payment in two years, and a $110 (coupon plus principal) payment in three years. The zero coupon yields corresponding to maturities of one, two, and three years are 10%, 11%, and 12% respectively.

From the zero coupon yield we determine the discount factor that should be used to discount each payment. For example, payments made in three years time should be discounted using a factor of 0.712. The PV of the $110 payment that is made in three years time is therefore 0.712 × $110 = $78.30.

The sum of the PVs of all payments is $95.50 which is the value of the bond.

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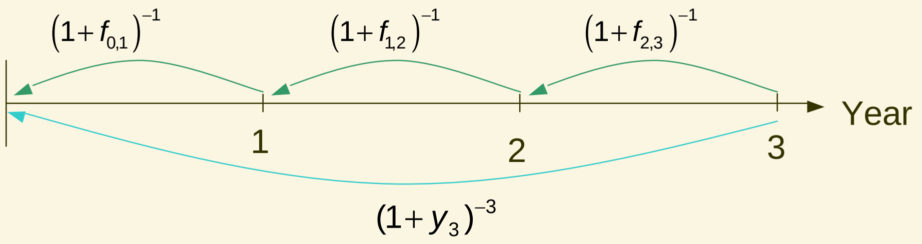
Valuing bullet bonds (using a series of forward short rates)

- $f_{n,n+1}$ is the forward short (one period) rate which can be fixed today for the period between *n* and *n+1* ($f_{0,1} = y_1$ is known today)



- The absence of arbitrage opportunities ensures that:

$$(1+y_3)^{-3} = (1+f_{0,1})^{-1} \times (1+f_{1,2})^{-1} \times (1+f_{2,3})^{-1}$$

Forward rates are the rates that can be locked in today for a future period. There is a fundamental (no arbitrage) relationship that exists between forward rates and the zero coupon yield curve. This relationship ensures that discounting a future cash flow by successively using the forward short rate is equivalent to using the corresponding zero coupon rate.

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Valuing bullet bonds (using a series of forward short rates)

- Forward short rates can be determined from today's zero coupon yield curve.
- Successively discount future payments using a series of forward short rates to get the PV of each future payment.

$$PV = \frac{CF_1}{(1+f_{0,1})} + \frac{CF_2}{(1+f_{0,1})(1+f_{1,2})} + \frac{CF_3}{(1+f_{0,1})(1+f_{1,2})(1+f_{2,3})} + \ldots$$

Compare this to:

$$PV = \frac{CF_1}{(1+y_1)^1} + \frac{CF_2}{(1+y_2)^2} + \frac{CF_3}{(1+y_3)^3} + \ldots$$

Equals

Absence of arbitrage arguments can be used to determine a series of forward short rates from the current zero coupon yield curve.

The no arbitrage relationship between the forward short rates and the zero coupon yield ensures that successive discounting using the forward short rate produces the same result as using the zero coupon yield.

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Valuing bullet bonds: an example

- Consider the same $10-coupon bond maturing in three years

Year	Zero coupon yield	Forward short rate	Short discount	Discount factor	Cash	PV
1	10%	10.00%	0.909	0.909	$10.00	$9.09
2	11%	12.01%	0.893	0.812	$10.00	$8.12
3	12%	14.03%	0.877	0.712	$110.00	$78.30
						$95.50

$$= \frac{1}{1+0.1403}$$

$$= 0.909 \times 0.893 \times 0.877$$

The column *zero coupon yield* shows the yield to be used to discount payments that occur at various future dates.

The column *forward short rate* shows the rate that should be used to discount payments to the previous year. For example: a rate of 14.03% should be used to discount payments from year 3 to year 2; a rate of 12.01% should be used to discount payments from year 2 to year 1; and a rate of 10% should be used to discount payments from year 1 to the present.

The forward short rate is determined from the zero coupon yield curve as follows:

$(1 + y_2)^2 = (1 + y_1) \times (1 + f_{1,2})$

$\Rightarrow (1 + f_{1,2}) = (1 + y_2)^2 / (1 + y_1) = (1.11)^2 / (1.1) = 1.1201$

$(1 + y_3)^3 = (1 + y_2)^2 \times (1 + f_{2,3})$

$\Rightarrow (1 + f_{2,3}) = (1 + y_3)^3 / (1 + y_2)^2 = (1 + 0.12)^3 / (1 + 0.11)^2 = 1.405 / 1.232 = 1.14025$

The short discount is the discount factor corresponding the the forward short rate. And the (total) discount factor is obtained by successive use of the short discount. The slide shows how the three year discount factor is obtained by successive use of three short discount factors.

Valuing bullet bonds

Q. Why use the forward short rate successively to discount future payments when we can simply use the zero coupon yield to discount future payments?

A. Because we will not always be able to use the zero coupon yield curve to discount future payments.

- Using the forward short rate points the way to using simulated short rates
  - This procedure will be necessary for the valuation of interest rate sensitive securities with embedded options

The use of the forward short rate may seem an unnecessary complication in the valuation of bullet bonds. Because of the no arbitrage relationship between the zero coupon yield curve and the forward short rates, the two methods are in effect equivalent.

Because the two methods are in effect the same: if we can use a series of forward short rates to discount future payments we can also use the zero coupon yield to discount these payments and there seems to be little reason to use the more complex procedure of successive use of the forward short rate. We will, however, extend the idea of using forward short rates to that of simulating the evolution of the short rate and then using these simulated short rates to value more complex fixed income securities.

Each simulation of the evolution of the short rate represents a possible path that may be followed by the (spot) short rate.

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Valuing bullet bonds: using an interest rate model

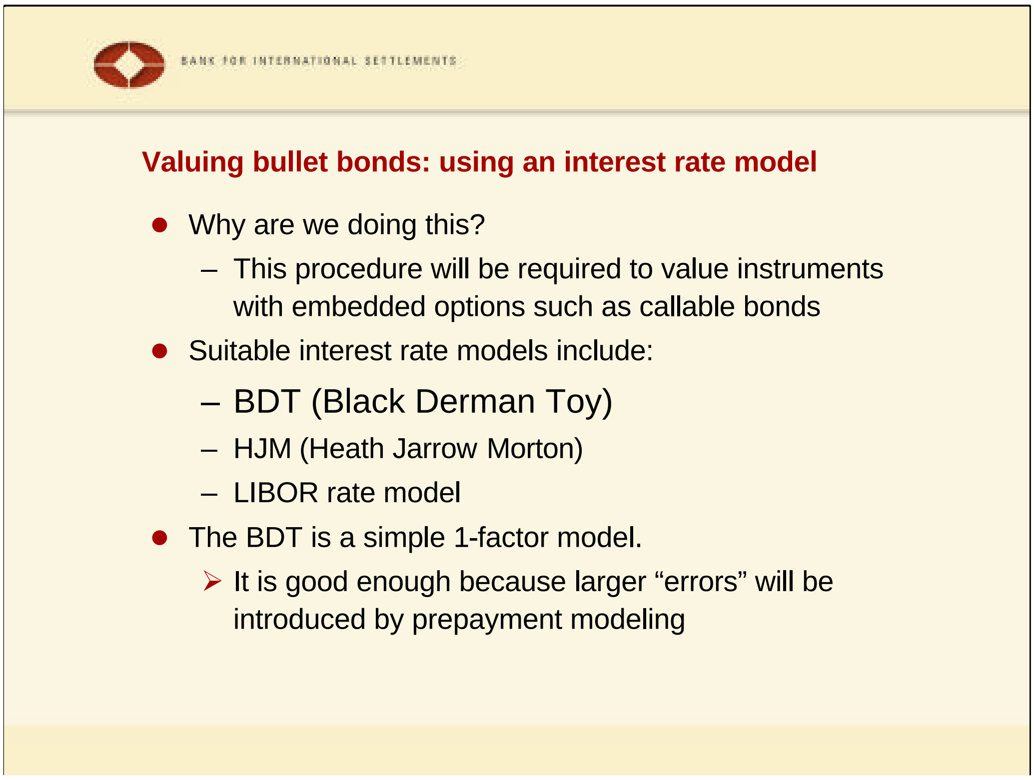
- Instead of using the forward short rate, use a model to predict how the short rate may evolve
  - Multiple runs result in different evolutions of the short rate
  - For each run successively use the simulated short rate to discount payments that are anticipated along the simulated path
  - Repeat over many runs to determine (average) value

<u>Technical comment</u>

In general no relationship exists between forward short rates and the expected value of the spot short rate for the corresponding period. The expected value of the short rate will be a function of the risk tolerance of investors.

We use a concept know as "risk-neutral valuation" to price options. Within this valuation framework investors are indifferent to the how risky a security is, and the expected return on all securities is the risk free rate of return. The security value obtained using risk-neutral valuation will, nevertheless, also be correct in the real, risk-averse, world.

The interest rate paths that we simulate are consistent with a risk-neutral interest rate process. For such a process the expected short discount is equal to the short discount corresponding to the forward short rate (see Appendix).



Our ultimate goal is to value MBS. We know that we will need to use an interest rate model to value these securities.

Before valuing interest rate derivatives with the help of an interest rate model we show how an interest rate model can be used to value a simple bullet bond.

We have chosen the Black, Derman and Toy (BDT) model to model the evolution of the short rate. This is a one factor model that is simple to implement. Because the evolution of the short rate is modeled in a binomial tree it is also relative easy for us to visualize and understand.

We could choose to use a more advanced interest rate model. In practice there is little benefit to be had from doing this as the largest errors in MBS valuation will be introduced by the prepayment model that is required to predict payments that get made along each possible interest rate path, and not by the interest rate model itself.

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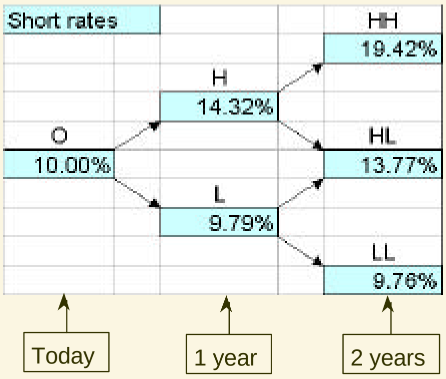
Valuing bullet bonds: using the BDT interest rate model

- BDT is a binomial tree model for the evolution of the short rate
- The tree is built in a manner that is consistent with the current term structure of the yield curve and a risk-neutral interest rate process

Year	Yield	Yield vol.
1	10%	-
2	11%	19%
3	12%	18%

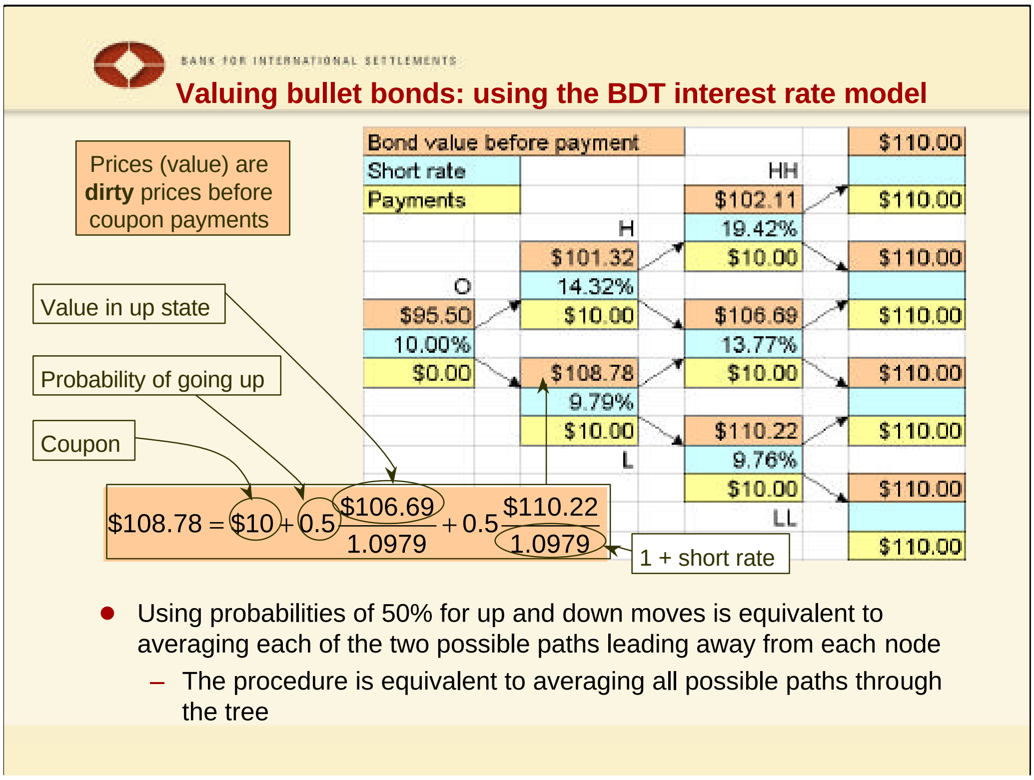
- $f_{2,1}$ = 12.01% ⇒ Short disc. = 0.893
- Possible model short rates
  1. 14.32% ⇒ Short disc. = 0.875
  2. 9.79% ⇒ Short disc. = 0.911
  - Av. model short disc. = 0.893



Here we do not worry about how to construct the binomial tree of short rates, we take the tree as given. Details on how to construct the tree are provided by Black, Derman and Toy, “A one-factor model of interest rates and its application to treasury bond options”, Financial Analyst Journal, Jan/Feb 1990.

The tree shows the short rate that should be used to discount payments that occur in one period (after one year in our case).

Today we know the rate that should be used to discount payments that occur in one year, it is 10%. In one year from now there are two possible rates (states of the world): 14.32% and 9.79%. If, in one year, we end up at node H the short rate is 14.32% and we should use this value to discount payments that will be made in one (further) year.



To value the bullet bond we work our way backwards through the binomial tree. We can do this because we know the value of the bond at all the terminal nodes of the tree and because the value of the bond at any node in the tree is a function of the payment made by the bond at that node, the value of the bond at the two nodes immediately to its right and the short rate that should be used to discount.

At node HH there is a 50% probability that we move higher and we end up with a bond value of $110. Discounting $110 at a short rate of 19.42% gives $92.11. The payoff of $110 in the upstate only occurs with a 50% probability the contribution of this payoff to the value of the bond at node HH is therefore $92.11/2. There is also a 50% probability that we move lower from node HH and end up at a node where the bond value (also) is $110, the contribution of this payoff to the value of the bond at node HH is therefore also $92.11/2. At node HH the probability weighted value of the future payments of the bond is therefore $92.11, if we add the $10 coupon that is made at node HH we get a value of the bond before coupon payment of $102.11.

We can repeat this process to value the bond at node LL and at the third node corresponding to 2 years. Once we have valued the bond at all the nodes that correspond to 2 years we can step back through the tree and value the bond at the two nodes that correspond to 1 year and then value the bond today.

It is noted that the process of working our way back through the tree is equivalent to selecting every possible path through the tree and discounting the payments that are made along that path by successively using the value of the short rate along the path.

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Valuing callable bonds

- Bond value depends on:
  1. Future payments
     - Depends on whether the bond gets called
     - Can the issuer refinance at a lower rate?
     - Depends on the the "state of the world" (interest rate environment)
  2. PV of these payments depends on "state of world" in which they are made

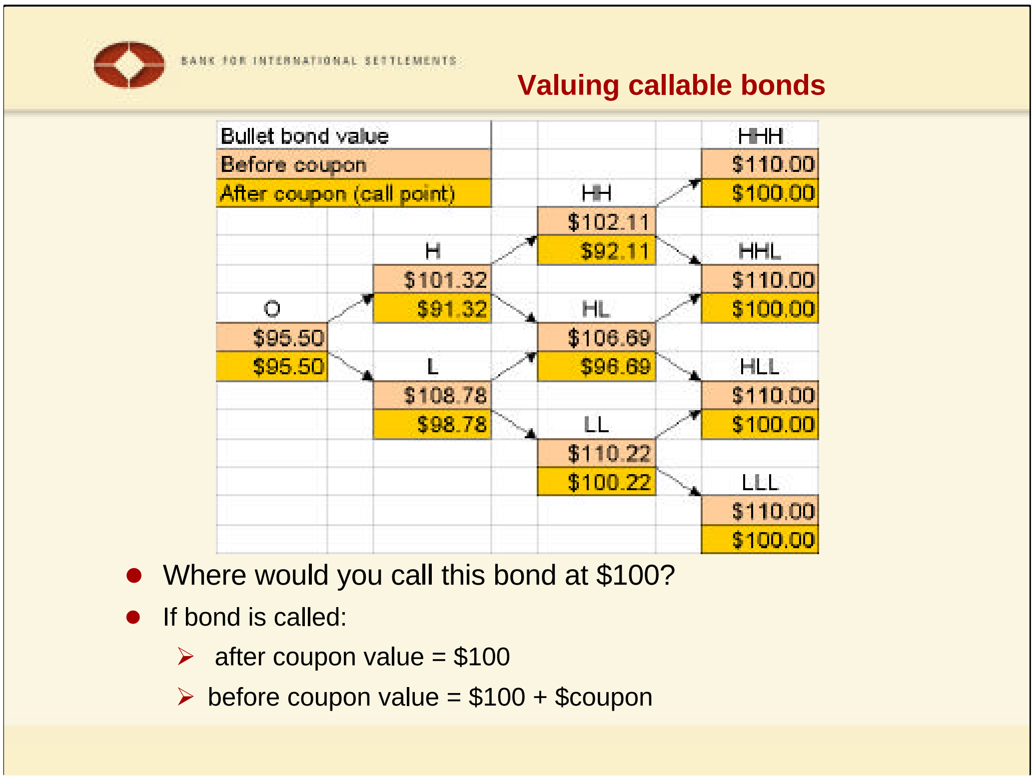
  - **We need an interest rate model (for both these steps)!**

- Assumptions:
  - The bond can be called at par (after each coupon payment)
  - The issuer behaves rationally

The valuation model that has been developed is much more complex than necessary for the valuation of bullet bonds. It has nevertheless been shown that the model prices bonds correctly. The model may now be used to value callable bonds.

An interest rate model is needed to value callable bonds because the payments made by the bond are not certain. These payments will be a function of whether the bond gets called, which in turn depends on whether the issuer can refinance the debt at a cheaper rate, and this will depend on the interest rate environment at the point where the issuer has to decide whether or not to call the bond.

An interest rate model can be used to determine the probability that the bond gets called. It will also tell us in which of the possible future interest rate states the bond gets called. As a result, it can be used to price the bond.



In order to determine the cash flows of the callable bond we assume that the issuer calls the bond whenever there is a financial incentive to do so. Whenever the value of a non-callable, but otherwise identical, bond is above the call price the issuer will call the bond and refinance at a cheaper rate. If the value of the bond is below par the issuer will not call the bond (if he wishes to de-leverage it is cheaper to buy the bond in the market rather than calling it at a higher price).

The figure shows a tree of bond values. At each node in the tree it is the after coupon value that the issuer can opt to call away from the bond holder at par.

The only node where it is in the issuer's interest to call the bond is at node LL. By calling at this node the bond issuer will no longer be under the obligation of making any future payments on the bond (the PV of which is $100.22), the issuer pays $100 to call the bond at node LL.

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Valuing callable bonds

- To see why the issuer will always call the bond at node LL consider these two options (transaction costs are zero):
  1. Don't call
  2. Call and re-issue a bond that pays $110 after 1 year

OPTION	Cashflow today	Cashflow in 1 year
Don't call	$0.00	-$110.00
Call	-$100.00	-$110.00
and re-issue	$100.22	

- The net effect of choosing option 2 instead of option 1 is an inflow of $0.22 on the call date
- The bond will never get called at any other node. If the issuer wants to de-leverage he can buy the bond cheaper in the market

To see why calling the bond at node LL is always in the interest of the issuer, we show that there is a strategy that involves calling the bond that "dominates" (is always better than) the option of not calling the bond.

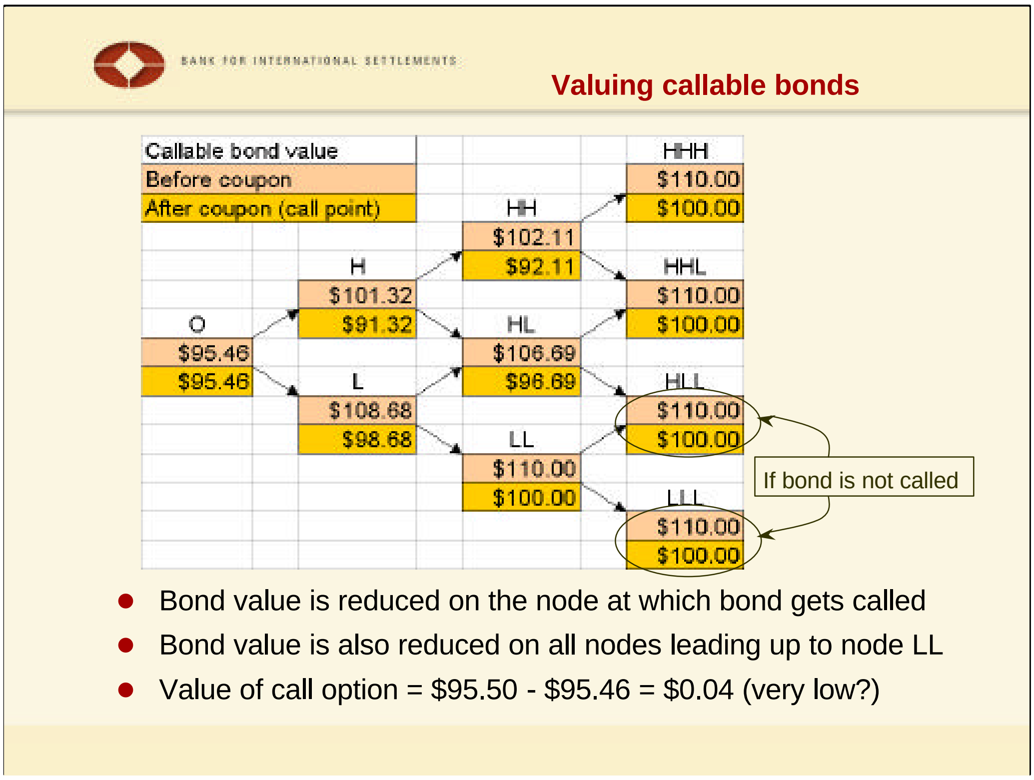
Option 1

Not calling the bond leaves the bond issuer with the obligation of paying principal ($100) and coupon ($10) to the bond holder in one year.

Option 2

The issuer issues a new $10-coupon bond that matures in one year. The proceeds from the issue of this bond is $100.22, of which he can use $100 to call the existing bond and keep the remaining $0.22. The net effect of these transactions, relative to adopting the strategy of not calling the bond, is a gain of $0.22 today.

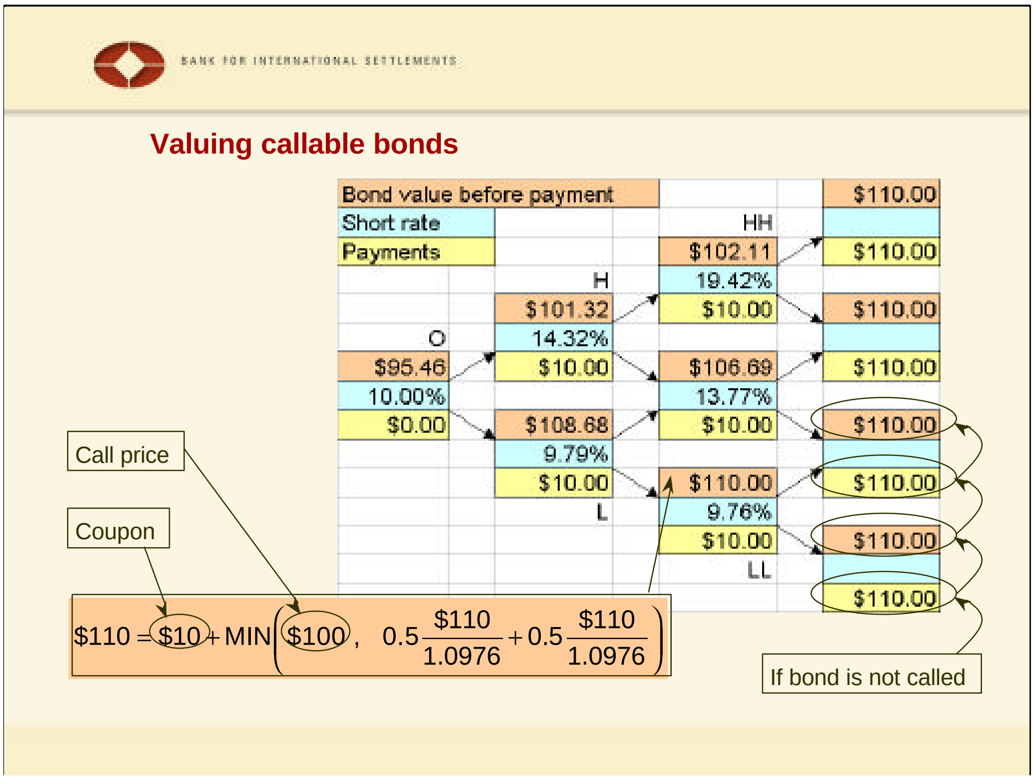
It is somewhat unfortunate that the gain the issuer realises by calling the bond is only $0.22. Transaction costs and issue costs will mean that in practice the issuer will probably not call the bond as these costs are more than the gain realised from calling the bond.



The Figure shows the binomial tree of the value the callable bonds. As described in the previous slides this value is capped at the price at which the bond can be called, which means that the value at node LL is $100.

The value of the callable bond is $95.46. This is only $0.04 below the value of the equivalent option-free bond. The value of the call option is therefore only 4 cents.

The value of the call option may seem rather low. We can however show that the computed value is consistent with the probability of the option being exercised and with financial gain to the issuer from exercising the option. The option is only exercised if node LL is reached, the probability of this occurring is 25% (a down followed by a down). The financial gain to the issuer from exercising the call option at node LL is $100.22-$100.00=$0.22. The financial gain of exercising the call option weighted by the probability of the option being exercised is therefore . This gain will occur in two periods time, if we present-value this gain we get , which is consistent with the computed option value.



We can now value the callable bond using a model for the evolution of the short rate in the same way that we used it to value the bullet bond. The only small modification that is required is that, on all nodes, the after-coupon value of the bond by capped at the call price.

Valuation of the bond proceeds as before by working backwards through the tree and discounting the expected (average) future value of the bond by the current short rate in the tree. At each node the value of the bond after coupon payment is capped at $100. The value of the bond before coupon payment is therefore capped at $100 + $coupon.

We see that capping the value of the bond at node LL affects the value of the bond at all nodes that are on a possible path leading to node LL.

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Valuing callable bonds : using Monte Carlo simulation

- For callable bond:
  - Payments are path-independent
  - We can work backwards through the tree
- For other securities:
  - Payments are path-dependent
  - We cannot work backwards through the tree
  - We need to select paths through the tree and discount payments along the selected path
- We now use this approach to value our callable bond:
  - Selecting every possible path through the tree is equivalent to working backwards through the tree

The bond values that have been assigned to the 2 bottom terminal nodes of the tree (in the previous slide) assume that the bond has not been called, if the bond gets called before this node is reached the value of the bond at the node will be zero. It is therefore not wrong to say that the value of a callable bond is also path dependent.

However, if the interest rate process takes a path along which the bond gets called we can consider this path to end. The value we assign to the nodes on which the bond gets called is dictated by the call price, the bond value on all subsequent nodes is immaterial. On nodes “beyond” a node on which the bond gets called we assign the non-called value to the bond, we do this because this is the value that should be used along paths on which the bond is not called and (as we have seen) along paths on which the bond gets called it does not matter what value we assign to these nodes. For example: the value assigned to node HLL is $100 (instead of $0).

Appropriately capping the value of the bond on nodes at which the bond gets called accounts for the fact that: if the interest rate process goes through this node the bond value on all subsequent nodes is zero.

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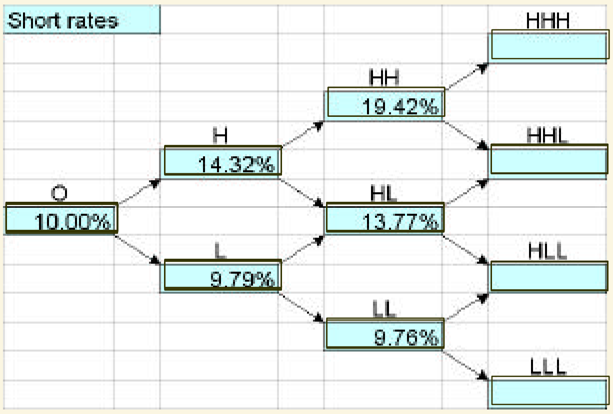
Valuing callable bonds: using Monte Carlo simulation

Q. What are the possible paths through the tree?

A. 2^N



- Adding one extra period doubles the number of paths
- In practice we need trees with may periods
  - A tree with 360 periods has more than 10^{108} possible paths!

At each node in the tree there is the possibility of moving up or of moving down to the next node. The number of possible paths through the tree therefore doubles for each additional period we add to the tree. As we need to consider payments that occur in 3 years time, and the duration of each period in the tree is of 1 year, we have a total of 3 periods in the tree and there are $2^3=8$ possible paths through the tree. For such a small tree it is possible for us to sample all possible paths in order to value the bond.

Since the payments at the terminal nodes of the tree are all the same there are, in effect, only 4 different paths through the tree. If we reach node HH it does not matter whether the path continues to node HHH or if it continues to node HHL: a payment of $110 is made at each of these two possible terminal nodes. The 8 possible paths through the binomial tree therefore come in pairs. For more complex interest rate derivatives we can have a different payoff at node HHH to the payoff at node HHL, for such a security the valuation along each of the 8 possible paths through the tree will be unique.

To price MBS we will often need to simulate the evolution of the short rate over a period of up to thirty years and because the borrowers may opt to prepay at the end of any month the interest rate process must be simulated on a monthly interval. These considerations mean that our binomial tree should simulate the evolution of the short rate over 360 monthly periods; and this in turn mean that there are 2^{360} (more than 10^{108}) possible paths through the tree. This is a **very** large number of possible paths through the tree, we cannot possibly sample each and every path (even on today's most powerful computer). Instead of sampling every possible path through the tree we can sample a large (say 10,000) but finite number of random paths through the tree, it is well know that as the number of random paths increases the result of the simulation converges to the correct value.

We can nevertheless easily work backwards (there is no computational limitation) through a binomial tree with 360 periods. This means that such an approach is preferable to value non-path-dependent instruments (such as callable bonds) for which we can work backwards through the tree.

BANK FOR INTERNATIONAL SETTLEMENTS

Valuing callable bonds: using Monte Carlo simulation

	Nodes crossed			Cash flow			
Path	t=1	t=2	t=3	t=1	t=2	t=3	PV
1	H	HH	HHH	\$10	\$10	\$110	\$90.29
2	H	HH	HHL	\$10	\$10	\$110	\$90.29
3	H	HL	HHL	\$10	\$10	\$110	\$93.93
4	H	HL	HLL	\$10	\$10	\$110	\$93.93
5	L	HL	HHL	\$10	\$10	\$110	\$97.43
6	L	HL	HLL	\$10	\$10	\$110	\$97.43
7	L	LL	HLL	\$10	\$110	\$0	\$100.17
8	L	LL	LLL	\$10	\$110	\$0	\$100.17
						Av.:	\$95.46

Before coupon value at node HHH

Before coupon value at node HH = \$102.11

Before coupon value at node H = \$99.32

$$\frac{\left(\frac{\left(\frac{\$110}{1.1942}+\$10\right)}{1.1432}+\$10\right)}{1.1}=\frac{\$10}{1.1}+\frac{\$10}{1.1\times 1.1432}+\frac{\$110}{1.1\times 1.1432\times 1.1942}=\$90.29$$

This table shows all 8 possible paths through the binomial tree, the payments that get made by the callable bond at each node of each path and the value of the callable bond for each of the 8 paths. Averaging the value of the callable bond across all 8 paths gives a value for the callable bond of \$95.46. This is the same value that was obtained earlier by working backwards through the tree.

For each path the value of the bond can be determined by working backwards in time. The left hand side of the equation in the slide shows this approach. For path 1 the value of the bond at node HHH is \$110; discounting this at the short rate of node HH (19.42%) and adding the coupon payment of \$10 that occurs at node HH gives a value of the bond at node HH of \$102.11 (=\$110 / 1.1942 + \$10); discounting this at the short rate of node H (14.32%) and adding the coupon payment of \$10 that occurs at node HH gives a value of the bond at node H of \$99.32 (=\$102.11 / 1.1432 + \$10); discounting this at the short rate of node O (10%) gives a value of \$90.29 (\$99.32 / 1.1) for the bond at node O.

The left hand side of the equation can be split into three terms: each term corresponds to one of the payments made by the bond along path 1. Looking at the three terms we see that each corresponds to the discounted value of one of the payments made by the bond along path 1, the discount factor for each payment is obtained by successively using the short rate up to the point at which the payment is made.

This methodology for valuing interest rate dependent payments is robust and can be used to value any interest rate derivative as long as we can anticipate the paymnets that get made along each possible interest rate path.

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Presentation outline

1. The need for valuation models
2. Factors affecting value
3. Valuing bullet bonds
4. Valuing callable bonds
5. Valuing mortgage-backed securities
6. Questions and answers
7. Appendix

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Valuing mortgage-backed securities

- Framework for valuing callable bonds:
  1. Simulate the evolution of the short rate
  2. Determine payments that occur along each path
  3. Discount payments by successively using the short rate
  4. Repeat many times to get expectation

- Framework for valuing MBS:
  1. Simulate the evolution of the short rate
  2. Determine payments that occur along each path
  3. Discount payments by successively using the short rate
  4. Repeat many times to get expectation

The payments that will be made by a MBS are not known in advance. This is because of the prepayment option: the homeowner has the option to prepay all or part of the outstanding loan balance at any moment of his choosing. The effect of exercising of the prepayment option is identical to that of the issuer of a callable bond exercising his option to call the bond: principal is returned at par to the security holder and the borrower has no further obligation to the security holder.

One factor influencing mortgage prepayments is the availability of cheaper finance. Homeowners will be tempted to refinance when rates fall and they have the possibility to refinance their mortgage at a cheaper rate. The availability of cheaper finance is the only factor that determines whether the issuer of a callable bond exercises his call option. One is tempted to conclude that valuing MBS is just as easy as valuing callable bonds.

Even though the framework that has been described for valuing callable bonds can also be used for valuing MBS, there is one component of this framework that is much more difficult to “get right” for MBS: Determining the payments that occur along each interest rate path.

The reason that it is so difficult to model the payments that get made by a pool of mortgages is because homeowner’s decisions to exercise (or not exercise) their prepayment option often appears to be irrational.

<u>Exercising the prepayment option when it may not appear to be in the homeowners interest to do so</u>

Apparent irrational exercise of the prepayment option occurs if there is no incentive to refinance and a borrower chooses to prepay. There are various possible reasons for such prepayments:

-The borrower defaults (the agency or insurer will repay the outstanding balance on the loan)

-The borrower sells the house (for non-assumable loans this results in full prepayment)

-The borrower wishes to access the equity in his home

<u>Failure to refinance (prepay) when it may appear to be in the homeowners interest to do so</u>

Failure to prepay under these conditions may be because of:

-The borrower is unaware of the benefit to be had from refinancing

-The borrower’s creditworthiness may have deteriorated and he may not be able to refinance at a cheaper rate

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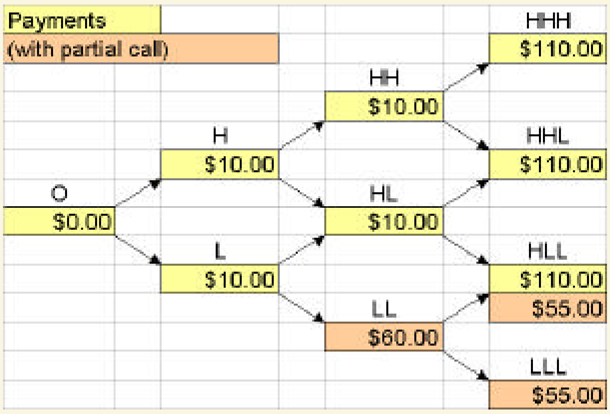
Valuing mortgage-backed securities: a hypothetical example



- This figure shows the payments made by our callable bond if the issuer only calls part of the bond at node LL
- In practice, such an outcome does not occur. It nevertheless sheds light on the behaviour of MBS

In this slide we consider a hypothetical example which sheds light on the issues we will need to consider when pricing MBS.

We consider once more the $10-coupon, 3-year, callable bond. We have seen that there is only one node (node LL) where it is in the issuer's interest to call the bond.

If the issuer acts in a rational manner he will call the bond at node LL and no further payments will be made by the bond. Let us assume that the issuer does not behave in a rational manner and we have a model to predict the "fraction" of the outstanding amount of the bond that the issuer decides to call at any node of the tree. This model predicts that 50% of the bond will be called at node LL and that the bond does not get called at any of the other nodes. This situation is analogous to half of the homeowners in the pool prepaying at node LL.

If half of the bond is called at node LL $50 of principal is paid back at node LL, with the $10 coupon that is also paid at this node, a total payment of $60 is made at node LL. The remaining $50 of principal is returned at nodes HLL and LLL, with the $5 coupon that is also made at these nodes (10% of $50 = $5), a total payment of $55 is mode at nodes HLL and LLL.

This example also shows why we cannot work backwards through the tree to value our hypothetical security. The payment made at node HLL depends on the path taken to get to the node, there are two possible values. If the interest rate process goes through node LL $55 is paid at this node, on the other hand, if the interest rate process goes through node HL $110 is made at node HLL.

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Valuing mortgage-backed securities: a hypothetical example

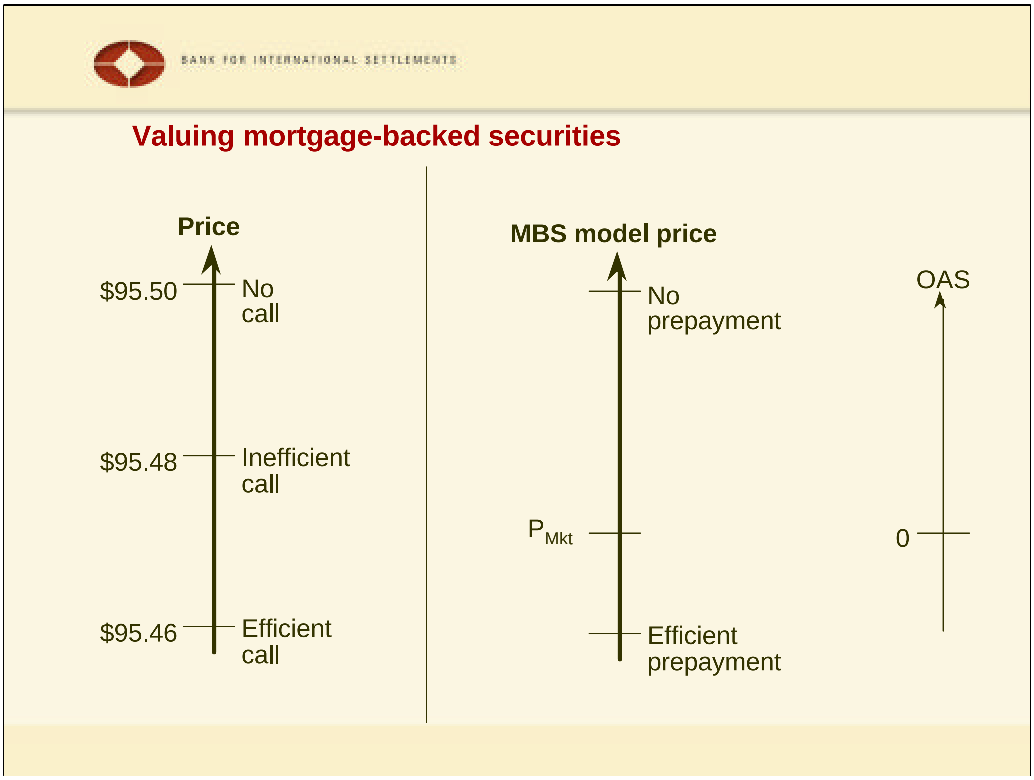
	Nodes crossed			Cash flow			
Path	t=1	t=2	t=3	t=1	t=2	t=3	PV
1	H	HH	HHH	$10	$10	$110	$90.29
2	H	HH	HHL	$10	$10	$110	$90.29
3	H	HL	HHL	$10	$10	$110	$93.93
4	H	HL	HLL	$10	$10	$110	$93.93
5	L	HL	HHL	$10	$10	$110	$97.43
6	L	HL	HLL	$10	$10	$110	$97.43
7	L	LL	HLL	$10	$60	$55	$100.26
8	L	LL	LLL	$10	$60	$55	$100.26
						Av.:	$95.48

- The value of this bond is now $0.02 higher than if the issuer optimally (fully) exercises its call option at node LL
  - Optimal exercise of an option transfers value from the seller to the buyer of the option (it is a “zero-sum” game)
  - The less optimal we expect the call option to be exercised the more we should be prepared to pay for the bond

The table in this slide shows the payments that get made along the 8 possible paths through the tree.

When a bond gets called it will be because the issuer has the possibility to refinance at a cheaper rate. When a bond gets called, the coupon on the called bond are higher than current market conditions dictate. The holder of the bond would be happy if the issuer “forgets” to call the bond.

The process of calling the bond transfers value for the bondholder to the issuer. The less optimal we expect the call option to be exercised the more we should be willing to pay for the bond.



The left hand side of the slide shows how the price of a callable bond varies as we change our assumption as to how efficiently the issuer exercises its call option. We have seen that if the issuer efficiently exercises its call option (by fully calling the bond at node LL) the bond price is $95.46. We have also seen that if we assume that the issuer calls half of the outstanding bonds at node LL then the bond price is $95.48. If we assume that the issuer is completely inefficient at exercising it call option (it never calls) then the value of the bond will be the same as that of a bullet bond: $95.50.

Efficient exercises of the call option therefore lowers the value of the bond and transfers value from the bond holder (the option writer) to the

bond issuer (the option holder).

We can view the effect, on MBS price, of changes in the efficiency with which borrowers exercise their prepayment option in a very similar fashion. We will see in the next presentation that homeowners often exercise their prepayment option in an inefficient manner. In modelling prepayments we need to determine how efficiently homeowners take advantage of the incentive to refinance into a lower rate (cheaper) mortgage.

We can consider two extremes:

First a (very unrealistic) prepayment model that assumes that borrowers never prepay their mortgage. Using this assumption in our MBS valuation framework will overvalue the MBS (the model price will be above the market price). This is because the market known that the homeowners will be more efficient in the exercise of their prepayment option, an action which normally transfers value from MBS holder to homeowner.

Next an (equally unrealistic) prepayment model that assumes that borrowers will fully prepay their mortgage whenever this detracts value from the MBS holder. Using this assumption in our MBS valuation framework will undervalue the MBS (the model price will be below the market price). This is because the market knows that the prepayment behaviour of homeowners will not be as bad for the MBS holder as the prepayment model assumes.

Between these two extreme prepayment behaviours there is a prepayment behaviour for which the model value equals market price.

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Valuing mortgage-backed securities

- Correctly predicting prepayments is by far the most difficult thing to “get right” when valuing MBS
- We will look at the necessary components of a prepayment model in the next presentation

The valuation framework that has been presented requires us to be able to predict the payments that get made by the security in different possible future states of the world. The payments that get made by a MBS depend on the prepayments that get made by the borrowers in the pool. There are various factors that affect the prepayment rate of a mortgage pool. To value MBS we therefore need a model to predicts the prepayments that get made by the MBS under different interest rate scenarios.

Refinancing is the component of prepayments that is driven by the borrower seeking to maximise “value”. Maximising value of the borrower occurs at the expense of the MBS holder. Therefore, when valuing MBS, the refinancing component of prepayments is the most important component to get right.

As a result of idiosyncratic factors we cannot possibly hope to accurately predict the prepayment behaviour of a single homeowner. Pools backing MBS will normally contain many hundreds of individual loans and we stand a better chance of predicting average prepayment characteristics of the mortgage pool.

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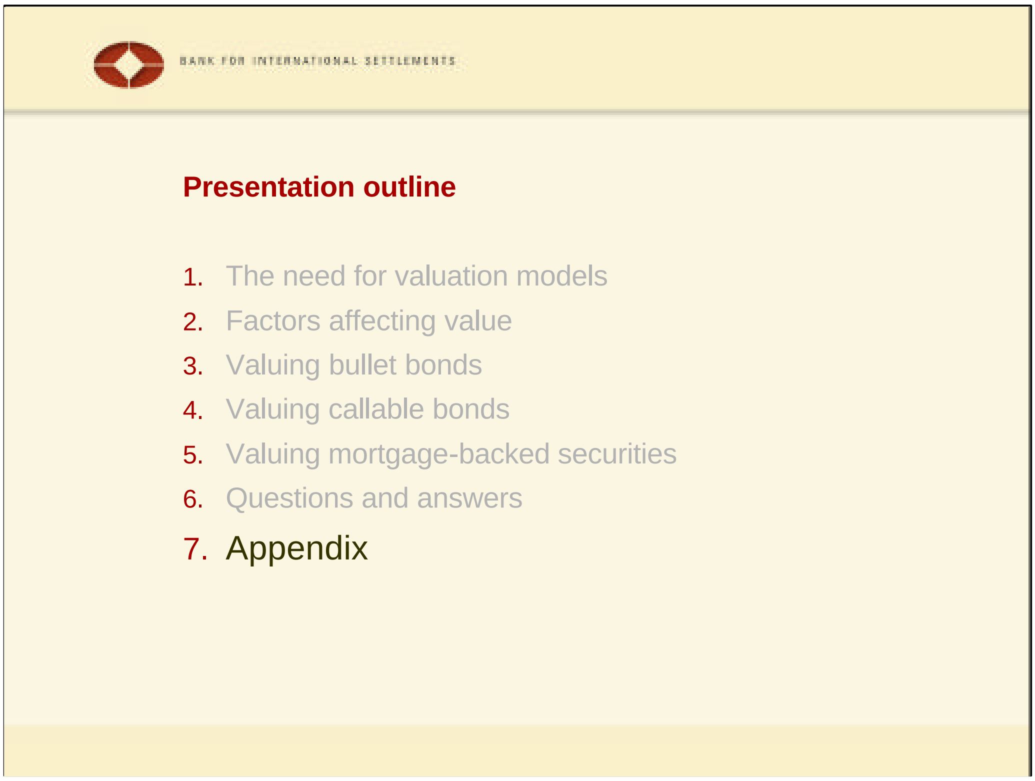

In this appendix we provide justification for the use of the risk-neutral valuation framework that we have used to value interest rate derivatives.

When interest rates are constant it is well known that we can use a procedure known as risk-neutral valuation to value options that do not depend on the level of interest rates, such as stock options. The procedure first computes the expected payoff of the option in a risk-neutral world (a world where the expected return on all securities is the risk-free rate); the expected payoff of the option is then discounted at the risk-free rate to obtain the current value of the option.

In the real world the expected payoff from holding the option will be higher than in the risk-neutral world. This is because investors demand a premium to hold the risky option. The appropriate rate to discount the expected payoff in the real world is also higher than the risk-free rate. The offsetting effects of a higher expected payoff and a higher discount rate means that the option value in the real world is identical to that computed in the risk-neutral world. The risk-neutral world simply provides a convenient alternative setting in which to determine expected payoffs and in which to discount these expected payoffs. Performing calculations in this alternative setting is much simpler than doing so in the real world. To determine the expected payoff of the option in the real world we need to make assumptions about the expected return of the instrument underlying the option (this will depend on investors' risk preferences). Furthermore, in the real world, the correct rate to discount the expected payoff of the option also depends on investors' risk preferences.

The framework that has been proposed in this presentation to value interest rate sensitive derivatives is very similar to this risk-neutral valuation methodology. The payoff of an interest rate derivative depends on the interest rate environment, which we allow to evolve in a manner consistent with a risk neutral world. The risk-free discount factor that is then used to discount future payments is that that corresponds to successive use of the short rate.

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Appendix

- The equivalent martingale measure result:
  - If we use the price of a traded security as the unit of measure then there is some market price of risk for which all security prices follow martingales

$$\frac{f_0}{g_0} = E_g\left(\frac{f_T}{g_T}\right) \quad \Rightarrow \quad f_0 = g_0 E_g\left(\frac{f_T}{g_T}\right)$$

- ➢ If we measure the value of security *f* in units of security *g*, its value today equals its expected future value in a world that is forward risk neutral with respect to *g*.
- ➢ Knowing the value g_0 and the ability to determine the expected future value of f_T/g_T enables us to determine f_0, today's value of security *f*
- Ref: J.C. Hull, "Options Futures and Other Derivatives", 5^{th} Ed.

The equivalent martingale measure result states that if we use the price of a traded security as the unit of measure then there is some market price of risk for which all security prices follow martingales. A martingale is a zero-drift stochastic process, the expected value of such a process at time *T* is its current value (time *0*).

If *g* is the price of traded security *G* in which the prices of all other securities are measured and σ_g is the volatility of *g* then the price of security *F* (measured in units of *g*) is *f/g*. The market price of risk that makes *f/g* follow a martingale is σ_g. A world where the market price of risk is σ_g is referred to as a world that is forward risk neutral with respect to *g*.

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Appendix

- The traditional risk-neutral world
  - *Numeraire g* is risk free: it pays the instantaneous risk-free rate

$$dg = r(t)g(t)dt \ ; \ g_0 = 1 \quad \Rightarrow \quad g_T = e^{\int_0^T rdt} = e^{\bar{r}T}$$

$$f_0 = g_0 \hat{E}\left(\frac{f_T}{g_T}\right) \quad \Rightarrow \quad f_0 = \hat{E}\left(e^{-\int_0^T rdt} f_T\right) = \hat{E}\left(e^{-\bar{r}T} f_T\right)$$

- We can determine the expectation through Monte Carlo simulation by:
  1. Allowing the short rate to evolves in a manner consistent with the traditional risk neutral world
  2. Identifying the future payments (f_T) that occur in such a world
  3. Discount using the instantaneous short rate
  4. Determine expectation (by repeating and averaging)

The traditional risk neutral world is a world where g is a security that pays the instantaneous risk free rate. The drift of g is stochastic but the volatility of g is zero.

The present value, f_0, of security f (f may be an interest rate sensitive security) is given by the expectation (taken in the traditional risk neutral world) of:

$f_T \times \mathrm{Exp}(-\int_0^T rdt)$

The exponential term is the discount factor to be used to discount a payment made at time t=T back to time t=0. This discount factor corresponds to a continuously compounded instantaneous rate r(t).

The discrete version of this term can be written as:

$f_T \times d_{0,1} \times d_{1,2} \times d_{2,3} \times \ldots \times d_{T-1,T}$

Where $d_{T-1,T}$ is the discount factor corresponding to the short rate appropriate for the period between time t=T-1 and time t=T. This shows that value of an interest rate-dependent option is equal to the expected value of the future payment made by the option discounted by successively using the short rate; where the interest rate process (and expectations) are determined in the traditional risk-neutral world.

In general there will be multiple possible future payments associated with an interest rate-sensitive security. The value of the security can be considered as the sum of the present values of the individual future payments of the security.

One straightforward way to determine this expectation (and the one we proposed in the presentation) is to simulate the risk-neutral interest rate process and then use Monte Carlo simulation to determine the expected value.

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Appendix

- Once the process followed by the short rate has been fully defined, the zero coupon yield curve, y_T, is also fully defined:
  - We have seen that: $f_0 = \hat{E}(e^{-\int_0^T rdt} f_T)$
- Define P_T to be the price of a zero coupon bond paying \$1 at time T
  - $$P_T = e^{-y_T T}$$
    $$P_T = \hat{E}(e^{-\int_0^T rdt}) = \hat{E}(e^{-\bar{r}\,T})$$
    $$\Rightarrow e^{-y_T T} = \hat{E}(e^{-\int_0^T rdt}) \qquad \left(= \hat{E}(e^{-\bar{r}\,T})\right)$$
    $$\Rightarrow y_T = -\frac{1}{T}\ln \hat{E}(e^{-\bar{r}\,T})$$

In general the payment made by an interest rate-sensitive security at time t does not only depend on the short rate at time t but it depends on the entire shape of the yield curve at time t. For example: the decision to call callable bond depends on whether the issuer can refinance the debt at a cheaper rate, this in turn depends on the entire zero coupon curve up to the maturity of the bond.

Fortunately knowing the current short rate and the risk-neutral process that dictates the evolution of the short rate enables us to determine the current shape of the entire zero coupon yield curve.

The last identity in the slide shows how to obtain the zero coupon yield at any maturity from the expected value of a function of the expected value of the short rate (up to the maturity at which we want to determine the zero coupon yield), which can be determined if we know the current value of the short rate and the risk-neutral process by which it evolves.

The second last equation in the slide shows that, by construction, discounting a certain, or known, future payment of \$1 using the zero coupon yield curve is equivalent to expected value of \$1 discounted by successively using the instantaneous short rate. By construction the two methods produce the same value for known future payments. Therefore both methods will value a bullet bond identically.