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Logarithmic multifractal spectrum of stable occupation measure

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Abstract

For a stable subordinator Y_t of index α, $0<\alpha<1$, the occupation measure

$$\mu(A)=|\{t\in[0,1]: Y_t\in A\}|$$

is known to have (with probability 1) the property that

$$\liminf_{r\downarrow 0}\frac{\ln\mu(x-r,x+r)}{\ln r}=\alpha,\quad \forall x\in Y[0,1].$$

In order to obtain an interesting spectrum for the large values of $\mu(x-r,x+r)$, we consider the set

$$B_\theta=\left\{x\in Y[0,1]: \limsup_{r\downarrow 0}\frac{\mu(x-r,x+r)}{c_\alpha r^\alpha(\ln(1/r))^{1-\alpha}}=\theta\right\},$$

where c_α is a suitable constant. It is shown that $B_\theta=\emptyset$ for $\theta>1$, and $B_\theta\neq\emptyset$ for $0\leqslant\theta\leqslant 1$; moreover, $\dim B_\theta=\operatorname{Dim} B_\theta=\alpha(1-\theta^{1/(1-\alpha)})$.

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1. Introduction

In this paper, we continue to study the detailed structure of the random probability measure on $\Re$ defined by ($|\cdot|$ denotes the Lebesgue measure)

$$\mu(A)=|\{t\in[0,1]: Y_t\in A\}|, \tag{1.1}$$

where Y_t is a nice version of a stable subordinator of index α, $0<\alpha<1$. Thus, $\mu(A)$ is the length of time(up to $t=1$) spent in A by the process Y_t. The topological support of

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μ is the random set $\overline{Y[0,1]}$. In a recent paper by Hu and Taylor (1997), they obtained the ordinary multifractal structure of μ. With probability 1, for μ-a.e. x we have

$$\lim_{r\downarrow 0} \frac{\ln \mu(x-r,x+r)}{\ln r} = \alpha, \tag{1.2}$$

but there are exceptional sets where (1.2) is false. In fact, it was shown in that paper that with probability 1

$$\liminf_{r\downarrow 0} \frac{\ln \mu(x-r,x+r)}{\ln r} = \alpha \quad \forall x \in Y[0,1]; \tag{1.3}$$

while, if we define

$$C_\beta = \left\{ x \in Y[0,1] : \limsup_{r\downarrow 0} \frac{\ln \mu(x-r,x+r)}{\ln r} \geqslant \beta \right\},$$

and

$$D_\beta = \left\{ x \in Y[0,1] : \limsup_{r\downarrow 0} \frac{\ln \mu(x-r,x+r)}{\ln r} = \beta \right\},$$

then $C_\beta = \emptyset$ for $\beta > 2\alpha$, $D_\beta \neq \emptyset$ for $\alpha \leqslant \beta \leqslant 2\alpha$, and in the latter case $\dim C_\beta = \dim D_\beta = 2\alpha^2/\beta - \alpha$.

The purpose of the present paper is to look in more detail at the large values of $\mu(x-r,x+r)$ as $r \downarrow 0$. The result (1.3) tells that, in terms of powers of r, the only relevant power is α. To obtain an interesting decomposition, we consider the sets

$$A_\theta = \left\{ x \in Y[0,1] : \limsup_{r\downarrow 0} \frac{\mu(x-r,x+r)}{c_\alpha r^\alpha (\ln(1/r))^{1-\alpha}} \geqslant \theta \right\},$$

$$B_\theta = \left\{ x \in Y[0,1] : \limsup_{r\downarrow 0} \frac{\mu(x-r,x+r)}{c_\alpha r^\alpha (\ln(1/r))^{1-\alpha}} = \theta \right\},$$

and show the existence of a suitable constant c_α (see Theorem 5.1) which ensures that, with probability 1, $A_\theta = \emptyset$ for $\theta > 1$, while $B_\theta \neq \emptyset$ for $0 \leqslant \theta \leqslant 1$. When the sets are non-empty, we compute their Hausdorff (and packing) dimensions for each value of θ.

There are several papers in the literature concerning both "fast" and "slow" points for Brownian motions or stable processes. There is a sense in which our exceptional sets A_θ and B_θ can be thought as the "two-sided slow"points for the subordinator Y_t, since large values of $\mu(x-r,x+r)$ imply larger than usual first passage times for Y_{t+h} and Y_{t-h}. The uniform result for "one-sided slow" points of Y_t is due to Hawkes (1971). However we do not make direct use of his result; instead we include a "two-sided" version of his result as a corollary of our analysis. We believe our results to be true for a general transient stable process of index α. However there are independence problems in the construction of the time sets in Section 4 (the construction is crucial for the lower bound of dimensions), so we do not claim to have proved the results beyond the subordinator case. We should remind the reader that, for $0 < \alpha \leqslant \frac{1}{2}$ our occupation measure can be thought of as the local time at zero of a strictly stable process of index

β, with $\beta = 1/(1-\alpha)$; in particular, for $\alpha = \frac{1}{2}$ the measure μ is the Brownian local time at 0.

Our paper is organized as follows. In Section 2 we collect definitions and the main probability estimates used in the later proofs. In Section 3 we obtain the range of θ for which A_θ is non-empty and calculate upper bounds for dimensions by standard first moment arguments. In Section 4 we give a construction of a Cantor-like random time set T_θ for which each point in $Y(T_\theta)$ is "two-sided slow" for Y_t. In the final Section 5, we bring together the results to complete the analysis. We will use the notations c or c' to denote a finite positive constant, whose value is unimportant (perhaps unknown) and may change from place to place. Special constants with known values will be denoted by $c_1, c_2, \ldots$. The notation $x \sim y$ for two positive numbers x, y means that $x/y \to 1$.

2. Preliminaries

We will only be concerned with an increasing stable process Y_t taking values in $\Re$. This process has index α satisfying $0 < \alpha < 1$. It has stationary independent increments and we normalize it so that the Laplace transform

$$E \exp\{-u(Y_{t+h} - Y_t)\} = \mathrm{e}^{-hu^\alpha}, \quad u > 0 \tag{2.1}$$

holds for each $t \in \Re$ and $h > 0$. As usual we can assume that Y_t is right continuous with left limits everywhere. The process satisfies the scaling property that, for any constant $u > 0$, $u^{-1/\alpha}(Y_{ut} - Y_0)$ is another version of $Y_t - Y_0$. Note that we do not tie down Y_0. We define two associated processes from each fixed t_0 by

$$Y^1_{t_0}(t) = Y_{t_0+t} - Y_{t_0}, \qquad Y^2_{t_0}(t) = Y_{t_0} - Y_{t_0-t}.$$

Both $Y^1_{t_0}$ and $Y^2_{t_0}$ are versions of Y, with $Y^1_{t_0}(0) = Y^2_{t_0}(0) = 0$, and they are independent. In looking at the local behaviour of Y_t near $x = Y_{t_0}$, we will use Y^1, resp. Y^2, to describe the behaviour of Y_t for $t > t_0$, resp. $t < t_0$. We will consider for the moment the σ-finite occupation measure $\tilde{\mu}$ instead of μ,

$$\tilde{\mu}(A) = |\{t \in \Re : Y_t \in A\}|.$$

Note that μ is the restriction of $\tilde{\mu}$ to the interval $[Y_0, Y_1]$, so that, for $0 < t < 1$, the local behaviour of $\tilde{\mu}$ and μ are identical at $x = Y_t$. The usage of $\tilde{\mu}$ rather than μ is to avoid end effects when $(x-r, x+r) \not\subset [Y_0, Y_1]$. Now, since the path is monotone increasing, for $x = Y_{t_0}$

$$\tilde{\mu}(x, x+r) = \inf\{s : Y_{t_0+s} \geqslant x + r\},$$

hence

$$\{\tilde{\mu}(x, x+r) \geqslant u\} = \{Y^1_{t_0}(u) \leqslant r\}. \tag{2.2}$$

Similar relation holds for $\tilde{\mu}(x-r, x)$ and $Y^2_{t_0}$. By the scaling property mentioned above, the probability

$$P\{\tilde{\mu}(x, x+r) \geqslant r^\alpha \lambda\}$$

is the same for all $r > 0$, for each $t_0 \in \Re$ and $\lambda > 0$.

Lemma 2.1. *If* $\tilde{\mu}$ *is the* σ*-finite occupation measure of a stable subordinator of index* α*, then, for* $x=Y_{t_0}$,

$$P\{\tilde{\mu}(x,x+r)\geqslant r^{\alpha}\lambda\}\sim c_1\lambda^{-1/2(1-\alpha)}\exp(-c_2\lambda^{1/(1-\alpha)}),\quad \text{as } \lambda\uparrow\infty,$$

where

$$c_1=[2\pi(1-\alpha)\alpha^{\alpha/2(1-\alpha)}]^{-1/2},\qquad c_2=(1-\alpha)\alpha^{\alpha/(1-\alpha)}.$$

Proof. In view of (2.2), the assertion is equivalent to Hawkes (1971, Lemma 1). □

Now, $\tilde{\mu}(x-r,x)$ and $\tilde{\mu}(x,x+r)$ are independent with the same distribution, so that $\tilde{\mu}(x-r,x+r)$ is the sum of two independent random variables for which we know the asymptotics for the large tail. It is ought to be possible to deduce the asymptotics for the large tail of $\tilde{\mu}(x-r,x+r)$, but we content ourselves with the estimate which we require for the subsequent analysis.

Lemma 2.2. *If* $\tilde{\mu}$ *is the* σ*-finite occupation measure of a stable subordinator of index* α*, then, for a given* $\varepsilon>0$*, there exist* λ_1 *and* λ_2 *such that, for all* $r>0$,

(i) *for any* $y\in\Re$,
$P\{\tilde{\mu}(y-r,y+r)\geqslant r^{\alpha}\lambda\}\leqslant c_4(1+\varepsilon)\lambda^{-1/2(1-\alpha)}\exp(-c_3\lambda^{1/(1-\alpha)})$
$\times P\{Y \text{ hits } [y-r,y+r]\}$,
$\lambda>\lambda_1$;

(ii) *for* $x=Y_t$*, some* t,
$P\{\tilde{\mu}(x-r,x+r)\geqslant r^{\alpha}\lambda\}\geqslant c_5(1-\varepsilon)\lambda^{-1/(1-\alpha)}\exp(-c_3\lambda^{1/(1-\alpha)})$,
$\lambda>\lambda_2$; *where*
$c_3=2^{1-1/(1-\alpha)}c_2,\quad c_4=2^{-\alpha/2(1-\alpha)}c_1,\quad c_5=2^{-(1+1/(1-\alpha))}c_1.$

Proof. (i) Conditioning on $Y(0)\in(y-r,y+r)$ we apply Lemma 2.1 forwards and backwards from $Y(0)$. The largest value of the product comes from $Y(0)=y$ and Y then certainly hits $[y-r,y+r]$. If $Y(0)\leqslant y-r$ then we run the process forwards until it hits $[y-r,y+r]$ at the time T, which is a Markov time. By the strong Markov property, we can apply Lemma 2.1 to the distribution of $\tilde{\mu}(Y_T,x+r)$ to obtain the assertion (since $y+r-Y_T\leqslant 2r$). A similar argument works for $Y(0)\geqslant y+r$, if we run the process backwards.
(ii) We apply Lemma 2.1, using the inclusion

$$\{\tilde{\mu}(x-r,x+r)\geqslant r^{\alpha}\lambda\}\supset[\{\tilde{\mu}(x-r,x)\geqslant \tfrac{1}{2}r^{\alpha}\lambda\}\cap\{\tilde{\mu}(x,x+r)\geqslant\tfrac{1}{2}r^{\alpha}\lambda\}].\qquad \square$$

In the formulation of Hawkes (1971, Theorem 2), he used the local time of a strictly stable process of index β, $1<\beta\leqslant 2$. This restricts the corresponding subordinator to be of index α, $0<\alpha\leqslant\frac{1}{2}$. However, his proof is valid for $0<\alpha<1$. We restate his display (6) as

Lemma 2.3. *If* μ *is the occupation measure of a stable subordinator of index* α*, then with probability* 1

$$\lim_{r\downarrow 0}\ \sup_{x\in Y[0,1]}\frac{\mu(x,x+r)}{r^{\alpha}(\ln(1/r))^{1-\alpha}}=\left(\frac{c_2}{\alpha}\right)^{\alpha-1},$$

where c_2 *is given in Lemma* 2.1.

In our analysis, we want more detailed behaviour about the size of the exceptional B_θ, and will obtain a "two-sided" version of the above Lemma 2.3 as a corollary (see Corollary 5.2). For the moment, we observe that Lemma 2.3 tells us that

$$\psi(r)=r^\alpha(\ln(1/r))^{1-\alpha}$$

is the correct order of the magnitude for the uniform result, as opposed to

$$\phi(r)=r^\alpha(\ln\ln(1/r))^{1-\alpha},$$

which gives the local upper asymptotic behaviour of $\mu(x-r,x+r)$, at $\mu-a.e.\ x$, and the correct Hausdorff measure function for the sample paths, since with probability 1

$$\phi-mY[0,1]=c;$$

see Taylor (1986).

Our results in this paper have the same flavour as those in Orey and Taylor (1974) for the "fast" points on a Brownian path. For the relation of our results to multifractal formalism, we refer the reader to Hu and Taylor (1997). In the present paper, we are mainly concerned with the set of those x in the range $Y[0,1]$ where the upper asymptotic growth rate of $\mu(x-r,x+r)$ is much larger than $\phi(r)$, the typical rate. The uniform result (1.3) implies that, for each $\varepsilon>0$,

$$\lim_{r\downarrow 0}\frac{\mu(x-r,x+r)}{r^{\alpha-\varepsilon}}=0,\quad \forall x\in Y[0,1],$$

but we want more precise information, using the measure function $\psi(r)$.

Detailed definitions and properties of the dimension indices defined for subset $A\subset\Re^d$ have been described in many papers, including Taylor (1986), which is a convenient reference. We use the notations established in that paper, and note that

$$0\leqslant\dim A\leqslant\operatorname{Dim}A\leqslant\Delta(A)\leqslant 1$$

for any $A\subset\Re$, where $\dim A$ denotes the Hausdorff dimension, $\operatorname{Dim}A$ the packing dimension, and $\Delta(A)$ the upper Minkowski dimension. We will not consider $\Delta(A_\theta)$ in the sequel, but note that, since each A_θ is dense in $Y[0,1]$ whenever it is not empty, it follows that $\Delta(A_\theta)=\Delta(Y[0,1])=\alpha$ for $0\leqslant\theta\leqslant 1$.

3. Upper bound for dimension indices

For notational convenience we will not work directly with A_θ in Sections 3 and 4. We temporarily fix $\theta>0$, $0<\varepsilon<1$, and $a>1$ in the following arguments. We consider the random set

$$\begin{aligned}E_{\theta,\varepsilon}=\ &\{x\in Y[0,1]:\ \text{there exist } r_n=r_n(x,\varepsilon)\downarrow 0,\ \text{such that}\\ &\mu(x-r_n,x+r_n)>\theta(1-\varepsilon)r_n^\alpha(\ln(1/r_n))^{1-\alpha}\quad\forall n\},\end{aligned}$$

and we will prove an exact upper bound for $\dim E_{\theta,\varepsilon}$. We may assume that $Y_0=0$; then it suffices to consider $x:0<x<1$. For positive integers i,j,k, we set

$$x_{i,j,k}=(i+j/k)a^{-k}$$

and

$$I_{i,j.k} = (x_{i,j,k} - (1+1/k)a^{-k}, x_{i,j,k} + (1+1/k)a^{-k}),$$

where $k=1,2,\ldots,\ i=1,\ldots,[a^k],\ j=1,\ldots,k$. Let $\mathscr{C}_k$ be the class of all those $I_{i.j.k}$ such that

$$\mu(I_{i,j.k}) > \frac{\theta}{a^\alpha}(1-\varepsilon)a^{-k\alpha}(\ln a^k)^{1-\alpha}. \tag{3.1}$$

When k is large enough, $\mathscr{C}_k$ is a cover of $E_{\theta,\varepsilon}\cap[0,1]$. Indeed, whenever $a^{-(k+1)} < r_n < a^{-k}$ and x satisfies $x_{i,j-1,k} < x < x_{i,j.k}$(so that $x\in I_{i,j,k}$), then

$$\mu(x-r_n, x+r_n) > \theta(1-\varepsilon)r_n^\alpha(\ln(1/r_n))^{1-\alpha}$$

will imply that $I_{i,j,k}$ has property (3.1). Now let N_k denote the number of $\mathscr{C}_k$, and we estimate the expectation of the random variable N_k as follows. Let $T_{i,j,k}$ be the hitting time for $Y(t)\in I_{i,j,k}$. By Lemma 2.2(i) and that $\mu\leqslant\tilde{\mu}$, we have

$$EN_k \leqslant \sum_{i,j} C_k \cdot P\{T_{i,j,k} < \infty\},$$

where

$$C_k = c_4(1+\varepsilon)a^{-k(1+(1/k))^{-\alpha/(1-\alpha)}c_3a^{-\alpha/(1-\alpha)}(1-\varepsilon)^{1/(1-\alpha)}\theta^{1/(1-\alpha)}}.$$

By Taylor (1967, Lemma 4),

$$\sum_{i,j} P\{T_{i,j,k} < \infty\} \leqslant c\sum_{i=2}^{[a^k]} k\left(\frac{(1+1/k)a^{-k}}{(i-1)a^{-k}}\right)^{1-\alpha} = cka^{k\alpha}.$$

Now, we impose the assumption

$$c_3\theta^{1/(1-\alpha)} < \alpha. \tag{3.2}$$

Then, for all $a>1,\ 0<\varepsilon<1$, and $\delta>0$,

$$\Delta_{a,\varepsilon,\delta} := \alpha - c_3a^{-\alpha/(1-\alpha)}(1-\varepsilon)^{1/(1-\alpha)}\theta^{1/(1-\alpha)} + \delta > \delta.$$

Then, by Chebyshev's inequality (first moments) we have, for $\delta'\leqslant\delta$,

$$\sum_k P\{N_k > a^{k\Delta_{a,\varepsilon,\delta}}\} \leqslant c\sum_k a^{-k\delta'} < \infty,$$

so that with probability 1 there exists $k_0(\omega)$ such that

$$N_k \leqslant a^{k\Delta_{a,\varepsilon,\delta}} \quad \forall k\geqslant k_0,$$

consequently, $\dim E_{\theta,\varepsilon}\leqslant\Delta_{a,\varepsilon,\delta}$.

Let

$$E_\theta := \bigcap_{\varepsilon>0} E_{\theta,\varepsilon}.$$

Then, letting $a\downarrow 1$, and ε, $\delta\downarrow 0$ through countably many positives, we have

$$\dim E_\theta \leqslant \alpha - c_3\theta^{1/(1-\alpha)} = \alpha\left(1-\frac{c_3}{\alpha}\theta^{1/(1-\alpha)}\right). \tag{3.3}$$

We remark that the above arguments indeed give the same upper bound estimate for $\operatorname{Dim} E_\theta$. Now, if we suppose, as opposed to (3.2), that

$$c_3\theta^{1/(1-\alpha)} > \alpha, \tag{3.4}$$

then we can choose $a>1$ and $\delta>0$ so that $\Delta_{a,\varepsilon,\delta}<0$ (for each fixed ε). Then, again using a first moment argument, we can prove that, with probability 1, there exists $k_1(\omega)$ such that the class $\mathscr{G}_k$ of all those possible $I_{i,j,k}$ which intersect that $Y[0,1]$ is void if $k\geqslant k_1$. The emptiness of $\mathscr{G}_k$ will imply that $E_{\theta,\varepsilon}=\emptyset$. Thus $E_\theta=\emptyset$ under (3.4).

4. Lower bound: a Cantor-like set construction

We aim to construct a Cantor-like random set $T_\theta\in[0,1]$, where θ satisfies (3.2), and for which if $x\in Y(T_\theta)$,

$$\limsup_{r\downarrow 0}\frac{\mu(x-r,x+r)}{r^\alpha(\ln(1/r))^{1-\alpha}}\geqslant\theta. \tag{4.1}$$

For this purpose, we define a sequence $r_n\downarrow 0$ inductively by

$$r_1=r,\quad 0<r<1,\quad r_{n+1}=\exp(-r_n^{1/(\alpha-1)}). \tag{4.2}$$

We also set

$$\eta_n=\tfrac{1}{2}\theta r_n^\alpha(\ln(1/r_n))^{1-\alpha},$$

$$\tilde\eta_n=\frac{\eta_n}{(\ln(1/r_n))^2}=\frac{1}{2}\theta r_n^\alpha(\ln(1/r_n))^{-(1+\alpha)},$$

$$t_{i,n}=i\eta_n,\quad i=1,3,5,\ldots,$$

$$x_{i,n}=Y(t_{i,n}),$$

$$I_{i,n}=[t_{i,n}-\eta_n,t_{i,n}+\eta_n],$$

$$\tilde I_{i,n}=[t_{i,n}-\tilde\eta_n,t_{i,n}+\tilde\eta_n].$$

We denote the class of all $\tilde I_{i,n}$ by $\mathscr{F}_n$. We call $I_{i,n}$, or equivalently $\tilde I_{i,n}$, a type-S interval ("S" stands for "slow") for the path $Y.(\omega)$, if the associated Y^j(see Section 2) satisfy

$$Y^j_{t_{i,n}}(\eta_n)-Y^j_{t_{i,n}}(\tilde\eta_n)\leqslant\frac{r_n}{a},\quad j=1,2,$$

where $a>1$ will be determined later; and in addition,

$$Y^1_{t_{i,n}}(\tilde\eta_n)+Y^2_{t_{i,n}}(\tilde\eta_n)\leqslant\left(1-\frac{1}{a}\right)r_n. \tag{4.4}$$

Since $Y.$ is monotone increasing, condition (4.4) implies that, for each $t\in\tilde I_{i,n}$, $x=Y_t$,

$$|x_{i,n}-x|\leqslant\left(1-\frac{1}{a}\right)r_n;$$

so that

$$Y_{t+\eta_n} - Y_t \leqslant \frac{r_n}{a} + \left(1 - \frac{1}{a}\right) r_n = r_n.$$

It follows that

$$\mu(x, x + r_n) \geqslant \eta_n.$$

A similar argument shows that

$$\mu(x - r_n, x) \geqslant \eta_n,$$

so we have proved that

$$\mu(x - r_n, x + r_n) \geqslant 2\eta_n \quad \forall x \in Y(\tilde{I}_{i,n}). \tag{4.5}$$

Now for any fixed $a > 1$, the distribution of $Y^1_{t_{i,n}}(\tilde{\eta}_n) + Y^2_{t_{i,n}}(\tilde{\eta}_n)$ is the same as that of $Y_{t_{i,n}+2\tilde{\eta}_n} - Y_{t_{i,n}}$, and

$$\tilde{\eta}_n^{1/\alpha} = o(r_n) \quad \text{as } n \to \infty.$$

Hence,

$$P\{(4.4)\} \to 1 \quad \text{as } n \to \infty.$$

We define

$$p_n = p_{i,n} = P\{\omega\colon \tilde{I}_{i,n} \text{ is type} - S \text{ for the path } Y_{\cdot}(\omega)\},$$

and obtain a lower bound for p_n as follows

Lemma 4.1. *A lower bound for p_n is given by*

$$p_n \geqslant \tfrac{1}{2} c_4 a^{-\alpha/(1-\alpha)} \theta^{-1/(1-\alpha)} \left(\ln\left(\frac{1}{r_n}\right)\right)^{-1} \left(\frac{1}{r_n}\right)^{-c_3 a^{\alpha/(1-\alpha)} \theta^{1/(1-\alpha)}} \qquad \textit{for } n > n_0.$$

Proof. By the property of stationary independent increments,

$$\begin{aligned} p_n &= P\left\{Y^j_{t_{i,n}}(\eta_n - \tilde{\eta}_n) < \frac{r_n}{a},\ j = 1, 2\right\} \times P\{(4.4)\} \\ &= P\left\{\mu\left(x_{i,n}, x_{i,n} + \frac{r_n}{a}\right) > \eta_n - \tilde{\eta}_n\right\}^2 \times P\{(4.4)\} \\ &= P\left\{\mu(x_{i,n}, x_{i,n} + r_n) > a^\alpha(\eta_n - \tilde{\eta}_n)\right\}^2 \times P\{(4.4)\} \\ &\geqslant P\left\{\mu(x_{i,n}, x_{i,n} + r_n) > a^\alpha \eta_n\right\}^2 \times P\{(4.4)\}. \end{aligned}$$

The estimate in Lemma 2.1 completes the proof. □

Remark. Similar arguments yield

$$p_n \leqslant \left(\frac{1}{r_n}\right)^{-c_3(1-\varepsilon) a_1^{\alpha/(1-\alpha)} \theta^{1/(1-\alpha)}} \qquad \text{for } n \geqslant n_1,$$

where $0 < \varepsilon < 1$ and $a_1 < a$.

For each $\tilde{I} \in \mathscr{F}_n$, let $M_{n+1}(\tilde{I})$ denote the number of those type-S intervals from $\mathscr{F}_{n+1}$ which are contained in $\tilde{I}$. Note that $M_{n+1}(\tilde{I})$ is a binomial random variable. Set

$$U_n := \left[\frac{|\tilde{I}_{0,\,n-1}|}{8|I_{0,\,n}|} p_n \right],$$

where $[\,\cdot\,]$ denotes the greatest integer part.

Lemma 4.2. *Assume that* (3.2) *holds, namely* $c_3\theta^{1/(1-\alpha)} < \alpha$. *Then with probability* 1 *there exists* $n_2 = n_2(\omega)$ *such that*

$$M_{n+1}(\tilde{I}) \geqslant U_{n+1} \quad \forall \tilde{I} \in \mathscr{F}_n \quad \forall n \geqslant n_2.$$

Proof. The definition of U_{n+1} and the fact that $M_{n+1}(\tilde{I})$ is binomially distributed together assert that $EM_{n+1}(\tilde{I}) \geqslant 2U_{n+1}$ and $\operatorname{Var} M_{n+1}(\tilde{I}) \leqslant cU_{n+1}$. We remark that $I_{i,n+1}$ and $I_{j,n+1}$ are disjoint time intervals when $i \neq j$ so that the behaviour of Y_t on these intervals is independent and $M_{n+1}(\tilde{I})$ counts independent events. By the Chebyshev's inequality (second moments) we have

$$\begin{aligned}
&\sum_n P\left\{ M_{n+1}(\tilde{I}) < U_{n+1} \quad \text{for some } \tilde{I} \in \mathscr{F}_n \right\} \\
&\quad \leqslant c \sum_n \sum_{\tilde{I} \in \mathscr{F}_n} \frac{\operatorname{Var} M_{n+1}(\tilde{I})}{(EM_{n+1}(\tilde{I}))^2} \\
&\quad \leqslant c \sum_n \eta_n^{-1} |\tilde{I}_{0,n+1}| |I_{0,n}|^{-1} p_{n+1}^{-1}.
\end{aligned}$$

By Lemma 4.1, the definition of $\tilde{I}_{i,n}$, and defining relations between η_{n+1}, r_{n+1}, r_n, we can see that whether the series in the above display converges depends on

$$r_{n+1}^{\alpha - c_3 a^{\alpha/(1-\alpha)} \theta^{1/(1-\alpha)}}.$$

Under the assumption (3.2) we can find a suitable $a > 1$ so that the power in the above is positive. Then we can apply Borel–Cantelli Lemma to obtain the assertion. □

We also note that $U_{n+1} \uparrow \infty$ as $n \uparrow \infty$. Now, we construct a Cantor-like random T_θ, under assumption (3.2), in a procedure adapted from Hu and Taylor (1997, Section 5). We start with $\mathscr{F}_{n_2}$, where n_2 is the index in Lemma 4.2. For each $\tilde{I} \in \mathscr{F}_{n_2}$ we pick up U_{n_2+1} type-S intervals from $\mathscr{F}_{n_2+1}$ which are contained in $\tilde{I}$. Then we form a union F_{n_2+1}. Next, for each member $\tilde{I}'$ in this F_{n_2+1} we pick U_{n_2+2} type-S intervals from $\mathscr{F}_{n_2+2}$ which are contained in $\tilde{I}'$; then we form a second union F_{n_2+2}. We proceed inductively and have a chain $F_{n_2+1} \supset F_{n_2+2} \supset \cdots$, then we set

$$T_\theta = \bigcap_{n > n_2} F_n.$$

For each $x \in Y(T_\theta)$, (4.5) shows that (4.1) holds. We can also construct a Borel measure v supported by T_θ by a procedure adapted from Hu and Taylor (1997, Section 5) too. We define v_1 to be a multiple of Lebesgue measure on F_{n_2+1} such that $v_1(F_{n_2+1}) = 1$ and each member in F_{n_2+1} has the equal v_1-mass. Then we define v_2 from v_1, keeping the total v_2-mass on F_{n_2+2} to be 1 and each member in F_{n_2+2} to have equal v_2-mass too. The limiting mesaure $v(I) = \lim_{i\to\infty} \gamma_i(I)$ is a Borel measure supported by T_θ, and $v(\tilde{I}) = v_j(\tilde{I})$ whenever $\tilde{I}$ is a member in F_{n_2+i} and $j \geqslant i$.

Under (3.2), the index

$$\gamma := \frac{c_3}{\alpha}\theta^{1/(1-\alpha)} < 1.$$

Let the measure function $h(s)$ be defined by

$$h(s) = s^{1-\gamma}(\ln(1/s))^b,$$

where $b > 1$ will be determined later. We will prove that with probability 1 the Hausdorff measure

$$h - m(T_\theta) = \infty. \tag{4.6}$$

We prove the assertion by showing that the energy integral

$$I_\gamma := \int\int_{T_\theta \times T_\theta} \frac{v(\mathrm{d}s)v(\mathrm{d}t)}{|s-t|^{1-\gamma}(\ln 1/(s-t))^b}$$

is finite a.s. For this purpose, we set

$$A = \bigcup_{n>n_2} \bigcup_{j=1}^{m_n} G_{j,n},$$

where

$$G_{j,n} = \{(s,t) \in T_\theta \times T_\theta \colon 2^j \eta_n < |s-t| \leqslant 2^{j+1}\eta_n\}$$

and

$$2^{m_n}\eta_n \leqslant \eta_{n-1} \leqslant 2^{m_n+1}\eta_n.$$

In the above energy integral, it suffices to consider the integration over A. For each $\tilde{I} \in \mathscr{F}_n$, the expected number of type-S $\tilde{J} \in \mathscr{F}_n$ satisfying $2^j\eta_n < |s-t| \leqslant 2^{j+1}\eta_n$ for all $s \in \tilde{I}$, $t \in \tilde{J}$ is at most $[2^j p_n] + 1$. This means that if K_n is the number of all $\tilde{I}_{i,n}$ in $\mathscr{F}_n$, $n > n_2$ then the expected number of squares $\tilde{I} \times \tilde{J}$, with both $\tilde{I}$, $\tilde{J}$ type-S intervals in $\mathscr{F}_n$, needed to cover $G_{j,n}$ is not more than $K_n \cdot ([2^{j+1} p_{0,n}] + 1)$. Hence,

$$E\int\int_{G_{j,n}} \frac{v(\mathrm{d}s)v(\mathrm{d}t)}{|s-t|^{1-\gamma}(\ln 1/(s-t))^b} \leqslant \frac{c}{K_n^2} \cdot \frac{K_n \cdot 2^{j+1} p_n}{(2^j\eta_n)^{1-\gamma}(\ln(2^j\eta_n)^{-1})^b}.$$

We observe that

$$\begin{aligned}\sum_{j=1}^{m_n} \frac{2^{j\gamma}}{(\ln \eta_n^{-1} - j)^b} &\leqslant \left(\sum_{j=1}^{m_n} 2^{2j\gamma}\right)^{1/2} \left(\sum_{j=1}^{m_n} \frac{1}{(\ln \eta_n^{-1} - j)^{2b}}\right)^{1/2}\\ &\leqslant c\left(\frac{\eta_{n-1}}{\eta_n}\right)^\gamma (\ln \eta_n^{-1} - 1)^{(1-2b)/2}.\end{aligned}$$

Moreover,

$$K_n \geqslant U_n \geqslant c\eta_{n-1}(\ln(1/r_{n-1}))^{-2}\eta_n^{-1} p_n.$$

Therefore,

$$E\int\int_{\bigcup_{j=1}^{m_n} G_{n,j}} \frac{v(\mathrm{d}s)v(\mathrm{d}t)}{|s-t|^{1-\gamma}(\ln 1/(s-t))^b}$$

$$\leqslant c\eta_{n-1}^{\gamma-1}(\ln \eta_n^{-1}-1)^{(1-2b)/2}(\ln(1/r_{n-1}))^2. \tag{4.7}$$

Using the defining relations between η_n, η_{n-1}, r_n and r_{n-1}, we see that the right-hand side of (4.7) is

$$\leqslant c r_{n-1}^{\alpha(\gamma-1)} r_{n-1}^{\frac{2b-1}{2(1-\alpha)}}(\ln(1/r_{n-1}))^{2-(1-\alpha)(1-\gamma)}.$$

We can choose $b>1$ large enough so that the power of r_{n-1} in the above display is positive, say $\delta>0$. Then, for a fixed $\delta'<\delta$

$$E\int\int_A \frac{v(\mathrm{d}s)v(\mathrm{d}t)}{|s-t|^{1-\gamma}(\ln 1/(s-t))^b} \leqslant c\sum_{n>n_1} r_{n-1}^{\delta'} < \infty.$$

Consequently $I_\gamma(v)<\infty$ a.s., and thus we have proved (4.6).

In view of (4.6) and Perkins and Taylor (1987, Theorem 3.1), we have

Lemma 4.3. *Suppose that T_θ is the Cantor-like random set constructed above, under the assumption* (3.2). *Define the measure function*

$$\phi_{\alpha,\gamma}(s) = s^{\alpha(1-\gamma)}(\ln(1/s))^{(1-\alpha)(1-\gamma)+b},$$

where γ is defined above and b is large enough (*determined as above*) *and fixed. Then, with probability* 1,

$$\phi_{\alpha,\gamma} - m(Y(T_\theta)) = \infty.$$

5. Conclusion: the log-multifractal spectrum

We are ready to state and prove the conclusion of our analysis in this section. Let Y_t be a stable subordinator of index α, $0<\alpha<1$. Let μ be the occupation measure of Y. Define

$$A_\theta = \left\{ x \in Y[0,1] : \limsup_{r\downarrow 0} \frac{\mu(x-r,x+r)}{c_\alpha r^\alpha (\ln(1/r))^{1-\alpha}} \geqslant \theta \right\},$$

$$B_\theta = \left\{ x \in Y[0,1] : \limsup_{r\downarrow 0} \frac{\mu(x-r,x+r)}{c_\alpha r^\alpha (\ln(1/r))^{1-\alpha}} = \theta \right\},$$

where

$$c_\alpha = 2^\alpha \left(\frac{(1-\alpha)\alpha^{\alpha/(1-\alpha)}}{\alpha} \right)^{\alpha-1}.$$

We note that $Y(T_\theta) \subset A_\theta$ since $x \in Y(T_\theta)$ implies (4.5), for all large n.

Theorem 5.1. *If* μ *is the occupation measure of a stable subordinator of index* α, $0<\alpha<1$, *then, with probability* 1,

(i) $A_\theta = \emptyset$, *if* $\theta > 1$.

(ii) A_θ *and* B_θ *are non-empty, if* $0 \leqslant \theta \leqslant 1$; *moreover,*

$$\dim A_\theta = \dim B_\theta = \operatorname{Dim} A_\theta = \operatorname{Dim} B_\theta = \alpha(1-\theta^{1/(1-\alpha)}).$$

Proof. In Sections 3 and 4 we work under the assumption (3.2) and its opposite (3.4), which are equivalent to, respectively, $\theta_1 := (c_3/\alpha)^{1-\alpha}\theta < 1$ and >1. Moreover, "$\geqslant \theta$" is simply equivalent to "$\geqslant (c_3/\alpha)^{\alpha-1}\theta_1$"; the latter one is just the content for A_{θ_1}, by definition of c_3 in Section 2. Thus the results which we have obtained in Sections 3 and 4 prove the assertions for A_θ, except for the critical cases $\theta = 0, 1$. As for B_θ, $0<\theta<1$, we note that

$$B_\theta = A_\theta \Big\backslash \bigcup_{n=1} A_{\theta+1/n}.$$

Let the measure function $\phi_{\alpha,\gamma}(s)$ now be defined by

$$\phi_{\alpha,\gamma}(s) = s^{\alpha(1-\theta^{1/(1-\alpha)})}(\ln(1/s))^{(1-\alpha)(1-\theta^{1/(1-\alpha)})+b},$$

where $b>1$ is large enough and fixed. Then Lemma 4.3 (change the index γ there) implies that, with probability 1, $\phi_{\alpha,\gamma} - m(A_\theta) = \infty$ while $\phi_{\alpha,\gamma} - m(A_{\theta+1/n}) = 0$ for all n, by Eq. (3.3). Thus $\phi_{\alpha,\gamma} - m(B_\theta) = \infty$ too, which again implies that $\dim B_\theta \geqslant \alpha(1 - \theta^{1/(1-\alpha)})$. Finally, the cases $\theta = 0, 1$ can be treated as follows. For $\theta = 0$, we simply note that $\dim Y[0,1] = \operatorname{Dim} Y[0,1] = \alpha$; see Taylor (1986). For the other extreme, we go back to the critical case in (3.2) of Section 3, namely $c_3\theta^{1/(1-\alpha)} = \alpha$. In this case, the construction of T_θ in Section 4 has to be modified. In the definition of type-S intervals, we require that, instead of some fixed $a>1$,

$$Y^j_{t_{i,n}}(\eta_n) - Y^j_{t_{i,n}}(\tilde\eta_n) \leqslant \frac{r_n}{a_n}, \quad j = 1, 2,$$

where $a_n = 1 + 1/n$. The key display (4.4) holds, with a being replaced by a_n. Because $r_n \downarrow 0$ very fast, we can still prove (4.5). We have a corresponding result to Lemma 4.1 and $U_{n+1} \uparrow \infty$ too. To construct T_θ, in the present case, we pick up *all* possible type-S intervals from $\mathscr{F}_{n+1}$ which are contained in each type-S $\tilde I \in \mathscr{F}_n$, $n = 1, 2, \ldots$. We thus form a chain of unions $F_1 \supset F_2 \supset \cdots$. Then $T_\theta = \bigcap_{n=1} F_n$ is non-empty since all F_n are compact and non-empty (In this case, $\dim T_\theta = 0$). □

Corollary 5.2. *With probability* 1,

$$\lim_{r\downarrow 0} \sup_{x\in Y[0,1]} \frac{\mu(x-r, x+r)}{r^\alpha(\ln(1/r))^{1-\alpha}} = c_\alpha.$$

Proof. This follows immediately from $A_\theta = \emptyset$ for $\theta > 1$, while $A_1 = B_1 \neq \emptyset$; both statements holding with probability 1. □

Added in proof. 1. We have recently received a preprint from Lawrence Marsalle entitled "Slow points and fast points of local times" which is based on results in her PhD thesis (1996). Her paper contains results related to this paper: her "local times" are more general than those arising from stable subordinators, and her "fast points" are one-sided while ours are two-sided. For this reason we could not deduce our results from hers, even though the dimensions of exceptional sets turn out to be the same.

2. Jay Rosen and Ofer Zeitouni pointed out that there is a computational error in the proof of (4.6). We are able to correct this, so that the results stated in the paper are still valid. Any reader wishing to see the corrected version should request it from the first author.

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