5. Geometry of Regular Finitely Ramified Fractals.

In Section 2 I introduced the Sierpinski gasket, and gave a direct "hands on" construction of a diffusion on it. Two properties of the SG played a crucial role: its symmetry and scale invariance, and the fact that it is finitely ramified. In this section we will introduce some classes of sets which preserve some of these properties, and such that a similar construction has a chance of working. (It will not always do so, as we will see).

There are two approaches to the construction of a family of well behaved regular finitely ramified fractals. The first, adopted by Lindstrøm [L1], and most of the mathematical physics literature, is to look at fractal subsets of $\mathbb{R}^d$ obtained by generalizations of the construction of the Cantor set. However when we come to study processes on F the particular embedding of F in $\mathbb{R}^d$ plays only a small role, and some quite natural sets (such as the "cut square" described below) have no simple embedding. So one may also choose to adapt an abstract approach, defining a collection of well behaved fractal metric spaces. This is the approach of Kigami [Ki2], and is followed in much of the subsequent mathematical literature on general fractal spaces. ("Abstract" fractals may also be defined as quotient spaces of product spaces – see [Kus2]).

The question of embedding has lead to confusion between mathematicians and physicists on at least one (celebrated) occasion. If G is a graph then the natural metric on G for a mathematician is the standard graph distance $d(x,y)$, which gives the length of the shortest path in G between x and y. Physicists call this the *chemical distance*. However, physicists, thinking in terms of the graph G being a model of a polymer, in which the individual strands are tangled up, are interested in the Euclidean distance between x and y in some embedding of G in $\mathbb{R}^d$. Since they regard each path in G as being a random walk path in $\mathbb{Z}^d$, they generally use the metric $d'(x,y) = d(x,y)^{1/2}$.

In this section, after some initial remarks on self-similar sets in $\mathbb{R}^d$, I will introduce the largest class of regular finitely ramified fractals which have been studied in detail. These are the *pc.f.s.s. sets* of Kigami [Ki2], and in what follows I will follow the approach of [Ki2] quite closely.

Definition 5.1. A map $\psi : \mathbb{R}^d \to \mathbb{R}^d$ is a *similitude* if there exists $\alpha \in (0,1)$ such that $|\psi(x) - \psi(y)| = \alpha|x-y|$ for all x, $y \in \mathbb{R}^d$. We call α the *contraction factor* of ψ.

Let $M \geq 1$, and let $\psi_1, \dots, \psi_M$ be similitudes with contraction factors α_i. For $A \subset \mathbb{R}^d$ set

$$\Psi(A) = \bigcup_{i=1}^{M} \psi_i(A). \tag{5.1}$$

Let $\Psi^{(n)}$ denote the n-fold composition of Ψ.

Definition 5.2. Let $\mathcal{K}$ be the set of non-empty compact subsets of $\mathbb{R}^d$. For $A \subset \mathbb{R}^d$ set $\delta_\varepsilon(A) = \{x : |x-a| \leq \varepsilon \text{ for some } a \in A\}$. The *Hausdorff metric* d on $\mathcal{K}$ is defined by

$$d(A,B) = \inf\{\varepsilon > 0 : A \subset \delta_\varepsilon(B) \text{ and } B \subset \delta_\varepsilon(A)\}.$$

Lemma 5.3. *(See [Fe, 2.10.21]). (a) d is a metric on $\mathcal{K}$.*
(b) $(\mathcal{K}, d)$ is complete.
(c) If $K_N = \{K \in \mathcal{K} : K \subset \overline{B}(0, N)\}$ then K_N is compact in K.

Theorem 5.4. *Let $(\psi_1, \dots, \psi_M)$ be as above, with $\alpha_i \in (0,1)$ for each $1 \le i \le M$. Then there exists a unique $F \in \mathcal{K}$ such that $F = \Psi(F)$. Further, if $G \in \mathcal{K}$ then $\Psi^n(G) \to F$ in d. If $G \in \mathcal{K}$ satisfies $\Psi(G) \subset G$ then $F = \cap_{n=0}^{\infty} \Psi^{(n)}(G)$.*

Proof. Note that $\Psi : \mathcal{K} \to \mathcal{K}$. Set $\alpha = \max_i \alpha_i < 1$. If A_i, $B_i \in \mathcal{K}$, $1 \le i \le M$ note that

$$d(\cup_{i=1}^M A_i, \cup_{i=1}^M B_i) \le \max_i d(A_i, B_i).$$

(This is clear since if $\varepsilon > 0$ and $B_i \subset \delta_\varepsilon(A_i)$ for each i, then $\cup B_i \subset \delta_\varepsilon(\cup A_i)$). Thus

$$\begin{aligned} d\big(\Psi(A), \Psi(B)\big) &\le \max_i d\big(\psi_i(A), \psi_i(B)\big) \\ &= \max_i \alpha_i \, d(A, B) = \alpha d(A, B). \end{aligned}$$

So Ψ is a contraction on $\mathcal{K}$, and therefore has a unique fixed point. For the final assertion, note that if $\Psi(G) \subset G$, then $\Psi^{(n)}(G)$ is decreasing. So $\cap_n \Psi^{(n)}(G)$ is non-empty, and must equal F. □

Examples 5.5. The fractal sets described in Section 2 can all be defined as the fixed point of a map Ψ of this kind.

1. *The Sierpinski gasket.* Let $\{a_1, a_2, a_3\}$ be the 3 corners of the unit triangle, and set

$$\psi_i(x) = a_i + \tfrac{1}{2}(x - a_i), \quad x \in \mathbb{R}^2, \quad 1 \le i \le 3. \tag{5.2}$$

2. *The Vicsek Set.* Let $\{a_1, \dots, a_4\}$ be the 4 corners of the unit square, let $M = 5$, let $a_5 = (\frac{1}{2}, \frac{1}{2})$, and let

$$\psi_i(x) = a_i + \tfrac{1}{3}(x - a_i), \quad 1 \le i \le 5. \tag{5.3}$$

It is possible to calculate the dimension of the limiting set F from $(\psi_1, \dots, \psi_M)$. However an "non-overlap" condition is necessary.

Definition 5.6. $(\psi_1, \dots, \psi_M)$ satisfies the *open set condition* if there exists an open set U such that $\psi_i(U)$, $1 \le i \le M$, are disjoint, and $\Psi(U) \subset U$. Note that, since $\Psi(\overline{U}) \subset \overline{U}$, then the fixed point F of Ψ satisfies $F = \cap \Psi^{(n)}(\overline{U})$.

For the Sierpinski gasket, if H is the convex hull of $\{a_1, a_2, a_3\}$, then one can take $U = \text{int}(H)$.

Theorem 5.7. *Let $(\psi_1, \dots, \psi_M)$ satisfy the open set condition, and let F be the fixed point of Ψ. Let β be the unique real such that*

$$\sum_{i=1}^M \alpha_i^\beta = 1. \tag{5.4}$$

Then $\dim_H(F) = \beta$, and $0 < \mathcal{H}^\beta(F) < \infty$.

Proof. See [Fa2, p. 119].

Remark. If $\alpha_i = \alpha$, $1 \le i \le M$, then (5.4) simplifies to $M\alpha^\beta = 1$, so that

$$\beta = \frac{\log M}{\log \alpha^{-1}}. \tag{5.5}$$

We now wish to set up an abstract version of this, so that we can treat fractals without necessarily needing to consider their embeddings in $\mathbb{R}^d$. Let (F, d) be a compact metric space, let $I = I_M = \{1, \dots, M\}$, and let

$$\psi_i : F \to F, \quad 1 \le i \le M$$

be continuous injections. We wish the copies $\psi_i(F)$ to be strictly smaller than F, and we therefore assume that there exists $\delta > 0$ such that

$$d\big(\psi_i(x), \psi_i(y)\big) \le (1-\delta) d(x, y), \quad x, y \in F, \qquad i \in I_M. \tag{5.6}$$

Definition 5.8. $(F, \psi_i, 1 \le i \le M)$ is a *self-similar structure* if (F, d) is a compact metric space, ψ_i are continuous injections satisfying (5.6) and

$$F = \bigcup_{i=1}^M \psi_i(F). \tag{5.7}$$

Let $(F, \psi_i, 1 \le i \le M)$ be a self-similar structure. We can use iterations of the maps ψ_i to give the 'address' of a point in F. Introduce the word spaces

$$\mathbb{W}_n = I^n, \quad \mathbb{W} = I^{\mathbb{N}}.$$

We endow $\mathbb{W}$ with the usual product topology. For $w \in \mathbb{W}_n$, v in $\mathbb{W}_n$ or $\mathbb{W}$, let $w \cdot v = (w_1, \dots, w_n, v_1, \dots)$, and define the left shift σ on $\mathbb{W}$ (or $\mathbb{W}_n$) by

$$\sigma w = (w_2, \dots).$$

For $w = (w_1, ..., w_n) \in \mathbb{W}_n$ define

$$\psi_w = \psi_{w_1} \circ \psi_{w_2} \circ \dots \circ \psi_{w_n}. \tag{5.8}$$

It is clear from (5.7) that for each $n \ge 1$,

$$F = \bigcup_{w \in \mathbb{W}_n} \psi_w(F).$$

If $a = (a_1, \dots, a_M)$ is a vector indexed by I, we write

$$a_w = \prod_{i=1}^n a_{w_i}, \quad w \in \mathbb{W}_n. \tag{5.9}$$

Write $A_w = \psi_w(A)$ for $w \in \cup_n \mathbb{W}_n$, $A \subset F$. If $n \ge 1$, and $w \in \mathbb{W}$ (or $\mathbb{W}_m$ with $m \ge n$) write

$$w|n = (w_1, \dots, w_n) \in \mathbb{W}_n. \tag{5.10}$$

Lemma 5.9. *For each $w \in \mathbb{W}$, there exists a $x_w \in F$ such that*

$$\bigcap_{n=1}^{\infty} \psi_{w|n}(F) = \{x_w\}. \tag{5.11}$$

Proof. Since $\psi_{w|(n+1)}(F) = \psi_{w|n}\big(\psi_{w_{n+1}}(F)\big) \subset \psi_{w|n}(F)$, the sequence of sets in (5.11) is decreasing. As ψ_i are continuous, $\psi_{w|n}(F)$ are compact, and therefore $A = \cap_n F_{w|n}$ is non-empty. But as $\operatorname{diam}(F_{w|n}) \le (1-\delta)^n \operatorname{diam}(F)$, we have $\operatorname{diam}(A) = 0$, so that A consists of a single point. □

Lemma 5.10. *There exists a unique map $\pi : \mathbb{W} \to F$ such that*

$$\pi(i \cdot w) = \psi_i\big(\pi(w)\big), \quad w \in \mathbb{W}, \quad i \in I. \tag{5.12}$$

π is continuous and surjective.

Proof. Define $\pi(w) = x_w$, where x_w is defined by (5.11). Let $w \in \mathbb{W}$. Then for any n,

$$\pi(i \cdot w) \in F_{(i \cdot w)|n} = F_{i \cdot (w|n-1)} = \psi_i(F_{w|n-1}).$$

So $\pi(i \cdot w) \in \cap_m \psi_i(F_m) = \{\psi_i(x_w)\}$, proving (5.12). If π' also satisfies (5.12) then $\pi'(v \cdot w) = \psi_v\big(\pi'(w)\big)$ for $v \in \mathbb{W}_n$, $w \in \mathbb{W}$, $n \ge 1$. Then $\pi'(w) \in F_{w|n}$ for any $n \ge 1$, so $\pi' = \pi$.

To prove that π is surjective, let $x \in F$. By (5.7) there exists $w_1 \in I_M$ such that $x \in F_{w_1} = \psi_{w_1}(F) = \cup_{w_2=1}^{M} F_{w_1 w_2}$. So there exists w_2 such that $x \in F_{w_1 w_2}$, and continuing in this way we obtain a sequence $w = (w_1, w_2, \dots) \in \mathbb{W}$ such that $x \in F_{w|n}$ for each n. It follows that $x = \pi(w)$.

Let U be open in F, and $w \in \pi^{-1}(U)$. Then $F_{w|n} \cap U^c$ is a decreasing sequence of compact sets with empty intersection, so there exists m with $F_{w|m} \subset U$. Hence $V = \{v \in \mathbb{W} : v|m = w|m\} \subset \pi^{-1}(U)$, and since V is open in $\mathbb{W}$, $\pi^{-1}(U)$ is open. Thus π is continuous. □

Remark 5.11. It is easy to see that (5.12) implies that

$$\pi(v \cdot w) = \psi_v\big(\pi(w)\big), \qquad v \in \mathbb{W}_n, \quad w \in \mathbb{W}. \tag{5.13}$$

Lemma 5.12. *For $x \in F$, $n \ge 0$ set*

$$N_n(x) = \bigcup \{F_w : w \in \mathbb{W}_n, x \in F_w\}.$$

Then $\{N_n(x), n \ge 1\}$ form a base of neighbourhoods of x.

Proof. Fix x and n. If $v \in \mathbb{W}_n$ and $x \notin F_v$ then, since F_v is compact, $d(x, F_v) = \inf\{d(x,y) : y \in F_v\} > 0$. So, as $\mathbb{W}_n$ is finite, $d(x, N_n(x)^c) = \min\{d(x, F_v) : x \notin F_v, v \in \mathbb{W}_n\} > 0$. So $x \in \operatorname{int}(N_n(x))$. Since $\operatorname{diam} F_w \le (1-\delta)^n \operatorname{diam}(F)$ for $w \in \mathbb{W}_n$ we have $\operatorname{diam} N_n(x) \le 2(1-\delta)^n \operatorname{diam}(F)$. So if $U \ni x$ is open, $N_n(x) \subset U$ for all sufficiently large n. □

The definition of a self-similar structure does not contain any condition to prevent overlaps between the sets $\psi_i(F)$, $i \in I_M$. (One could even have $\psi_1 = \psi_2$ for example). For sets in $\mathbb{R}^d$ the open set condition prevents overlaps, but relies on the existence of a space in which the fractal F is embedded. A general, abstract, non-overlap condition, in terms of dimension, is given in [KZ1]. However, for finitely ramified sets the situation is somewhat simpler.

For a self-similar structure $\mathcal{S} = (F, \psi_i, i \in I_M)$ set

$$B = B(\mathcal{S}) = \bigcup_{i,j,i\neq j} F_i \cap F_j.$$

As one might expect, we will require $B(\mathcal{S})$ to be finite. However, this on its own is not sufficient: we will require a stronger condition, in terms of the word space $\mathbb{W}$. Set

$$\Gamma = \pi^{-1}\big(B(\mathcal{S})\big),$$
$$P = \bigcup_{n=1}^{\infty} \sigma^n(\Gamma).$$

Definition 5.13. A self-similar structure (F, ψ) is *post critically finite*, or p.c.f., if P is finite. A metric space (F, d) is a *p.c.f.s.s. set* if there exists a p.c.f. self-similar structure $(\psi_i, 1 \le i \le M)$ on F.

Remarks 5.14. 1. As this definition is a little impenetrable, we will give several examples below. The definition is due to Kigami [Ki2], who called Γ the *critical set* of $\mathcal{S}$, and P the *post critical set.*
2. The definition of a self-similar structure given here is slightly less general than that given in [Ki2]. Kigami did not impose the constraint (5.6) on the maps ψ_i, but made the existence and continuity of π an axiom.
3. The initial metric d on F does not play a major role. On the whole, we will work with the natural structure of neighbourhoods of points provided by the self-similar structure and the sets $F_w, w \in \mathbb{W}_n$, $n \ge 0$.

Examples 5.15. 1. *The Sierpinski gasket.* Let a_1, a_2, a_3 be the corners of the unit triangle in $\mathbb{R}^d$, and let

$$\psi_i(x) = a_i + \tfrac{1}{2}(x - a_i), \quad x \in \mathbb{R}^2, \quad 1 \le i \le 3.$$

Write G for the Sierpinski gasket; it is clear that $(G, \psi_1, \psi_2, \psi_3)$ is a self-similar structure. Writing $\dot{s} = (s, s, \ldots)$, we have

$$\pi(\dot{s}) = a_s, \quad 1 \le s \le 3.$$

So

$$B(\mathcal{S}) = \left\{\tfrac{1}{2}(a_3 + a_1), \quad \tfrac{1}{2}(a_1 + a_2), \quad \tfrac{1}{2}(a_2 + a_3)\right\},$$
$$\Gamma = \left\{(1\dot{3}), (3\dot{1}), (1\dot{2}), (2\dot{1}), (2\dot{3}), (3\dot{2})\right\},$$

and

$$P = \sigma(\Gamma) = \left\{(\dot{1}), (\dot{2}), (\dot{3})\right\}.$$

2. *The cut square.* This is an example of a p.c.f.s.s. set which has no convenient embedding in Euclidean space. (Though of course such an embedding can certainly be found).

Start with the unit square $C_0 = [0,1]^2$. Now make 'cuts' along the line $L_1 = \{(\frac{1}{2}, y) : 0 < y < \frac{1}{2}\}$, and the 3 similar lines (L_2, L_3, L_4 say) obtained from L_1

by rotation. So the set C_1 consists of C_0, but with the points in the line segment $(\frac{1}{2}, y-)$, $(\frac{1}{2}, y+)$, viewed as distinct, for $0 < y < \frac{1}{2}$. (And similarly for the 3 similar sets obtained by rotation). Alternatively, C_1 is the closure of $A = C_0 - \cup_{i=1}^4 L_i$ in the geodesic metric d_A defined in Section 2. One now repeats this construction on each of the 4 squares of side $\frac{1}{2}$ which make up C_1 to obtain successively C_2, $C_3, \ldots$; the cut square C is the limit.

This is a p.c.f.s.s. set; one has $M = 4$, and if $a_1, \ldots, a_4$ are the 4 corners of $[0,1]^2$, then the maps ψ_i agree at all points with irrational coordinates with the maps $\varphi_i(x) = a_i + \frac{1}{2}(x - a_i)$. We have

$$B = \{(0, \tfrac{1}{2}), (\tfrac{1}{2}, \tfrac{1}{2}), (1, \tfrac{1}{2}), (\tfrac{1}{2}, 0), (\tfrac{1}{2}, 1)\}$$
$$\Gamma = \{(1\dot{2}), (2\dot{1}), (2\dot{3}), (3\dot{2}), (3\dot{4}), (4\dot{3}), (4\dot{1}), (1\dot{4}), (1\dot{3}), (3\dot{1}), (2\dot{4}), (4\dot{2})\},$$

so that

$$P = \{(\dot{1}), (\dot{2}), (\dot{3}), (\dot{4})\}.$$

Note also that $\pi(1\dot{2}) = \pi(2\dot{1})$, and $\pi(1\dot{3}) = \pi(3\dot{1}) = \pi(2\dot{4}) = \pi(4\dot{2}) = z$, the centre of the square.

In both the examples above we had $P = \{(\dot{s}), s \in I_M\}$, and $P = \sigma^n P$ for all $n \geq 1$. However P can take a more complicated form if the sets $\psi_i(F)$, $\psi_j(F)$ overlap at points which are sited at different relative positions in the two sets.

3. *Sierpinski gasket with added triangle.* (See [Kum2]). We describe this set as a subset of $\mathbb{R}^2$. Let $\{a_1, a_2, a_3\}$ be the corners of the unit triangle in $\mathbb{R}^2$, and let $\psi_i(x) = \frac{1}{2}(x - a_i) + a_i$, $1 \leq i \leq 3$. Let $a_4 = \frac{1}{3}(a_1 + a_2 + a_3)$ be the centre of the triangle, and let $\psi_4(x) = a_4 + \frac{1}{4}(x - a_4)$. Of course (ψ_1, ψ_2, ψ_3) gives the Sierpinski gasket, but $\Psi = (\psi_1, \psi_2, \psi_3, \psi_4)$ still satisfies the open set condition, and if $F = F(\Psi)$ is the fixed point of Ψ then $(F, \psi_1, \ldots, \psi_4)$ is a self-similar structure. Writing b_1, b_2, b_3 for the mid-points of (a_2, a_3), (a_3, a_1), (a_1, a_2) respectively, and $c_i = \frac{1}{2}(a_i + b_i)$, $1 \leq i \leq 3$, we have

$$B = \{b_1, b_2, b_3, c_1, c_2, c_3\},$$

$\pi^{-1}(b_1) = \{(2\dot{3}), (3\dot{2})\}$, while $\pi^{-1}(c_1) = \{(12\dot{3}), (13\dot{2}), (4\dot{1})\}$, with similar expressions for $\pi^{-1}(b_j)$, $\pi^{-1}(c_j)$, $j = 2, 3$. So $\#(\Gamma) = 15$, and

$$\sigma(\Gamma) = \{(\dot{1}), (\dot{2}), (\dot{3}), (2\dot{3}), (3\dot{2}), (3\dot{1}), (1\dot{3}), (1\dot{2}), (2\dot{1})\},$$
$$\sigma^2(\Gamma) = \{(\dot{1}), (\dot{2}), (\dot{3})\}.$$

Then $P = \sigma(\Gamma)$ consists of 9 points in $\mathbb{W}$, and $\#(\pi(P)) = 6$.

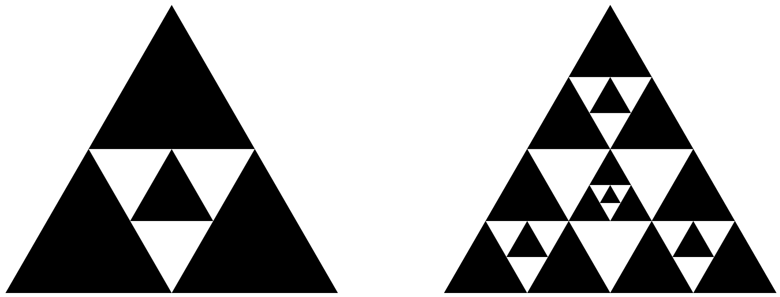

Fig. 5.1 : Sierpinski gasket with added triangle.

4. (*Rotated triangle*). Let a_i, b_i, ψ_i, $1 \leq i \leq 3$, be as above. Let $\lambda \in (0,1)$, and let $p_1 = \lambda b_2 + (1-\lambda)b_3$, with p_2, p_3 defined similarly. Evidently $\{p_1, p_2, p_3\}$ is an equilateral triangle; let ψ_4 be the similitude such that $\psi_4(a_i) = p_i$. Let $F = F(\Psi)$ be the fixed point of Ψ. If H is the convex hull of $\{a_1, a_2, a_3\}$, then $\Psi(H) \subset H$, so clearly F is finitely ramified, and

$$B = \{b_1, b_2, b_3, p_1, p_2, p_3\}.$$

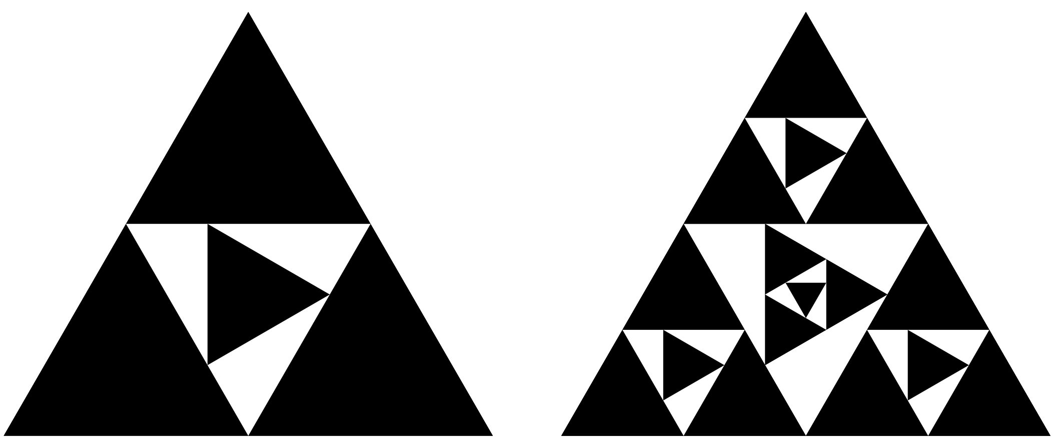

Fig. 5.2 : Rotated triangle with $\lambda = 2/3$.

As before, $\pi^{-1}(b_1) = \{(2\dot{3}), (3\dot{2})\}$. Let $y_1 = \psi_1^{-1}(p_1)$; then y_1 lies on the line segment connecting a_2 and a_3. If $A = \pi^{-1}(y_1)$ then A consists of one or two points, according to whether λ is a dyadic rational or not. Let $A = \{v, w\}$, where $v = w$ if $\lambda \notin \mathbb{D}$. Note that for each element $u \in A$, we have, writing $u = (u_1, u_2, \ldots)$, that $u_k \in \{2,3\}$, $k \geq 1$. Then $\pi^{-1}(p_1) = \{(4\dot{1}), (1 \cdot v), (1 \cdot w)\}$. If $\theta : \mathbb{W} \to \mathbb{W}$ is defined by $\theta(w) = w'$, where $w'_i = w_i + 1 \pmod 3$, and

$$A_n = \{(\dot{1}), \sigma^n v, \sigma^n w\},$$

then $\sigma^n(\Gamma) = A_n \cup \theta(A_n) \cup \theta^2(A_n)$.

(a) $\lambda = \frac{1}{2}$ gives Example 3 above.

(b) If λ is irrational, then $P = \cup_{n\geq 1}\sigma^n(\Gamma)$ is infinite. This example therefore shows that the "p.c.f." condition in Definition 5.13 is strictly stronger than the requirement that the set F be finitely ramified and self-similar.
(c) Let $\lambda = \frac{2}{3}$. Then $v = w = (\dot{2}\dot{3})$. Therefore B consists of p_1 and b_1, with their rotations, and $\sigma(L)$ consists of $(2\dot{3})$, $(3\dot{2})$, $(4\dot{1})$, $(123\dot{2}\dot{3})$ and their "rotations" by θ. Hence

$$P = \left\{(\dot{1}), (\dot{2}), (\dot{3}), (\dot{2}\dot{3}), (\dot{3}\dot{2}), (\dot{3}\dot{1}), (\dot{1}\dot{3}), (\dot{1}\dot{2}), (\dot{2}\dot{1})\right\}.$$

So $\lambda = \frac{2}{3}$ does give a p.c.f.s.s. set.
(d) In general, as is clear from the examples above, while F is finitely ramified for any $\lambda \in (0,1)$, F is a p.c.f.s.s. set if and only if $\lambda \in \mathbb{Q} \cap (0,1)$.

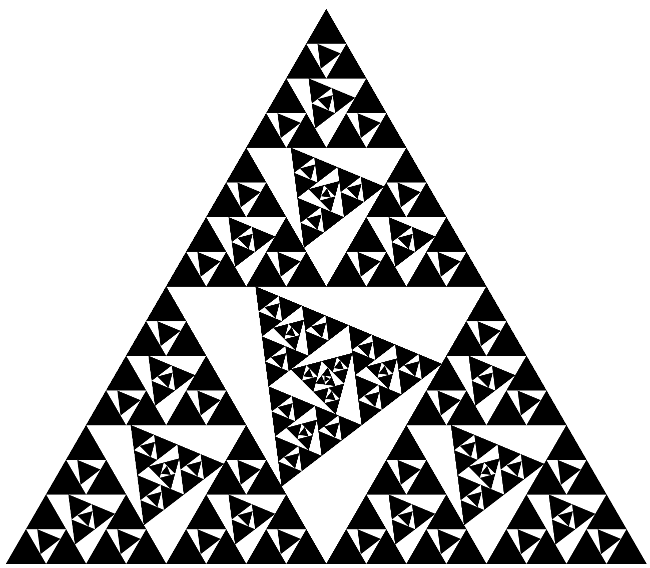

Fig. 5.3 : Rotated triangle with $\lambda = 0.721$.

We now introduce some more notation.

Definition 5.16. Let $(F, \psi_1, \dots, \psi_M)$ be a p.c.f.s.s. set. Set for $n \geq 0$,

$$P^{(n)} = \{w \in \mathbb{W} : \sigma^n w \in P\},$$
$$V^{(n)} = \pi(P^{(n)}).$$

Any set of the form F_w, $w \in \mathbb{W}_n$, we call an *n-complex*, and any set of the form $\psi_w(V^{(0)}) = V_w^{(0)}$ we call a *n-cell*.

Lemma 5.17. *(a) Let $x \in V^{(n)}$. Then $x = \psi_w(y)$, where $y \in V^{(0)}$ and $w \in \mathbb{W}_n$.*
(b) $V^{(n)} = \cup_{w\in\mathbb{W}_n} V_w^{(0)}$.

Proof. (a) From the definition, $x = \pi(w \cdot v)$, for $w \in \mathbb{W}_n$, $v \in \mathbb{W}$. Then if $y = \pi(v)$, $y \in V^{(0)}$, and by (5.13), $x = \pi(w \cdot v) = \psi_w(y)$.
(b) Let $x \in V_w^{(0)}$. Then $x = \psi_w\big(\pi(v)\big)$, where $v \in P$. Hence $x = \pi(w \cdot v)$, and since $w \cdot v \in P^{(n)}$, $x \in V^{(n)}$. The other inclusion follows from (a). $\square$

We think of $V^{(0)}$ as being the "boundary" of the set F. The set F consists of the union of M^n n-complexes F_w (where $w \in \mathbb{W}_n$), which intersect only at their boundary points.

Lemma 5.18. *(a) If $w,\, v \in \mathbb{W}_n$, $w \neq v$, then $F_w \cap F_v = V_w^{(0)} \cap V_v^{(0)}$.*
(b) If $n \geq 0$, $\pi^{-1}\big(\pi(P^{(n)})\big) = \pi^{-1}(V^{(n)}) = P^{(n)}$.

Proof. (a) Let $n \geq 1$, $v,\, w \in \mathbb{W}_n$, and $x \in F_w \cap F_v$. So $x = \pi(w \cdot u) \neq \pi(v \cdot u')$ for $u, u' \in \mathbb{W}$. Suppose first that $w_1 \neq v_1$. Then as $F_w \subset F_{w_1}$, we have $x \in F_{w_1} \cap F_{v_1} \subset B$. So $w \cdot u,\ v \cdot u' \in \Gamma$, and thus $u = \sigma^{n-1}\sigma(w \cdot u) \in P$. Therefore $\pi(u) \in V^{(0)}$, and $x = \psi_w\big(\pi(u)\big) \in V_w^{(0)}$. If $w_1 = v_1$ then let k be the largest integer such that $w|k = v|k$. Applying $\psi_{w|k}^{-1}$ we can then use the argument above.
(b) It is elementary that $P^{(n)} \subset \pi^{-1}\big(\pi(P^{(n)})\big)$. Let $n = 0$ and $w \in \pi^{-1}\big(\pi(P)\big)$. Then there exists $v \in P$ such that $\pi(w) = \pi(v)$. As $v \in P$, $v \in \sigma^m(\Gamma)$ for some $m \geq 1$. Hence there exists $u \in \mathbb{W}_m$ such that $u \cdot v \in \pi^{-1}(B)$. However $\pi(u \cdot w) = \psi_u\big(\pi(w)\big) = \pi(u \cdot v) \in B$, and thus $u \cdot v \in \sigma$. Hence $v \in P$.

If $n \geq 1$, and $\pi(w) \in \pi\big(P^{(n)}\big) = V^{(n)}$, then $\pi(w) \in V_v^{(0)}$ for some $v \in \mathbb{W}_n$. So $\pi(w) \in V_v^{(0)} \cap F_{w|n} = V_v^{(0)} \cap V_{w|n}^{(0)}$ by (a). Therefore $\pi(w) \in V_{w|n}^{(0)}$, and thus $\pi(w) = \psi_{w|n}\big(\pi(v)\big)$, where $v \in P$. So $\pi(w) = \pi(w|n \cdot v)$, and thus $\pi(\sigma^n w) = \pi(v)$. By the case $n = 0$ above $\sigma^n w \in P$, and hence $w \in P^{(n)}$. □

Remark 5.19. Note we used the fact that $\pi(v \cdot w) = \pi(v \cdot w')$ implies $\pi(w) = \pi(w')$, which follows from the fact that ψ_v is injective.

Lemma 5.20. *Let $s \in \{1, \ldots, M\}$. Then $\pi(\dot{s})$ is in exactly one n-complex, for each $n \geq 1$.*

Proof. Let $n = 1$, and write $x_s = \pi(\dot{s})$. Plainly $x_s \in F_s$; suppose $x_s \in F_i$ where $i \neq s$. Then $x_s = \psi_i\big(\pi(w)\big)$ for some $w \in \mathbb{W}$. Since $x_s = \psi_s^k(x_s)$ for any $k \geq 1$, $x_s = \psi_s^k\big(\pi(i \cdot w)\big) = \pi(s^k \cdot i \cdot w)$, where $s^k = (s, s, \ldots, s) \in \mathbb{W}_k$. Since $x_s \in F_i \cap F_s \subset B$, $\pi^{-1}(x_s) \in C$. But therefore $s^k \cdot i \cdot w \in C$ for each $k \geq 1$, and since $i \neq s$, C is infinite, a contradiction.

Now let $n \geq 2$, and suppose $x_s = \pi(\dot{s}) \in F_w$, where $w \in \mathbb{W}_n$ and $w \neq s^n$. Let $0 \leq k \leq n-1$ be such that $w = s^k \cdot \sigma^k w$, and $w_{k+1} \neq s$. Then applying ψ_s^{-k} to F_{s^k} we have that $x_s \in F_{\sigma^k w} \cap F_{s^{n-k}}$, which contradicts the case $n = 1$ above. □

Let $(F, \psi_1, \ldots, \psi_M, \pi)$ be a p.c.f.s.s. set. For $x \in F$, let

$$m_n(x) = \#\left\{w \in \mathbb{W}_n \,:\, x \in F_w\right\}$$

be the n-multiplicity of x, that is the number of distinct n-complexes containing x. Plainly, if $x \notin \cup_n V^{(n)}$, then $m_n(x) = 1$ for all n. Note also that $m_\cdot(x)$ is increasing.

Proposition 5.21. *For all $x \in F$, $n \geq 1$,*

$$m_n(x) \leq M\#(P).$$

Proof. Suppose $x \in F_{w^1} \cap \ldots \cap F_{w^k}$, where w^i, $1 \leq i \leq k$ are distinct elements of $\mathbb{W}_n$. Suppose first that $w_1^i \neq w_1^j$ for some $i \neq j$. Then $x \in B$, and therefore there exist $v^1, \ldots, v^k \in \mathbb{W}$ such that $\pi(w^l \cdot v^l) = x$, $1 \leq l \leq k$. Hence $w^l \cdot v^l \in \Gamma$ for each l, and so $\#(\Gamma) \geq k$. But $\#(P) \geq M^{-1}\#(\Gamma)$, and thus $k \leq M\#(P)$.

If all the w^l contain a common initial string v, then applying ψ_v^{-1} we can use the argument above. $\square$

Nested Fractals and Affine Nested fractals.

Nested fractals were introduced by Lindstrøm [L1], and affine nested fractals (ANF) by [FHK]. These are of p.c.f.s.s. sets, but have two significant additional properties:

(1) They are embedded in Euclidean space,
(2) They have a large symmetry group.

I will first present the definition of an ANF, and then relate it to that for p.c.f.s.s. sets. Let $\psi_1, \dots, \psi_M$ be similitudes in $\mathbb{R}^d$, and let F be the associated compact set. Writing ψ_i also for the restrictions of ψ_i to F, $(F, \psi_1, \dots, \psi_M)$ is a self similar structure. Let $\mathbb{W}$, π, $V^{(0)}$, etc. be as above. For $x, y \in V^{(0)}$ let $g_{xy} : \mathbb{R}^d \to \mathbb{R}^d$ be reflection in the hyperplane which bisects the line segment connecting x and y. As each ψ_i is a contraction, it has a unique fixed point, z_i say. Let $\overline{V} = \{z_1, ..., z_M\}$ be the set of fixed points. Call $x \in \overline{V}$ an *essential fixed point* if there exists $y \in \overline{V}$, and $i \neq j$ such that $\psi_i(x) = \psi_j(y)$. Write $\overline{V}^{(0)}$ for the set of essential fixed points. Set also

$$\overline{V}^{(n)} = \bigcup_{w \in \mathbb{W}_n} \overline{V}^{(0)}.$$

Definition 5.22. $(F, \psi_1, \dots, \psi_M)$ is an *affine nested fractal* if $\psi_1, \dots, \psi_M$ satisfy the open set condition, $\#(\overline{V}^{(0)}) \geq 2$, and

(A1) (Connectivity) For any i, j there exists a sequence of 1-cells $V_{i_0}^{(0)}, \dots, V_{i_k}^{(0)}$ such that $i_0 = i$, $i_k = j$ and $\overline{V}_{i_{r-1}}^{(0)} \cap \overline{V}_{i_r}^{(0)} \neq \emptyset$ for $1 \leq r \leq k$.

(A2) (Symmetry) For each $x, y \in \overline{V}^{(0)}$, $n \geq 0$, g_{xy} maps n cells to n cells.

(A3) (Nesting) If $w, v \in \mathbb{W}_n$ and $w \neq v$ then

$$F_w \cap F_v = \overline{V}_w^{(0)} \cap \overline{V}_v^{(0)}.$$

In addition $(F, \psi_1, \dots, \psi_M)$ is a *nested fractal* if the ψ_i all have the same contraction factor.

If ψ_i has contraction factor α_i, then by (5.4) $\dim_H(F) = \beta$, where β solves

$$\sum_{i=1}^{M} \alpha_i^\beta = 1. \tag{5.14}$$

If $\alpha_i = \alpha$, so that F is a nested fractal, then

$$\dim_H(F) = \frac{\log M}{\log(1/\alpha)}. \tag{5.15}$$

Following Lindstrøm we will call M the *mass scale factor*, and $1/\alpha$ the *length scale factor*, of the nested fractal F.

Lemma 5.23. *Let $(F, \psi_1, \dots, \psi_M)$ be an affine nested fractal. Write z_i for the fixed point of ψ_i. Then $z_i \notin F_j$ for any $j \neq i$.*

Proof. Suppose that $z_1 \in F_2$. Then by Definition 5.22(A3) $F_1 \cap F_2 = \overline{V}_1^{(0)} \cap \overline{V}_2^{(0)}$, so $z_1 \in \overline{V}_2^{(0)}$, and $z_1 = \psi_2(z_i)$, for some $z_i \in \overline{V}^{(0)}$. We cannot have $i = 2$, as $\psi_2(z_2) = z_2 \neq z_1$. Also, if $i = 1$ then ψ_2 would fix both z_1 and z_2, so could not be a contraction. So let $i = 3$. Therefore for any $k \geq 0$, $i \geq 0$,

$$z_1 = \psi_1^k \circ \psi_2 \circ \psi_3^i(z_3) \in F_{1^k \cdot 2 \cdot 3^i}.$$

Write $D_n = \{w \in \mathbb{W}_n : z_1 \in F_w\}$: by the above $\#(D_n) \geq n$. Let U be the open set given by the open set condition. Since $F \subset \overline{U}$ we have $z_i \in \overline{U}$ for each i. So $z_1 \in \overline{U}_w$ for each $w \in D_n$, while the open set condition implies that the sets $\{U_w, w \in D_n\}$ are disjoint. So z_1 is on the boundary of at least n disjoint open sets. If (as is true for nested fractals) all these sets are congruent then a contradiction is almost immediate.

For the general case of affine nested fractals we need to work a little harder to obtain the same conclusion. Let $a > 0$ be such that

$$|B(z_i, 1) \cap U| > a \quad \text{for each } i.$$

Let α_i, $1 \leq i \leq M$ be the contraction factors of the ψ_i. Recall the notation $\alpha_w = \Pi_{i=1}^n \alpha_{w_i}$, $w \in \mathbb{W}_n$. Set $\delta = \min_{w \in D_n} \alpha_w$, and let $\beta = \min_i \alpha_i$. For each $w \in D_n$ let $w' = w \cdot 1...1$ be chosen so that $\beta\delta < \alpha_{w'} \leq \delta$. Then $z_1 \in F_{w'} \subset \overline{U}_{w'}$, for each $w \in D_n$, and the sets $\{U_{w'}, w \in D_n\}$ are still disjoint. (Since $\Psi(U) \subset U$ we have $U_{w'} \subset U_w$ for each $w \in D_n$).

Now if $w \in D_n$ then if j is such that $z_1 = \psi_{w'}(z_j)$

$$|B(z_1, \delta) \cap U_{w'}| = \alpha_{w'}^d |B(z_j, \delta/\alpha_{w'}) \cap U| \geq (\beta\delta)^d |B(z_j, 1) \cap U| \geq a(\beta\delta)^d.$$

So

$$c_d \delta^d = |B(z_1, \delta)| \geq \sum_{w \in D_n} |B(z_1, \delta) \cap U_{w'}| \geq na(\beta\delta)^d.$$

Choosing n large enough this gives a contradiction. □

Proposition 5.24. *Let $(F, \psi_1, \dots, \psi_M)$ be an affine nested fractal. Write z_i for the fixed point of ψ_i. Then $(F, \psi_1, \dots, \psi_M)$ is a p.c.f.s.s. set, and*
(a) $\overline{V}^{(0)} = V^{(0)}$,
(b) $P = \left\{ (\dot{s}) : z_s \in \overline{V}^{(0)} \right\}$.
(c) If $z \in V^{(0)}$ then z is in exactly one n-complex for each $n \geq 1$.
(d) Each 1-complex contains at most one element of $V^{(0)}$.

Proof. It is clear that $(F, \psi_1, \dots, \psi_M)$ is a self-similar structure. Relabelling the ψ_i, we can assume $\overline{V}^{(0)} = \{z_1, \dots, z_k\}$ where $2 \leq k \leq M$. We begin by calculating B, Γ and P. It is clear from (A3) that

$$B = \bigcup_{s \neq t} (\overline{V}_s^{(0)} \cap \overline{V}_t^{(0)}).$$

Let $w \in \Gamma$. Then $\pi(w) \in B$, so (as $\pi(w) \in F_{w_1}$) $\pi(w) \in \overline{V}^{(0)}_{w_1}$, and therefore $\pi(\sigma w) \in \overline{V}^{(0)}$. Say $\pi(\sigma w) = z_s$, where $s \in \{1,..,k\}$. Then since $z_s \in F_{w_2}$, by Lemma 5.23 we must have $w_2 = s$. So $\psi_s\big(\pi(\sigma^2 w)\big) = \pi(s \cdot \sigma^2 w) = \pi(\sigma w) = z_s$, and therefore $\pi(\sigma^2 w) = z_s$. So $w_3 = s$, and repeating we deduce that $\sigma w = (\dot{s})$. Therefore $\{\sigma w, w \in \Gamma\} = \{(\dot{s}), 1 \le s \le k\}$. This proves (b); as P is finite $(F, \psi_1, \dots, \psi_M)$ is a p.c.f.s.s. set. (a) is immediate, since $\pi(P) = V^{(0)} = \{\pi(\dot{s})\} = \overline{V}^{(0)}$.
(c) This is now immediate from (a), (b) and Lemma 5.23.
(d) Suppose F_i contains z_s and z_t, where $s \neq t$. Then one of s, t is distinct from i – suppose it is s. Then $z_s \in F_s \cap F_i$, which contradicts (c). □

Remarks 5.25. 1. Of the examples considered above, the SG is a nested fractal and the SG with added triangle is an ANF. The cut square is not an ANF, since if it were, the maps $\psi_i : \mathbb{R}^d \to \mathbb{R}^d$ would preserve the plane containing its 4 corners, and then the nesting axiom fails. The rotated triangle fails the symmetry axiom unless $\lambda = 1/2$. The Vicsek set defined in Section 2 is a nested fractal, but the Sierpinski carpet fails the nesting axiom.

2. The simplest examples of p.c.f.s.s. sets, and nested fractals can be a little misleading. Note the following points:
(a) Proposition 5.24(c) fails for p.c.f.s.s. sets. See for example the SG with added triangle, where $V^{(0)}$ contains the points $\{b_1, b_2, b_3\}$ as well as the corners $\{a_1, a_2, a_3\}$, and each of the points b_i lies in 2 distinct 1-cells.
(b) This example also shows that for a general p.c.f.s.s. set it is possible to have $F - V^{(0)}$ disconnected even if F is connected.
(c) Let $V_i^{(0)}$ and $V_j^{(0)}$ be two distinct 1-cells in a p.c.f.s.s. set. Then one can have $\#\big(V_i^{(0)} \cap V_j^{(0)}\big) \ge 2$. (The cut square is an example of this). For nested fractals, I do not know whether it is true that

$$\#(V_i^{(0)} \cap V_j^{(0)}) \le 1 \quad \text{if } i \neq j. \tag{5.16}$$

In [FHK, Prop. 2.2(4)] it is asserted that (5.16) holds for affine nested fractals, quoting a result of J. Murai: however, the result of Murai was proved under stronger hypotheses. While much of the work on nested fractals has assumed that (5.16) holds, this difficulty is not a serious one, since only minor modifications to the definitions and proofs in the literature are needed to handle the general case.

3. The symmetry hypothesis (A2) is very strong. We have

$$g_{xy} : V^{(0)} \to V^{(0)} \quad \text{for all} \quad x \neq y, \qquad x, y \in V^{(0)}. \tag{5.17}$$

The question of which sets $V^{(0)}$ satisfy (5.17) leads one into questions concerning reflection groups in $\mathbb{R}^d$. It is easy to see that $V^{(0)}$ satisfies (5.17) if $V^{(0)}$ is a regular planar polygon, a d-dimensional tetrahedron or a d-dimensional simplex. (That is, the set $V^{(0)} = \{e_i, -e_i, 1 \le i \le d\} \subset \mathbb{R}^d$, where $e_i = (\delta_{1i}, \dots, \delta_{di})$. I have been assured by two experts in this area that these are the only possibilities, and my web page see (`http://www.math.ubc.ca/`) contains a letter from G. Maxwell with a sketch of a proof of this fact.

Note that the cube in $\mathbb{R}^3$ fails to satisfy (5.17).

4. Note also that if F is a nested fractal in $\mathbb{R}^d$, and $V^{(0)} \subset H$ where H is a k-dimensional subspace, one does not necessarily have $F \subset H$. This is the case of the Koch curve, for example. (See [L1, p. 39]).

Example 5.26. (Lindstrøm snowflake). This nested fractal is the "classical example", used in [L1] as an illustration of the axioms. It may be defined briefly as follows. Let z_i, $1 \leq i \leq 6$ be the vertices of a regular hexagon in $\mathbb{R}^2$, and let $z_7 = \frac{1}{6}(z_1 + \ldots z_6)$ be the centre. Set

$$\psi_i(x) = z_i + \tfrac{1}{3}(x - z_i), \qquad 1 \leq i \leq 7.$$

It is easy to verify that this set satisfies the axioms (A1)–(A3) above.

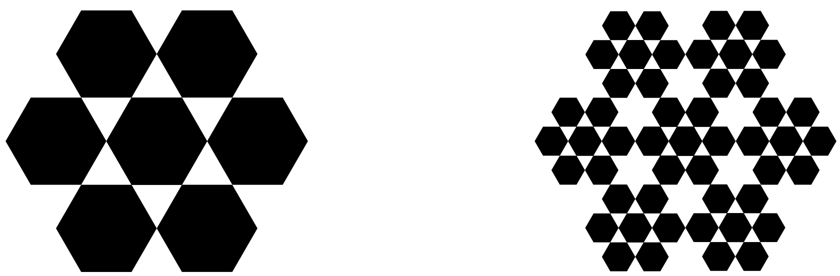

Fig. 5.4. Lindstrøm snowflake.

Measures on p.c.f.s.s. sets.

The structure of these sets makes it easy to define measures which have good properties relative to the maps ψ_i. We begin by considering measures on $\mathbb{W}$. Let $\theta = (\theta_1, \ldots, \theta_M)$ satisfy

$$\sum_{i=1}^{M} \theta_i = 1, \qquad 0 < \theta_i < 1 \quad \text{for each} \quad i \in I_M.$$

Recall the notation $\theta_w = \prod_{i=1}^{n} \theta_{w_i}$ for $w \in \mathbb{W}_n$. We define the measure $\tilde{\mu}_\theta$ on $\mathbb{W}$ to be the natural product measure associated with the vector θ. More precisely, let $\xi_n : \mathbb{W} \to I_M$ be defined by $\xi_n(w) = w_n$; then $\tilde{\mu}_\theta$ is the measure which makes (ξ_n) i.i.d. random variables with distribution given by $\mathbb{P}(\xi_n = r) = \theta_r$. Note that for any $n \geq 1$, $w \in \mathbb{W}_n$,

$$\tilde{\mu}_\theta \left(\{v \in \mathbb{W} : v|n = w\}\right) = \prod_{i=1}^{n} \theta_{w_i}. \tag{5.18}$$

Definition 5.27. Let $\mathcal{B}(F)$ be the σ-field of subsets of F generated by the sets $\{F_w, w \in \mathbb{W}_n, n \geq 1\}$. (By Lemma 5.12 this is the Borel σ-field). For $A \in \mathcal{B}(F)$, set

$$\mu(A) = \tilde{\mu}\big(\pi^{-1}(A)\big).$$

Then for $w \in \mathbb{W}_n$

$$\mu_\theta(F_w) = \tilde{\mu}_\theta\big(\pi^{-1}(F_w)\big) = \tilde{\mu}_\theta\left(\{v : v|n = w\}\right) = \theta_w = \prod_{i=1}^{n} \theta_{w_i}. \tag{5.19}$$

In contexts when θ is fixed we will write μ for μ_θ.

Remark. If $(F, \psi_1, \dots, \psi_M)$ is a nested fractal, then the sets $\psi_i(F)$, $1 \le i \le M$ are congruent, and it is natural to take $\theta_i = M^{-1}$. More generally, for an ANF, the 'natural' θ is given by

$$\theta_i = \alpha_i^\beta,$$

where β is defined by (5.4).

The following Lemma summarizes the self-similarity of μ in terms of the space $L^1(F, \mu)$.

Lemma 5.28. *Let $f \in L^1(F, \mu)$. Then for $n \ge 1$*

$$\int_F f\, d\mu = \sum_{w \in \mathbb{W}_n} \theta_w \int (f \circ \psi_w)\, d\mu, \qquad n \ge 1. \tag{5.20}$$

Proof. It is sufficient to prove (5.20) in the case $n = 1$: the general case then follows by iteration. Write $G = F - V^{(0)}$. Note that $G_v \cap G_w = \emptyset$ if $v, w \in \mathbb{W}_n$ and $v \ne w$. As μ is non-atomic we have $\mu(F_w) = \mu(G_w)$ for any $w \in \mathbb{W}_n$. Let $f = 1_{G_w}$ for some $w \in \mathbb{W}_n$. Then $f \circ \psi_i = 0$ if $i \ne w_1$, and $f \circ \psi_{w_1} = 1_{G_{\sigma w}}$. Thus

$$\int (f \circ \psi_i)\, d\mu = \mu(G_{\sigma w}) = \theta_{w_1}^{-1} \mu(G_w) = \theta_{w_1}^{-1} \int f d\mu,$$

proving (5.20) for this particular f. The equality then extends to L^1 by a standard argument. □

We will also need related measures on the sets $V^{(n)}$. Let $N_0 = \#V^{(0)}$. Fix θ and set

$$\mu_n(x) = N_0^{-1} \sum_{w \in \mathbb{W}_n} \theta_w 1_{V_w^{(0)}}(x), \quad x \in V^{(n)}. \tag{5.21}$$

Lemma 5.29. *μ_n is a probability measure on $V^{(n)}$ and*

$$\mathrm{wlim}_{n \to \infty} \mu_n = \mu_\theta.$$

Proof. Since $\#V_w^{(0)} = N_0$ we have

$$\mu_n(V^{(n)}) = \sum_{x \in V^{(n)}} N_0^{-1} \sum_{w \in \mathbb{W}_n} \theta_w 1_{V_w^{(0)}}(x) = \sum_{w \in \mathbb{W}_n} \theta_w = 1,$$

proving the first assertion.

We may regard μ_n as being derived from μ by shifting the mass on each n-complex F_w to the boundary $V_w^{(0)}$, with an equal amount of mass being moved to each point. (So a point $x \in V_w^{(0)}$ obtains a contribution of θ_w from each n-complex it belongs to). So if $f : F \to \mathbb{R}$ then

$$\left| \int_F f\, d\mu - \int_F f\, d\mu_n \right| \le \max_{w \in \mathbb{W}_n} \sup_{x,y \in F_w} |f(x) - f(y)| \tag{5.22}$$

It follows that $\mu_n \xrightarrow{w} \mu_\theta$. □

Symmetries of p.c.f.s.s. sets.

Definition 5.30. Let $\mathcal{G}$ be a group of continuous bijections from F to F. We call $\mathcal{G}$ a *symmetry group* of F if
(1) $g : V^{(0)} \to V^{(0)}$ for all $g \in \mathcal{G}$.
(2) For each $i \in I$, $g \in \mathcal{G}$ there exists $j \in I$, $g' \in \mathcal{G}$ such that

$$g \circ \psi_i = \psi_j \circ g'. \tag{5.23}$$

Note that if g, h satisfy (5.23) then

$$\begin{aligned}(g \circ h) \circ \psi_i &= g \circ (h \circ \psi_i) = g \circ (\psi_j \circ h') = (g \circ \psi_j) \circ h' \\ &= (\psi_k \circ g') \circ h' = \psi_k \circ g'',\end{aligned}$$

for some $j, k \in I$, $g', h', g'' \in \mathcal{G}$. The calculation above also shows that if $\mathcal{G}_1$ and $\mathcal{G}_2$ are symmetry groups then the group generated by $\mathcal{G}_1$ and $\mathcal{G}_2$ is also a symmetry group. Write $\mathcal{G}(F)$ for the largest symmetry group of F. If $\mathcal{G}$ is a symmetry group, and $g \in \mathcal{G}$ write $\tilde{g}(i)$ for the unique element $j \in I$ such that (5.23) holds.

Lemma 5.31. *Let $g \in \mathcal{G}$. Then for each $n \ge 0$, $w \in \mathbb{W}_n$, there exist $v \in \mathbb{W}_n$, $g' \in \mathcal{G}$ such that $g \circ \psi_w = \psi_v \circ g'$. In particular $g : V^{(n)} \to V^{(n)}$.*

Proof. The first assertion is just (5.23) if $n = 1$. If $n \ge 1$, and the assertion holds for all $v \in \mathbb{W}_n$ then if $w = i \cdot v \in \mathbb{W}_{n+1}$ then

$$g \circ \psi_w = g \circ \psi_i \circ \psi_v = \psi_j \circ g' \circ \psi_v = \psi_j \circ \psi_{v'} \circ g'',$$

for $j \in I$, $g', g'' \in \mathcal{G}$. □

Proposition 5.32. *Let $(F, \psi_1, \dots, \psi_M)$ be an ANF. Let $\mathcal{G}_1$ be the set of isometries of $\mathbb{R}^d$ generated by reflections in the hyperplanes bisecting the line segments $[z_i, z_j]$, $i \ne j$, $z_i, z_j \in V^{(0)}$. Let $\mathcal{G}_0$ be the group generated by $\mathcal{G}_1$. Then $\mathcal{G}_R = \{g|_F : g \in \mathcal{G}_0\}$ is a symmetry group of F.*

Proof. If $g \in \mathcal{G}_1$ then $g : V^{(n)} \to V^{(n)}$ for each n and hence also $g : F \to F$. Let $i \in I$: by the symmetry axiom (A2) $g(V_i^{(0)}) = V_j^{(0)}$ for some $j \in I$. For each of the possible forms of $V^{(0)}$ given in Remark 5.25(3), the symmetry group of $V^{(0)}$ is generated by the reflections in $\mathcal{G}_1$. So, there exists $g' \in \mathcal{G}_0$ such that $g \circ \psi_i = \psi_j \circ g'$. Thus (5.23) is verified for each $g \in \mathcal{G}_1$, and hence (5.23) holds for all $g \in \mathcal{G}_0$. □

Remark 5.33. In [BK] the collection of 'p.c.f. morphisms' of a p.c.f.s.s. set was introduced. These are rather different from the symmetries defined here since the definition in [BK] involved 'analytic' as well as 'geometric' conditions.

Connectivity Properties.

Definition 5.34. Let F be a p.c.f.s.s. set. For $n \geq 0$, define a graph structure on $V^{(n)}$ by taking $\{x, y\} \in \mathbf{E}_n$ if $x \neq y$, and $x, y \in V_w^{(0)}$ for some $w \in \mathbb{W}_n$.

Proposition 5.35. *Suppose that $(V^{(1)}, \mathbf{E}_1)$ is connected. Then $(V^{(n)}, \mathbf{E}_n)$ is connected for each $n \geq 2$, and F is pathwise connected.*

Proof. Suppose that $(V^{(n)}, \mathbf{E}_n)$ is connected, where $n \geq 1$. Let $x, y \in V^{(n+1)}$. If $x, y \in V_w^{(1)}$ for some $w \in \mathbb{W}_n$, then, since $(V^{(1)}, \mathbf{E}_1)$ is connected, there exists a path $\psi_w^{-1}(x) = z_0, z_1, \ldots, z_k = \psi_w^{-1}(y)$ in $\left(V^{(1)}, \mathbf{E}_1\right)$ connecting $\psi_w^{-1}(x)$ and $\psi_w^{-1}(y)$. We have $z_{i-1},\ z_i \in V_{w_i}^{(0)}$ for some $w_i \in \mathbb{W}_1$, for each $1 \leq i \leq k$. Then if $z_i' = \psi_w(z_i)$, $z_{i-1}',\ z_i' \in F_{w_i \cdot w}$ and so $\{z_{i-1}', z_i\} \in \mathbf{E}_{n+1}$. Thus x, y are connected by a path in $(V^{(n+1)}, \mathbf{E}_{n+1})$.

For general $x,\ y \in V^{(n+1)}$, as $(V^{(n)}, \mathbf{E}_n)$ is connected there exists a path $y_0, \ldots, y_m$ in $(V^{(n)}, \mathbf{E}_n)$ such that $\{y_{i-1}, y_i\} \in \mathbf{E}_n$ and x, y_0, and y, y_m, lie in the same $n+1$-cell. Then, by the above, the points $x, y_0, y_1, \ldots, y_m, y$ can be connected by chains of edges in $\mathbf{E}_{n+1}$.

To show that F is path-connected we actually construct a continuous path $\gamma : [0,1] \to F$ such that $F = \{\gamma(t), t \in [0,1]\}$. Let $x_0, \ldots, x_N$ be a path in $(V^{(1)}, \mathbf{E}_1)$ which is "space-filling", that is such that $V^{(1)} \subset \{x_0, \ldots, x_N\}$. Define $\gamma(i/N) = x_i$, $A_1 = \{i/N,\ 0 \leq i \leq N\}$. Now $x_0, x_1 \in V_w^{(0)}$, for some $w \in \mathbb{W}_1$. Let $x_0 = y_0, y_1, \ldots, y_m = x_1$ be in a space-filling path in $\left(V_w^{(1)}, \mathbf{E}_2\right)$. Define $\gamma(k/Nm) = y_k$, $0 \leq k \leq m$. Continuing in this way we fill each of the sets $V_w^{(1)}$, $w \in \mathbb{W}_1$, and so can define $A_2 \subset [0,1]$ such that $A_1 \subset A_2$, and $\gamma(t)$, $t \in A_2$ is a space filling path in the graph $(V^{(2)}, \mathbf{E}_2)$. Repeating this construction we obtain an increasing sequence (A_n) of finite sets such that $\gamma(t)$, $t \in A_n$ is a space filling path in $(V^{(n)}, \mathbf{E}_n)$, and $\cup_n A_n$ is dense in $[0,1]$. If $t \in A_n$, and $t' < t < t''$ are such that $(t', t'') \cap A_n = \{t\}$, then $\gamma(s)$ is in the same n-complex as $\gamma(t)$ for $s \in (t', t'')$. So, if $t \in [0,1] - A$, and $s_n, t_n \in A_n$ are chosen so that $s_n < t < t_n$, $(s_n, t_n) \cap A_n = \emptyset$, then the points $\gamma(u)$, $u \in A \cap (s,t)$ all lie in the same n-complex. So defining $\gamma(t) = \lim_n \gamma(t_n)$, we have that the limit exists, and γ is continuous. The construction of γ also gives that γ is space filling; if $w \in \mathbb{W}$ then for any $n \geq 1$ a section of the path, $\gamma(s)$, $a_n \leq s \leq b_n$, $s \in A_n$, fills $V_{w|n}^{(0)}$.

It follows immediately from the existence of γ that F is pathwise connected. □

Remark. This proof returns to the roots of the subject – the original papers of Sierpinski [Sie1, Sie2] regarded the Sierpinski gasket and Sierpinski carpet as "curves".

Corollary 5.36. *Any ANF is pathwise connected.*

Remark 5.37. If F is a p.c.f.s.s. set, and the graph $(V^{(1)}, \mathbf{E}_1)$ is not connected, then it is easy to see that F is not connected.

For the case of ANFs, we wish to examine the structure of the graphs $(V^{(n)}, \mathbf{E}_n)$ a little more closely. Let $(F, \psi_1, \ldots, \psi_M)$ be an ANF. Then let

$$a = \min\left\{ |x - y| : x, y \in V^{(0)},\ x \neq y \right\},$$

and set

$$\mathbf{E}'_0 = \big\{\{x,y\} \in V^{(0)} : |x-y| = a\big\},$$
$$\mathbf{E}'_n = \Big\{\{x,y\} \in \mathbf{E}_n : x = \psi_w(x'), y = \psi_w(y') \quad \text{for some} \quad w \in \mathbb{W}_n,\ \{x',y'\} \in \mathbf{E}'_0\Big\},\ n \geq 1.$$

Proposition 5.38. *Let F be an ANF.*
(a) Let $x,y,z \in V^{(0)}$ be distinct points. Then there exists a path in $(V^{(0)}, \mathbf{E}'_0)$ connecting x and y and not containing z.
(b) Let $x,y \in V^{(0)}$. There exists a path in $(V^{(1)}, \mathbf{E}'_1)$ connecting x,y which does not contain any point in $V^{(0)} - \{x,y\}$.
(c) Let $x,y,x',y' \in V^{(0)}$ with $|x-y| = |x'-y'|$. Then there exists $g \in \mathcal{G}_R$ such that $g(x') = x$, $g(y') = y$.

Proof. If $\#\big(V^{(0)}\big) = 2$ then $\mathbf{E}_0 = \mathbf{E}'_0$, so (a) is vacuous and (b) is immediate from Corollary 5.36. So suppose $\#\big(V^{(0)}\big) \geq 3$.
(a) Since (see Remark 5.25(3)) $V^{(0)}$ is either a d-dimensional tetrahedron, or a d-dimensional simplex, or a regular polygon, this is evident. (For a proof which does not use this fact, see [L1, p. 34–35]).
(b) This now follows from (a) by the same kind of argument as that given in Proposition 5.35.
(c) Write $g[x,y]$ for the reflection in the hyperplane bisecting the line segment $[x,y]$. Let $g_1 = g[y,y']$, and $z = g_1(x')$. Then if $z = x$ we are done. Otherwise note that $|x-y| = |x'-y'| = |z-y|$, so if $g_2 = g[x,z]$ then $g_2(y) = y$. Hence $g_1 \circ g_2$ works. □

Metrics on Nested Fractals.

Nested fractals, and ANFs, are subsets of $\mathbb{R}^d$, and so of course are metric spaces with respect to the Euclidean metric. Also, p.c.f.s.s. sets have been assumed to be metric spaces. However, these metrics do not necessarily have all properties we would wish for, such as the mid-point property that was used in Section 3. We saw in Section 2 that the geodesic metric on the Sierpinski gasket was equivalent to the Euclidean metric, but for a general nested fractal there may be no path of finite length between distinct points. (It is easy to construct examples). It is however, still possible to construct a geodesic metric on a ANF.

For simplicity, we will just treat the case of nested fractals. Let $(F, (\psi_i)_{i=1}^M)$ be a nested fractal, with length scale factor L. Write $d_n(x,y)$ for the natural graph distance in the graph $(V^{(n)}, \mathbf{E}_n)$. Fix $x_0,\ y_0 \in V^{(0)}$ such that $\{x_0, y_0\} \in \mathbf{E}'_0$, and let $a_n = d_n(x_0, y_0)$, and b_0 be the maximum distance between points in $(V^{(0)}, \mathbf{E}'_0)$.

Lemma 5.39. *If $x,y \in V^{(0)}$ then $a_n \leq d_n(x,y) \leq b_0 a_n$.*

Proof. Since x,y are connected by a path of length at most b_0 in $(V^{(0)}, \mathbf{E}'_0)$, the upper bound is evident. Fix x,y, and let $k = d_n(x,y)$. If $\{x,y\} \in \mathbf{E}'_0$ then $d_n(x,y) = d_n(x_0,y_0) = a_n$, so suppose $\{x,y\} \notin \mathbf{E}'_0$. Choose $y' \in V^{(0)}$ such that $\{x,y'\} \in \mathbf{E}'_0$, let H be the hyperplane bisecting $[y,y']$ and let g be reflection in H. Write A, A' for the components of $\mathbb{R}^d - H$ containing y,y' respectively. As $|x-y'| < |x-y|$ we have $x \in A'$. Let $x = z_0, z_1, \ldots, z_k = y$ be the shortest path in $(V^{(n)}, \mathbf{E}_n)$ connecting x and y. Let $j = \min\{i : z_i \in A\}$, and write $z'_i = z_i$ if $i < j$, $z'_i = g(z_i)$

if $i \geq j$. Then z'_i, $0 \leq i \leq k$ is a path in $(V^{(n)}, \mathbf{E}_n)$ connecting x and y', and so $d_n(x,y) = k \geq d_n(x,y') = a_n$. □

Lemma 5.40. *Let $x, y \in V^{(n)}$. Then for $m \geq 0$*

$$a_m d_n(x,y) \leq d_{n+m}(x,y) \leq b_0 a_m d_n(x,y). \tag{5.24}$$

In particular

$$a_n a_m \leq a_{n+m} \leq b_0 a_n a_m, \quad n \geq 0,\, m \geq 0. \tag{5.25}$$

Proof. Let $k = d_n(x,y)$, and let $x = z_0, z_1, \dots, z_k = y$ be a shortest path connecting x and y in $(V^{(n)}, \mathbf{E}_n)$. Then since by Lemma 5.39 $d_m(z_{i-1}, z_i) \leq b_0 a_m$, the upper bound in (5.24) is clear.

For the lower bound, let $r = d_{n+m}(x,y)$, and let $(z_i)_{i=0}^r$ be a shortest path in $(V^{(n+m)}, \mathbf{E}_{n+m})$ connecting x, y. Let $0 = i_0, i_1, \dots, i_s = r$ be successive disjoint hits by this path on $V^{(n)}$. (Recall the definition from Section 2: of course it makes sense for a deterministic path as well as a process). We have $s = d_n(x,y) \geq a_n$. Then since $z_{i_{j-1}}, z_{i_j}$ lie in the same n-cell, $i_j - i_{j-1} = d_m(z_{i_{j-1}}, z_{i_j}) \geq a_m$, by Lemma 5.39. So $r = \sum_{j=1}^s (i_j - i_{j-1}) \geq a_n a_m$. □

Corollary 5.41. *There exists $\gamma \in [L, b_0 a_1]$ such that*

$$b_0^{-1} \gamma^n \leq a_n \leq \gamma^n. \tag{5.26}$$

Proof. Note that $\log(b_0 a_n)$ is a subadditive sequence, and that $\log a_n$ is superadditive. So by the general theory of these sequences there exist θ_0, θ_1 such that

$$\theta_0 = \lim_{n\to\infty} n^{-1} \log(b_0 a_n) = \inf_{n \geq 0} n^{-1} \log(b_0 a_n),$$

$$\theta_1 = \lim_{n\to\infty} n^{-1} \log(a_n) = \sup_{n \geq 0} n^{-1} \log(a_n).$$

So $\theta_0 = \theta_1$, and setting $\gamma = e^{\theta_0}$, (5.26) follows.

To obtain bounds on γ note first that as $a_n \leq b_0 a_1 a_{n-1}$ we have $\gamma \leq b_0 a_1$. Also,

$$|x_0 - y_0| \leq a_n L^{-n} |x_0 - y_0|,$$

so $\gamma \geq L$. □

Definition 5.42. We call $d_c = \log \gamma / \log L$ the *chemical exponent* of the fractal F, and γ the *shortest path scaling factor.*

Theorem 5.43. *There exists a metric d_F on F with the following properties.*
(a) There exists $c_1 < \infty$ such that for each $n \geq 0$, $w \in \mathbb{W}_n$,

$$d_F(x,y) \leq c_1 \gamma^{-n} \;\text{ for } x, y \in F_w, \tag{5.27}$$

and

$$d_F(x,y) \geq c_2 \gamma^{-n} \;\text{ for } x \in V^{(n)},\, y \in N_n(x)^c. \tag{5.28}$$

(b) d_F induces the same topology on F as the Euclidean metric.

(c) d_F has the midpoint property.
(d) The Hausdorff dimension of F with respect to the metric d_F is

$$d_f(F) = \frac{\log M}{\log \gamma}. \tag{5.29}$$

Proof. Write $V = \cup_n V^{(n)}$. By Lemma 5.41 for $x, y \in V$ we have

$$b_0^{-1} \gamma^m d_n(x,y) \le d_{n+m}(x,y) \le b_0 \gamma^m d_n(x,y). \tag{5.30}$$

So $(\gamma^{-m} d_{n+m}(x,y), m \ge 0)$ is bounded above and below. By a diagonalization argument we can therefore find a subsequence $n_k \to \infty$ such that

$$d_F(x,y) = \lim_{k\to\infty} \gamma^{-n_k} d_{n_k}(x,y) \text{ exists for each } x, y \in V.$$

So, if $x, y \in V_w^{(0)}$ where $w \in \mathbb{W}_n$ then

$$c_0^{-1} \gamma^{-n} \le d_F(x,y) \le c_0 \gamma^{-n}. \tag{5.31}$$

It is clear that d_F is a metric on V.

Let $n \ge 0$ and $y \in V^{(n)}$. For $m = n-1, n-2, \ldots, 0$ choose inductively $y_m \in V^{(m)}$ such that y_m is in the same m-cell as $y_{m+1}, ..., y_n$. Then

$$d_{m+1}(y_m, y_{m+1}) \le \max\{d_1(x', y') : x', y' \in V^{(1)}\} = c < \infty.$$

So by (5.30) $d_n(y_m, y_{m+1}) \le b_0 \gamma^{n-(m+1)} c$, and therefore

$$d(y_k, y) \le c \sum_{i=k}^{\infty} \gamma^{-i-1} = c' \gamma^{-k}.$$

So if $x, y \in V$ are in the same k-cell, choosing x_k in the same way we have

$$d_F(x,y) \le d_F(x, x_k) + d_F(x_k, y_k) + d_F(y_k, y) \le c_1 \gamma^k, \tag{5.32}$$

since $d_k(x_k, y_k) \le b_0$. Thus d_F is uniformly continuous on $V \times V$, and so extends by continuity to a metric d_F on F. (a) is immediate from (5.31).

If $x, y \in V^{(n)}$ and $x \ne y$ then $d_F(x,y) \ge b_0^{-1} \gamma^{-n}$. This, together with (5.30), implies (b).

If $x, y \in V^{(n)}$ then there exists $z \in V^{(n)}$ such that

$$|d_n(u,z) - \tfrac{1}{2} d_n(x,y)| \le 1, \quad u = x, y.$$

So the metrics d_n have an approximate midpoint property: (c) follows by an easy limiting argument.

Let μ be the measure on F associated with the vector $\theta = (M^{-1}, ..., M^{-1})$. Thus $\mu(F_w) = M^{-|w|}$ for each $w \in \mathbb{W}_n$. Since we have $\operatorname{diam}_{d_F}(F_w) \asymp \gamma^{-|w|}$, it follows that, writing $d_f = \log M / \log \gamma$,

$$c_5 r^{d_f} \le \mu(B_{d_F}(x,r)) \le c_6 r^{d_f}, \quad x \in F$$

and the conclusion then follows from Corollary 2.8. □

Remark 5.44. The results here on the metric d_F are not the best possible. The construction here used a subsequence, and did not give a procedure for finding the scale factor γ. See [BS], [Kum2], [FHK], [Ki6] for more precise results.

6. Renormalization on Finitely Ramified Fractals.

Let $(F, \psi_1, \dots, \psi_M)$ be a p.c.f.s.s. set. We wish to construct a sequence $Y^{(n)}$ of random walks on the sets $V^{(n)}$, nested in a similar fashion to the random walks on the Sierpinski gasket considered in Section 2. The example of the Vicsek set shows that, in general, some calculation is necessary to find such a sequence of walks. As the random walks we treat will be symmetric, we will find it convenient to use the theory of Dirichlet forms, and ideas from electrical networks, in our proofs.

Fix a p.c.f.s.s. set $\big(F, (\psi_i)_{i=1}^M\big)$, and a Bernouilli measure $\mu = \mu_\theta$ on F, where each $\theta_i > 0$. We also choose a vector $r = (r_1, \dots, r_M)$ of positive "weights": loosely speaking r_i is the size of the set $\psi_i(F) = F_i$, for $1 \le i \le M$. We call r a *resistance vector.*

Definition 6.1. Let $\mathbb{D}$ be the set of Dirichlet forms $\mathcal{E}$ defined on $C\big(V^{(0)}\big)$. From Section 4 we have that each element $\mathcal{E} \in \mathbb{D}$ is of the form $\mathcal{E}_A$, where A is a conductance matrix. Let also $\mathbb{D}_1$ be the set of Dirichlet forms on $C\big(V^{(1)}\big)$.

We consider two operations on $\mathbb{D}$:

(1) Replication – i.e. extension of $\mathcal{E} \in \mathbb{D}$ to a Dirichlet form $\mathcal{E}^R \in \mathbb{D}_1$.
(2) Decimation/Restriction/Trace. Reduction of a form $\mathcal{E} \in \mathbb{D}_1$ to a form $\widetilde{\mathcal{E}} \in \mathbb{D}$.

Note. In Section 4, we defined a Dirichlet form $(\mathcal{E}, \mathcal{D})$ with domain $\mathcal{D} \subset L^2(F, \mu)$. But for a finite set F, as long as μ charges every point in the set it plays no role in the definition of $\mathcal{E}$. We therefore will find it more convenient to define $\mathcal{E}$ on $C(F) = \{f : F \to \mathbb{R}\}$.

Definition 6.2. Given $\mathcal{E} \in \mathbb{D}$, define for $f, g \in C\big(V^{(1)}\big)$,

$$\mathcal{E}^R(f, g) = \sum_{i=1}^M r_i^{-1} \mathcal{E}(f \circ \psi_i, g \circ \psi_i). \tag{6.2}$$

(Note that as $\psi_i : V^{(0)} \to V^{(1)}$, $f \circ \psi_i \in C\big(V^{(0)}\big)$.) Define $R : \mathbb{D} \to \mathbb{D}_1$ by $R(\mathcal{E}) = \mathcal{E}^R$.

Lemma 6.3. *Let $\mathcal{E} = \mathcal{E}_A$, and let*

$$a_{xy}^R = \sum_{i=1}^M 1_{\left(x \in V_i^{(0)}\right)} 1_{\left(y \in V_i^{(0)}\right)} r_i^{-1} a_{\psi_i^{-1}(x), \psi_i^{-1}(y)}. \tag{6.3}$$

Then

$$\mathcal{E}^R(f, g) = \tfrac{1}{2} \sum_{x,y} a_{xy}^R \big(f(x) - f(y)\big)\big(g(x) - g(y)\big). \tag{6.4}$$

$A^R = (a^R_{xy})$ *is a conductance matrix, and* $\mathcal{E}^R$ *is the associated Dirichlet form.*

Proof. As the maps ψ_i are injective, it is clear that $a^R_{xy} \geq 0$ if $x \neq y$, and $a^R_{xx} \leq 0$. Also $a^R_{xy} = a^R_{yx}$ is immediate from the symmetry of A. Writing $x_i = \psi_i^{-1}(x)$ we have

$$\sum_{y \in V^{(1)}} a^R_{xy} = \sum_i r_i^{-1} 1_{V_i^{(0)}}(x) \sum_{y \in V^{(1)}} 1_{V_i^{(0)}}(y) a_{\psi_i^{-1}(x), \psi_i^{-1}(y)}$$

$$= \sum_i r_i^{-1} 1_{V_i^{(0)}}(x) \sum_{y \in V_i^{(0)}} a_{x,y} = 0,$$

so A^R is a conductance matrix.

To verify (6.4), it is sufficient by linearity to consider the case $f = g = \delta_z$, $z \in V^{(1)}$. Let $B = \{i \in \mathbb{W}_1 : z \in V_i^{(0)}\}$. If $i \notin B$, then $f \circ \psi_i(x) = 0$, since $\psi_i(x)$ cannot equal z. If $i \in B$, then $f \circ \psi_i(x) = \delta_{z_i}(x)$, where $z_i = \psi_i^{-1}(z)$. So,

$$\mathcal{E}(f \circ \psi_i, f \circ \psi_i) = \mathcal{E}(\delta_{z_i}, \delta_{z_i}) = -a_{z_i z_i}.$$

Thus

$$\mathcal{E}^R(f,f) = -\sum_{i \in B} r_i^{-1} a_{z_i z_i} = -\sum_{i=1}^{M} r_i^{-1} 1_{V_i^{(0)}}(z) a_{\psi_i^{-1}(z), \psi_i^{-1}(z)} = -a^R_{zz},$$

while

$$\tfrac{1}{2} \sum_{x,y} a^R_{xy} \big(f(x) - f(y)\big)^2 = -f^T A^R f = -a^R_{zz}.$$

So (6.4) is verified. □

The most intuitive explanation of the replication operation is in terms of electrical networks. Think of $V^{(0)}$ as an electric network. Take M copies of $V^{(0)}$, and rescale the ith one by multiplying the conductance of each wire by r_i^{-1}. (This explains why we called r a resistance vector). Now assemble these to form a network with nodes $V^{(1)}$, using the ith network to connect the nodes in $V_i^{(0)}$. Then $\mathcal{E}^R$ is the Dirichlet form corresponding to the network $V^{(1)}$.

As we saw in the previous section, for $x, y \in V^{(1)}$ there may in general be more than one 1-cell which contains both x and y: this is why the sum in (6.3) is necessary. If x and y are connected by k wires, with conductivities $c_1, \ldots, c_k$ then this is equivalent to connection by one wire of conductance $c_1 + \ldots + c_k$.

Remark 6.4. The replication of conductivities defined here is not the same as the replication of transition probabilities discussed in Section 2. To see the difference, consider again the Sierpinski gasket. Let $V^{(0)} = \{z_1, z_2, z_3\}$, and y_3 be the midpoint of $[z_1, z_2]$, and define y_1, y_2 similarly. Let A be a conductance matrix on $V^{(0)}$, and write $a_{ij} = a_{z_i z_j}$. Take $r_1 = r_2 = r_3 = 1$. While the continuous time Markov Chains $X^{(0)}$, $X^{(1)}$ associated with $\mathcal{E}_A$ and $\mathcal{E}_A^R$ will depend on the choice of a measure on $V^{(0)}$ and $V^{(1)}$, their discrete time skeletons that is, the processes $X^{(i)}$

sampled at their successive jump times do not – see Example 4.21. Write $Y^{(i)}$ for these processes. We have

$$\mathbb{P}^{y_3}\left(Y_1^{(1)} \in \{z_2, y_1\}\right) = \frac{a_{12} + a_{31}}{2a_{12} + a_{31} + a_{23}}.$$

On the other hand, if we replicate probabilities as in Section 2,

$$\mathbb{P}^{y_3}\left(Y_1^{(1)} \in \{z_2, y_1\}\right) = \mathbb{P}^{y_3}\left(Y_1^{(1)} \in \{z_1, y_2\}\right) = \tfrac{1}{2};$$

in general these expressions are different. So, even when we confine ourselves to symmetric Markov Chains, replication of conductivities and transition probabilities give rise to different processes.

Since the two replication operations are distinct, it is not surprising that the dynamical systems associated with the two operations should have different behaviours. In fact, the simple symmetric random walk on $V^{(0)}$ is stable fixed point if we replicate conductivities, but an unstable one if we replicate transition probabilities.

The second operation on Dirichlet forms, that of restriction or trace, has already been discussed in Section 4.

Definition 6.5. For $\mathcal{E} \in \mathbb{D}_1$ let

$$T(\mathcal{E}) = Tr(\mathcal{E}|V^{(0)}). \tag{6.5}$$

Define $\Lambda : \mathbb{D} \to \mathbb{D}$ by $\Lambda(\mathcal{E}) = T\big(R(\mathcal{E})\big)$. Note that Λ is homogeneous in the sense that if $\theta > 0$,

$$\Lambda(\theta\mathcal{E}) = \theta\Lambda(\mathcal{E}).$$

Example 6.6. (The Sierpinski gasket). Let A be the conductance matrix corresponding to the simple random walk on $V^{(0)}$, so that

$$a_{xy} = 1, \quad x \neq y, \quad a_{xx} = -2.$$

Then A^R is the network obtained by joining together 3 symmetric triangular networks. If $\Lambda(\mathcal{E}_A) = \mathcal{E}_B$, then B is the conductance matrix such that the networks $(V^{(1)}, A^R)$ and $(V^{(0)}, B)$ are electrically equivalent on $V^{(0)}$. The simplest way to calculate B is by the $\Delta - Y$ transform. Replacing each of the triangles by an (upside down) Y, we see from Example 4.24 that the branches in the Y each have conductance 3. Thus $(V^{(1)}, A^R)$ is equivalent to a network consisting of a central triangle of wires of conductance $3/2$, and branches of conductance 3. Applying the transform again, the central triangle is equivalent to a Y with branches of conductance $9/2$. Thus the whole network is equivalent to a Y with branches of conductance $9/5$, or a triangle with sides of conductance $3/5$.

Thus we deduce

$$\Lambda(\mathcal{E}_A) = \mathcal{E}_B, \qquad \text{where} \quad B = \tfrac{3}{5}A.$$

The example above suggests that to find a decimation invariant random walk we need to find a Dirichlet form $\mathcal{E} \in \mathbb{D}$ such that for some $\lambda > 0$

$$\Lambda(\mathcal{E}) = \lambda \mathcal{E}. \tag{6.6}$$

Thus we wish to find an eigenvector for the map Λ on $\mathbb{D}$. Since however (as we will see shortly) Λ is non-linear, this final formulation is not particularly useful. Two questions immediately arise: does there always exist a non-zero $(\mathcal{E}, \lambda)$ satisfying (6.6) and if so, is this solution (up to constant multiples) unique? We will abuse terminology slightly, and refer to an $\mathcal{E} \in \mathbb{D}$ such that (6.6) holds as a *fixed point* of Λ. (In fact it is a fixed point of Λ defined on a quotient space of $\mathcal{D}$.)

Example 6.7. ("*abc* gaskets" – see [HHW1]).

Let m_1, m_2, m_3 be integers with $m_i \geq 1$. Let z_1, z_2, z_3 be the corners of the unit triangle in $\mathbb{R}^2$, H be the closed convex hull of $\{z_1, z_2, z_3\}$. Let $M = m_1 + m_2 + m_3$, and let ψ_i, $1 \leq i \leq M$ be similitudes such that (writing for convenience $\psi_{M+j} = \psi_j$, $1 \leq j \leq M$) $H_i = \psi_i(H) \subset H$, and the M triangles H_i are arranged round the edge of H, such that each triangle H_i touches only H_{i-1} and H_{i+1}. (H_1 touches H_M and H_2 only). In addition, let $z_1 \in H_1$, $z_2 \in H_{m_3+1}$, $z_3 \in H_{m_3+m_1+1}$. So there are $m_3 + 1$ triangles along the edge $[z_1, z_2]$, and $m_1 + 1$, $m_2 + 1$ respectively along $[z_2, z_3]$, $[z_3, z_1]$. We assume that ψ_i are rotation-free. Note that the triangles H_2 and H_M do not touch, unless $m_1 = m_2 = m_3 = 1$.

Let F be the fractal obtained by Theorem 5.4 from $(\psi_1, \ldots, \psi_M)$. To avoid unnecessarily complicated notation we write ψ_i for both ψ_i and $\psi_i|_F$.

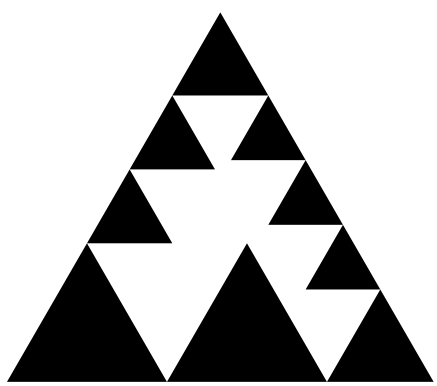

Figure 6.1: *abc* gasket with $m_1 = 4$, $m_2 = 3$, $m_3 = 2$.

It is easy to check that $(F, \psi_1, \ldots, \psi_M)$ is a p.c.f.s.s. set. Write $r = 1$, $s = m_3+1$, $t = m_3 + m_1 + 1$. We have $\pi(i\dot{s}) = \pi\big((i+1)\dot{r}\big)$ for $1 \leq i \leq m_3$, $\pi(i\dot{t}) = \pi\big((i+1)\dot{s}\big)$ for $m_3 + 1 \leq i \leq m_3 + m_1$, $\pi(i\dot{r}) = \pi\big((i+1)\dot{t}\big)$ for $m_3 + m_1 + 1 \leq i \leq M - 1$, and $\pi(M\dot{r}) = \pi(1\dot{t})$. The set $B = \cup(H_i \cap H_j)$ consists of these points. Hence

$$P = \{(\dot{r}), (\dot{s}), (\dot{t})\}, \quad V^{(0)} = \{z_1, z_2, z_3\}.$$

While it is easier to define F in $\mathbb{R}^2$, rather than abstractly, doing so has the misleading consequence that it forces the triangles H_i to be of different sizes. However, we will view F as an abstract metric space in which all the triangles H_i are of equal size, and so we will take $r_i = 1$ for $1 \le i \le M$.

We now study the renormalization map Λ. If $\mathcal{E} = \mathcal{E}_A \in \mathbb{D}$, then A is specified by the conductivities

$$\alpha_1 = a_{z_2,z_3}, \quad \alpha_2 = a_{z_3,z_1}, \quad \alpha_3 = a_{z_1,z_2}.$$

Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the renormalization map acting on $(\alpha_1, \alpha_2, \alpha_3)$. (So if $A = A(\alpha)$ then $\Lambda(\mathcal{E}) = \mathcal{E}_{A\left(f(\alpha)\right)}$).

It is easier to compute the action of the renormalization map on the variables β_i given by the $\triangle - Y$, transform. So let $\varphi : (0,\infty)^3 \to (0,\infty)^3$ be the $\triangle - Y$ map given in Example 4.24. Note that φ is bijective. Let $\beta = \varphi(\alpha)$ be the $Y-$conductivities, and write $\widetilde{\beta} = (\widetilde{\beta}_1, \widetilde{\beta}_2, \widetilde{\beta}_3)$ for the renormalized $Y-$conductivities: then $\widetilde{\beta} = \varphi(f(\alpha))$.

Applying the $\triangle - Y$ transform on each of the small triangles, we obtain a network with nodes $z_1, z_2, z_3, y_1, y_2, y_3$, where $\{z_i, y_i\}$ has conductivity β_i, and if $i \neq j$ $\{y_i, y_j\}$ has conductivity β_i, and if $i \neq j$, $\{y_i, y_i\}$ has conductivity

$$\gamma_k = \frac{\beta_i \beta_j}{(\beta_i + \beta_j) m_k},$$

where $k = k(i,j)$ is such that $k \in \{1,2,3\} - \{i,j\}$.

Apply the $\triangle - Y$ transform again to $\{y_1, y_2, y_3\}$, to obtain a Y, with conductivities δ_1, δ_2, δ_3, in the branches where

$$\delta_i \gamma_i = S = \gamma_1 \gamma_2 + \gamma_2 \gamma_3 + \gamma_3 \gamma_1, \qquad 1 \le i \le 3.$$

Then

$$\widetilde{\beta}_1^{-1} = \beta_1^{-1} + \delta_1^{-1} = \beta_1^{-1} + \tfrac{\beta_2 \beta_3}{(\beta_2 + \beta_3) m_1 S}. \tag{6.7}$$

Suppose that $\alpha \in (0,\infty)^3$ is such that $\varphi(\alpha) = \lambda \alpha$ for some $\lambda > 0$. Then since $\varphi(\theta\alpha) = \theta\,\varphi(\alpha)$ for any $\theta > 0$, we deduce that $\widetilde{\beta} = \varphi(f(\alpha)) = \lambda\beta$. So, from (6.7),

$$\lambda^{-1} \beta_1^{-1} = \beta_1^{-1} + \tfrac{\beta_2 \beta_2}{(\beta_2 + \beta_3) m_1 S},$$

which implies that $\lambda^{-1} > 1$. Writing $T = \beta_1 \beta_2 \beta_3 / S$, and $\theta = T\lambda(1-\lambda)^{-1}$, we therefore have

$$m_1(\beta_2 + \beta_3) = \theta,$$

and (as S, T are symmetric in the β_i) we also obtain two similar equations. Hence

$$\beta_2 + \beta_3 = \theta/m_1, \quad \beta_3 + \beta_1 = \theta/m_2, \quad \beta_1 + \beta_2 = \theta/m_3, \tag{6.8}$$

which has solution

$$2\beta_1 = \theta(m_2^{-1} + m_3^{-1} - m_1^{-1}), \qquad \text{etc.} \tag{6.9}$$

Since, however we need the $\beta_i > 0$, we deduce that a solution to the conductivity renormalization problem exists only if m_i^{-1} satisfy the triangle condition, that is that

$$(6.10) \qquad m_2^{-1} + m_3^{-1} > m_1^{-1}, \quad m_3^{-1} + m_1^{-1} > m_2^{-1}, \quad m_1^{-1} + m_2^{-1} > m_3^{-1}.$$

If (6.10) is satisfied, then (6.9) gives β_i such that the associated $\alpha = \varphi^{-1}(\beta)$ does satisfy the eigenvalue problem.

In the discussion above we looked for strictly positive α such that $\varphi(\alpha) = \lambda\alpha$. Now suppose that just one of the α_i, α_3 say, equals 0. Then while z, and z_2 are only connected via z_3 in the network $V^{(0)}$ they are connected via an additional path in the network $V^{(1)}$. So, $\varphi(\alpha)_3 > 0$, and α cannot be a fixed point. If now $\alpha_1 > 0$, and $\alpha_2 = \alpha_3 = 0$ then we obtain $\varphi(\alpha)_2 = \varphi(\alpha)_3 = 0$. So $\alpha = (1, 0, 0)$ satisfies $\varphi(\alpha) = \lambda\alpha$ for some $\lambda > 0$. Similarly $(0, 1, 0)$ and $(0, 0, 1)$ are also fixed points. Note that in these cases the network $(V^{(0)}, A(\alpha))$ is not connected.

The example of the *abc* gaskets shows that, even if fixed points exist, they may correspond to a reducible (ie non-irreducible) $\mathcal{E} \in \mathbb{D}$. The random walks (and limiting diffusion) corresponding to such a fixed point will be restricted to part of the fractal F. We therefore wish to find a *non-degenerate fixed point* of (6.6), that is an $\mathcal{E}_A \in \mathbb{D}$ such that the network $(V^{(0)}, A)$ is connected.

Definition 6.8. Let $\mathbb{D}^i$ be the set of $\mathcal{E} \in \mathbb{D}_0$ such that $\mathcal{E}$ is irreducible – that is the network $(V^{(0)}, A)$ is connected. Call $\mathcal{E} \in \mathbb{D}$ *strongly irreducible* if $\mathcal{E} = \mathcal{E}_A$ and $a_{xy} > 0$ for all $x \neq y$. Write $\mathbb{D}^{si}$ for the set of strongly irreducible Dirichlet forms on $V^{(0)}$.

The existence problem therefore takes the form:

Problem 6.9. (Existence). Let $(F, \psi_1, ..., \psi_M)$ be a p.c.f.s.s. set and let $r_i > 0$. Does there exist $\mathcal{E} \in \mathbb{D}^i$, $\lambda > 0$, such that

$$(6.12 \qquad \Lambda(\mathcal{E}) = \lambda\mathcal{E}?$$

Before we pose the uniqueness question, we need to consider the role of symmetry. Let $(F, (\psi_i))$ be a p.c.f.s.s. set, and let $\mathcal{H}$ be a symmetry group of F.

Definition 6.10. $\mathcal{E} \in \mathbb{D}$ is $\mathcal{H}$*-invariant* if for each $h \in \mathcal{H}$

$$\mathcal{E}(f \circ h, g \circ h) = \mathcal{E}(f, g), \quad f, g \in C(V^{(0)}).$$

r is $\mathcal{H}$*-invariant* if $r_{\tilde{h}(i)} = r_i$ for all $h \in \mathcal{H}$. (Here $\tilde{h}$ is the bijection on I associated with h).

Lemma 6.11. *(a) Let $\mathcal{E} = \mathcal{E}_A$. Then $\mathcal{E}$ is $\mathcal{H}$-invariant if and only if:*

$$(6.13) \qquad a_{h(x)\,h(y)} = a_{xy} \text{ for all } x, y \in V^{(0)}, \; h \in \mathcal{H}.$$

(b) Let $\mathcal{E}$ and r be $\mathcal{H}$-invariant. Then $\Lambda\mathcal{E}$ is $\mathcal{H}$-invariant.

Proof. (a) This is evident from the equation $\mathcal{E}(1_x, 1_y) = -a_{xy}$.

(b) Let $f \in C(V^{(1)})$. Then if $h \in \mathcal{H}$,

$$\begin{aligned}\mathcal{E}^R(f\circ h, f\circ h) &= \sum_i r_i^{-1}\, \mathcal{E}(f\circ h\circ \psi_i, f\circ h\circ\psi_i)\\ &= \sum r_i^{-1}\, \mathcal{E}(f\circ \psi_{\tilde h(i)}\circ h, f\circ\psi_{\tilde h(i)}\circ h,)\\ &= \sum r_{\tilde h(i)}^{-1}\, \mathcal{E}(f\circ\psi_{\tilde h(i)}, f\circ\psi_{\tilde h(i)}) = \mathcal{E}^R(f,f).\end{aligned}$$

If $g \in C(V^{(0)})$ then writing $\widetilde{\mathcal{E}} = \Lambda(\mathcal{E})$, if $f|_{V^{(0)}} = g$ then as $f\circ h|_{V^{(0)}} = g\circ h$,

$$\widetilde{\mathcal{E}}(g\circ h, g\circ h) \le \mathcal{E}^R(f\circ h, f\circ h) = \mathcal{E}^R(f,f),$$

and taking the infimum over f, we deduce that for any $h \in \mathcal{H}$, $\widetilde{\mathcal{E}}(g\circ h, g\circ h) \le \widetilde{\mathcal{E}}(g,g)$. Replacing g by $g\circ h$ and h by h^{-1} we see that equality must hold. □

If the fractal F has a non-trivial symmetry group $\mathcal{G}(F)$ then it is natural to restrict our attention to $\mathcal{G}(F)$-symmetric diffusions. We can now pose the uniqueness problem.

Problem 6.12. (Uniqueness). Let $(F, (\psi_i))$ be a p.c.f.s.s. set, let $\mathcal{H}$ be a symmetry group of F, and let r be $\mathcal{H}$-invariant. Is there at most one $\mathcal{H}$-invariant $\mathcal{E} \in \mathbb{D}^i$ such that $\Lambda(\mathcal{E}) = \lambda\mathcal{E}$?

(Unless otherwise indicated, when I refer to fixed points for nested fractals, I will assume they are invariant under the symmetry group $\mathcal{G}_R$ generated by the reflections in hyperplanes bisecting the lines $[x,y]$, $x, y \in V^{(0)}$).

The following example shows that uniqueness does not hold in general.

Example 6.13. (Vicsek sets – see [Me3].) Let $(F, \psi_i, 1 \le i \le 5)$ be the Vicsek set – see Section 2. Write $\{z_1, z_2, z_3, z_4,\}$ for the 4 corners of the unit square in $\mathbb{R}^2$. For $\alpha, \beta, \gamma > 0$ let $A(\alpha,\beta,\gamma)$ be the conductance matrix given by

$$a_{12} = a_{23} = a_{34} = a_{41} = \alpha, \quad \alpha_{13} = \beta, \quad a_{24} = \gamma,$$

where $a_{ij} = a_{z_i\, z_j}$. If $\mathcal{H}$ is the group on F generated by reflections in the lines $[z_1, z_3]$ and $[z_2, z_4]$ then A is clearly $\mathcal{H}$-invariant. Define $\tilde\alpha$, $\tilde\beta$, $\tilde\gamma$ by

$$\Lambda(\mathcal{E}_A) = \mathcal{E}_{A(\widetilde\alpha,\, \widetilde\beta,\, \widetilde\gamma)}.$$

Then several minutes calculation with equivalent networks shows that

$$\begin{aligned}\widetilde\alpha &= \frac{\alpha(\alpha+\beta)(\alpha+\gamma)}{5\alpha^2 + 3\alpha\beta + 3\alpha\gamma + \beta\gamma},\\ \widetilde\beta &= \tfrac{1}{3}(\alpha+\beta) - \widetilde\alpha,\\ \widetilde\gamma &= \tfrac{1}{3}(\alpha+\gamma) - \widetilde\alpha.\end{aligned} \tag{6.14}$$

If $(1, \beta, \gamma)$ is a fixed point then $(\widetilde\alpha, \widetilde\beta, \widetilde\gamma) = (\theta, \theta\beta, \theta\gamma)$ for some $\theta \ge 0$, so that $\widetilde\beta = \widetilde\alpha\beta$, $\widetilde\gamma = \widetilde\alpha\gamma$. So $\widetilde\alpha = \frac{1}{3}$, and this implies that $\beta\gamma = 1$. We therefore have that $(1,\, \beta, \beta^{-1})$ is a fixed point (with $\lambda = \frac{1}{3}$) for any $\beta \in (0,\infty)$ Thus for the group $\mathcal{H}$ uniqueness does not hold.

However if we replace $\mathcal{H}$ by the group $\mathcal{G}_R = \mathcal{G}(F)$, generated by all the symmetries of the square then for $\mathcal{E}_A$ to be $\mathcal{G}_R$-invariant we have to have $\beta = \gamma$. So in this case we obtain

$$(6.15) \qquad \begin{aligned} \widetilde{\alpha}(\alpha,\beta) &= \frac{\alpha(\alpha+\beta)^2}{5\alpha^2+6\alpha\beta+\beta^2}, \\ \widetilde{\beta}(\alpha,\beta) &= \tfrac{1}{3}(\alpha+\beta)-\widetilde{\alpha}. \end{aligned}$$

This has fixed points $(0,\beta)$, $\beta > 0$, and (α,α), $\alpha > 0$. The first are degenerate, the second not, so in this case, as we already saw in Section 2, uniqueness does hold for Problem 6.12.

This example also shows that Λ is in general non-linear.

As these examples suggest, the general problem of existence and uniqueness is quite hard. For all but the simplest fractals, explicit calculation of the renormalization map Λ is too lengthy to be possible without computer assistance – at least for 20th century mathematicians. Lindstrøm [L1] proved the existence of a fixed point $\mathcal{E} \in \mathbb{D}^{si}$ for nested fractals, but did not treat the question of uniqueness. After the appearance of [L1], the uniqueness of a fixed point for Lindstrøm's canonical example, the snowflake (Example 5.26) remained open for a few years, until Green [Gre] and Yokai [Yo] proved uniqueness by computer calculations.

The following analytic approach to the uniqueness problem, using the theory of quadratic forms, has been developed by Metz and Sabot – see [Me2-Me5, Sab1, Sab2]. Let $\mathbb{M}_+$ be set of symmetric bilinear forms $Q(f,g)$ on $C(V^{(0)})$ which satisfy

$$\begin{aligned} &Q(1,1) = 0, \\ &Q(f,f) \geq 0 \quad \text{for all } f \in C(V^{(0)}). \end{aligned}$$

For Q_1, $Q_2 \in \mathbb{M}_+$ we write $Q_1 \geq Q_2$, if $Q_2 - Q_1 \in \mathbb{M}_+$ or equivalently if $Q_2(f,f) \geq Q_1(f,f)$ for all $f \in C(V^{(0)})$. Then $\mathbb{D} \subset \mathbb{M}_+$; it turns out that we need to consider the action of Λ on $\mathbb{M}_+$, and not just on $\mathbb{D}$. For $Q \in \mathbb{M}_+$, the replication operation is defined exactly as in (6.2)

$$(6.16) \qquad Q^R(f,g) = \sum_{i=1}^{M} r_i^{-1}\, Q(f\circ\psi_i, g\circ\psi_i), \quad f,g \in C(V^{(1)}).$$

The decimation operation is also easy to extend to $\mathbb{M}_+$:

$$T(Q^R)(g,g) = \inf\{Q^R(f,f) : f \in C(V^{(0)}), f|_{V^{(0)}} = g\};$$

we can write $T(Q^R)$ in matrix terms as in (4.24). We set $\Lambda(Q) = T(Q^R)$.

Lemma 6.14. *The map* Λ *on* $\mathbb{M}_+$ *satisfies:*
(a) $\Lambda : \mathbb{M}_+ \to \mathbb{M}_+$, *and is continuous on* $\operatorname{int}(\mathbb{M}_+)$.
(b) $\Lambda(Q_1+Q_2) \geq \Lambda(Q_1) + \Lambda(Q_2)$.
(c) $\Lambda(\theta Q) = \theta\Lambda(Q)$

Proof. (a) is clear from the formulation of the trace operation in matrix terms.

Since the replication operation is linear, we clearly have $Q^R = Q_1^R + Q_2^R$, $(\theta Q)^R = \theta Q^R$. (c) is therefore evident, while for (b), if $g \in C(V^{(0)})$,

$$\begin{aligned} T(Q^R)(g,g) &= \inf\{Q_1^R\,(f,f) + Q_2^R\,(f,f):\ f|_{V^{(0)}} = g\} \\ &\geq \inf\{Q_1^R\,(f,f):\ f|_{V^{(0)}} = g\} + \inf\{Q_2^R\,(f,f):\ f|_{V^{(0)}} = g\} \\ &= T(Q_1^R)(g,g) + T(Q_2^R)(g,g). \end{aligned}$$

□

Note that for $\mathcal{E} \in \mathbb{D}^i$, we have $\mathcal{E}(f,f) = 0$ if only if f is constant.

Definition 6.15. For $\mathcal{E}_1, \mathcal{E}_2\ \in \mathbb{D}^i$ set

$$\begin{aligned} m(\mathcal{E}_1/\mathcal{E}_2) &= \sup\{\alpha \geq 0;\ \mathcal{E}_1 - \alpha\mathcal{E}_2 \in \mathbb{M}_+\}. \\ &= \inf\{\frac{\mathcal{E}_1(f,f)}{\mathcal{E}_2(f,f)}\ : f \text{ non constant}\}. \end{aligned}$$

Similarly let

$$M(\mathcal{E}_1/\mathcal{E}_2) = \sup\{\frac{\mathcal{E}_1(f,f)}{\mathcal{E}_2(f,f)}\ : f \text{ non constant}\}.$$

Note that

(6.18) $$M(\mathcal{E}_1/\mathcal{E}_2) = m(\mathcal{E}_2/\mathcal{E}_1)^{-1}.$$

Lemma 6.16. *(a) For $\mathcal{E}_1, \mathcal{E}_2\ \in \mathbb{D}^i$, $0 < m(\mathcal{E}_1, \mathcal{E}_2) < \infty$.*
(b) If $\mathcal{E}_1,\ \mathcal{E}_2\ \in \mathbb{D}^i i$ then $m(\mathcal{E}_1/\mathcal{E}_2) = M(\mathcal{E}_1/\mathcal{E}_2)$ if and only if $\mathcal{E}_2 = \lambda\mathcal{E}_1$ for some $\lambda > 0$.
(c) If $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\ \in \mathbb{D}^i$ then

$$m(\mathcal{E}_1/\mathcal{E}_3) \geq m(\mathcal{E}_1/\mathcal{E}_2)\, m(\mathcal{E}_2/\mathcal{E}_3),$$

$$M(\mathcal{E}_1/\mathcal{E}_3) \leq M(\mathcal{E}_1/\mathcal{E}_2)\, M(\mathcal{E}_2/\mathcal{E}_3).$$

Proof. (a) This follows from the fact that $\mathcal{E}_i$ are irreducible, and so vanish only on the subspace of constant functions.
(b) is immediate from the definition of m and M.
(c) We have

$$m(\mathcal{E}_1/\mathcal{E}_3) = \inf_f\ \frac{\mathcal{E}_1(f,f)}{\mathcal{E}_2(f,f)}\frac{\mathcal{E}_2(f,f)}{\mathcal{E}_3(f,f)} \geq m(\mathcal{E}_1/\mathcal{E}_2)m(\mathcal{E}_2/\mathcal{E}_3);$$

while the second assertion is immediate from (6.18). □

Definition 6.17. Define

$$d_H(\mathcal{E}_1, \mathcal{E}_2) = \log\ \frac{M(\mathcal{E}_1\mathcal{E}_2)}{m(\mathcal{E}_1\,\mathcal{E}_2)}, \quad \mathcal{E}_1, \mathcal{E}_2 \in \mathbb{D}^i.$$

Let $p\,\mathbb{D}^i$ be the projective space $\mathbb{D}^i/\sim$, where $\mathcal{E}_1 \sim \mathcal{E}_2$ if $\mathcal{E}_1 = \lambda\mathcal{E}_2$. d_H is called *Hilbert's projective metric* – see [Nus], [Me4].

Proposition 6.18. *(a) $d_H(\mathcal{E}_1, \mathcal{E}_2) = 0$ if and only if $\mathcal{E}_1 = \lambda \mathcal{E}_2$ for some $\lambda > 0$.*
(b) d_H is a pseudo-metric on $\mathbb{D}^i$, and a metric on $p\,\mathbb{D}^i$.
(c) If $\mathcal{E}, \mathcal{E}_0, \mathcal{E}_1 \in \mathbb{D}^i$ then for $\alpha_0, \alpha_1 > 0$,

$$d_H(\mathcal{E}, \alpha_0\, \mathcal{E}_0 + \alpha_1 \mathcal{E}_1) \leq \max(d_H(\mathcal{E}, \mathcal{E}_0), d_H(\mathcal{E}, \mathcal{E}_1)).$$

In particular open balls in d_H are convex.
(d) $(p\,\mathbb{D}^i, d_H)$ is complete.

Proof. (a) is evident from Lemma 6.17(b). To prove (b) note that $d_H(\mathcal{E}_1, \mathcal{E}_2) \geq 0$, and that $d_H(\mathcal{E}_1, \mathcal{E}_2) = d_H(\mathcal{E}_2, \mathcal{E}_1)$ from (6.18). The triangle inequality is immediate from Lemma 6.17(c). So d_H is a pseudo metric on $\mathbb{D}^i$.

To see that d_H is a metric on $p\,\mathbb{D}^i$, note that

$$m(\lambda \mathcal{E}_1 / \mathcal{E}_2) = \lambda m(\mathcal{E}_1/\mathcal{E}_2), \quad \lambda > 0,$$

from which it follows that $d_H(\lambda \mathcal{E}_1, \mathcal{E}_2) = d_H(\mathcal{E}_1, \mathcal{E}_2)$ and thus d_H is well defined on $p\,\mathbb{D}^i$. The remaining properties are now immediate from those of d_H on $\mathbb{D}^i$.
(c) Replacing $\mathcal{E}_1$ by $\big(m(\mathcal{E}_1/\mathcal{E}_0)/m(\mathcal{E}/\mathcal{E}_1)\big)\mathcal{E}_1$ we can suppose that

$$m(\mathcal{E}/\mathcal{E}_0) = m(\mathcal{E}\,/\mathcal{E}_1) = m.$$

Write $M_i = M(\mathcal{E}/\mathcal{E}_i)$. Then if $\mathcal{F} = \alpha_0\, \mathcal{E}_0 + \alpha_1 \mathcal{E}_1$,

$$\begin{aligned} M(\mathcal{E}/\mathcal{F}) &= \inf_f \frac{\alpha_0 \mathcal{E}_0(f,f) + \alpha_1 \mathcal{E}_1(f,f)}{\mathcal{E}(f,f)} \\ &\geq \alpha_0 m(\mathcal{E}/\mathcal{E}_0) + \alpha_1 m(\mathcal{E}/\mathcal{E}_1) = \alpha_0 + \alpha_1\,. \end{aligned}$$

Similarly $M(\mathcal{E}/\mathcal{F}) \leq \alpha_0 M_0 + \alpha_1 M_1$. Therefore

$$\begin{aligned} \exp d_H(\mathcal{E}, \mathcal{F}) &\leq (\alpha_0/(\alpha_0+\alpha_1))(M_0/m) + (\alpha_1/(\alpha_0+\alpha_1))(M_1/m) \\ &\leq \max\,(M_0/m, M_1/m). \end{aligned}$$

It is immediate that if $\mathcal{E}_i \in B(\mathcal{E}, r)$ then $d_H(\mathcal{E}, \lambda \mathcal{E}_0 + (1-\lambda)\mathcal{E}_1) < r$, so that $B(\mathcal{E}, r)$ is convex. For (d) see [Nus, Thm. 1.2]. □

Theorem 6.19. *Let $\mathcal{E}_1, \mathcal{E}_2 \in \mathbb{D}^i$. Then*

$$m(\Lambda(\mathcal{E}_1), \Lambda(\mathcal{E}_2)) \geq m(\mathcal{E}_1, \mathcal{E}_2), \tag{6.19}$$

$$M(\Lambda(\mathcal{E}_1), \Lambda(\mathcal{E}_2)) \leq M(\mathcal{E}_1, \mathcal{E}_2). \tag{6.20}$$

In particular Λ is non-expansive in d_H :

$$d_H(\Lambda(\mathcal{E}_1), \Lambda(\mathcal{E}_2)) \leq d_H(\mathcal{E}_1, \mathcal{E}_2). \tag{6.21}$$

Proof. Suppose $\alpha < m(\mathcal{E}_1, \mathcal{E}_2)$. Then $Q = \mathcal{E}_1 - \alpha\, \mathcal{E}_2 \in \mathbb{M}_+$, and $Q(f,f) > 0$, for all non-constant $f \in C(V^{(0)})$. So by Lemma 6.14

$$\Lambda(\mathcal{E}_1) = \Lambda(Q + \alpha \mathcal{E}_2) \geq \Lambda(Q) + \alpha \Lambda(\mathcal{E}_2),$$

and since $\Lambda(Q) \geq 0$, this implies that $\Lambda(\mathcal{E}_1) - \alpha\Lambda(\mathcal{E}_2) \geq 0$. So $\alpha < m(\Lambda(\mathcal{E}_1), \Lambda(\mathcal{E}_2))$, and thus $m(\mathcal{E}_1, \mathcal{E}_2) \leq m(\Lambda(\mathcal{E}_1), \Lambda(\mathcal{E}_2))$, proving (6.19). (6.20) and (6.21) then follow immediately from (6.19), and the definition of d_H. □

A strict inequality in (6.21) would imply the uniqueness of fixed points. Thus the example of the Vicsek set above shows that strict inequality cannot hold in general. So this Theorem gives us much less than we might hope. Nevertheless, we can obtain some useful information.

Corollary 6.20. *(See [HHW1, Cor. 3.7]) Suppose $\mathcal{E}_1, \mathcal{E}_2$ are fixed points satisfying $\Lambda(\mathcal{E}_i) = \lambda_i \mathcal{E}_i$, $i = 1, 2$. Then $\lambda_1 = \lambda_2$.*

Proof. From (6.19)

$$m(\mathcal{E}_1/\mathcal{E}_2) \leq m\big(\Lambda(\mathcal{E}_1)/\Lambda(\mathcal{E}_2)\big) = (\lambda_1/\lambda_2) m(\mathcal{E}_1/\mathcal{E}_2),$$

so that $\lambda_1 \geq \lambda_2$. Interchanging $\mathcal{E}_1$ and $\mathcal{E}_2$ we obtain $\lambda_1 = \lambda_2$. □

We can also deduce the existence of $\mathcal{H}$-invariant fixed points.

Proposition 6.21. *Let $\mathcal{H}$ be a symmetry group of F. If Λ has a fixed point $\mathcal{E}_1$ in $\mathbb{D}^i$ then Λ has an $\mathcal{H}$-invariant fixed point in $\mathbb{D}^i$.*

Proof. Let $A = \{\mathcal{E} \in \mathbb{D}^i : \mathcal{E}$ is $\mathcal{H}$-invariant.$\}$. (It is clear from Lemma 6.11 that A is non-empty). Then by Lemma 6.11(b) $\Lambda : A \to A$. Let $\mathcal{E}_0 \in A$, and write $r = d_H(\mathcal{E}_1, \mathcal{E}_0)$, $B = B_{d_H}(\mathcal{E}_1, 2r)$. By Theorem 6.20 $\Lambda : B \to B$. So $\Lambda : A \cap B \to A \cap B$. Each of A, B is convex (A is convex as the sum of two $\mathcal{H}$-invariant forms is $\mathcal{H}$-invariant, B by Proposition 6.18(c)), and so $A \cap B$ is convex. Since Λ is a continuous function on a convex space, by the Brouwer fixed point theorem Λ has a fixed point $\mathcal{E}' \in A \cap B$, and $\mathcal{E}'$ is $\mathcal{H}$-invariant. □

We will not make use of the following result, but is useful for understanding the general situation.

Corollary 6.22. *Suppose Λ has two distinct fixed points $\mathcal{E}_1$ and $\mathcal{E}_2$ (with $\mathcal{E}_1 \neq \lambda\mathcal{E}_2$ for any λ). Then Λ has uncountably many fixed points.*

Proof. (Note that the example of the Vicsek set shows that $\frac{1}{2}(\mathcal{E}_1 + \mathcal{E}_2)$ is not necessarily a fixed point). Let $\mathbb{F} \subset \mathbb{D}^i$ be the set of fixed points. Let $\mathcal{E}_0, \mathcal{E}_1 \in \mathbb{F}$; multiplying $\mathcal{E}_1$ by a scalar we can take $m(\mathcal{E}_0, \mathcal{E}_1) = 1$. Write $R = d_H(\mathcal{E}_0, \mathcal{E}_1)$. If $\mathcal{E}_\lambda = \lambda\mathcal{E}_1 + (1-\lambda)\mathcal{E}_0$ then as in Proposition 6.19(c)

$$\exp d_{\mathcal{H}}(\mathcal{E}_\lambda, \mathcal{E}_0) \leq (1-\lambda) + \lambda M(\mathcal{E}_1, \mathcal{E}_0)$$

and so

$$d_{\mathcal{H}}(\mathcal{E}_{1/2}, \mathcal{E}_0) \leq \log((1 + e^R)/2).$$

Thus there exists δ, depending only on R, such that

$$A = \{\mathcal{E} \in \mathbb{D}^i : \mathcal{E} \in B(\mathcal{E}_0, (1-\delta)R) \bigcap B(\mathcal{E}_1, (1-\delta)R)\}$$

is non-empty. Since Λ preserves A, Λ has a fixed point in A. $\mathbb{F}$ thus has the property:

if $\mathcal{E}_1, \mathcal{E}_2$ are distinct elements of $\mathbb{F}$ then there exists $\mathcal{E}_3 \in \mathbb{F}$
such that $0 < d_{\mathcal{H}}(\mathcal{E}_3, \mathcal{E}_1) < d_{\mathcal{H}}(\mathcal{E}_2, \mathcal{E}_1)$.

As $\mathbb{F}$ is closed (since Λ is continuous) we deduce that $\mathbb{F}$ is perfect, and therefore uncountable. $\square$

This if as far as we will go in general. For nested fractals the added structure – symmetry and the embedding in $\mathbb{R}^d$, enables us to obtain stronger results. If $(F,(\psi_i))$ is a nested fractal, or an ANF, we only consider the set $\mathbb{D}^i \cap \{\mathcal{E} : \mathcal{E}$ is $\mathcal{G}_R$-invariant$\}$, so that in discussing the existence and uniqueness of fixed points we will be considering only $\mathcal{G}_R$-invariant ones.

Let $(F,(\psi_i))$ be a nested fractal, write $\mathcal{G} = \mathcal{G}_R$ and let $\mathcal{E}_A$ be a ($\mathcal{G}$-invariant) Dirichlet form on $C(V^{(0)})$. $\mathcal{E}_A$ is determined by the conductances on the equivalence classes of edges in $(V^{(0)},\mathcal{E}_0)$ under the action of $\mathcal{G}$. By Proposition 5.38(c) if $|x-y| = |x'-y'|$ then the edges $\{x,y\}$ and $\{x',y'\}$ are equivalent, so that $A_{xy} = A_{x'y'}$.

List the equivalence classes in order of increasing Euclidean distance, and write $\alpha_1, \alpha_2 ..., \alpha_k$ for the common conductances of the edges. Since $\widetilde{A} = \Lambda(A)$ is also $\mathcal{G}$-invariant, Λ induces a map $\Lambda' : \mathbb{R}^k_+ \to \mathbb{R}^k_+$ such that, using obvious notation, $\Lambda(A(\alpha)) = A(\Lambda'(\alpha))$.

Set $\mathbb{D}^* = \{\alpha : \alpha_1 \geq \alpha_2 \geq ... \geq \alpha_k > 0\}$. Clearly we have $\mathbb{D}^* \subset \mathbb{D}^{si}$. We have the following existence theorem for nested fractals.

Theorem 6.23. *(See [L1, p. 48]). Let $(F,(\psi_i))$ be a nested fractal (or an ANF). Then Λ has a fixed point in $\mathbb{D}^*$.*

Proof. Let $\mathcal{E}_A \in \mathbb{D}^*$, and let $\alpha_1, ...\alpha_k$ be the associated conductivities. Let $(Y_t, t \geq 0, \mathbb{Q}^x, x \in V^{(0)})$ be the continuous time Markov chain associated with $\mathcal{E}_A$, and let $(\widehat{Y}_n, n \geq 0,\ \mathbb{Q}^x, x \in V^{(0)})$ be the discrete time skeleton of Y.

Let $E_0^{(1)}, ..., E_0^{(k)}$ be the equivalence classes of edges in $(V^{(0)}, E_0)$, so that $A_{xy} = \alpha_j$ if $\{x,y\} \in E_0^{(j)}$. Then if $\{x,y\} \in E_0^{(j)}$,

$$\mathbb{Q}^x(\widehat{Y}_1 = y) = \frac{\alpha_j}{\sum_{y \neq x} A_{xy}}.$$

As $c_1 = \sum_{y \neq x} A_{xy}$ does not depend on x (by the symmetry of $V^{(0)}$) the transition probabilities of $\widehat{Y}$ are proportional to the α_j.

Now let $R(A)$ be the conductivity matrix on $V^{(1)}$ attained by replication of A. Let $(X_t, t \geq 0,\ \mathbb{P}^x,\ x \in V^{(1)})$ and $(\widehat{X}_n, n \geq 0,\ \mathbb{P}^x, x \in V^{(1)})$ be the associated Markov Chains. Let $T_0, T_1, ...$ be successive disjoint hits (see Definition 2.14) on $V^{(0)}$ by $\widehat{X}_n$.

Write $\widetilde{A} = \Lambda(A)$, and $\widetilde{\alpha}$ for the edge conductivities given by A. Using the trace theorem,

$$\mathbb{P}^x(\widehat{X}_{T_1} = y) = \widetilde{\alpha}_j / c_1 \quad \text{if } \{x,y\} \in E_0^{(j)}.$$

Now let $x_1, y_1, y_2 \in V^{(0)}$, with $|x - y_1| < |x - y_2|$. We will prove that

$$\mathbb{P}^{x_1}(\widehat{X}_{T_1} = y_2) < \mathbb{P}^{x_1}(\widehat{X}_{T_1} = y_1). \tag{6.23}$$

Let H be the hyperplane bisecting $[y_1, y_2]$, let g be reflection in H, and $x_2 = g(x_1)$. Let

$$T = \min\{n \geq 0 :\ \widehat{X}_n \in V^{(0)} - \{x_1\}\},$$

so that $T_1 = T$ $\mathbb{P}^{x_1}$-almost surely. Set

$$f_n(x) = \mathbb{E}^x \, 1_{(T \le n)}(1_{y_1}(\widehat{X}_T) - 1_{y_2}(\widehat{X}_T)).$$

Let $p(x,y),\ x,y \in V^{(0)}$ be the transition probabilities of $\widehat{X}$. Then

$$f_{n+1}(x) = 1_A(x)\, f_0(x) + 1_{A^c}(x) \sum_y p(x,y) f_n(y). \qquad (6.24$$

Let $J_{12} = \{x \in V^{(1)} : |x - y_1| \le |x - y_2|\}$, and define J_{21} analogously. We prove by induction that f_n satisfies

$$f_n(x) \ge 0, \qquad x \in J_{12}, \qquad (6.25a)$$

$$f_n(x) + f_n(g(x)) \ge 0, \qquad x \in J_{12}. \qquad (6.25b)$$

Since $f_0 = 1_{y_1} - 1_{y_2}$, and $y_1 \in J_{12}$, f_0 satisfies (6.25). Let $x \in A^c \cup J_{12}$ and suppose f_n satisfies (6.25). If $p(x,y) > 0$, and $y \in J_{12}^c$, then x, y are in the same 1-cell so if $y' = g(y)$, y' is also in the same 1-cell as x_1 and $|x - y'| \le |x - y|$. So (since $\mathcal{E}_A \in \mathbb{D}^*$), $p(x,y') \ge p(x,y)$ and using (6.25b), as $f_n(y') \ge 0$,

$$p(x,y)f_n(y) + p(x,y')f_n(y') \ge p(x,y)(f_n(y) + f_n(g(y)) \ge 0.$$

Then by (6.24), $f_{n+1}(x) \ge 0$. A similar argument implies that f_{n+1} satisfies (6.25b).

So (f_n) satisfies (6.25) for all n, and hence its limit f_∞ does. Thus $f_\infty(x_1) = \mathbb{P}^x(\widehat{X}_T = y_1) - \widehat{\mathbb{P}}(\widehat{X}_T = y_2) \ge 0$, proving (6.23).

From (6.23) we deduce that $\widetilde{\alpha}_1 \ge \widetilde{\alpha}_2 \ge \ldots \ge \widetilde{\alpha}_k$, so that $\Lambda : \mathbb{D}^* \to \mathbb{D}^*$. As $\Lambda'(\theta\alpha) = \theta\Lambda'(\alpha)$, we can restrict the action of Λ' to the set

$$\{\alpha \in \mathbb{R}_+^k : \alpha_1 \ge \ldots \ge \alpha_k \ge 0, \sum \alpha_i = 1\}.$$

This is a closed convex set, so by the Brouwer fixed point theorem, Λ' has a fixed point in $\mathbb{D}^*$. $\square$

Remark 6.24. The proof here is essentially the same as that in Lindstrøm [L1]. The essential idea is a kind of reflection argument, to show that transitions along shorter edges are more probable. This probabilistic argument yields (so far) a stronger existence theorem for nested fractals than the analytic arguments used by Sabot [Sab1] and Metz [Me7]. However, the latter methods are more widely applicable.

It does not seem easy to relax any of the conditions on ANFs without losing some link in the proof of Theorem 6.23. This proof used in an essential fashion not only the fact that $V^{(0)}$ has a very large symmetry group, but also the Euclidean embedding of $V^{(0)}$ and $V^{(1)}$.

The following uniqueness theorem for nested fractals was proved by Sabot [Sab1]. It is a corollary of a more general theorem which gives, for p.c.f.s.s. sets, sufficient conditions for existence and uniqueness of fixed points. A simpler proof of this result has also recently been obtained by Peirone [Pe].

Theorem 6.25. *Let $(F, (\psi_i))$ be a nested fractal. Then Λ has a unique $\mathcal{G}_R$-invariant non-degenerate fixed point.*

Definition 6.26. Let $\mathcal{E}$ be a fixed point of Λ. The *resistance scaling factor* of $\mathcal{E}$ is the unique $\rho > 0$ such that

$$\Lambda(\mathcal{E}) = \rho^{-1}\, \mathcal{E}.$$

Very often we will also call ρ the resistance scaling factor of F: in view of Corollary 6.21, ρ will have the same value for any two non-degenerate fixed points.

Proposition 6.27. *Let $(F, (\psi_i))$ be a p.c.f.s.s. set, let (r_i) be a resistance vector, and let $\mathcal{E}_A$ be a non-degenerate fixed point of Λ. Then for each $s \in \{1, ...M\}$ such that $\pi(\dot{s}) \in V^{(0)}$,*

$$r_s\, \rho^{-1} < 1. \tag{6.27}$$

Proof. Fix $1 \le s \le M$, let $x = \pi(\dot{s})$, and let $f = 1_x \in C(V^{(0)})$. Then

$$\mathcal{E}_A(f, f) = \sum_{y \in V^{(0)},\, y \ne x} A_{xy} = |A_{xx}|.$$

Let $g = 1_x \in C(V^{(1)})$. As $\Lambda(\mathcal{E}_A) = \rho^{-1}\mathcal{E}_A$,

$$\rho^{-1}|A_{xx}| = \Lambda(\mathcal{E}_A)(f, f) < \mathcal{E}_A^R(g, g) : \tag{6.28}$$

since g is not harmonic with respect to $\mathcal{E}_A^R$, strict inequality holds in (6.28). By Proposition 5.24(c), x is in exactly one 1-complex. So

$$\mathcal{E}_A^R(g, g) = \sum_i r_i^{-1} \mathcal{E}_A(g \circ \psi_i, g \circ \psi_i) = r_s^{-1}|A_{xx}|,$$

and combining this with (6.28) gives (6.27). □

Since $r_s = 1$ for nested fractals, we deduce

Corollary 6.28. *Let $(F, (\psi_i))$ be a nested fractal. Then $\rho > 1$.*

For nested fractals, many properties of the process can be summarized in terms of certain scaling factors.

Definition 6.29. Let $(F, (\psi_i))$ be a nested fractal, and $\mathcal{E}$ be the (unique) non-degenerate fixed point. See Definition 5.22 for the length and mass scale factors L and M. The *resistance scale factor* ρ of F is the resistance scaling factor of $\mathcal{E}$. Let

(6.29 $\qquad \tau = M\rho;$

we call τ the *time scaling factor*. (In view of the connection between resistances and crossing times given in Theorem 4.27, it is not surprising that τ should have a connection with the space-time scaling of processes on F.)

It may be helpful at this point to draw a rough distinction between two kinds of structure associated with the nested fractal (F, ψ). The quantities introduced in Section 5, such as L, M, the geodesic metric d_F, the chemical exponent γ and the dimension $d_w(F)$ are all *geometric* – that is, they can be determined entirely by a geometric inspection of F. On the other hand, the resistance and time scaling factors ρ and τ are *analytic* or *physical* – they appear in some sense to lie deeper than the geometric quantities, and arise from the solution to some kind of equation on the space. On the Sierpinski gasket, for example, while one obtains $L = \gamma = 2$, and $M = 3$ almost immediately, a brief calculation (Lemma 2.16) is needed to obtain ρ. For more complicated sets, such as some of the examples given in Section 5, the calculation of ρ would be very lengthy.

Unfortunately, while the distinction between these two kinds of constant arises clearly in practice, it does not seem easy to make it precise. Indeed, Corollary 6.20 shows that the geometry does in fact determine ρ: it is not possible to have one nested fractal (a geometric object) with two distinct analytic structures which both satisfy the symmetry and scale invariance conditions.

We have the following general inequalities for the scaling factors.

Proposition 6.30. *Let $(F,(\psi_i))$, be a nested fractal with scaling factors L, M, ρ, τ. Then*

$$L > 1, \qquad M \geq 2, \qquad M \geq L, \qquad \tau = M\rho \geq L^2. \tag{6.30}$$

Proof. $L > 1$, $M \geq 2$ follow from the definition of nested fractals. If $\theta = \text{diam}(V^{(0)})$, then, as $V^{(1)}$ consists of M copies of $V^{(0)}$ each of diameter $L^{-1}\theta$, by the connectivity axiom we deduce $ML^{-1}\theta \geq \theta$. Thus $M \geq L$.

To prove the final inequality in (6.30) we use the same strategy as in Proposition 6.27, but with a better choice of minimizing function.

Let $\mathcal{H}$ be the set of functions f of the form $f(x) = Ox + a$, where $x \in \mathbb{R}^d$ and O is an orthogonal matrix. Set $\mathcal{H}_n = \{f|_{V^{(n)}},\, f \in \mathcal{H}\}$. Let $\theta = \sup\{\mathcal{E}(f,f) : f \in \mathcal{H}_0\}$: clearly $\theta < \infty$. Choose f to attain the supremum, and let $g \in \mathcal{H}$ be such that $f = g|_{V^{(0)}}$. Then if $f_1 = g|_{V^{(1)}}$

$$\rho^{-1}\,\theta = \rho^{-1}\mathcal{E}(f,f) = \Lambda(\mathcal{E})(f,f) \leq \mathcal{E}^R(g_1,g_1) = \sum_{i=1}^{M} \mathcal{E}(g_1 \circ \psi_i, g_1 \circ \psi_i).$$

However, $g_1 \circ \psi_i$ is the restriction to $V^{(0)}$ of a function of the form $L^{-1}Ox + a_i$, and so $\mathcal{E}(g \circ \psi_i, g \circ \psi_i) \leq L^{-2}\theta$. Hence $\rho^{-1}\theta \leq ML^{-2}\theta$, proving (6.30). □

The following comparison theorem provides a technique for bounding ρ in certain situations.

Proposition 6.31. *Let $(F_1, \{\psi_i, 1 \leq i \leq M_1\})$ be a p.c.f.s.s. set. Let $F_0 \subset F_1$, $M_0 \leq M_1$, and suppose that $(F_0, \{\psi_i, 1 \leq i \leq M_0\})$ is also a p.c.f.s.s. set, and that $V^{(0)}_{F_1} = V^{(0)}_{F_0}$. Let $(r_i^{(k)}, 1 \leq i \leq M_k)$ be resistance vectors for $k = 0, 1$, and suppose that $r_i^{(0)} \geq r_i^{(1)}$ for $1 \leq i \leq M_0$. Let Λ_k be the renormalization map for $(F_k, (\psi_i)_{i=1}^{M_k}, (r_i^{(k)})_{i=1}^{M_k})$. If $\mathcal{E}_k$ are non-degenerate Dirichlet forms satisfying $\Lambda_k(\mathcal{E}_k) = \rho_k^{-1}\mathcal{E}_k$, $k = 0, 1$, then $\rho_1 \leq \rho_0$.*

Proof. Since $V^{(0)}_{F_1} \subset V^{(1)}_{F_1}$, we have, writing R_i for the replication maps associated with F_i,

$$R_1\mathcal{E}(f,f) \geq R_0\mathcal{E}(f,f), \quad f \in C(V^{(1)}_{F_1}).$$

So $\Lambda_1(\mathcal{E}) \geq \Lambda_0(\mathcal{E})$ for any $\mathcal{E} \in \mathbb{D}$. If $m = m(\mathcal{E}_1/\mathcal{E}_0)$, then

$$\rho_1^{-1}\mathcal{E}_1 = \Lambda_1(\mathcal{E}_1) \geq \Lambda_1(m\,\mathcal{E}_0) \geq \Lambda_0(m\,\mathcal{E}_0) = m\rho_0^{-1}\mathcal{E}_0 \geq \rho_0^{-1}\mathcal{E}_1\,,$$

which implies that $\rho_0 \geq \rho_1$. □

7. Diffusions on p.c.f.s.s. sets.

Let $\big(F,(\psi_i)\big)$ be a p.c.f.s.s. set, and r_i be a resistance vector. We assume that the graph $(V^{(1)}, \mathbf{E}_1)$ is connected. Suppose that the renormalization map Λ has a non-degenerate fixed point $\mathcal{E}^{(0)} = \mathcal{E}_A$, so that $\Lambda(\mathcal{E}^{(0)}) = \rho^{-1}\mathcal{E}^{(0)}$. Fixing F, r, and $\mathcal{E}_A$, in this section we will construct a diffusion X on F, as a limit of processes on the graphical approximations $V^{(n)}$. In Section 2 this was done probabilistically for the Sierpinski gasket, but here we will use Dirichlet form methods, following [Kus2, Fu1, Ki2].

Definition 7.1. For $f \in C(V^{(n)})$, set

$$\mathcal{E}^{(n)}(f,f) = \rho^n \sum_{w\in\mathbb{W}_n} r_w^{-1}\mathcal{E}^{(0)}(f\circ\psi_w, f\circ\psi_w). \tag{7.1}$$

This is the Dirichlet form on $V^{(n)}$ obtained by replication of scaled copies of $\mathcal{E}^{(0)}$, where the scaling associated with the map ψ_w is $\rho^n r_w^{-1}$.

These Dirichlet forms have the following nesting property.

Proposition 7.2. *(a) For $n \geq 1$, $Tr(\mathcal{E}^{(n)}|V^{(n-1)} = \mathcal{E}^{(n-1)}$.*
(b) If $f \in C(V^{(n)})$, and $g = f|_{V^{(n-1)}}$ then $\mathcal{E}^{(n)}(f,f) \geq \mathcal{E}^{(n-1)}(g,g)$.
(c) $\mathcal{E}^{(n)}$ is non-degenerate.

Proof. (a) Let $f \in C(V^{(n)})$. Then decomposing $w \in \mathbb{W}_n$ into $v\cdot i$, $v \in \mathbb{W}_{n-1}$,

$$\begin{aligned}\mathcal{E}^{(n)}(f,f) &= \rho^n \sum_{v\in\mathbb{W}_{n-1}} r_v^{-1}\sum_i r_i^{-1}\mathcal{E}^{(0)}(f\circ\psi_v\circ\psi_i, f\circ\psi_v\circ\psi_i)\\ &= \rho^{n-1}\sum_{v\in\mathbb{W}_{n-1}} r_v^{-1}\mathcal{E}^{(1)}(f_v,f_v),\end{aligned} \tag{7.2}$$

where $f_v = f\circ\psi_v \in C(V^{(1)})$. Now let $g \in C(V^{(n-1)})$. If $f|_{V^{(n-1)}} = g$ then $f_v|_{V^{(0)}} = g\circ\psi_v = g_v$. As $\mathcal{E}^{(0)}$ is a fixed point of Λ,

$$\begin{aligned}\inf\Big\{\mathcal{E}^{(1)}(h,h) : h|_{V^{(0)}} = g_v\Big\} &= \rho\inf\Big\{R\mathcal{E}^{(0)}(h,h) : h|_{V^{(0)}} = g_v\Big\}\\ &= \rho\Lambda(\mathcal{E}^{(0)})(g_v,g_v) = \mathcal{E}^{(0)}(g_v,g_v).\end{aligned} \tag{7.3}$$

Summing over $v \in \mathbb{W}_{n-1}$ we deduce therefore

$$\inf\Big\{\mathcal{E}^{(n)}(f,f) : f|_{V^{(n-1)}} = g\Big\} \leq \rho^{n-1}\sum_v r_v^{-1}\mathcal{E}^{(0)}(g,g) = \mathcal{E}^{(n-1)}(g,g).$$

For each $v \in \mathbb{W}_{n-1}$, let $h_v \in C(V^{(1)})$ be chosen to attain the infimum in (7.3). We wish to define $f \in C(V^{(n)})$ such that

$$(7.4) \qquad f \circ \psi_v = h_v, \qquad v \in \mathbb{W}_{n-1}.$$

Let $v \in \mathbb{W}_{n-1}$. We define

$$f\big(\psi_v(y)\big) = h_v(y), \qquad y \in V^{(1)}.$$

We need to check f is well-defined; but if v, u are distinct elements of $\mathbb{W}_{n-1}$ and $x = \psi_v(y) = \psi_u(z)$, then $x \in V^{(n-1)}$ by Lemma 5.18, and so y, $z \in V^{(0)}$. Therefore

$$f\big(\psi_v(y)\big) = h_v(y) = g_v(y) = g(x) = f\big(\psi_u(z)\big),$$

so the definitions of f at x agree. (This is where we use the fact that F is finitely ramified: it allows us to minimize separately over each set of the form $V_v^{(1)}$).

So

$$\mathcal{E}^{(n)}(f,f) = \mathcal{E}^{(n-1)}(g,g),$$

and therefore $Tr\left(\mathcal{E}^{(n)}|V^{(n-1)}\right) = \mathcal{E}^{(n-1)}$.
(b) is evident from (a).
(c) We prove this by induction. $\mathcal{E}^{(0)}$ is non-degenerate by hypothesis. Suppose $\mathcal{E}^{(n-1)}$ is non-degenerate, and that $\mathcal{E}^{(n)}(f,f) = 0$. From (7.2) we have

$$\mathcal{E}^{(n)}(f,f) = \rho \sum_{v \in \mathbb{W}_1} r_v^{-1} \mathcal{E}^{(n-1)}(f \circ \psi_v, f \circ \psi_v),$$

and so $f \circ \psi_v$ is constant for each $v \in \mathbb{W}_1$. Thus f is constant on each 1-complex, and as $(V^{(1)}, \mathbf{E}_1)$ is connected this implies that f is constant. □

To avoid clumsy notation we will identify functions with their restrictions, so, for example, if $f \in C(V^{(n)})$, and $m < n$, we will write $\mathcal{E}^{(m)}(f,f)$ instead of $\mathcal{E}^{(m)}\left(f|_{V^{(m)}}, f|_{V^{(m)}}\right)$.

Definition 7.3. Set $V^{(\infty)} = \cup_{n=0}^{\infty} V^{(n)}$. Let $U = \{f : V^{(\infty)} \to \mathbb{R}\}$. Note that the sequence $\left(\mathcal{E}^{(n)}(f,f)\right)_{n=1}^{\infty}$ is non-decreasing. Define

$$\mathcal{D}' = \{f \in U : \sup_n \mathcal{E}^{(n)}(f,f) < \infty\},$$

$$\mathcal{E}'(f,g) = \sup_n \mathcal{E}^{(n)}(f,g); \quad f, g \in \mathcal{D}'.$$

$\mathcal{E}'$ is the initial version of the Dirichlet form we are constructing.

Lemma 7.4. *$\mathcal{E}'$ is a symmetric Markov form on $\mathcal{D}'$.*

Proof. $\mathcal{E}'$ clearly inherits the properties of symmetry, bilinearity, and positivity from the $\mathcal{E}^{(n)}$. If $f \in \mathcal{D}'$, and $g = (0 \vee f) \wedge 1$ then $\mathcal{E}^{(n)}(g,g) \le \mathcal{E}^{(n)}(f,f)$, as the $\mathcal{E}^{(n)}$ are Markov. So $\mathcal{E}'(g,g) \le \mathcal{E}'(f,f)$. □

What we have done here seems very easy. However, more work is needed to obtain a 'good' Dirichlet form $\mathcal{E}$ which can be associated with a diffusion on F. Note the following scaling result for $\mathcal{E}'$.

Lemma 7.5. *For $n \ge 1$, $f \in \mathcal{D}'$,*

$$\mathcal{E}'(f,f) = \sum_{w \in \mathbb{W}_n} \rho^n r_w^{-1} \mathcal{E}'(f \circ \psi_w, f \circ \psi_w). \tag{7.5}$$

Proof. We have, for $m \ge n$, $f \in \mathcal{D}'$,

$$\mathcal{E}^{(m)}(f,f) = \sum_{w \in \mathbb{W}_n} \rho^n r_w^{-1} \mathcal{E}^{(m-n)}(f \circ \psi_w, f \circ \psi_w).$$

Letting $m \to \infty$ it follows, first that $f \circ \psi_w \in \mathcal{D}'$, and then that (7.5) holds. □

If H is a set, and $f : H \to \mathbb{R}$, we write

$$\mathrm{Osc}(f,B) = \sup_{x,y \in B} |f(x) - f(y)|, \quad B \subset H. \tag{7.6}$$

Lemma 7.6. There exists a constant c_0, depending only on $\mathcal{E}$, such that

$$\mathrm{Osc}(f, V^{(0)}) \le c_0 \mathcal{E}^{(0)}(f,f), \qquad f \in C(V^{(0)}).$$

Proof. Let $\widetilde{E}_0 = \big\{\{x,y\} : A_{xy} > 0\big\}$. As $\mathcal{E}^{(0)}$ is non-degenerate, $(V^{(0)}, \widetilde{E}_0)$ is connected; let N be the maximum distance between points in this graph. Set $\alpha = \min\{A_{xy}, \{x,y\} \in \widetilde{E}_0\}$. If x, $y \in V^{(0)}$, there exists a chain $x = x_0, x_1, \dots, x_n = y$ connecting x, y with $n \le N$, and therefore,

$$\begin{aligned} |f(x) - f(y)|^2 &\le \left(\sum_{i=1}^n |f(x_i) - f(x_{i-1})| \right)^2 \\ &\le n \sum_{i=1}^n |f(x_i) - f(x_{i-1})|^2 \\ &\le n\alpha^{-1} \sum_{i=1}^n A_{x_{i-1},x_i} |f(x_i) - f(x_{i-1})|^2 \\ &\le N\alpha^{-1} \mathcal{E}^{(0)}(f,f). \end{aligned}$$

□

Since $V^{(1)}$ consists of M copies of $V^{(0)}$ we deduce a similar result for $V^{(1)}$.

Corollary 7.7. *There exists a constant $c_1 = c_1(F, r, A)$ such that*

$$\mathrm{Osc}(f, V^{(1)}) \le c_1 \mathcal{E}^{(1)}(f,f), \qquad f \in \mathcal{D}'. \tag{7.7}$$

Proof. For $i \in \mathbb{W}_1$, $f \in C(V^{(1)})$,

$$\mathrm{Osc}(f, V_i^{(0)}) = \mathrm{Osc}(f \circ \psi_i, V^{(0)}) \le c_0 \mathcal{E}^{(0)}(f \circ \psi_i, f \circ \psi_i).$$

So, as $V^{(1)}$ is connected,

$$\begin{aligned} \mathrm{Osc}(f, V^{(1)}) &\le \sum_i \mathrm{Osc}(f, V_i^{(0)}) \\ &\le \sum_i c_0 \mathcal{E}^{(0)}\left(f \circ \psi_i, f \circ \psi_i\right) \le c_1 \mathcal{E}^{(1)}(f,f), \end{aligned}$$

where c_1 is chosen so that $c_0 \le c_1 \rho r_i^{-1}$ for each $i \in \mathbb{W}_1$. □

Corollary 7.8. *Let $w \in \mathbb{W}_n$, and $x, y \in V_w^{(1)}$. Then*

$$\operatorname{Osc}(f, V_w^{(1)}) \le c_1 r_w \rho^{-n} \mathcal{E}'(f,f), \qquad f \in \mathcal{D}'.$$

Proof. We have $\operatorname{Osc}(f, V_w^{(1)}) = \operatorname{Osc}(f \circ \psi_w, V^{(1)}) \le c_1 \mathcal{E}^{(1)}(f \circ \psi_w, f \circ \psi_w)$. Since $\mathcal{E}^{(1)} \le \mathcal{E}'$, and by (7.5)

$$\mathcal{E}'(f \circ \psi_w, f \circ \psi_w) \le r_w \rho^{-n} \mathcal{E}'(f,f),$$

the result is immediate. □

Definition 7.9. We will call the fixed point $\mathcal{E}^{(0)}$ a *regular fixed point* if

$$r_i < \rho \qquad \text{for} \quad 1 \le i \le M. \tag{7.8}$$

Proposition 6.27 implies that (7.8) holds for any $s \in \{1, \dots, M\}$ such that $\pi(\dot{s}) \in V^{(0)}$. In particular therefore, for nested fractals, where every point in $V^{(0)}$ is of this form and r is constant, any fixed point is regular.

It is not hard to produce examples of non-regular fixed points. Consider the Lindstrøm snowflake, but with $r_i = 1$, $1 \le i \le 6$, $r_7 = r > 1$. Writing $\rho(r)$ for the resistance scale factor, we have (by Proposition 6.31) that $\rho(r)$ is increasing in r. However, also by Proposition 6.31, $\rho(r) \le \rho_0$, where ρ_0 is the resistance scale factor of the nested fractal obtained just from ψ_i, $1 \le i \le 6$. So if we choose $r_7 > \rho_0$, then as $r_7 > \rho_0 \ge \rho(r_7)$, we have an example of an affine nested fractal with a non-regular fixed point.

From now on we take $\mathcal{E}^{(0)}$ to be a regular fixed point. (See [Kum3] for the general situation). Write $\gamma = \max_i r_i/\rho < 1$. For $x, y \in F$, set $w(x,y)$ to be the longest word w such that $x, y \in F_w$.

Proposition 7.10. *(Sobolev inequality). Let $f \in \mathcal{D}'$. Then if $\mathcal{E}^{(0)}$ is a regular fixed point*

$$|f(x) - f(y)|^2 \le c_2 r_{w(x,y)} \rho^{-|w(x,y)|} \mathcal{E}'(f,f), \qquad x, y \in V^{(\infty)}. \tag{7.8}$$

Proof. Let $x, y \in V^{(n)}$, let $w = w(x,y)$ and let $|w| = m$. We prove (7.8) by a standard kind of chaining argument, similar to those used in continuity results such as Kolmogorov's lemma. (But this argument is deterministic and easier). We may assume $n \ge m$.

Let $u \in \mathbb{W}_n$ be an extension of w, such that $x \in V_u^{(0)}$: such a u certainly exists, as $x \in V_n^{(0)} \cap F_w$. Write $u_k = u|k$ for $m \le k \le n$. Now choose a sequence z_k, $m \le k \le n$ such that $z_n = x$, and $z_k \in V_{u_k}^{(0)}$ for $k \le m \le n-1$. For each $k \in \{m, \dots, n-1\}$ we have $z_k, z_{k+1} \in V_{u_k}^{(1)}$. So

$$\begin{aligned} |f(z_n) - f(z_m)| &\le \sum_{k=m}^{n-1} |f(z_{k+1}) - f(z_k)| \\ &\le \sum_{k=m}^{n-1} \left(c_1 r_{u_k} \rho^{-k} \mathcal{E}(f,f) \right)^{1/2} \end{aligned} \tag{7.9}$$

$$= \left(c_1 r_w \rho^{-m} \mathcal{E}(f,f)\right)^{1/2} \Big(\sum_{k=m}^{n-1} \frac{r_{u_k}}{r_w} \rho^{-k+m} \Big)^{1/2}.$$

As $\mathcal{E}$ is a regular fixed point, $\gamma = \max_i r_i/\rho < 1$, so the final sum in (7.9) is bounded by $(\sum_{k=m}^{\infty} \gamma^{k-m})^{1/2} = c_3 < \infty$. Thus we have

$$|f(x) - f(z_m)|^2 \le c_1 c_3 r_w \rho^{-n} \mathcal{E}'(f,f),$$

and as a similar bound holds for $|f(y) - f(z_m)|^2$, this proves (7.8). □

We have not so far needed a measure on F. However, to define a Dirichlet form we need some L^2 space in which the domain of $\mathcal{E}$ is closed. Let μ be a probability measure on $(F, \mathcal{B}(F))$ which charges every set of the form F_w, $w \in \mathbb{W}_n$. Later we will take μ to be the Bernouilli measure μ_θ associated with a vector of weights $\theta \in (0,\infty)^M$, but for now any measure satisfying the condition above will suffice.

As $\mu(F) = 1$, $C(F) \subset L^2(F,\mu)$. Set

$$\begin{aligned}\mathcal{D} &= \{f \in C(F) : f|_{V(\infty)} \in \mathcal{D}'\}\\ \mathcal{E}(f,f) &= \mathcal{E}'\left(f|_{V(\infty)}, f|_{V(\infty)}\right), \quad f \in \mathcal{D}.\end{aligned}$$

Proposition 7.11. *$(\mathcal{E}, \mathcal{D})$ is a closed symmetric form on $L^2(F,\mu)$.*

Proof. Note first that the condition on μ implies that if $f, g \in \mathcal{D}$ then $||f-g||_2 = 0$ implies that $f = g$. We need to prove that $\mathcal{D}$ is complete in the norm $\|f\|^2_{\mathcal{E}_1} = \mathcal{E}(f,f) + \|f\|_2^2$. So suppose (f_n) is Cauchy in $\|\cdot\|_{\mathcal{E}_1}$. Since (f_n) is Cauchy in $\|\cdot\|_2$, passing to a subsequence there exists $\widetilde{f} \in L^2(F,\mu)$ such that $f_n \to \widetilde{f}$ μ–a.e. Fix $x_0 \in F$ such that $f_n(x_0) \to \widetilde{f}(x)$. Then since $f_n - f_m$ is continuous, (7.8) extends to an estimate on the whole of F and so

$$\begin{aligned}|f_n(x) - f_m(x)| &\le |(f_n - f_m)(x) - (f_n - f_m)(x_0)| + |(f_n - f_m)(x_0)|\\ &\le c_2^{1/2} \mathcal{E}(f_n - f_m, f_n - f_m)^{1/2} + |f_n(x_0) - f_m(x_0)|.\end{aligned}$$

So (f_n) is Cauchy in the uniform norm, and thus there exists $f \in C(F)$ such that $f_n \to f$ uniformly.

Let $n \ge 1$. Then as $\mathcal{E}^{(n)}(g,g)$ is a finite sum,

$$\begin{aligned}\mathcal{E}^{(n)}(f,f) = \lim_{m\to\infty} \mathcal{E}^{(n)}(f_m, f_m) &\le \limsup_{m\to\infty} \mathcal{E}(f_m, f_m)\\ &\le \sup_m \|f_m\|_{\mathcal{E}_1} < \infty.\end{aligned}$$

Hence $\mathcal{E}^{(n)}(f,f)$ is bounded, so $f \in \mathcal{D}$. Finally, by a similar calculation, for any $N \ge 1$,

$$\mathcal{E}^{(N)}(f_n - f, f_n - f) \le \lim_{m\to\infty} \mathcal{E}(f_n - f_m, f_n - f_m).$$

So $\mathcal{E}(f_n - f, f_n - f) \to 0$ as $n \to \infty$, and thus $\|f - f_n\|^2_{\mathcal{E}_1} \to 0$. □

To show that $(\mathcal{E}, \mathcal{D})$ is a Dirichlet form, it remains to show that $\mathcal{D}$ is dense in $L^2(F,\mu)$. We do this by studying the harmonic extension of a function.

Definition 7.12. Let $f \in C(V^{(n)})$. Recall that $\mathcal{E}^{(n)}(f,f) = \inf\{\mathcal{E}^{(n+1)}(g,g) : g|_{V^{(n)}} = f\}$. Let $\widetilde{H}_{n+1}f \in C(V^{(n+1)})$ be the (unique, as $\mathcal{E}^{(n+1)})$ is non-degenerate) function which attains the infimum.

For $x \in V^{(\infty)}$ set

$$\widehat{H}_n f(x) = \lim_{m\to\infty} \widetilde{H}_m \widetilde{H}_{m-1} \dots \widetilde{H}_{n+1} f(x);$$

note that (as $\widetilde{H}_{n+1}f = f$ on $V^{(n)}$) this limit is ultimately constant.

Proposition 7.13. *Let $\mathcal{E}$ be a regular fixed point.*
(a) $\widehat{H}_n f$ has a continuous extension to a function $H_n f \in \mathcal{D} \cap C(F)$, which satisfies

$$\mathcal{E}(H_n f, H_n f) = \mathcal{E}^{(n)}(f,f).$$

(b) If f, $g \in C(F)$

$$\mathcal{E}(H_n f, g) = \mathcal{E}^{(n)}(f,g). \tag{7.10}$$

Proof. From the definition of $\widetilde{H}_{n+1}$, $\mathcal{E}^{(n+1)}(\widetilde{H}_{n+1}f, \widetilde{H}_{n+1}f) = \mathcal{E}^{(n)}(f,f)$. Thus $\mathcal{E}^{(m)}(\widehat{H}_n f, \widehat{H}_n f) = \mathcal{E}^{(n)}(f,f)$ for any m, so that $\widehat{H}_n f \in \mathcal{D}'$ and

$$\mathcal{E}(\widehat{H}_n f, \widehat{H}_n f) = \mathcal{E}^{(n)}(f,f), \qquad f \in C(V^{(n)}).$$

If $w \in \mathbb{W}_m$, and x, $y \in V^{(\infty)} \cap F_w$ then by Proposition 7.10

$$|\widehat{H}_n f(x) - \widehat{H}_n f(y)|^2 \le c_2 r_w \rho^{-m} \mathcal{E}^{(n)}(f,f). \tag{7.11}$$

Since $r_w \rho^{-m} \le \gamma^m$, (7.11) implies that $\mathrm{Osc}(\widehat{H}_n f, V^{(\infty)} \cap F_w)$ converges to 0 as $|w| = m \to \infty$. Thus $\widehat{H}_n f$ has a continuous extension $H_n f$, and $H_n f \in \mathcal{D}$ since $\widehat{H}_n f \in \mathcal{D}'$.
(b) Note that, by polarization, we have

$$\mathcal{E}^{(n+1)}(\widetilde{H}_{n+1}f, \widetilde{H}_{n+1}g) = \mathcal{E}^{(n)}(f,g).$$

Since $\mathcal{E}^{(n+1)}(\widetilde{H}_{n+1}f, h) = 0$ for any h such that $h|_{V^{(n)}} = 0$, it follows that

$$\mathcal{E}^{(n+1)}(\widetilde{H}_{n+1}f, g) = \mathcal{E}^{(n)}(f,g).$$

Iterating, we obtain (7.10). □

Theorem 7.14. *$(\mathcal{E}, \mathcal{D})$ is an irreducible, regular, local Dirichlet form on $L^2(F,\mu)$.*

Proof. Let $f \in C(F)$. Since for any $n \ge 1$, $w \in \mathbb{W}_n$ we have

$$\inf_{F_w} f \le H_n f(x) \le \sup_{F_w} f, \qquad x \in F_w$$

it follows that $H_n f \to f$ uniformly. As $H_n f \in \mathcal{D}$, we deduce that $\mathcal{D}$ is dense in $C(F)$ in the uniform norm. Hence also $\mathcal{D}$ is dense in $L^2(F,\mu)$. As (4.5) is immediate, we deduce that $\mathcal{D}$ is a regular Dirichlet form. If $\mathcal{E}(f,f) = 0$ then $\mathcal{E}^{(n)}(f,f) = 0$ for each n. Since $\mathcal{E}^{(n)}$ is irreducible, $f|_{V^{(n)}}$ is constant for each n. As f is continuous, f is therefore constant. Thus $\mathcal{E}$ is irreducible.

To prove that $\mathcal{E}$ is local, let f, g be functions in $\mathcal{D}$ with disjoint closed supports, S_f, S_g say. If $\mathcal{E}^{(n)}(f,g) \neq 0$ then one of the terms in the sum (7.1) must be non-zero, so there exists $w_n \in \mathbb{W}_n$, and points $x_n \in S_f \cap V^{(0)}_{w_n}$, $y_n \in S_g \cap V^{(0)}_{w_n}$. Passing to a subsequence, there exists z such that $x_n \to z$, $y_n \to z$, and as therefore $z \in S_f \cap S_g$, this is a contradiction. □

By Theorem 4.8 there exists a continuous μ-symmetric Hunt process $(X_t, t \geq 0, \mathbb{P}^x, x \in F)$ associated with $(\mathcal{E},\mathcal{D})$ and $L^2(F,\mu)$.

Remark 7.15. Note that we have constructed a process $X = X^{(\mu)}$ for each Radon measure μ on F. So, at first sight, the construction given here has built much more than the probabilistic construction outlined in Section 2. But this added generality is to a large extent an illusion: Theorem 4.17 implies that these processes can all be obtained from each other by time-change.

On the other hand the regularity of $(\mathcal{E},\mathcal{D})$ was established without much pain, and here the advantage of the Dirichlet form approach can be seen: all the probabilistic approaches to the Markov property are quite cumbersome.

The general probabilistic construction, such as given in [L1] for example, encounters another obstacle which the Dirichlet form construction avoids. As well as finding a decimation invariant set of transition probabilities, it also appears necessary (see e.g. [L1, Chapter VI]) to find associated transition times. It is not clear to me why these estimates appear essential in probabilistic approaches, while they do not seem to be needed at all in the construction above.

We collect together a number of properties of $(\mathcal{E},\mathcal{D})$.

Proposition 7.16. *(a) For each $n \geq 0$*

$$\mathcal{E}(f,g) = \sum_{w \in \mathbb{W}_n} \rho^n r_w^{-1} \mathcal{E}(f \circ \psi_w, g \circ \psi_w). \tag{7.12}$$

(b) For $f \in \mathcal{D}$,

$$|f(x) - f(y)|^2 \leq c_1 r_w \rho^{-n} \mathcal{E}(f,f) \quad \text{if} \quad x,y \in F_w,\ w \in \mathbb{W}_n \tag{7.13}$$

$$\int f^2 d\mu \leq c_2 \mathcal{E}(f,f) + \Big(\int f d\mu\Big)^2, \tag{7.14}$$

$$f(x)^2 \leq 2 \int f^2 d\mu + 2c_1 \mathcal{E}(f,f), \qquad x \in F. \tag{7.15}$$

Proof. (a) is immediate from Lemma 7.5, while (b) follows from Proposition 7.10 and the continuity of f. Taking $n = 0$ in (7.13) we deduce that

$$\big(f(x) - f(y)\big)^2 \leq c_1 \mathcal{E}(f,f), \qquad f \in \mathcal{D}.$$

So as $\mu(F) = 1$,

$$\begin{aligned}\int\!\!\int c_1 \mathcal{E}(f,f)\mu(dx)\mu(dy) &= c_1\mathcal{E}(f,f) \\ &\leq \int\!\!\int \big(f(x) - f(y)\big)^2 \mu(dx)\mu(dy) \\ &= 2\int f^2\,d\mu - 2\Big(\int f\,d\mu\Big)^2,\end{aligned}$$

proving (7.14).

Since $f(x)^2 \leq 2f(y)^2 + 2|f(x) - f(y)|^2$ we have from (7.13) that

$$\begin{aligned}f(x)^2 &= \int f(x)^2\mu(dy) \\ &\leq 2\int f(y)^2\mu(dy) + 2c_1\int \mathcal{E}(f,f)\mu(dy),\end{aligned}$$

which proves (7.15). □

We need to examine further the resistance metric introduced in Section 4.

Definition 7.17. Let $R(x,x) = 0$, and for $x \neq y$ set

$$R(x,y)^{-1} = \inf\left\{\mathcal{E}(f,f) : f(x) = 0, f(y) = 1, f \in \mathcal{D}\right\}.$$

Note that

(7.16) $$R(x,y) = \sup\Big\{\frac{|f(x) - f(y)|^2}{\mathcal{E}(f,f)} : f \in \mathcal{D}, \quad f \text{ non constant}\Big\}.$$

Proposition 7.18. *(a) If $x \neq y$ then $0 < R(x,y) \leq c_1 < \infty$.*
(b) If $w \in \mathbb{W}_n$ then

(7.17) $$R(x,y) \leq c_1 r_w \rho^{-n}, \qquad x, y \in F_w.$$

(c) For $f \in \mathcal{D}$

(7.18) $$|f(x) - f(y)|^2 \leq R(x,y)\mathcal{E}(f,f).$$

(d) R is a metric on F, and the topology induced by R is equal to the original topology on F.

Proof. Let x, y be distinct points in F. As $\mathcal{D}$ is dense in $C(F)$, there exists $f \in \mathcal{D}$ with $f(x) \geq 1$, $f(y) \leq 0$. Since $\mathcal{E}$ is irreducible, $\mathcal{E}(f,f) > 0$, and so by (7.16) $R(x,y) > 0$. (7.17) is immediate from Proposition 7.16, proving (b). Taking $n = 0$, and w to be the empty word in (7.17) we deduce $R(x,y) \leq c_1$ for any x, $y \in F$, completing the proof of (a).
(c) is immediate from (7.16).
(d) R is clearly symmetric. The triangle inequality for R is proved exactly as in Proposition 4.25, by considering the trace of $\mathcal{E}$ on the set $\{x,y,z\}$.

It remains to show that the topologies induced by R and d (the original metric on F) are the same. Let $R(x_n,x)\to 0$. If $\varepsilon>0$, there exists $f\in\mathcal{D}$ with $f(x)=1$ and $\operatorname{supp}(f)\subset B_d(x,\varepsilon)$. By (7.16) $R(x,y)\geq \mathcal{E}(f,f)^{-1}>0$ for any $y\in B_d(x,\varepsilon)^c$. So $x_n\in B_d(x,\varepsilon)$ for all sufficiently large n, and hence $d(x_n,x)\to 0$.

If $d(x_n,x)\to 0$ then writing

$$N_m(x)=\bigcup\{F_w : w\in\mathbb{W}_m,\quad x\in F_w\}$$

we have by Lemma 5.12 that $x_n\in N_m(x)$ for all sufficiently large n. However if $\gamma=\max_i r_i/\rho<1$ we have by, (7.17), $R(x,y)\leq c_1\gamma^m$ for $y\in N_m(x)$. Thus $R(x_n,x)\to 0$. □

Remark 7.19. The resistance metric R on F is quite well adapted to the study of the diffusion X on F. Note however that $R(x,y)$ is obtained by summing (in a certain sense) the resistance of all paths from x to y. So it is not surprising that R is not a geodesic metric. (Unless F is a tree).

Also, R is not a geometrically natural metric on F. For example, on the Sierpinski gasket, since $r_i=1$, and $\rho=5/3$, we have that if x, y are neighbours in $(V^{(n)},\mathbf{E}_n)$ then

$$R(x,y)\asymp(3/5)^n.$$

However, for general p.c.f.s.s. sets it is not easy to define a metric which is well-adapted to the self-similar structure. (And, if one imposes strict conditions of exact self-similarity, it is not possible in general – see the examples in [Ki6]). So, for these general sets the resistance metric plays an extremely useful role. The next section contains some additional results on R.

It is also worth remarking that the balls $B_R(x,r)=\{y: R(x,y)<r\}$ need not in general be connected. For example, consider the wire network corresponding to the graph consisting of two points x, y, connected by n wires each of conductivity 1. Let z be the midpoint of one of the wires. Then $R(x,y)=1/n$, while the conductivities in the network $\{x,y,z\}$ are given by $C(x,z)=C(z,y)=2$, $C(x,y)=n-1$. So, after some easy calculations,

$$R(x,z)=\frac{n+1}{4n-1}>\tfrac14.$$

So if $n=4$, $R(x,y)=\frac14$ while $R(x,z)=\frac13$. Hence if $\frac14<r<\frac13$ the ball $B_R(x,r)$ is not connected. (In fact, y is an isolated point of $\overline{B}_R(x,\frac14)=\{x': d(x,x')\leq\frac14\}$). (Are the balls $B_R(x,r)$ in the Sierpinski gasket connected? I do not know).

Recall the notation $\mathcal{E}_\alpha(f,g)=\mathcal{E}(f,g)+\alpha(f,g)$. Let $(U_\alpha,\alpha>0)$ be the resolvent of X. Since by (4.8) we have

$$\mathcal{E}_\alpha(U_\alpha f,g)=(f,g),$$

if U_α has a density $u_\alpha(x,y)$ with respect to μ, then a formal calculation suggests that

$$\mathcal{E}_\alpha\big(u_\alpha(x,\cdot),g\big)=\mathcal{E}_\alpha(U_\alpha\delta_x,g)=(\delta_x,g)=g(x).$$

We can use this to obtain the existence and continuity of the resolvent density u_α. (See [FOT, p. 73]).

Theorem 7.20. *(a) For each $x \in F$ there exists $u_\alpha^x \in \mathcal{D}$ such that*

$$\mathcal{E}_\alpha(u_\alpha^x, f) = f(x) \qquad \text{for all} \quad f \in \mathcal{D}. \tag{7.19}$$

(b) Writing $u_\alpha(x,y) = u_\alpha^x(y)$, we have

$$u_\alpha(x,y) = u_\alpha(y,x) \qquad \text{for all} \quad x, y \in F.$$

(c) $u_\alpha(\cdot,\cdot)$ is continuous on $F \times F$ and in particular

$$|u_\alpha(x,y) - u_\alpha(x,y')|^2 \le R(y,y')u_\alpha(x,x). \tag{7.20}$$

(d) $u_\alpha(x,y)$ is the resolvent density for X: for $f \in C(F)$,

$$E^x \int_0^\infty e^{-\alpha t} f(X_t)dt = U_\alpha f(x) = \int u_\alpha(x,y)f(y)\mu(dy).$$

(e) There exists $c_2(\alpha)$ such that

$$u_\alpha(x,y) \le c_2(\alpha), \qquad x, y \in F. \tag{7.21}$$

Proof. (a) The existence of u_α^x is given by a standard argument with reproducing kernel Hilbert spaces. Let $x \in F$, and for $f \in \mathcal{D}$ let $\phi(f) = f(x)$. Then by (7.15)

$$|\phi(f)|^2 = |f(x)|^2 \le 2\|f\|_2^2 + 2c_1\mathcal{E}(f,f) \le c_\alpha \mathcal{E}_\alpha(f,f),$$

where $c_\alpha = 2\max(c_1, \alpha^{-1})$. Thus ϕ is a bounded linear functional on the Hilbert space $(\mathcal{D}, \| \ \|_{\mathcal{E}_\alpha})$, and so there exists a $u_\alpha^x \in \mathcal{D}$ such that

$$\phi(f) = \mathcal{E}_\alpha(u_\alpha^x, f) = f(x), \qquad f \in \mathcal{D}.$$

(b) This is immediate from (a) and the symmetry of $\mathcal{E}$:

$$u_\alpha^y(x) = \mathcal{E}_\alpha(u_\alpha^x, u_\alpha^y) = \mathcal{E}_\alpha(u_\alpha^y, u_\alpha^x) = u_\alpha^x(y).$$

(c) As $u_\alpha^x \in \mathcal{D}$, $u_\alpha(x,x) < \infty$. Since $\mathcal{E}(u_\alpha^x, u_\alpha^x) = u_\alpha(x,x) < \infty$, the estimate (7.20) follows from (7.18). It follows immediately that u is jointly continuous on $F \times F$.
(d) This follows from (7.19) and linearity. For a measure ν on F set

$$V_\nu f(x) = \int u_\alpha(x,y)f(y)\nu(dy), \qquad f \in C(F).$$

As u_α is uniformly continuous on $F \times F$, we can choose $\nu_n \xrightarrow{w} \mu$ so that $V_{\nu_n} f \to Vf$ uniformly, and ν_n are atomic with a finite number of atoms. Write $V_n = V_{\nu_n}$, $V = V_\mu$. Since by (7.19)

$$\begin{aligned}\mathcal{E}_\alpha(V_n f, g) &= \sum_x \nu_n(\{x\}) f(x) \mathcal{E}_\alpha(u_a^x, g) \\ &= \sum_x f(x)g(x)\nu_n(\{x\}) = \int fg\, d\nu_n,\end{aligned}$$

we have

$$\mathcal{E}_\alpha(V_n f - V_m f, V_n f - V_m f) = \int f(V_n f - V_m f)\, d\nu_n - \int f(V_n f - V_m f)\, d\nu_m.$$

Thus $\mathcal{E}_\alpha(V_n f - V_m f, V_n f - V_m f) \to 0$ as $m,\ n \to \infty$, and so, as $\mathcal{E}$ is closed, we deduce that $Vf \in \mathcal{D}$ and $\mathcal{E}_\alpha(Vf, g) = \lim_n \mathcal{E}_\alpha(V_n f, g) = \lim_n \int fg\, d\nu_n = \int fg\, d\mu$. So $\mathcal{E}_\alpha(Vf, g) = \mathcal{E}_\alpha(U_\alpha f, g)$ for all g, and hence $Vf = U_\alpha f$.
(e) As $R(y, y') \le c_1$ for $y,\ y' \in F$, we have from (7.20) that

$$u_\alpha(x, y) \ge u_\alpha(x, x) - \big(c_1 u_\alpha(x, x)\big)^{1/2}. \tag{7.22}$$

Since $\int u_\alpha(x, y)\mu(dy) = \alpha^{-1}$, integrating (7.22) we obtain

$$u_\alpha(x, x) \le \big(c_1 u_\alpha(x, x)\big)^{1/2} + \alpha^{-1},$$

and this implies that $u_\alpha(x, x) \le c_2(\alpha)$, where $c(\alpha)$ depends only on α and c_1. Using (7.20) again we obtain (7.21). □

Theorem 7.21. *(a) For each $x \in F$, x is regular for $\{x\}$.*
(b) X has a jointly continuous local time $(L_t^x, x \in F, t \ge 0)$ such that for all bounded measurable f

$$\int_0^t f(X_s)\, ds = \int f(a) L_t^a \mu(da), \quad \text{a.s.}$$

Proof. These follow from the estimates on the resolvent density u_α. As u_α is bounded and continuous, we have that x is regular for $\{x\}$. Thus X has jointly measurable local times $(L_t^x, x \in F, t \ge 0)$.

Since X is a symmetric Markov process, by Theorem 8.6 of [MR], L_t^x is jointly continuous in (x, t) if and only if the Gaussian process Y_x, $x \in F$ with covariance function given by

$$EY_aY_b = u_1(a, b), \qquad a, b \in F$$

is continuous. Necessary and sufficient conditions for continuity of Gaussian processes are known (see [Tal]), but here a simple sufficient condition in terms of metric entropy is enough. We have

$$E(Y_a - Y_b)^2 = u_1(a, a) - 2u_1(a, b) + u_1(b, b) \le c_1 R(a, b)^{1/2}.$$

Set $r(a, b) = R(a, b)^{1/2}$: r is a metric on F. Write $N_r(\varepsilon)$ for the smallest number of sets of r-diameter ε needed to cover F. By (7.17) we have $R(a, b) \le c\gamma^n$ if $a,\ b \in F_w$ and $w \in \mathbb{W}_n$. So $N_r(c'\gamma^{n/2}) \le \#\mathbb{W}_n = M^n$, and it follows that

$$N_r(\varepsilon) \le c_2 \varepsilon^{-\beta},$$

where $\beta = 2\log M / \log \theta^{-1}$. So

$$\int_{0+} \big(\log N_r(\varepsilon)\big)^{1/2}\, d\varepsilon < \infty,$$

and thus by [Du, Thm. 2.1] Y is continuous. □

We can use the continuity of the local time of X to give a simple proof that X is the limit of a natural sequence of approximating continuous time Markov chains. For simplicity we take μ to be a Bernouilli measure of the form $\mu = \mu_\theta$, where $\theta_i > 0$. Let μ_n be the measure on $V^{(n)}$ given in (5.21). Set

$$\begin{aligned} A_t^n &= \int_F L_t^x \mu_n(dx), \\ \tau_t^n &= \inf\{s : A_s^n > t\}, \\ X_t^n &= X_{\tau_t^n}. \end{aligned}$$

Theorem 7.22. *(a) $(X_t^n, t \geq 0, \mathbb{P}^x, x \in V^{(n)})$ is the symmetric Markov process associated with $\mathcal{E}^{(n)}$ and $L^2(V^{(n)}, \mu_n)$.*
(b) $X_t^n \to X_t$ a.s. and uniformly on compacts.

Proof. (a) By Theorem 7.21(a) points are non-polar for X. So by the trace theorem (Theorem 4.17) X^n is the Markov process associated with the trace of $\mathcal{E}$ on $L^2(V^{(n)}, \mu_n)$. But for $f \in \mathcal{D}$, by the definition of $\mathcal{E}$,

$$Tr\big(\mathcal{E}|V^{(n)}\big)(f,f) = \mathcal{E}^{(n)}\big(f|_{V^{(n)}}, f|_{V^{(n)}}\big).$$

(b) As F is compact, for each $T > 0$, $(L_t^x, 0 \leq t \leq T, x \in F)$ is uniformly continuous. So, using (5.22), if $T_2 < T_1 < T$ then $A_t^n \to t$ uniformly in $[0, T_1]$, and so $\tau_t^n \to t$ uniformly on $[0, T_2]$. As X is continuous, $X_t^n \to X$ uniformly in $[0, T_2]$. □

Remark 7.23. As in Example 4.21, it is easy to describe the generator L_n of X^n. Let $a^{(n)}(x,y)$, x, $y \in V^{(n)}$ be the conductivity matrix such that

$$\mathcal{E}^{(n)}(f,f) = \tfrac{1}{2} \sum_{x,y} a^{(n)}(x,y)\big(f(x) - f(y)\big)^2.$$

Then by (7.1) we have

$$a^{(n)}(x,y) = \sum_{w \in \mathbb{W}_n} 1_{(x,y \in V_w^{(0)})} \rho^n r_w^{-1} A\big(\psi_w^{-1}(x), \psi_w^{-1}(y)\big), \tag{7.23}$$

where A is such that $\mathcal{E}^{(0)} = \mathcal{E}_A$, and $A(x,y) = A_{xy}$. Then for $f \in L^2(V^{(n)}, \mu_n)$,

$$L_n f(x) = \mu_n(\{x\})^{-1} \sum_{y \in V^{(n)}} a^{(n)}(x,y)\big(f(y) - f(x)\big). \tag{7.24}$$

Of course Theorem 7.22 implies that if (Y^n) is a sequence of continuous time Markov chains, with generators given by (7.24), then $Y^n \xrightarrow{w} X$ in $D([0,\infty), F)$.

8. Transition Density Estimates.

In this section we fix a connected p.c.f.s.s. set $\big(F,(\psi_i)\big)$, a resistance vector r_i, and a non-degenerate regular fixed point $\mathcal{E}_A$ of the renormalization map Λ. Let $\mu = \mu_\theta$ be a measure on F, and let $X = (X_t, t \geq 0, \mathbb{P}^x, x \in F)$ be the diffusion process constructed in Section 7. We investigate the transition densities of the process X: initially in fairly great generality, but as the section proceeds, I will restrict the class of fractals.

We begin by fixing the vector θ which assigns mass to the 1-complexes $\psi_i(F)$, in a fashion which relates $\mu_\theta\big(\psi_i(F)\big)$ with r_i. Let $\beta_i = r_i \rho^{-1}$: by (7.8) we have

$$\beta_i < 1, \quad 1 \leq i \leq M. \tag{8.1}$$

Let $\alpha > 0$ be the unique positive real such that

$$\sum_{i=1}^{M} \beta_i^\alpha = 1. \tag{8.2}$$

Set

$$\theta_i = \beta_i^\alpha, \qquad 1 \leq i \leq M, \tag{8.3}$$

and let $\mu = \mu_\theta$ be the associated Bernouilli type measure on F. Write $\beta_+ = \max_i \beta_i$, $\beta_- = \min_i \beta_i$: we have $0 < \beta_- \leq \beta_i \leq \beta_+ < 1$.

We wish to split the set F up into regions which are, "from the point of view of the process X", all roughly the same size. The approximation Theorem 7.22 suggests that if $w \in \mathbb{W}_n$ then the 'crossing time' of the region F_w is of the order of $\rho^{-n} r_w \theta_w^{-1} = \beta_w \theta_w^{-1} = \beta_w^{1-\alpha}$. (See Proposition 8.10 below for a more precise statement of this fact). So if $r.$ is non-constant the decomposition $F = \cup \{F_w,\, w \in \mathbb{W}_n\}$ of F into n complexes is unsuitable; instead we need to use words w of different lengths. (This idea is due to Hambly – see [Ham2]).

Let $\mathbb{W}_\infty = \cup_{n=0}^\infty \mathbb{W}_n$ be the space of all words of finite length. $\mathbb{W}_\infty$ has a natural tree structure: if $w \in \mathbb{W}_n$ then the parent of w is $w|n-1$, while the offspring of w are the words $w \cdot i$, $1 \leq i \leq M$. (We define the truncation operator τ on $\mathbb{W}_\infty$ by $\tau w = w|\big(|w|-1\big)$.) Write also for $w \in \mathbb{W}_\infty$

$$w \cdot \mathbb{W} = \{w \cdot v, v \in \mathbb{W}\} = \{v \in \mathbb{W} : v_i = w_i,\ 1 \leq i \leq |w|\}.$$

Lemma 8.1. *(a) For $\lambda > 0$ let*

$$\mathbb{W}_\lambda = \{w \in \mathbb{W}_\infty : \beta_w \leq \lambda,\ \beta_{\tau w} > \lambda\}.$$

Then the sets $\{w \cdot \mathbb{W}, w \in \mathbb{W}_\lambda\}$ are disjoint, and

$$\bigcup_{w \in \mathbb{W}_\lambda} w \cdot \mathbb{W} = \mathbb{W}.$$

(b) For $f \in L^1(F,\mu)$,

$$\int f\,d\mu = \sum_{w\in\mathbb{W}_\lambda} \theta_w \int f_w\,d\mu$$

$$\mathcal{E}(f,f) = \sum_{w\in\mathbb{W}_\lambda} \beta_w^{-1}\mathcal{E}(f_w,f_w).$$

Proof. (a) Suppose w, $w' \in \mathbb{W}_\lambda$ and $v \in (w\cdot\mathbb{W})\cap(w'\cdot\mathbb{W})$. Then there exist u, $u' \in \mathbb{W}$ such that $v = w\cdot u = w'\cdot u'$. So one of w, w' (say w) is an ancestor of the other. But if $\beta_w \le \lambda$, $\beta_{\tau w} > \lambda$ then as $\beta_i < 1$ we can only have $\beta_{\tau w'} > \lambda$ if $w' = w$. So if $w \ne w'$, $w\cdot\mathbb{W}$ and $w'\cdot\mathbb{W}$ are disjoint.

Let $v \in \mathbb{W}$. Then $\beta_{v|n} = \prod_{i=1}^n \beta_{v_i} \to 0$ as $n\to\infty$. So there exists m such that $v|m \in \mathbb{W}_\lambda$, and then $v \in (v|m)\cdot\mathbb{W}$, completing the proof of (a).

(b) This follows in a straightforward fashion from the decompositions given in (7.12) and Lemma 5.28. □

Note that $\beta_- > 0$ and that

(8.4) $$\beta\lambda \le \beta_w \le \lambda, \qquad (\beta_-)^\alpha\lambda^\alpha \le \theta_w \le \lambda^\alpha, \quad w\in\mathbb{W}_\lambda.$$

Definition 8.2. The *spectral dimension* of F is defined by

$$d_s = d_s(F,\mathcal{E}_A) = 2\alpha/(1+\alpha).$$

Theorem 8.3. *For* $f \in \mathcal{D}$,

(8.5) $$\|f\|_2^{2+4/d_s} \le c_1\left(\mathcal{E}(f,f) + \|f\|_2^2\right)\|f\|_1^{4/d_s}.$$

Proof. It is sufficient to consider the case f non-negative, so let $f \in \mathcal{D}$ with $f \ge 0$. Let $0 < \lambda < 1$: by Lemma 8.1, (7.14) and (8.4) we have

(8.6) $$\begin{aligned}\|f\|_2^2 &= \sum_{w\in\mathbb{W}_\lambda}\theta_w\int f_w^2\,d\mu\\ &\le \sum_w \theta_w\left(c_1\mathcal{E}(f_w,f_w) + \left(\int f_w\,d\mu\right)^2\right)\\ &\le c_2\sum_w \lambda^\alpha\mathcal{E}(f_w,f_w) + c_2\sum_w\lambda^\alpha\left(\int f_w\,d\mu\right)^2\\ &\le c_3\lambda^{\alpha+1}\sum_w\beta_w^{-1}\mathcal{E}(f_w,f_w) + c_2\lambda^\alpha\left(\sum_w\int f_w\,d\mu\right)^2\\ &\le c_3\lambda^{\alpha+1}\mathcal{E}(f,f) + c_4\lambda^{-\alpha}\left(\sum_w\theta_w\int f_w\,d\mu\right)^2\\ &= c_3\lambda^{\alpha+1}\mathcal{E}(f,f) + c_4\lambda^{-\alpha}\|f\|_1^2.\end{aligned}$$

The final line of (8.6) is minimized if we take $\lambda^{2\alpha+1} = c_5\|f\|_1^2/\mathcal{E}(f,f)$. If $\mathcal{E}(f,f) \geq c_5\|f\|_1^2$ then $\lambda < 1$ and so we obtain from (8.6) that

$$(8.7) \qquad \|f\|_2^2 \leq c\mathcal{E}(f,f)^{\alpha/(2\alpha+1)}\left(\|f\|_1^2\right)^{(\alpha+1)/(2\alpha+1)},$$

which implies that that

$$(8.8) \qquad \|f\|_2^{2+4/d_s} \leq c\mathcal{E}(f,f)\|f\|_1^{4/d_s} \qquad \text{if} \quad \mathcal{E}(f,f) \geq c_5\|f\|_1^2.$$

If $\mathcal{E}(f,f) \leq c_5\|f\|_1^2$ then by (7.14)

$$\|f\|_2^2 \leq c_1\left(\mathcal{E}(f,f) + \|f\|_1^2\right) \leq c\|f\|_1^2,$$

and so

$$(8.9) \qquad \|f\|_2^{2+4/d_s} \leq c\|f\|_2^2\|f\|_1^{4/d_s} \qquad \text{if} \quad \mathcal{E}(f,f) \leq c_5\|f\|_1^2.$$

Combining (8.8) and (8.9) we obtain (8.5). □

From the results in Section 4 we then deduce

Theorem 8.4. *X has a transition density $p(t,x,y)$ which satisfies*

$$(8.10) \qquad p(t,x,y) \leq c_1 t^{-d_s/2}, \qquad 0 < t \leq 1, \quad x,y \in F,$$

$$(8.11) \quad \left|p(t,x,y) - p(t,x,y')\right|^2 \leq c_2 t^{-1-d_s/2} R(y,y'), \quad 0 \leq t \leq 1, \quad x,y,y' \in F.$$

Proof. By Proposition 4.14 X has a jointly measurable transition density, and by Corollary 4.15 we have for x, $y \in F$, $0 < t \leq 1$,

$$p(t,x,y) \leq ct^{-d_s/2}e^{ct} \leq c't^{-d_s/2}.$$

By (4.17) the function $q_{t,x} = p(t,x,\cdot)$ satisfies $\mathcal{E}(q_{t,x},q_{t,x}) \leq ct^{-1-d_s/2}$, and so $q_{t,x} \in \mathcal{D}$ and is continuous. Further, by Proposition 7.18

$$|p(t,x,y) - p(t,x,y')|^2 \leq cR(y,y')t^{-d_s/2-1}, \qquad x,y,y' \in F.$$

Thus $p(t,\cdot,\cdot)$ is jointly Hölder continuous in the metric R on F. □

Remarks 8.5. 1. As $\alpha > 0$, we have $0 < d_s = 2\alpha(1+\alpha)^{-1} < 2$.
2. The estimate (8.10) is good if $t \in (0,1]$ and x close to y. It is poor if t is small compared with $R(x,y)$, and in this case we can obtain a better estimate by chaining, as was done for fractional diffusions in Section 3. For this we need some additional properties of the resistance metric.

Lemma 8.6. *If v, $w \in \mathbb{W}_\lambda$ and $v \neq w$ then $F_v \cap F_w = V_v^{(0)} \cap V_w^{(0)}$.*

Proof. This follows easily from the corresponding property for $\mathbb{W}_n$. Let v, $w \in \mathbb{W}_\lambda$, with $|v| = m \leq |w| = n$, $v \neq w$. Let $x \in F_v \cap F_w$. Set $w' = w|m$; then as $F_w \subset F_{w'}$, $x \in F_v \cap F_w$, and so by Lemma 5.17(a) $x \in V_v^{(0)} \cap V_{w'}^{(0)}$. Further, as $x \in F_v$ there exists $v' \in \mathbb{W}_n$ such that $v'|m = v$, and $x \in F_{v'}$. Then $x \in F_{v'} \cap F_w = V_{v'}^{(0)} \cap V_w^{(0)}$. So $x \in V_v^{(0)} \cap V_w^{(0)}$. □

Definition 8.7. Set

$$V_\lambda^{(0)} = \bigcup_{w \in \mathbb{W}_\lambda} V_w^{(0)}.$$

Let $G_\lambda = \left(V_\lambda^{(0)}, \mathbf{E}_\lambda\right)$ be the graph with vertex set $V_\lambda^{(0)}$, and edge set $\mathbf{E}_\lambda$ such that $\{x, y\}$ is an edge if and only if $x,\ y \in V_w^{(0)}$ for some $w \in \mathbb{W}_\lambda$. For $A \subset F$ set

$$\begin{aligned} N_\lambda(A) &= \bigcup \{F_w : w \in \mathbb{W}_\lambda,\ F_w \cap A \neq \emptyset\}, \\ \widetilde{N}_\lambda(x) &= N_\lambda\left(N_\lambda(\{x\})\right). \end{aligned}$$

As we will see, $\widetilde{N}_\lambda(x)$ is a neighbourhood of x with a structure which is well adapted to the geometry of F in the metric R. We write $N_\lambda(y) = N_\lambda\left(\{y\}\right)$.

Lemma 8.8. *(a) If $x,\ y \in V_\lambda^{(0)}$ and $x \neq y$ then*

$$R(x, y) \geq c_1 \lambda.$$

(b) If $\{x, y\} \in \mathbf{E}_\lambda$ then $R(x, y) \leq c_2 \lambda$.

Proof. (b) is immediate from the definition of $\mathbb{W}_\lambda$ and Proposition 7.18(b). For (a), note first that if $x \in F$ then by Proposition 5.21 x can belong to at most $M_1 = M\#(P)$ n–complexes, for any n. So there are at most M_1 distinct elements $w \in \mathbb{W}_\lambda$ such that $x \in F_w$.

As $V^{(0)}$ is a finite set, and $\mathcal{E}_A^{(0)}$ is non-degenerate, there exists $c_3,\ c_4 > 0$ such that,

$$c_4 \geq R(x, V^{(0)} - \{x\}) \geq c_3, \qquad x \in V^{(0)}. \tag{8.12}$$

(Recall that this resistance is, by the construction of $\mathcal{E}$, the same in $(F, \mathcal{E})$ as in $(V^{(0)}, \mathcal{E}_A^{(0)})$). Now fix $x \in V_\lambda^{(0)}$. If $w \in \mathbb{W}_\lambda$, and $x \in V_w^{(0)}$, let $x' = \psi_w^{-1}(x)$, and g_w be the function on F such that $g_w(x') = 1$, $g_w(y) = 0$, $g \in V^{(0)} - \{x'\}$, and

$$\mathcal{E}(g_w, g_w)^{-1} = R\left(x', V^{(0)} - \{x'\}\right) \geq c_3.$$

Define g'_w on F_w by $g'_w = g_w \circ \psi_w^{-1}$, and extend g'_w to F by setting $g'_w = 0$ on $F - F_w$.

Now let $g'_v = 0$ if $x \notin V_v^{(0)}$, $V \in \mathbb{W}_\lambda$, and set

$$g = \sum_{v \in \mathbb{W}_\lambda} g'_v.$$

Then $g(x) = 1$, $g(y) = 0$ if $y \in V_\lambda^{(0)}$, $y \neq x$, and

$$\begin{aligned} \mathcal{E}(g, g) &= \sum_{w \in \mathbb{W}_\lambda} \beta_w^{-1} \mathcal{E}(g \circ \psi_w, g \circ \psi_w) \\ &= \sum_w \beta_w^{-1} 1_{(x \in F_w)} \mathcal{E}(g_w, g_w) \leq c_5 \lambda^{-1} M_1. \end{aligned}$$

Hence if $y \neq x$, $y \in V_\lambda^{(0)}$, we have

$$R(x,y)^{-1} \leq \mathcal{E}(g,g) \leq \lambda^{-1} M_1 c_5^{-1},$$

so that $R(x,y) \geq c_6 \lambda$. □

Remark. For $x \in V_\lambda^{(0)}$ the function g constructed above is zero outside $N_\lambda(\{x\})$. So we also have

$$R(x,y) \geq c_6\lambda, \qquad x \in V_\lambda^{(0)}, \quad y \in N_\lambda(\{x\})^c. \tag{8.13}$$

Proposition 8.9. *There exist constants c_i such that for $x \in F$, $\lambda > 0$,*

$$B_R(x,c_1\lambda) \subset \widetilde{N}_\lambda(x) \subset B_R(x,c_2\lambda), \tag{8.14}$$

$$c_3\lambda^\alpha \leq \mu\big(B_R(x,\lambda)\big) \leq c_4\lambda^\alpha \tag{8.15}$$

$$c_5\lambda \leq R\big(x,\widetilde{N}_\lambda(x)^c\big) \leq c_6\lambda, \tag{8.16}$$

$$c_7\lambda \leq R\big(x,B_R(x,\lambda)^c\big) \leq c_8\lambda. \tag{8.17}$$

Proof. Let $x \in F$. If $y \in N_\lambda(\{x\})$ then by (7.17), $R(x,y) \leq c\lambda$. So if $z \in \widetilde{N}_\lambda(x)$, since there exists $y \in N_\lambda(\{x\})$ with $z \in N_\lambda(\{y\})$, $R(x,z) \leq c'\lambda$, proving the right hand inclusion in (8.14).

If $x \in V_\lambda^{(0)}$ then by (8.13), if $c_9 = c_{8\cdot 7\cdot 6}$,

$$B_R(x,c_9\lambda) \subset N_\lambda(x).$$

Now let $x \notin V_\lambda^{(0)}$, so that there exists a unique $w \in \mathbb{W}_\lambda$ with $x \in F_w$. For each $y \in V_w^{(0)}$ let $f_y(\cdot)$ be the function constructed in Lemma 8.8, which satisfies $f_y(y) = 1$, $f_y = 0$ outside $N_\lambda(y)$, $f_y(z) = 0$ for each $z \in V_\lambda^{(0)} - \{y\}$, and $\mathcal{E}(f_y,f_y) \leq c_{10}\lambda^{-1}$. Let $f = \sum_y f_y$: then $f(y) = 1$ for each $y \in V_w^{(0)}$. So if

$$g = 1_{F_w} + 1_{F_w^c} f,$$

$\mathcal{E}(g,g) \leq \mathcal{E}(f,f) \leq \#(V_w^{(0)})c_{10}\lambda^{-1} \leq c_{11}\lambda^{-1}$. As $g(x) = 1$, and $g(z) = 0$ for $z \notin \widetilde{N}_\lambda(x)$, we have for $z \notin \widetilde{N}_\lambda(x)$ that $R(x,z)^{-1} \leq \mathcal{E}(g,g) \leq c_{11}\lambda^{-1}$. So $B_R(x,c_{11}\lambda) \subset N_\lambda(x)$. This proves (8.14), and also that $R\big(x,\widetilde{N}_\lambda(x)^c\big) \geq c_{11}^{-1}\lambda$.

The remaining assertions now follow fairly easily. For $w \in \mathbb{W}_\lambda$ we have $c_{12}\lambda^\alpha \leq \mu(F_w) \leq c_{13}\lambda^\alpha$. As $\widetilde{N}_\lambda(x)$ contains at least one λ–complex, and at most $M^2\#(P)^2$ λ–complexes, we have

$$\mu\big(\widetilde{N}_\lambda(x)\big) \asymp \lambda^\alpha,$$

and using (8.14) this implies (8.15).

If $A \subset B$ then it is clear that $R(x,A) \geq R(x,B)$. So (provided λ is small enough) if $x \in F$ we can find a chain x, y_1, y_2, y_3 where $y_i \in V_\lambda^{(0)}$, $\{y_i,y_{i+1}\}$ is an edge in $\mathbf{E}_\lambda$, $y_3 \notin \widetilde{N}_\lambda(x)$, and x and y are in the same λ–complex. Then $R(x,y_3) \leq c\lambda$ by (7.17), and so, using Lemma 8.8(b) we have $R(x,y_3) \leq c'\lambda$. Thus $R\big(x,\widetilde{N}_\lambda(x)^c\big) \leq R(x,y_3) \leq c'\lambda$ proving the right hand side of (8.16): the left hand side was proved above.

(8.17) follows easily from (8.14) and (8.16). □

Corollary 8.10. *In the metric R, the Hausdorff dimension of F is α, and further*

$$0 < \mathcal{H}_R^\alpha(F) < \infty.$$

Proof. This is immediate from Corollary 2.8 and (8.15). □

Proposition 8.11. *For $x \in F$, $r > 0$ set $\tau(x,r) = T_{B_R(x,r)^c}$. Then*

$$c_1 r^{\alpha+1} \le \mathbb{E}^x \tau(x,r) \le c_2 r^{\alpha+1}, \qquad x \in F, \quad r > 0. \tag{8.18}$$

Proof. Let $B = B_R(x,r)$. Then by Theorem 4.25 and the estimates (8.15) and (8.17)

$$E^x \tau(x,r) \le \mu(B) R(x, B^c) \le c_3 r^{\alpha+1},$$

which proves the upper bound in (8.18).

Let $(X_t^B, t \ge 0)$ be the process X killed at $\tau = T_{B^c}$, and let $g(x,y)$ be the Greens' function for X^B. In view of Theorem 7.19, we can write

$$g(x,y) = \mathbb{E}^x L_\tau^y, \qquad x, y \in F.$$

Then if $f(y) = g(x,y)/g(x,x)$, $f \in \mathcal{D}$ and by the reproducing kernel property of g we have

$$\mathcal{E}(f,f) = g(x,x)^{-2} \mathcal{E}\big(g(x,\cdot), g(x,\cdot)\big) = g(x,x)^{-1},$$

and as in Theorem 4.25 $g(x,x) = R(x,B^c) \ge c_4 r$. By (7.18)

$$|f(x) - f(y)|^2 \le R(x,y)\mathcal{E}(f,f) \le R(x,y)(c_4 r)^{-1} \le \tfrac{1}{4}$$

if $R(x,y) \le \frac{1}{4} c_4 r$. Thus $f(y) \ge \frac{1}{2}$ on $B_R(x, \frac{1}{4} c_4 r)$, and hence

$$\begin{aligned} \mathbb{E}^x \tau &= \int_B g(x,y) \mu(dy) \\ &\ge \tfrac{1}{2} g(x,x) \mu\left(B_R\left(x, \tfrac{1}{4} c_4 r\right)\right) \ge c_5 r^{1+\alpha}, \end{aligned}$$

proving (8.18). □

We have a spectral decomposition of $p(t,x,y)$. Write $(f,g) = \int_F f g \, d\mu$.

Theorem 8.12. *There exist functions $\varphi_i \in \mathcal{D}$, $\lambda_i \ge 0$, $i \ge 0$, such that $(\varphi_i, \varphi_i) = 1$, $0 = \lambda_0 < \lambda_1 \le \cdots$, and*

$$\mathcal{E}(\varphi_i, f) = \lambda_i (\varphi_i, f), \qquad f \in \mathcal{D}.$$

The transition density $p(t,x,y)$ of X satisfies

$$p(t,x,y) = \sum_{i=0}^\infty e^{-\lambda_i t} \varphi_i(x) \varphi_i(y), \tag{8.19}$$

where the sum in (8.19) converges uniformly and absolutely. So p is jointly continuous in (t,x,y).

Proof. This follows from Mercer's Theorem, as in [DaS]. Note that $\varphi_0 = 1$ as $\mathcal{E}$ is irreducible and $\mu(F) = 1$. □

The following is an immediate consequence of (8.19)

Corollary 8.13. *(a) For $x,\, y \in F$, $t > 0$,*

$$p(t,x,y)^2 \le p(t,x,x)p(t,y,y).$$

(b) For each $x, y \in F$

$$\lim_{t\to\infty} p(t,x,y) = 1.$$

Lemma 8.14.

$$p(t,x,y) \ge c_0 t^{-d_s/2}, \qquad 0 \le t \le 1, \quad R(x,y) \le c_1 t^{1/(1+\alpha)}. \tag{8.20}$$

Proof. We begin with the case $x = y$. From Proposition 8.11 and Lemma 3.16 we deduce that there exists $c_2 > 0$ such that

$$\mathbb{P}^x\big(\tau(x,r) \le t\big) \le (1 - 2c_2) + c_3 t r^{-\alpha-1}.$$

Choose $c_4 > 0$ such that $c_3 t r_0^{-\alpha-1} = c_2$ if $r_0 = c_4 t^{1/(1+\alpha)}$. Then

$$\mathbb{P}^x\big(X_t \in B_R(x,r_0)\big) \ge \mathbb{P}^x\big(\tau(x,r_0) \le t\big) \ge c_2.$$

So using Cauchy-Schwarz and the symmetry of p, and writing $B = B_R(x,r_0)$,

$$\begin{aligned}
0 < c_2^2 &\le \Big(\int_B p(t,x,y)\mu(dy)^2\Big) \\
&\le \int_{B(x,r_0)} \mu(dy) \int_B p(t,x,y)p(t,y,x)\mu(dy) \\
&\le \mu\big(B)\big)\ p(2t,x,x) \\
&\le c_5 t^{\alpha/(1+\alpha)} p(2t,x,x).
\end{aligned}$$

Replacing t by $t/2$ we have

$$p(t,x,x) \ge c_0 t^{-d_s/2}.$$

Fix t, x, and write $q(y) = p(t,x,y)$. By (4.16) and (8.5) $\mathcal{E}(q,q) \le c_6 t^{-1-d_s/2}$ for $t \le 1$, so using (7.18), if $R(x,y) \le c_7 t^{1/(1+\alpha)}$ then, as $1 + d_s/2 = (1+2\alpha)/(1+\alpha)$,

$$\begin{aligned}
q(y) &\ge q(x) - |q(x) - q(y)| \\
&\ge c_0 t^{-\alpha/(1+\alpha)} - \big(R(x,y)\mathcal{E}(q,q)\big)^{1/2} \\
&\ge c_0 t^{-\alpha/(1+\alpha)} - \Big(c_7 c_6 t^{-2\alpha/(1+\alpha)}\Big)^{1/2} \\
&= t^{-\alpha/(1+\alpha)}(c_0 - (c_7 c_6)^{1/2}).
\end{aligned}$$

Choosing c_7 suitably gives (8.20). □

We can at this point employ the chaining arguments used in Theorem 3.11 to extend these bounds to give upper and lower bounds on $p(t,x,y)$. However, as R is not in general a geodesic metric, the bounds will not be of the form given in Theorem 3.11. The general case is given in a paper of Hambly and Kumagai [HK2], but since the proof of Theorem 3.11 does not use the geodesic property for the upper bound we do obtain:

Theorem 8.15. *The transition density* $p(t,x,y)$ *satisfies*

$$p(t,x,y) \le c_1 t^{-\alpha/(1+\alpha)} \exp\left(-c_2 \left(R(x,y)^{1+\alpha}/t\right)^{1/\alpha}\right). \tag{8.21}$$

Note. The power $1/\alpha$ in the exponent is not in general best possible.

Theorem 8.16. *Suppose that there exists a metric* ρ *on* F *with the midpoint property such that for some* $\theta > 0$

$$c_1 \rho(x,y)^\theta \le R(x,y) \le c_2 \rho(x,y)^\theta \; x,y \in F. \tag{8.22}$$

Then if $d_w = \theta(1+\alpha)$, $d_f = \alpha\theta$, (F,ρ,μ) *is a fractional metric space of dimension* d_f, *and* X *is a fractional diffusion with indices* d_f, d_w.

Proof. Since $B_\rho(x,(r/c_2)^\theta) \subset B_R(x,r) \subset B_\rho(x,(r/c_1)^\theta)$, it is immediate from (8.15) that (F,ρ) is a $FMS(d_f)$. Write $\tau_\rho(x,r) = \inf\{t : X_t \notin B_\rho(x,r)\}$. Then from (8.18) and (8.22)

$$cr^{\theta(1+\alpha)} \le \mathbb{E}^x \tau_\rho(x,r) \le c_2 r^{\theta(1+\alpha)}.$$

So, by (8.10) and (8.20), X satisfies the hypotheses of Theorem 3.11, and so X is a $FD(d_f,d_w)$. □

Remark. Note that in this case the estimate (7.20) on the Hölder continuity of $u_\lambda(x,y)$ implies that

$$|u_\lambda(x,y) - u_\lambda(x',y)| \le cR(x,x')^{\frac{1}{2}} \le c' \rho(x,x')^{\theta/2}, \tag{8.23}$$

while by Theorem 3.40 we have

$$|u_\lambda(x,y) - u_\lambda(x',y)| \le c\rho(x,x')^\theta. \tag{8.24}$$

The difference is that (8.23) used only the fact that $u_\lambda(.,y) \in \mathcal{D}$, while the proof of (8.24) used the fact that it is the λ-potential density.

Diffusions on nested fractals.

We conclude by treating briefly the case of nested fractals. Most of the necessary work has already been done. Let $(F,(\psi_i))$ be a nested fractal, with length, mass, resistance and shortest path scaling factors L, M, ρ, γ. Recall that in this context we take $r_i = 1$, $\theta_i = 1/M$, $1 \le i \le M$, and $\mu = \mu_\theta$ for the measure associated with θ. Write $d = d_F$ for the geodesic metric on F defined in Section 5.

Lemma 8.17. *Set* $\theta = \log\rho/\log\gamma$. *Then*

$$c_1 d(x,y)^\theta \le R(x,y) \le c_2 d(x,y)^\theta, \quad x,y \in F. \tag{8.24}$$

Proof. Let $\lambda \in (0,1)$. Since all the r_i are equal, $\widetilde{N}_\lambda(x)$ is a union of n-complexes, where $\rho^{-n} \le \lambda \le \rho^{-n+1}$. So by Theorem 5.43 and Proposition 8.8, since $\gamma^{-n} = (\rho^{-n})^\theta$,

$$y \in \widetilde{N}_\lambda(x) \text{ implies that } R(x,y) \le c_1\lambda, \text{ and } d(x,y) \le c_2\lambda^\theta, \tag{8.26}$$

(8.27) $$y \notin \widetilde{N}_\lambda(x) \text{ implies that } R(x,y) \ge c_3\lambda, \quad \text{and } d(x,y) \ge c_4\lambda^\theta.$$

The result is immediate from (8.26) and (8.27). □

Applying Lemma 8.17 and Theorem 8.15 we deduce:

Theorem 8.18. *Let F be a nested fractal, with scaling factors L, M, ρ, γ. Set*

$$d_f = \log M/\log\gamma, \quad d_w = \log M\rho/\log\gamma.$$

Then (F, d_F, μ) is a fractional metric space of dimension d_f, and X is a $FD(d_f, d_w)$. In particular, the transition density $p(t,x,y)$ of X is jointly continuous in (t,x,y) and satisfies

(8.28) $$c_1 t^{-d_f/d_w} \exp\left(-c_2\left(d(x,y)^{d_w}/t\right)^{1/(d_w-1)}\right) \le p(t,x,y) \le c_3 t^{-d_f/d_w} \exp\left(-c_4\left(d(x,y)^{d_w}/t\right)^{1/(d_w-1)}\right).$$

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