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LIMSUP RANDOM FRACTALS

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Abstract Orey and Taylor (1974) introduced sets of "fast points" where Brownian increments are exceptionally large, $\mathrm{F}(\lambda) := \{t \in [0,1] : \limsup_{h\to 0} |X(t+h) - X(t)|/\sqrt{2h|\log h|} \geqslant \lambda\}$. They proved that for $\lambda \in (0,1]$, the Hausdorff dimension of $\mathrm{F}(\lambda)$ is $1-\lambda^2$ a.s. We prove that for any analytic set $E \subset [0,1]$, the supremum of all λ's for which E intersects $\mathrm{F}(\lambda)$ a.s. equals $\sqrt{\dim_{\mathrm{P}} E}$, where $\dim_{\mathrm{P}} E$ is the *packing dimension* of E. We derive this from a general result that applies to many other random fractals defined by limsup operations. This result also yields extensions of certain "fractal functional limit laws" due to Deheuvels and Mason (1994). In particular, we prove that for any absolutely continuous function f such that $f(0) = 0$ and the energy $\int_0^1 |f'|^2\,dt$ is lower than the packing dimension of E, there a.s. exists some $t \in E$ so that f can be uniformly approximated in $[0,1]$ by normalized Brownian increments $s \mapsto [X(t+sh) - X(t)]/\sqrt{2h|\log h|}$; such uniform approximation is a.s. impossible if the energy of f is higher than the packing dimension of E.

Keywords Limsup random fractals, Packing dimension, Hausdorff dimension

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1 Introduction

The connections between hitting probabilities of stochastic processes, capacity and Hausdorff dimension are key tools in the analysis of sample paths; see Taylor (1986).

In this paper we show that for certain random fractals that are naturally defined by limsup operations (Brownian fast points being the canonical example), hitting probabilities are determined by the **packing dimension** $\dim_{\mathrm{P}}(E)$ of the target set E, rather than its Hausdorff dimension $\dim_{\mathrm{H}}(E)$. See Mattila (1995) for definitions of these dimension indices.

For $\lambda \in (0,1]$, Orey and Taylor (1974) considered the set of λ-**fast points** for linear Brownian motion X, defined by

$$\mathrm{F}(\lambda) := \Big\{ t \in [0,1] : \limsup_{h \to 0^+} \frac{|X(t+h) - X(t)|}{\sqrt{2h|\log h|}} \geqslant \lambda \Big\}. \tag{1.1}$$

Orey and Taylor proved that

$$\forall \lambda \in (0,1], \quad \dim_{\mathrm{H}}(\mathrm{F}(\lambda)) = 1 - \lambda^2 \quad \text{a.s.} \tag{1.2}$$

Kaufman (1975) subsequently showed that any analytic set E with $\dim_{\mathrm{H}}(E) > \lambda^2$, a.s. intersects $\mathrm{F}(\lambda)$. The next theorem shows that packing dimension is the right index for deciding which sets intersect $\mathrm{F}(\lambda)$. It is a special case of a general result that we state in §2.

Theorem 1.1 *Let X denote linear Brownian motion. For any analytic set $E \subset \mathbb{R}^1_+$,*

$$\sup_{t \in E} \limsup_{h \to 0+} \frac{|X(t+h) - X(t)|}{\sqrt{2h|\log h|}} = (\dim_{\mathrm{P}}(E))^{1/2}, \quad a.s.$$

Equivalently,

$$\forall \lambda > 0 \quad \mathbb{P}(\mathrm{F}(\lambda) \cap E \neq \varnothing) = \begin{cases} 1, & \text{if } \dim_{\mathrm{P}}(E) > \lambda^2 \\ 0 & \text{if } \dim_{\mathrm{P}}(E) < \lambda^2 \end{cases}. \tag{1.3}$$

Moreover, if $\dim_{\mathrm{P}}(E) > \lambda^2$ then $\dim_{\mathrm{P}}(F(\lambda) \cap E) = \dim_{\mathrm{P}}(E)$, a.s.

Remark 1.2 For compact sets E, we can sharpen (1.3) to a necessary and sufficient criterion for E to contain λ-fast points:

$\mathbb{P}(\mathrm{F}(\lambda) \cap E \neq \varnothing) = 1$ *if and only if E is not a union of countably many Borel sets E_n with $\dim_{\mathrm{P}}(E_n) < \lambda^2$.* (In this case, we say that $E \notin \{\dim_{\mathrm{P}} < \lambda^2\}_\sigma$.) See Theorem 2.1 and its application to fast points in §2.

Remark 1.3 For traditional random fractals such as the range and the level sets of Brownian motion (as well as those of many other stable processes), it is well known that Hausdorff and packing dimensions a.s. coincide. An interesting feature of limsup random fractals is that generally their Hausdorff and packing dimensions differ. For instance, as regards to fast points, this follows from comparing the result (1.2) of Orey and Taylor with a result of Dembo, Peres, Rosen and Zeitouni (1998). In fact, Corollary 2.4 of the latter paper implies that $\dim_{\mathrm{P}}(\mathrm{F}(\lambda)) = 1$ a.s. for every $\lambda \in [0,1)$. This equality is a special case of the final assertion of Theorem 1.1 with $E = [0,1]$.

Remark 1.4 Theorem 1.1 can be viewed as a probabilistic interpretation of the packing dimension of an analytic set $E \subset \mathbb{R}^1_+$. Similarly, we can provide a probabilistic interpretation for the packing dimension of a multi-dimensional set $E \subset \mathbb{R}^d_+$. Let $\mathbb{W}$ denote white noise on $\mathbb{R}^d_+$, viewed as an $L^2(\mathbb{P})$–valued random measure; see Walsh (1986) for details. For any $t \in \mathbb{R}^d_+$ and $h \geqslant 0$, let $[t, t+h]$ denote the Cartesian product $\prod_{i=1}^d [t^i, t^i + h]$. That is, $[t, t+h]$ designates the hyper-cube of side h with 'left endpoint' t. Then, for any analytic set $E \subset \mathbb{R}^d_+$,

$$\sup_{t \in E} \limsup_{h \to 0+} \frac{|\mathbb{W}[t, t+h]|}{\sqrt{2h^d |\log h|}} = \left(\dim_{\mathrm{P}}(E)\right)^{1/2}, \quad \text{a.s.} \tag{1.4}$$

Remark 1.5 By reversing the order of sup and lim sup in Theorem 1.1, we obtain the following probabilistic interpretations of the upper and lower Minkowski dimensions of E, denoted $\overline{\dim}_{\mathrm{M}}(E)$ and $\underline{\dim}_{\mathrm{M}}(E)$, respectively; see Mattila (1995) for definitions.

$$\limsup_{h \to 0+} \sup_{t \in E} \frac{|X(t+h) - X(t)|}{\sqrt{2h|\log h|}} = (\overline{\dim}_{\mathrm{M}}(E))^{1/2}, \quad \text{a.s.} \tag{1.5}$$

$$\liminf_{h \to 0+} \sup_{t \in E} \frac{|X(t+h) - X(t)|}{\sqrt{2h|\log h|}} = (\underline{\dim}_{\mathrm{M}}(E))^{1/2}, \quad \text{a.s.} \tag{1.6}$$

Together with Theorem 1.1, equations (1.5) and (1.6) complete the bounds in Theorem 1.1 of Khoshnevisan and Shi (1998).

Remark 1.6 It is interesting to note the following intersection property of the set of λ-fast points. Let $F'(\lambda)$ be an independent copy of $F(\lambda)$. Then,

$$\dim_{\mathrm{H}}(F(\lambda) \cap F'(\lambda)) = 1 - \lambda^2, \quad \text{a.s.}$$

This is different from the intersection properties of the images of Brownian motion or stable Lévy processes; see e.g. Hawkes (1971).

Another application of our method is to functional extensions of Theorem 1.1. Let $C_{\mathrm{ac}}[0,1]$ denote the collection of all absolutely continuous functions $f : [0,1] \mapsto \mathbb{R}^1$ with $f(0) = 0$. We endow $C_{\mathrm{ac}}[0,1]$ with the supremum norm $\|f\|_\infty := \sup_{0 \leqslant s \leqslant 1} |f(s)|$. Define the *Sobolev norm*

$$\|f\|_{\mathbb{H}} := \left(\int_0^1 \{f'(s)\}^2 ds \right)^{1/2}, \quad f \in C_{\mathrm{ac}}[0,1].$$

(The Sobolev norm squared is sometimes called the *energy* of f.) Let $\mathbb{H}$ denote the Hilbert space of all $f \in C_{\mathrm{ac}}[0,1]$ of finite energy. Strassen (1964) proved that for all $f \in \mathbb{H}$ with $\|f\|_{\mathbb{H}} \leqslant 1$, and for all $0 \leqslant t \leqslant 1$,

$$\liminf_{h \to 0+} \sup_{0 \leqslant s \leqslant 1} \left| \frac{X(t+hs) - X(t)}{\sqrt{2h \log|\log h|}} - f(s) \right| = 0, \qquad \text{a.s.}$$

The null set in question depends on the choice of t. Deheuvels and Mason (1994) analyzed the exceptional t's, and obtained functional extensions of the Orey-Taylor Theorem (1.2). To explain their result, consider the normalized increment process

$$\Delta_h[t](s) := (X(t+sh) - X(t))/\sqrt{2h|\log h|}\,,$$

and note the single logarithm in the normalization, in contrast with the iterated logarithm in Strassen's theorem. For any $f \in \mathbb{H}$, let

$$\mathfrak{D}(f) := \left\{ t \in [0,1] \,\middle|\, \liminf_{h \to 0+} \|\Delta_h[t] - f\|_\infty = 0 \right\}. \tag{1.7}$$

Deheuvels and Mason (1994) proved that $\dim_{\mathrm{H}}(\mathfrak{D}(f)) = 1 - \|f\|_{\mathbb{H}}^2$ a.s., for any $f \in \mathbb{H}$ such that $\|f\|_{\mathbb{H}} \leqslant 1$. Our next theorem extends their result as well as Theorem 1.1 above, by giving a hitting criterion for $\mathfrak{D}(f)$.

Theorem 1.7 *Let X denote linear Brownian motion. For any $f \in \mathbb{H}$ with $\|f\|_{\mathbb{H}} \in]0,1]$, and for any analytic set $E \subset [0,1]$,*

$$\mathbb{P}(\mathfrak{D}(f) \cap E \neq \varnothing) = \begin{cases} 1, & \text{if } \dim_{\mathrm{P}}(E) > \|f\|_{\mathbb{H}}^2 \\ 0, & \text{if } \dim_{\mathrm{P}}(E) < \|f\|_{\mathbb{H}}^2 \end{cases}.$$

Moreover, if $\dim_{\mathrm{P}}(E) > \|f\|_{\mathbb{H}}^2$, then $\dim_{\mathrm{P}}(\mathfrak{D}(f) \cap E) = \dim_{\mathrm{P}}(E)$, a.s., while

$$\dim_{\mathrm{H}}(E) - \|f\|_{\mathbb{H}}^2 \leqslant \dim_{\mathrm{H}}(\mathfrak{D}(f) \cap E) \leqslant \dim_{\mathrm{P}}(E) - \|f\|_{\mathbb{H}}^2, \quad \text{a.s.} \tag{1.8}$$

In words, given an analytic set $E \subset [0,1]$ and a function $f \in \mathbb{H}$, if $\|f\|_{\mathbb{H}}^2 < \dim_{\mathrm{P}}(E)$, then there a.s. exists a time $t \in E$ such that the (normalized) Brownian increments process $\Delta_h[t]$ converges uniformly to f along a sequence of h values tending to 0, but if $\|f\|_{\mathbb{H}}^2 > \dim_{\mathrm{P}}(E)$ then a.s. there is no such time in E.

We conclude the Introduction with a final application to extensions of Chung's law of the iterated logarithm for Brownian motion. Let X_d be standard Brownian motion in $\mathbb{R}^d$, and denote by

$$R_d(t,h) := \max_{0 \leqslant s \leqslant 1} |X_d(t+sh) - X_d(t)| \tag{1.9}$$

the radius of the smallest ball centered at $X_d(t)$ that contains $X_d[t, t+h]$. Theorem 4 of Ciesielski and Taylor (1962) states that for all $t \in [0,1]$,

$$\liminf_{h \to 0+} \frac{R_d(t,h)}{\sqrt{h/\log|\log h|}} = 2^{-1/2} q_d, \qquad \text{a.s.,} \tag{1.10}$$

where q_d is the smallest positive root of the Bessel function $J_{(d-2)/2}$; in particular, $q_1 = \pi/2$. The null set in question depends on t, and we show that any set in $\mathbb{R}_+$ of positive packing dimension contains random times t at which the behavior of the Brownian path is markedly different:

Theorem 1.8 *Let X_d denote Brownian motion in $\mathbb{R}^d$. For any analytic set $E \subset \mathbb{R}_+$,*

$$\inf_{t \in E} \liminf_{h \to 0+} \frac{R_d(t,h)}{q_d \sqrt{h/(2|\log h|)}} = (\dim_{\mathrm{P}}(E))^{-1/2} \quad \text{a.s.} \tag{1.11}$$

Remark 1.9 For all $\lambda \geqslant 1$, define

$$\mathrm{S}(\lambda) := \left\{ t \in [0,1] : \liminf_{h \to 0+} \frac{R_d(t,h)}{q_d \sqrt{h/2|\log h|}} \leqslant \lambda \right\}. \tag{1.12}$$

One can think of $\mathrm{S}(\lambda)$ as the collection of all points of (sporadic) slow escape of order λ. Theorem 1.8 means that for any analytic set $E \subset [0,1]$ and every $\lambda > 0$,

$$\mathbb{P}(\mathrm{S}(\lambda) \cap E \neq \varnothing) = \begin{cases} 1, & \text{if } \dim_{\mathrm{P}}(E) > \lambda^{-2} \\ 0, & \text{if } \dim_{\mathrm{P}}(E) < \lambda^{-2} \end{cases}. \tag{1.13}$$

Furthermore, we will show in §2 that $\dim_{\mathrm{P}}(\mathrm{S}(\lambda)) = 1$ while $\dim_{\mathrm{H}}(\mathrm{S}(\lambda)) = 1 - \lambda^{-2}$ for all $\lambda \geqslant 1$. In particular, $\dim_{\mathrm{P}}(\mathrm{S}(1)) = 1$ while $\dim_{\mathrm{H}}(\mathrm{S}(1)) = 0$. We will present a functional version of Theorem 1.8 and Remark 1.9 in Section 6.

Remark 1.10 Denote by $\tau_d(t,r) := \min\{h > 0 : |X(t+h) - X(t)| = r\}$ the hitting time of a sphere of radius r centered at $X(t)$. Since $R_d(t,h) \leqslant r$ if and only if $\tau_d(t,r) \geqslant h$,

$$\mathrm{S}(\lambda) = \left\{ t \in [0,1] : \limsup_{r \to 0+} \frac{\tau_d(t,r)}{r^2 |\log r|} \geqslant \frac{4}{q_d^2 \lambda^2} \right\},$$

and (1.11) can be restated in the form

$$\sup_{t \in E} \limsup_{r \to 0+} \frac{\tau_d(t,r)}{r^2 |\log r|} = \frac{4}{q_d^2} \dim_{\mathrm{P}}(E). \tag{1.14}$$

Remark 1.11 For all $\lambda \geqslant 0$, define

$$\mathrm{F}_=(\lambda) := \left\{ t \in [0,1] : \limsup_{h \to 0+} \frac{|X(t+h) - X(t)|}{\sqrt{2h|\log h|}} = \lambda \right\} \tag{1.15}$$

$$\mathrm{S}_=(\lambda) := \left\{ t \in [0,1] : \liminf_{h \to 0+} \frac{R_d(t,h)}{q_d \sqrt{h/2|\log h|}} = \lambda \right\}. \tag{1.16}$$

Compare these to Equations (1.1) and (1.12), respectively. In Section 2, we will show that the aforementioned dimension results as well as the hitting results hold for $\mathrm{F}_=(\lambda)$ and $\mathrm{S}_=(\lambda)$ in place of $\mathrm{F}(\lambda)$ and $\mathrm{S}(\lambda)$, respectively. In particular, we will see that for $G(\lambda)$ denoting either $\mathrm{F}_=(\lambda)$ or $\mathrm{S}_=(\lambda^{-1})$ for $\lambda \in]0,1]$, any analytic set $E \subset [0,1]$ satisfies $\mathbb{P}(E \cap G(\lambda) \neq \varnothing) = 1$ if $\dim_{\mathrm{P}}(E) > \lambda^2$, while this probability is 0 if $\dim_{\mathrm{P}}(E) < \lambda^2$. Moreover, for all $\lambda \in [0,1]$, we have $\dim_{\mathrm{P}}(G(\lambda)) = 1$ while $\dim_{\mathrm{H}}(G(\lambda)) = 1 - \lambda^2$.

2 The General Results

In this section, we state a general theorem for a class of multi-parameter, $\mathbb{R}^1_+$–valued stochastic process $Y := \{Y(t,h);\ t \in \mathbb{R}^N_+,\ h > 0\}$, where N is a fixed positive integer. We then use this theorem to derive the results announced in the Introduction.

We impose three conditions on Y; the first two are distributional, while the last is a condition on the sample functions. Throughout, $\mathbb{R}^N_+$ is endowed with the coordinatewise partial order: $s, t \in \mathbb{R}^N_+$ satisfy $s \prec t$ if and only if $s_i \leqslant t_i$ for all $1 \leqslant i \leqslant N$.

Condition 1: stationarity and tail power law. For each $t \in \mathbb{R}^N_+$ and $h > 0$, the random variables $Y(t,h)$ and $Y(0,h)$ have the same distribution. Moreover, there exist $y_1 > y_0 \geqslant 0$ such that $Y(t,h) \geqslant y_0$ for all t, h, and

$$\forall t \in \mathbb{R}^N_+, \quad \forall \gamma \in (y_0, y_1], \quad \lim_{h \to 0+} \frac{\log \mathbb{P}(Y(t,h) > \gamma)}{\log h} = \gamma.$$

Condition 2: tail correlation bound. For all $\varepsilon > 0$ and $M > 0$, there exists a function $\psi = \psi_{\varepsilon,M} : [0,\infty[\mapsto [0,\infty[$, regularly varying of order 1 at 0, such that for all $h \geqslant 0$ small enough and all $s,t \in [0,M]^N$ satisfying $|s-t| \geqslant \psi(h)$ and for all $0 \leqslant \gamma \leqslant N$,

$$\mathbb{P}(Y(t,h) > \gamma \mid Y(s,h) > \gamma) < (1+\varepsilon)\mathbb{P}(Y(t,h) > \gamma).$$

Condition 3: modulus of continuity. For all $\varepsilon > 0$ and all $\tau \in \mathbb{R}_+^N$,

$$\limsup_{h\to 0+} \sup_{t \prec \tau} \sup_{\substack{0 \leqslant h' \leqslant 1: \\ |h-h'| \leqslant h^{1+\varepsilon}}} |Y(t,h) - Y(t,h')| = 0 \quad \text{a.s.}$$

and

$$\limsup_{h\to 0+} \sup_{\substack{t,t' \prec \tau: \\ |t-t'| \leqslant h^{1+\varepsilon}}} |Y(t,h) - Y(t',h)| = 0 \quad \text{a.s.}$$

Definition. Given a set $E \subset \mathbb{R}^d$ and $\lambda > 0$, we say that $E \notin \{\dim_{\mathrm{P}} < \lambda\}_\sigma$ if E is not a union of countably many Borel sets E_n with $\dim_{\mathrm{P}}(E_n) < \lambda$.

Each of the following conditions is sufficient for $E \notin \{\dim_{\mathrm{P}} < \lambda\}_\sigma$:

- E has positive packing measure in dimension λ.
  (More generally, it suffices that the packing measure of E is positive in some gauge φ that satisfies $\lim_{r\to 0+} r^{\varepsilon-\lambda}\varphi(r) = 0$ for some $\varepsilon > 0$.)

- E is compact and $\overline{\dim}_{\mathrm{M}}(E \cap V) \geqslant \lambda$ for any open set V that intersects E.

Indeed, the sufficiency of the first condition is obvious; to see the sufficiency of the second condition, suppose that it holds and $E = \cup_n E_n$ with $\dim_{\mathrm{P}}(E_n) < \lambda$. By regularization, we can represent E as a countable union $E = \cup_{n,j} E_{n,j}$ with $\overline{\dim}_{\mathrm{M}}(\overline{E}_{n,j}) < \lambda$. Each $E_{n,j}$ is nowhere dense in E, so Baire's theorem yields the desired contradiction.

It follows from Joyce and Preiss (1995) that any analytic set $E \subset \mathbb{R}^d$ with positive λ-dimensional packing measure, contains a compact subset E' such that $E' \notin \{\dim_{\mathrm{P}} < \lambda\}_\sigma$.

We are ready to state the main result of this section (see §4 for a proof). Denote

$$\Gamma(\lambda) := \Big\{ t \in [0,1]^N : \limsup_{h\to 0+} Y(t,h) \geqslant \lambda \Big\}.$$

Theorem 2.1 *Suppose that the* $\mathbb{R}_+^1$*–valued stochastic process* Y *satisfies Conditions 1,2 and 3. Then, for any analytic set* $E \subset [0,1]^N$ *with* $\dim_{\mathrm{P}}(E) \in [y_0, y_1]$,

$$\sup_{t\in E} \limsup_{h\to 0+} Y(t,h) = \dim_{\mathrm{P}}(E), \quad \text{a.s.,} \tag{2.1}$$

If E is compact, we can determine when the supremum in (2.1) is attained:

$$\forall \lambda \in (y_0, y_1), \quad \mathbb{P}(\Gamma(\lambda) \cap E \neq \varnothing) = \begin{cases} 1, & \text{if } E \notin \{\dim_{\mathrm{P}} < \lambda\}_\sigma \\ 0, & \text{otherwise.} \end{cases}$$

Finally, suppose that $y_0 < \lambda \leqslant \dim_{\mathrm{P}}(E) \leqslant y_1$. *If* $\lambda < \dim_{\mathrm{P}}(E)$ *(or alternatively, if* E *is a compact set and* $E \notin \{\dim_{\mathrm{P}} < \lambda\}_\sigma$*), then*

$$\dim_{\mathrm{P}}(\Gamma(\lambda) \cap E) = \dim_{\mathrm{P}}(E) \quad \text{a.s.} \tag{2.2}$$

and

$$\dim_{\mathrm{H}}(E) - \lambda \leqslant \dim_{\mathrm{H}}(\Gamma(\lambda) \cap E) \leqslant \dim_{\mathrm{P}}(E) - \lambda, \quad \text{a.s.} \tag{2.3}$$

We record that the upper bound in Theorem 2.1 does not require any correlation bound.

Proposition 2.2 *Suppose that the* $\mathbb{R}^1_+$*–valued stochastic process* Y *satisfies Conditions 1 and 3. Then, for any analytic set* $E \subset [0,1]^N$ *with* $\dim_{\mathrm{P}}(E) \in [y_0, y_1]$,

$$\sup_{t \in E} \limsup_{h \to 0+} Y(t,h) \leqslant \dim_{\mathrm{P}}(E), \quad \text{a.s.} \tag{2.4}$$

Next, we determine the hitting properties for the level sets of the limsup considered in (2.1).

Theorem 2.3 *Suppose that the* $\mathbb{R}^1_+$*–valued stochastic process* Y *satisfies Conditions 1,2 and 3. For each* $\lambda > 0$*, define the random set*

$$\Gamma_=(\lambda) := \left\{ t \in [0,1]^N : \ \limsup_{h \to 0+} Y(t,h) = \lambda \right\}.$$

Then, for any compact set $E \subset [0,1]^N$ *and for all* $\lambda \in (y_0, y_1]$,

$$\mathbb{P}(\Gamma_=(\lambda) \cap E \neq \varnothing) = \begin{cases} 1, & \text{if } E \notin \{\dim_{\mathrm{P}} < \lambda\}_\sigma \\ 0, & \text{otherwise.} \end{cases}$$

Finally, if E *is a compact set with packing dimension in* $[y_0, y_1]$ *such that* $E \notin \{\dim_{\mathrm{P}} < \lambda\}_\sigma$*, then* $\dim_{\mathrm{P}}(\Gamma_=(\lambda) \cap E) = \dim_{\mathrm{P}}(E)$ *a.s. and*

$$\dim_{\mathrm{H}}(E) - \lambda \leqslant \dim_{\mathrm{H}}(\Gamma_=(\lambda) \cap E) \leqslant \dim_{\mathrm{P}}(E) - \lambda, \quad \text{a.s.}$$

Our next corollary gives the Hausdorff dimension and packing dimension of the intersection of $\Gamma(\lambda)$ (or $\Gamma_=(\lambda)$) with an independent copy of itself. Its proof follows easily from Step Three of our proof of Theorem 2.1 and Lemma 3.4.

Corollary 2.4 *Suppose that the* $\mathbb{R}^1_+$*–valued stochastic process* Y *satisfies Conditions 1,2 and 3. For each* $\lambda > 0$*, let* $\Gamma'(\lambda)$ *be an independent copy of* $\Gamma(\lambda)$*. If* E *is a compact set with packing dimension in* $[y_0, y_1]$ *such that* $E \notin \{\dim_{\mathrm{P}} < \lambda\}_\sigma$*, then* $\dim_{\mathrm{P}}(\Gamma(\lambda) \cap \Gamma'(\lambda) \cap E) = \dim_{\mathrm{P}}(E)$ *a.s. and*

$$\dim_{\mathrm{H}}(E) - \lambda \leqslant \dim_{\mathrm{H}}(\Gamma(\lambda) \cap \Gamma'(\lambda) \cap E) \leqslant \dim_{\mathrm{P}}(E) - \lambda, \quad \text{a.s.}$$

We conclude this section by using Theorems 2.1 and 2.3 to prove the assertions of the Introduction.

2.1 Proof of Theorem 1.1

Recall that X is linear Brownian motion. We let $N = 1$ and define the process Y as follows:

$$Y(t,h) := \frac{|X(t+h) - X(t)|^2}{2h|\log h|}, \qquad t \in \mathbb{R}^1_+ \ , \ h > 0.$$

It follows from the stationarity of increments and the scaling property of Brownian motion that Y satisfies Condition 1 with $y_0 = 0$ and any y_1. Condition 2 also holds for $\psi(h) := h$ due to the independence of increments of Brownian motion. Finally, by Lévy's modulus of continuity (cf. Orey and Taylor (1974)), we can deduce Condition 3. According to (2.1) and Theorem 2.1, Theorem 1.1 and Remark 1.2 both follow. This proves Theorem 1.1. □

We conclude this subsection by sketching some proofs for the Remarks following Theorem 1.1.

First, we prove (1.4) of Remark 1.4. Take $Y(t,h)$ to be $|\mathbb{W}[t,t+h]|^2/(2h^d|\log h|)$ and use the modulus of continuity of Orey and Pruitt (1973, Theorem 2.1):

$$\limsup_{h\to 0} \sup_{t\in[0,1]^d} \frac{|\mathbb{W}[t,t+h]|}{\sqrt{2h^d|\log h|}} = \sqrt{d}, \quad \text{a.s.}$$

instead of that of Lévy.

Equations (1.5) and (1.6) of Remark 1.5 are proved by direct and elementary means. We verify (1.5) to illustrate the basic idea. In light of Theorem 1.1 of Khoshnevisan and Shi (1998), it suffices to demonstrate that with probability one,

$$\limsup_{h\to 0} \sup_{t\in E} \frac{|X(t+h) - X(t)|}{\sqrt{2h|\log h|}} \geqslant (\overline{\dim}_{\rm M}(E))^{1/2}. \tag{2.5}$$

We can assume $\overline{\dim}_{\rm M}(E) > 0$ and fix a $0 < \delta < \overline{\dim}_{\rm M}(E)$. Then, there exist at least $[n_k^\delta]$ points σ_i $(1 \leqslant i \leqslant [n_k^\delta])$ of E such that $|\sigma_i - \sigma_j| \geqslant n_k^{-1}$ for all $i \neq j$, where $\{n_k\}$ is some sequence which tends to infinity. We note that for any constant $\eta > 0$ we can choose $\{n_k\}$ such that $\sum_k n_k^{-\eta} < \infty$.

Elementary properties of Brownian motion can be used to show that for any $0 < \beta < \delta$ and for all $k \geqslant 1$,

$$\inf_{1\leqslant i\leqslant [n_k^\delta]} \mathbb{P}\Big(|X(\sigma_i + n_k^{-1}) - X(\sigma_i)| \geqslant \sqrt{2\beta n_k^{-1}\log n_k}\Big) = n_k^{-\beta(1+\varepsilon_k)},$$

where $\lim_{k\to\infty} \varepsilon_k = 0$. Let

$$\mathsf{N}_k := \sum_{i=1}^{[n_k^\delta]} \mathbb{1}\Big\{|X(\sigma_i + n_k^{-1}) - X(\sigma_i)| \geqslant \sqrt{2\beta n_k^{-1}\log n_k}\Big\}.$$

We have shown that $\mathbb{E}[\mathsf{N}_k] = n_k^{\delta-\beta(1+\varepsilon_k)}$, which goes to infinity. Moreover, $\mathrm{Var}(\mathsf{N}_k) \leqslant \mathbb{E}[\mathsf{N}_k]$. By the Chebyshev's inequality, we have for any $1/2 < \varepsilon < 1$,

$$\mathbb{P}\Big(|\mathsf{N}_k - \mathbb{E}[\mathsf{N}_k]| \geqslant (\mathbb{E}[\mathsf{N}_k])^\varepsilon\Big) \leqslant (\mathbb{E}[\mathsf{N}_k])^{1-2\varepsilon}.$$

Suppose we have chosen the sequence $\{n_k\}$ such that $\sum_k \{\mathbb{E}[\mathsf{N}_k]\}^{1-2\varepsilon} < \infty$. Then, by the Borel–Cantelli lemma, $\lim_k \mathsf{N}_k/\mathbb{E}[\mathsf{N}_k] = 1$, almost surely. In particular, for all $0 < \beta < \overline{\dim}_{\mathrm{M}}(E)$,

$$\limsup_{h\to 0} \sup_{t\in E} \frac{|X(t+h)-X(t)|}{\sqrt{2h|\log h|}} \geqslant \beta^{1/2}, \quad \text{a.s.}$$

This verifies (2.5) and completes our proof of (1.5) of Remark 1.5.

Finally, Remark 1.6 follows directly from Corollary 2.4. □

2.2 Proof of Theorem 1.7

As usual, extend the Sobolev norm to the space $C[0,1]$ of all continuous functions, by setting $\|g\|_{\mathbb{H}} := \infty$ for every continuous function $g : [0,1] \mapsto \mathbb{R}^1$ which is not in $\mathbb{H}$. We will use the following form of Schilder's theorem (See Chapter 5 of Dembo and Zeitouni 1998): *Let X be linear Brownian motion. For any Borel set $F \subset C[0,1]$, write $I(F) = \inf\{\|g\|_{\mathbb{H}}^2 : g \in F\}$ and denote by F^o and $\overline{F}$ the interior and closure of F, respectively. Then*

$$-I(F^o) \leqslant \liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\Big(\sqrt{\frac{\varepsilon}{2}}\; X(\cdot) \in F\Big) \leqslant \limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\Big(\sqrt{\frac{\varepsilon}{2}}\; X(\cdot) \in F\Big) \leqslant -I(\overline{F}).$$

Fix $f \in \mathbb{H}$ with $\|f\|_{\mathbb{H}} > 0$ and define $R(\gamma) := \min \|g\|_{\mathbb{H}}^2$, where the minimum is taken over the $\|\cdot\|_{\mathbb{H}}$-closed convex set of functions g such that $\|g - f\|_\infty \leqslant \gamma$. Obviously, $R(\gamma) \leqslant \|f\|_{\mathbb{H}}^2$, and the map $\gamma \mapsto R(\gamma)$ is continuous on the interval $[0,\infty)$. Moreover, $R(\cdot)$ is strictly decreasing on $[0, \|f\|_\infty]$, and $R(\gamma) \equiv 0$ for $\gamma \geqslant \|f\|_\infty$. Let $L : [0, \|f\|_{\mathbb{H}}^2] \to [0, \|f\|_\infty]$ denote the inverse function to R. Define

$$Y(t,h) := R\Big(\|\Delta_h[t] - f\|_\infty\Big), \qquad t \geqslant 0,\ h > 0.$$

Then, Condition 1 with $y_0 = 0$ and $y_1 = \|f\|_{\mathbb{H}}^2$ follows from Schilder's theorem. Condition 2 follows with $\psi(h) := h$, due to the independence of the increments of X. Finally, Condition 3 follows from Lévy's uniform modulus of continuity. Note that for $\lambda \geqslant 0$,

$$\Gamma(\lambda) = \Big\{t \in [0,1] :\ \liminf_{h\to 0+} \|\Delta_h[t] - f\|_\infty \leqslant L(\lambda)\Big\}.$$

For all $\eta \geqslant 0$, define

$$\mathcal{F}(\eta) := \Big\{t \in [0,1] :\ \liminf_{h\to 0+} \|\Delta_h[t] - f\|_\infty \leqslant \eta\Big\} = \Gamma(R(\eta)).$$

By Theorem 2.1, for any compact set $E \subset [0,1]$ and all $\eta \in [0, \|f\|_\infty]$,

$$\mathbb{P}(\mathcal{F}(\eta) \cap E \neq \varnothing) = \begin{cases} 1, & \text{if } E \notin \{\dim_{\mathrm{P}} < R(\eta)\}_\sigma \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

Moreover, if $E \notin \{\dim_{\mathrm{P}} < R(\eta)\}_\sigma$, then $\dim_{\mathrm{P}}(\mathcal{F}(\eta) \cap E) = \dim_{\mathrm{P}}(E)$, a.s. and

$$\dim_{\mathrm{H}}(E) - R(\eta) \leqslant \dim_{\mathrm{H}}(\mathcal{F}(\eta) \cap E) \leqslant \dim_{\mathrm{P}}(E) - R(\eta) \quad \text{a.s.}$$

In particular, we can take $\eta = 0$, and note that $R(0) = \|f\|_{\mathbb{H}}^2$, while

$$\mathfrak{D}(f) = \mathcal{F}(0) = \Gamma(\|f\|_{\mathbb{H}}^2)\,.$$

This establishes Theorem 1.7. □

2.3 Proof of Theorem 1.8

Let X_d denote Brownian motion in $\mathbb{R}^d$, and for all $t \geqslant 0$ and $h > 0$, define

$$Y(t,h) := \frac{q_d^2 h}{2|\log h| R_d(t,h)^2}\,.$$

In order to verify Condition 1 we use the 'small-ball' estimate of Lévy (1953) (see also (3.2) in Ciesielski and Taylor (1962)),

$$\mathbb{P}\Big(\max_{0 \leqslant t \leqslant 1} |X_d(t)| < a\Big) = \sum_{k=1}^{\infty} \xi_{d,k} \exp\Big(-\frac{q_{d,k}^2}{2a^2}\Big),$$

where $\xi_{d,k}$ are positive constants and $q_{d,k}$ are the positive zeros of the Bessel function $J_{(d-2)/2}$ with $q_{d,1} = q_d$. Note that as $a \to 0$, the above probability is equivalent to $\xi_{d,1} \exp(-q_d^2/(2a^2))$. Since

$$\mathbb{P}\Big(Y(t,h) > \gamma\Big) = \mathbb{P}\Big(R_d(t,h) < \frac{q_d h^{1/2}}{\sqrt{2\gamma|\log h|}}\Big)\,,$$

we see that Condition 1 is satisfied with $y_0 = 0$ and any y_1. As in our proofs of Theorems 1.1 and 1.6, Conditions 2 and 3 follow from the independence of increments and Lévy's modulus of continuity. Therefore the conclusion of Theorem 1.8 follows from (2.1), and Remarks 1.9 and 1.11 follow from Theorems 2.1 and 2.3, respectively. $\square$

3 Discrete Limsup Random Fractals

In this section, we establish a general result pertaining to "discrete limsup random fractals". This will be used in the proof of the results of Section 2.

Throughout, let us fix an integer $N \geqslant 1$. For every integer $n \geqslant 1$, let $\mathcal{D}_n$ denote the collection of all hyper-cubes of the form $[k^1 2^{-n}, (k^1+1)2^{-n}] \times \cdots \times [k^N 2^{-n}, (k^N+1)2^{-n}]$, where $k \in \mathbb{Z}_+^N$ is any N–dimensional positive integer. In words, $\mathcal{D}_n$ denotes the totality of all N–dimensional dyadic hyper-cubes. Suppose for each integer $n \geqslant 1$, $\{Z_n(I);\ I \in \mathcal{D}_n\}$ denotes a collection of random variables, each taking values in $\{0,1\}$. By a discrete **limsup random fractal**, we mean a random set of the form $A := \limsup_n A(n)$, where,

$$A(n) := \bigcup_{I \in \mathcal{D}_n : Z_n(I)=1} I^o,$$

where I^o denotes the interior of I. Discrete limsup random fractals and some of their dimension properties can be found in Orey and Taylor (1974), in Deheuvels and Mason (1998) and in Dembo, Peres, Rosen and Zeitouni (1998). The goal of this section is the determination of hitting probabilities for a discrete limsup random fractal A, under some conditions on the random variables $\{Z_n(I);\ I \in \mathcal{D}_n\}$.

Condition 4: the index assumption. Suppose that for each $n \geqslant 1$, the mean $p_n := \mathbb{E}[Z_n(I)]$ is the same for all $I \in \mathcal{D}_n$ and that

$$\lim_{n \to \infty} \frac{1}{n} \log_2 p_n = -\gamma,$$

for some $\gamma > 0$, where $\log_2$ is the logarithm in base 2.

We refer to γ as the **index** of the limsup random fractal A.

Condition 5: a bound on correlation length. For each $\varepsilon > 0$, define

$$f(n, \varepsilon) := \max_{I \in \mathcal{D}_n} \# \Big\{ J \in \mathcal{D}_n : \ \mathrm{Cov}(Z_n(I), Z_n(J)) \geqslant \varepsilon \mathbb{E}[Z_n(I)] \mathbb{E}[Z_n(J)] \Big\}.$$

Suppose that $\delta > 0$ satisfies

$$\forall \varepsilon > 0, \quad \limsup_{n \to \infty} \frac{1}{n} \log_2 f(n, \varepsilon) \leqslant \delta \, .$$

If Condition 5 holds for all $\delta > 0$, then we say that **Condition** 5_* holds.

We are ready to state and prove the main result of this section.

Theorem 3.1 *Suppose $A = \limsup_n A(n)$ is a discrete limsup random fractal which satisfies Condition 4 with index γ, and Condition 5 for some $\delta > 0$. Then, for any analytic set $E \subset \mathbb{R}_+^N$,*

$$\mathbb{P}(A \cap E \neq \varnothing) = \begin{cases} 1, & \text{if } \dim_{\mathrm{P}}(E) > \gamma + \delta \\ 0, & \text{if } \dim_{\mathrm{P}}(E) < \gamma \end{cases}.$$

Proof. First, we show that $\dim_{\mathrm{P}}(E) < \gamma$ implies that $A \cap E = \varnothing$, a.s. By regularization (see Mattila 1995), it suffices to show that whenever $\overline{\dim}_{\mathrm{M}}(E) < \gamma$, then $A \cap E = \varnothing$, a.s. Fix an arbitrary but small $\eta > 0$ such that $\overline{\dim}_{\mathrm{M}}(E) < \gamma - \eta$. By the definition of upper Minkowski dimension, we can find $\theta \in]0, \gamma - \eta[$, such that for all $n \geqslant 1$,

$$\# \Big\{ I \in \mathcal{D}_n : \ I \cap E \neq \varnothing \Big\} \leqslant 2^{n\theta}. \tag{3.1}$$

On the other hand, by Condition 4, for all n large enough,

$$p_n \leqslant 2^{-n(\gamma - \eta)}. \tag{3.2}$$

It follows from (3.1) and (3.2) that for each $n \geqslant 1$

$$\begin{aligned} \mathbb{P}\Big(E \cap A(n) \neq \varnothing \Big) &\leqslant 2^{n\theta} \max_{I \in \mathcal{D}_n} \mathbb{P}(I \cap A(n) \neq \varnothing) \\ &= 2^{n\theta} p_n \\ &\leqslant 2^{-n(\gamma - \eta - \theta)}. \end{aligned}$$

Since $\theta < \gamma - \eta$, the Borel-Cantelli lemma implies that there exists a random variable n_0, such that $E \cap A(n) = \varnothing$ a.s. for all $n \geqslant n_0$. This shows that $A \cap E = \varnothing$, a.s.

It remains to show that if $\dim_{\mathrm{P}}(E) > \gamma+\delta$, then $A\cap E \neq \varnothing$, a.s. Indeed, suppose $\dim_{\mathrm{P}}(E) > \gamma+\delta$. By Joyce and Preiss (1995), we can find a closed $E_\star \subset E$, such that for all open sets V, whenever $E_\star \cap V \neq \varnothing$, then $\overline{\dim}_{\mathrm{M}}(E_\star\cap V) > \gamma+\delta$. It suffices to show that with probability one, $A\cap E_\star \neq \varnothing$. Define the open sets $B(n) := \cup_{k=n}^\infty A(k)$, $n\geqslant 1$. We claim that for all $n\geqslant 1$, the open set $B(n)\cap E_\star$ is a.s. dense in (the complete metric space) $E_\star$. If so, Baire's category theorem (see Munkres 1975) implies that $E_\star \cap \bigcap_{n=1}^\infty B(n)$ is dense in $E_\star$ and in particular, nonempty. Since $A = \cap_{n=1}^\infty B(n)$, the result follows. Fix an open set V such that $V\cap E_\star \neq \varnothing$. It suffices to show that $A(n)\cap V\cap E_\star \neq \varnothing$ for infinitely many n, a.s. Indeed, this will imply that $B(n)\cap V \cap E_\star \neq \varnothing$ for all n a.s.; by letting V run over a countable base for the open sets, we will conclude that $B(n)\cap E_\star$ is a.s. dense in $E_\star$.

Thus fix an open set V such that $V\cap E_\star \neq \varnothing$. Let $\mathcal{N}_n$ denote the total number of hyper-cubes $I\in\mathcal{D}_n$ such that $I\cap V\cap E_\star \neq \varnothing$. Since $\overline{\dim}_{\mathrm{M}}(V\cap E_\star) > \gamma+\delta$, there exists $\gamma_1 > \gamma+\delta$, such that $\mathcal{N}_n \geqslant 2^{n\gamma_1}$ for infinitely many integers n. In other words, $\#(\mathfrak{N}) = \infty$, where

$$\mathfrak{N} := \Big\{ n\geqslant 1 :\ \mathcal{N}_n \geqslant 2^{n\gamma_1} \Big\}. \tag{3.3}$$

Define $S_n := \sum Z_n(I)$, where the sum is taken over all $I\in\mathcal{D}_n$ such that $I\cap E_\star\cap V\neq\varnothing$. In words, S_n is the total number of hyper–cubes $I\in\mathcal{D}_n$ such that $I\cap V\cap E_\star\cap A(n)\neq\varnothing$. We need only show that $S_n>0$ for infinitely many n. We want to estimate

$$\mathrm{Var}(S_n) = \sum_{\substack{I\in\mathcal{D}_n:\\ I\cap V\cap E_\star\neq\varnothing}} \ \sum_{\substack{J\in\mathcal{D}_n:\\ J\cap V\cap E_\star\neq\varnothing}} \mathrm{Cov}(Z_n(I), Z_n(J)).$$

Fix $\varepsilon>0$ and for each $I\in\mathcal{D}_n$, let $\mathcal{G}_n(I)$ denote the collection of all $J\in\mathcal{D}_n$ such that

(i) $J\cap V\cap E_\star\neq\varnothing$, and

(ii) $\mathrm{Cov}(Z_n(I), Z_n(J)) \leqslant \varepsilon p_n^2$.

If $J\in\mathcal{D}_n$ satisfies (i) but not (ii), then we say that J is in $\mathcal{B}_n(I)$. (The notation is meant to indicate 'good' and 'bad' choices of J.) Thus,

$$\mathrm{Var}(S_n) \leqslant \varepsilon\mathcal{N}_n^2 p_n^2 + \sum_{\substack{I\in\mathcal{D}_n:\\ J\in\mathcal{B}_n(I)}} \mathrm{Cov}(Z_n(I), Z_n(J)).$$

For the remaining covariance, use the fact that all $Z_n(I)$'s are either 0 or 1. In particular, $\mathrm{Cov}(Z_n(I), Z_n(J)) \leqslant \mathbb{E}[Z_n(I)] = p_n$. Thus,

$$\mathrm{Var}(S_n) \leqslant \varepsilon\mathcal{N}_n^2 p_n^2 + \mathcal{N}_n \max_{I\in\mathcal{D}_n} \#\mathcal{B}_n(I) p_n.$$

Recalling the notation of Condition 5, we can deduce that

$$\mathrm{Var}(S_n) \leqslant \mathcal{N}_n p_n\{ f(n,\varepsilon) + \varepsilon\mathcal{N}_n p_n\}.$$

Combining this with the Chebyshev's inequality, we obtain:

$$\mathbb{P}(S_n = 0) \leqslant \frac{\mathrm{Var}(S_n)}{\left(\mathbb{E}[S_n]\right)^2} \leqslant \varepsilon + \frac{f(n,\varepsilon)}{\mathcal{N}_n p_n},$$

since $\mathbb{E}[S_n] = \mathcal{N}_n p_n$. By Conditions 4 and 5, there exists sequences a_n and b_n such that $\lim_n a_n = \lim_n b_n = 0$ and $p_n = 2^{-n\gamma(1+a_n)}$ and $f(n,\varepsilon) \leqslant 2^{n(\delta+b_n)}$. Thus, by (3.3) and the inequality $\gamma_1 > \gamma + \delta$,,

$$\limsup_{n\to\infty: n\in\mathfrak{N}} \mathbb{P}(S_n = 0) \leqslant \varepsilon + \limsup_{n\to\infty} \frac{2^{n(\delta+b_n)}}{2^{n(\gamma_1-\gamma-a_n)}} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we see that $\mathbb{P}(S_n = 0) \to 0$ as $n \to \infty$ in $\mathfrak{N}$. Finally,

$$\mathbb{P}(S_n > 0 \text{ i.o.}) \geqslant \limsup_{n\to\infty} \mathbb{P}(S_n > 0) = 1\,.$$

□

Next, we derive an extension of Theorem 3.1.

Theorem 3.2 *Suppose A is a discrete limsup random fractal which satisfies Conditions 4 and 5_*. On the same probability space, consider A', an independent discrete limsup random fractal which satisfies Conditions 4 and 5_* and has exponent γ'. Then for any analytic set $E \subset \mathbb{R}^N_+$ satisfying $\dim_{\mathrm{P}}(E) > \gamma \vee \gamma'$, we have $\mathbb{P}(A \cap A' \cap E \neq \varnothing) = 1$. In particular, if $\dim_{\mathrm{P}}(E) > \gamma$, then $\dim_{\mathrm{P}}(A \cap E) = \dim_{\mathrm{P}}(E)$, a.s.*

Proof. Consider the closed set $E_\star$ and the open sets $B(n)$ of the described proof of Theorem 3.1, and let $\{B'(n)\}$ be the sequence of open sets corresponding to the limsup random fractal A'. Then, Theorem 3.1 shows that for any open set V such that $V \cap E_\star \neq \varnothing$,

$$\mathbb{P}(B(n) \cap E_\star \cap V \neq \varnothing \ \text{ for all } n) = \mathbb{P}(B'(n) \cap E_\star \cap V \neq \varnothing \ \text{ for all } n) = 1.$$

By independence, there exists one null set outside which for all open hyper-cubes V of 'rational end-points' with $V \cap E_\star \neq \varnothing$, we have

$$B(n) \cap E_\star \cap V \neq \varnothing \ \text{ and } \ B'(n) \cap E_\star \cap V \neq \varnothing \ \text{ for all } n \geqslant 1.$$

That is, $\{B(n) \cap E_\star\}_{n\geqslant 1} \cup \{B'(n) \cap E_\star\}_{n\geqslant 1}$ is a countable collection of open, dense subsets of the complete metric space $E_\star$. Baire's theorem implies that a.s., $A \cap A' \cap E_\star$ is dense in $E_\star$. In particular, $A \cap A' \cap E \neq \varnothing$, a.s. To conclude, suppose $\dim_{\mathrm{P}}(E) > \gamma \vee \gamma'$. By what we have so far, $\mathbb{P}(A \cap A' \cap E \neq \varnothing) = 1$. By conditioning on A and applying Theorem 3.1 to the random fractal A' and the target set $A \cap E$, we see that $\dim_{\mathrm{P}}(A \cap E) \geqslant \gamma'$ a.s. Letting γ' increase to $\dim_{\mathrm{P}}(E)$, we conclude that $\dim_{\mathrm{P}}(A \cap E) \geqslant \dim_{\mathrm{P}}(E)$, a.s. The proof is complete. □

Corollary 3.3 *Suppose A is a discrete limsup random fractal satisfying Condition 4 with index γ and condition 5_*. Then, for any analytic set $E \subset \mathbb{R}^N_+$,*

$$\dim_{\mathrm{H}}(E) - \gamma \leqslant \dim_{\mathrm{H}}(A \cap E) \leqslant \dim_{\mathrm{P}}(E) - \gamma \quad a.s. \tag{3.1}$$

In particular, $\dim_{\mathrm{H}}(A) = N - \gamma$, a.s.

Proof. The right-hand inequality in (3.1) does not require condition 5_*, and can be verified by a direct first-moment calculation:

By regularization, it suffices to prove that

$$\dim_{\mathrm{H}}(A \cap E) \leqslant \overline{\dim}_{\mathrm{M}}(E) - \gamma \quad a.s. \tag{3.2}$$

Let $\mathcal{N}_n$ denote the total number of hyper-cubes $I \in \mathcal{D}_n$ such that $I \cap E \neq \varnothing$. Define $S_n := \sum Z_n(I)$, where the sum is taken over all $I \in \mathcal{D}_n$ such that $I \cap E \neq \varnothing$. Then,

$$\mathbb{E}(S_n) = \mathcal{N}_n p_n \leqslant 2^{n(\xi+\varepsilon_n)} 2^{n(\varepsilon_n - \gamma)}$$

where $\xi = \overline{\dim}_{\mathrm{M}}(E)$ and $\varepsilon_n \to 0$. Thus $\mathbb{E}\sum_n S_n 2^{-n\theta} < \infty$ for any $\theta > \xi - \gamma$. Finally, for any n_0, the intersection $A \cap E$ has a cover consisting of S_n intervals in $\mathcal{D}_n$ for each $n \geqslant n_0$. By picking n_0 large, we see that the θ-dimensional Hausdorff measure of $A \cap E$ vanishes, whence (3.2) follows.

The left-hand inequality in (3.1) is not as easy to prove from scratch, but it follows from Theorem 3.1 by a co–dimension argument; similar arguments can be found, e.g., in Peres (1996b) or Khoshnevisan and Shi (1998). For later use, we state it in greater generality.

Lemma 3.4 *Equip $[0,1]^N$ with the Borel σ-field. Suppose $A = A(\omega)$ is a random set in $[0,1]^N$ (*i.e., *$\mathbf{1}_{A(\omega)}(x)$ is jointly measurable) such that for any compact $E \subset [0,1]^N$ with $\dim_{\mathrm{H}}(E) > \gamma$, we have $\mathbb{P}(A \cap E \neq \emptyset) = 1$. Then, for any analytic set $E \subset [0,1]^N$,*

$$\dim_{\mathrm{H}}(E) - \gamma \leqslant \dim_{\mathrm{H}}(A \cap E) \quad a.s. \tag{3.3}$$

Proof. For $\alpha < \dim_{\mathrm{H}}(E) - \gamma$, let Υ_α be a random closed set (independent of A) in the cube $[0,1]^N$, that has Hausdorff dimension $N - \alpha$ a.s., and satisfies $\mathbb{P}(\Upsilon_\alpha \cap F \neq \emptyset) > 0$ for any Borel set $F \subset [0,1]^N$ that has $\dim_{\mathrm{H}}(F) > \alpha$, but $\mathbb{P}(\Upsilon_\alpha \cap F \neq \emptyset) = 0$ if $\dim_{\mathrm{H}}(F) < \alpha$. Such a random set can be obtained, *e.g.*, as the closed range of an $N - \alpha$ stable process if $\alpha > N - 2$, and as a fractal-percolation limit set in general; see Hawkes (1971), Peres (1996a) and the references therein. With the latter choice, the intersection of Υ_α with an independent copy of Υ_β has the same distribution as $\Upsilon_{\alpha+\beta}$ for any $\alpha, \beta \geqslant 0$. Consequently,

$$\| \dim_{\mathrm{H}}(E \cap \Upsilon_\alpha) \|_\infty = \dim_{\mathrm{H}}(E) - \alpha,$$

where the L^∞ norm is taken in the underlying probability space; see Peres (1996b). Let $\widehat{\Upsilon}_\alpha$ be a union of countably many i.i.d. copies of Υ_α. Then $\mathbb{P}(\widehat{\Upsilon}_\alpha \cap E \neq \emptyset) = 1$ for any analytic set $E \subset [0,1]^N$ with $\dim_{\mathrm{H}}(E) > \alpha$, and

$$\dim_{\mathrm{H}}(E \cap \widehat{\Upsilon}_\alpha) = \dim_{\mathrm{H}}(E) - \alpha > \gamma \quad a.s.,$$

whence $A \cap E \cap \widehat{\Upsilon}_\alpha \neq \emptyset$ a.s. in the product space; here, we used the fact that any analytic set of Hausdorff dimension $> \gamma$ contains a compact set of Hausdorff dimension $> \gamma$. Therefore, $\dim_{\mathrm{H}}(A \cap E) \geqslant \alpha$ a.s. Taking $\alpha \to \dim_{\mathrm{H}}(E) - \gamma$ completes our proof of the lemma and the corollary. □

Remark 3.5 The sets Υ_α used in the previous proof may be constructed as follows. Consider the natural tiling of the unit cube $[0,1]^N$ by 2^N closed cubes of side $1/2$. Let Ξ_1 be a random subcollection of these cubes, where each cube has probability $2^{-\alpha}$ of belonging to Ξ_1, and these events are mutually independent. At the k'th stage, if Ξ_k is nonempty, tile each cube $Q \in \Xi_k$ by 2^N closed subcubes of side 2^{-k-1} (with disjoint interiors) and include each of these subcubes in Ξ_{k+1} with probability $2^{-\alpha}$ (independently). Finally, define

$$\Upsilon_\alpha = \bigcap_{k=1}^{\infty} \bigcup_{Q \in \Xi_k} Q \,.$$

We record the next lemma for use in the next section.

Lemma 3.6 *Let $E \subset [0,1]^N$ be an analytic set. Then, for $\alpha \leqslant \dim_{\mathrm{P}}(E)$, the sets Υ_α defined above satisfy*

$$\dim_{\mathrm{P}}(E \cap \Upsilon_\alpha) \leqslant \dim_{\mathrm{P}}(E) - \alpha \quad \text{a.s.} \tag{3.4}$$

Proof. By regularization, it suffices to prove that if $E \subset [0,1]^N$ is compact, then

$$\overline{\dim}_{\mathrm{M}}(E \cap \Upsilon_\alpha) \leqslant \overline{\dim}_{\mathrm{M}}(E) - \alpha \quad \text{a.s.}, \tag{3.5}$$

for $\alpha \leqslant \overline{\dim}_{\mathrm{M}}(E)$. Let $\mathcal{N}_n(E)$ denote the total number of hyper-cubes $I \in \mathcal{D}_n$ that intersect E. Fix $\beta > \overline{\dim}_{\mathrm{M}}(E)$. By the definition of Minkowski dimension, $\sum_n \mathcal{N}_n(E) 2^{-n\beta} < \infty$. Clearly $\mathbb{E}\mathcal{N}_n(E \cap \Upsilon_\alpha) \leqslant 2^{-n\alpha}\mathcal{N}_n(E)$, and therefore

$$\mathbb{E}\Big(\sum_n 2^{n(\alpha-\beta)} \mathcal{N}_n(\Upsilon_\alpha \cap E) \Big) < \infty.$$

Thus the sum inside the expectation is finite a.s., whence $\overline{\dim}_{\mathrm{M}}(E \cap \Upsilon_\alpha) \leqslant \beta - \alpha$ a.s., and (3.5) follows. □

As a consequence of Theorem 3.2, Corollary 3.3 and Lemma 3.4, the following result gives the Hausdorff dimension and packing dimension of the intersection of two independent limsup random fractals.

Corollary 3.7 *Let A and A' be two independent discrete limsup random fractals which satisfy Conditions 4 and 5_* with indices γ and γ' respectively. Then, for any analytic set $E \subset \mathbb{R}_+^N$, we have*

$$\dim_{\mathrm{H}}(E) - \gamma \vee \gamma' \leqslant \dim_{\mathrm{H}}(A \cap A' \cap E) \leqslant \dim_{\mathrm{P}}(E) - \gamma \vee \gamma' \quad a.s. \tag{3.6}$$

In particular, $\dim_{\mathrm{H}}(A \cap A') = N - \gamma \vee \gamma'$, a.s. Furthermore, for any analytic set $E \subset \mathbb{R}_+^N$ satisfying $\dim_{\mathrm{P}}(E) > \gamma \vee \gamma'$, we have $\dim_{\mathrm{P}}(A \cap A' \cap E) = \dim_{\mathrm{P}}(E)$, a.s.

4 Proofs of general theorems from Section 2

4.1 Proof of Theorem 2.1

The proof is divided into three steps.

Step One: the upper bound

Our strategy is to show that for all **compact** sets $E \subset \mathbb{R}_+^N$, if $\overline{\dim}_{\rm M}(E) < \lambda \leqslant y_1$, then

$$\limsup_{h\to 0+} \sup_{t\in E} Y(t,h) \leqslant \lambda \vee y_0, \quad \text{a.s.} \tag{4.1}$$

From this, we will deduce that

$$\sup_{t\in E} \limsup_{h\to 0+} Y(t,h) \leqslant \overline{\dim}_{\rm M}(E) \vee y_0, \quad \text{a.s.}$$

By regularization, for all analytic sets $E \subset \mathbb{R}_+^N$ with $\dim_{\rm P}(E) \leqslant y_1$,

$$\sup_{t\in E} \limsup_{h\to 0+} Y(t,h) \leqslant \dim_{\rm P}(E) \vee y_0, \quad \text{a.s.},$$

and this will constitute the first step of our proof; see Mattila (1995) for details on regularization. We now prove (4.1). Fix $\rho > 2$ and for all $n \geqslant 1$ and all $k \in \mathbb{Z}_+^N$, define

$$I_{k,n} := [k^1\rho^{-n}, (k^1+1)\rho^{-n}] \times \cdots \times [k^N\rho^{-n}, (k^N+1)\rho^{-n}].$$

Without loss of generality, we will prove the validity of (4.1) when $E \subset [0,1]^N$. Let $\mathcal{G}_n$ denote the collection of all points of the form $k\rho^{-n}$ such that (i) $k \in \mathbb{Z}_+^N$ satisfies $k^i \leqslant \rho^n + 1$ for all $1 \leqslant i \leqslant N$; and (ii) $I_{k,n} \cap E \neq \varnothing$. For brevity, write $\xi := \overline{\dim}_{\rm M}(E)$. By definitions, there exists a sequence c_n such that $\lim_n c_n = 0$ and $\#\mathcal{G}_n \leqslant \rho^{n\xi(1+c_n)}$. Hence, for any $\gamma \in (y_0, y_1]$,

$$\begin{aligned}
&\mathbb{P}\Big(\max_{t\in\mathcal{G}_n} \max_{\{m:\ 2^{-n-1}\leqslant m\rho^{-n}\leqslant 2^{-n}\}} Y(t, m\rho^{-n}) > \gamma\Big) \\
&\leqslant \rho^{n\xi(1+c_n)} \frac{1}{2}\Big(\frac{\rho}{2}\Big)^n \max_{t\in\mathcal{G}_n} \max_{\{m:\ 2^{-n-1}\leqslant m\rho^{-n}\leqslant 2^{-n}\}} \mathbb{P}\Big(Y(t, m\rho^{-n}) > \gamma\Big) \\
&\leqslant \frac{1}{2}\Big(\frac{\rho}{2}\Big)^n \rho^{n\xi(1+c_n)} 2^{-n\gamma(1+d_n)}
\end{aligned} \tag{4.2}$$

where $\lim_n d_n = 0$. We used Condition 1 in obtaining the last inequality. If we choose γ satisfying $\lambda > \gamma > \xi = \overline{\dim}_{\rm M}(E)$, then we can pick $\rho > 2$ sufficiently close to 2 such that the sum (4.2) is finite. By the Borel–Cantelli lemma,

$$\limsup_{n\to\infty} \max_{t\in\mathcal{G}_n} \max_{\{m:\ 2^{-n-1}\leqslant m\rho^{-n}\leqslant 2^{-n}\}} Y(t, m\rho^{-n}) \leqslant \gamma, \qquad \text{a.s.} \tag{4.3}$$

Now we use Condition 3 to show (4.1). For any $0 < \varepsilon < \lambda - \gamma$, $h > 0$ small and $s \in E$, there exist a positive integer n and $t \in \mathcal{G}_n$ such that

$$2^{-n-1} \leqslant h < 2^{-n} \quad \text{and} \quad |t - s| \leqslant \sqrt{N}\rho^{-n}.$$

Furthermore, we can find a positive integer m such that such that $2^{-n-1} \leqslant m\rho^{-n} \leqslant 2^{-n}$ and $|h - m\rho^{-n}| \leqslant \rho^{-n}$. Combining the triangle inequality, (4.3) and Condition 3, we see that for $h > 0$ small enough, or equivalently, for n large enough

$$\begin{aligned}
Y(s,h) &\leqslant Y(t, m\rho^{-n}) + |Y(s,h) - Y(s, m\rho^{-n})| + |Y(s, m\rho^{-n}) - Y(t, m\rho^{-n})| \\
&\leqslant \gamma + \varepsilon/2 + \varepsilon/2 \\
&\leqslant \lambda\,.
\end{aligned}$$

This proves (4.1).

Step Two: the lower bound

By regularization, it suffices to show that for compact $E \subset [0,1]^N$ with $\dim_{\rm P}(E) \in (y_0, y_1]$,

$$\sup_{t\in E} \limsup_{h\to 0+} Y(t,h) \geqslant \dim_{\rm P}(E), \qquad \text{a.s.}$$

Fix γ such that $y_0 < \gamma < \dim_{\rm P}(E)$. It is enough to show that

$$\sup_{t\in E} \limsup_{h\to 0+} Y(t,h) \geqslant \gamma, \qquad \text{a.s.} \tag{4.4}$$

Choose $\delta > 0$ such that $\gamma + \delta < \dim_{\rm P}(E)$. Recall the notation of Section 3 and for all $I \in \mathcal{D}_n$, define $\pi_n(I)$ to be the element in I which is coordinatewise smaller than all other elements of I. For $I \in \mathcal{D}_n$, let $Z_n(I)$ denote the indicator function of the event $(Y(\pi_n(I), 2^{(\delta-1)n}) > \gamma)$. By Condition 1, the distribution of $Z_n(I)$ does not depend on the choice of $I \in \mathcal{D}_n$. Moreover, if we let $p_n := \mathbb{E}[Z_n(I)]$ for $I \in \mathcal{D}_n$, then $\lim_n n^{-1}\log_2 p_n = -\gamma(1-\delta)$. In other words, we have verified Condition 4 with index $\gamma(1-\delta)$, for the discrete limsup random fractal obtained from the $Z_n(I)$. Similarly, by Condition 2, for each $\varepsilon > 0$, there exists a regularly varying function ψ of order 1 at 0, such that whenever $I, J \in \mathcal{D}_n$ satisfy $|\pi_n(I) - \pi_n(J)| \geqslant \psi(2^{(\delta-1)n})$, then $\mathrm{Cov}(Z_n(I), Z_n(J)) < \varepsilon \mathbb{E}[Z_n(I)]\mathbb{E}[Z_n(J)]$. In other words,

$$f(n,\varepsilon) \leqslant \max_{I\in\mathcal{D}_n} \#\Big\{ J \in \mathcal{D}_n:\ |\pi_n(I) - \pi_n(J)| \leqslant \psi(2^{(\delta-1)n}) \Big\} \leqslant \Big[2^n \psi(2^{(\delta-1)n}) \Big]^N.$$

Since ψ is regularly varying of order 1, it follows that Condition 5 holds, with the same δ. Since $\dim_{\rm P}(E) > \gamma + \delta$, Theorem 3.1 implies that there almost surely exists $t \in E$, such that $Y(2^{-n}[t2^n], 2^{(\delta-1)n}) \geqslant \gamma$ for infinitely many n. In particular,

$$\sup_{t\in E} \limsup_{n\to\infty} Y(2^{-n}[t2^n], 2^{(\delta-1)n}) \geqslant \gamma, \qquad \text{a.s.}$$

By Condition 3,

$$\lim_{n\to\infty} \sup_{t\in I:\ I\in\mathcal{D}_n} \Big| Y(t, 2^{(\delta-1)n}) - Y(2^{-n}[2^n t], 2^{(\delta-1)n}) \Big| = 0, \qquad \text{a.s.}$$

Thus, if $\dim_{\rm P}(E) > \gamma > y_0$, then (4.4) holds. $\square$

Step Three: dimension estimates

We can now complete our proof of Theorem 2.1. The results of steps one and two are equivalent to the following:

Let $y_0 < \lambda \leqslant y_1$. for any analytic set $E \subset [0,1]^N$,

$$\mathbb{P}(\Gamma(\lambda) \cap E \neq \varnothing) = \begin{cases} 1, & \text{if } \dim_{\rm P}(E) > \lambda \\ 0, & \text{if } \dim_{\rm P}(E) < \lambda. \end{cases}$$

This implies that if $E = \cup_{n=1}^\infty E_n$ with $\dim_{\rm P}(E_n) < \lambda$ for all $n \geqslant 1$, then $\mathbb{P}(\Gamma(\lambda) \cap E \neq \varnothing) = 0$.

Now suppose that the compact set $E \subset [0,1]^N$ is not in $\{\dim_{\rm P} < \lambda\}_\sigma$. Let $\{U_i\}$ be a countable basis for the open sets in $[0,1]^N$. Define

$$E_\star = E \setminus \bigcup \{E \cap U_i \,:\, E \cap U_i \in \{\dim_{\rm P} < \lambda\}_\sigma\}\,. \tag{4.5}$$

Then, $E_\star$ is compact, and every open set V that intersects $E_\star$, satisfies $\dim_{\rm P}(E_\star \cap V) \geqslant \lambda$. (Otherwise, for some set U_j from the basis, we would have $\emptyset \neq E_\star \cap U_j \in \{\dim_{\rm P} < \lambda\}_\sigma$ whence $E \cap U_j \in \{\dim_{\rm P} < \lambda\}_\sigma$, a contradiction.) For $n, m \geqslant 1$, denote

$$U(n,m) := \{t \in E_\star : \exists h \in (0, m^{-1}),\ Y(t,h) > \lambda - n^{-1}\}\,.$$

By continuity of $Y(t,h)$, the set $U(n,m)$ is relatively open in $E_\star$; by the argument in step two above, $U(n,m)$ is also dense in $E_\star$. Since $E_\star$ is compact, Baire's theorem implies that the set

$$E_\star \cap \Gamma(\lambda) = \bigcap_{n,m=1}^{\infty} U(n,m)$$

is dense in $E_\star$. This completes the characterization of hitting probabilities for $\Gamma(\lambda)$. The remainder of our proof of Theorem 2.1 follows that of Corollary 3.3: let Y' be an independent copy of Y. In analogy with the definition of $\Gamma(\lambda)$, we can define for all $\lambda > 0$,

$$\Gamma'(\lambda) := \Big\{ t \in [0,1]^N :\ \limsup_{h \to 0+} Y'(t,h) \geqslant \lambda \Big\}.$$

Suppose $E \subset [0,1]^N$ is a compact set such that $E \notin \{\dim_{\rm P} < \lambda \vee \lambda'\}_\sigma$, yet $\dim_{\rm P}(E) \leqslant y_1$. Then, by Theorem 3.2 and the arguments of step two, $\mathbb{P}(\Gamma(\lambda) \cap \Gamma'(\lambda') \cap E \neq \varnothing) = 1$. Using the presented proof of Theorem 3.2 and applying our characterization of hitting probabilities, we can deduce now that $\dim_{\rm P}(\Gamma(\lambda) \cap E) = \dim_{\rm P}(E)$, a.s.

The Hausdorff dimension estimate from below in (2.3) follows immediately from Lemma 3.4. To prove the upper estimate in (2.3), we will use Lemma 3.6. For $\alpha > \dim_{\rm P}(E) - \lambda$, that lemma implies that $\dim_{\rm P}(E \cap \Upsilon_\alpha) < \lambda$ a.s. By step one of our proof, $\Gamma(\lambda) \cap E \cap \Upsilon_\alpha$ is a.s. empty, whence the intersection properties of Υ_α yield that $\dim_{\rm H}(\Gamma(\lambda) \cap E) \leqslant \alpha$ a.s. Since this holds for all $\alpha > \dim_{\rm P}(E) - \lambda$, the upper estimate in (2.3) follows. □

4.2 Proof of Theorem 2.3

If E is a countable union of Borel sets E_n with $\dim_{\rm P}(E_n) < \lambda$, it follows from Theorem 2.1 that $\Gamma_=(\lambda) \cap E = \varnothing$ a.s. Suppose that E is compact and not in $\{\dim_{\rm P} < \lambda\}_\sigma$. By Joyce and Preiss (1995), there exists a compact set $E_0 \subset E$ that is not in $\{\dim_{\rm P} < \lambda\}_\sigma$ and satisfies $\dim_{\rm P}(E_0) = \lambda$. Using the argument in step three of §4.1, we can find a compact set $E_\star \subset E_0$, such that $\dim_{\rm P}(E_\star \cap V) = \lambda$ for any open set V that intersects $E_\star$. By Theorem 2.1, $\Gamma(\lambda) \cap E_\star$ is dense in $E_\star$ a.s. On the other hand, $\Gamma(\lambda + n^{-1}) \cap E_\star = \varnothing$ a.s. for any $n \geqslant 1$. Thus,

$$\Gamma(\lambda) \cap E_\star \cap \bigcap_{n=1}^{\infty} \{\Gamma(\lambda + n^{-1})\}^c \neq \varnothing \qquad \text{a.s.}$$

As this is $\Gamma_=(\lambda) \cap E_\star$, the first part of the theorem is proved. Finally, the equality involving packing dimension follows as in Theorem 2.1. The asserted Hausdorff dimension estimate from below follows from Lemma 3.4, while the upper estimate follows from Theorem 2.1. □

5 Fast Points for Gaussian Processes

In this section, we apply the general Theorems 2.1 and 2.3 to obtain information on the fast points of a large class of Gaussian processes with stationary increments. Throughout, suppose $\{G(t);\ t \geqslant 0\}$ is a mean zero Gaussian process with stationary increments such that for all $s, t \geqslant 0$,

$$\sigma^2(|t-s|) := \mathbb{E}\Big[\big(G(t)-G(s)\big)^2\Big].$$

We shall impose the following condition on the function σ.

Condition 6: There exist $\alpha \in]0,1[$, a function L which is slowly varying at 0 and positive constants c_1 and c_2 such that

$$c_1 s^{2\alpha} L(s) \leqslant \sigma^2(s) \leqslant c_2 s^{2\alpha} L(s) \qquad \text{for } s \geqslant 0 \text{ small enough.} \tag{5.1}$$

Furthermore $\sigma^2(s)$ is twice continuously differentiable and there exist positive constants c_3 and c_4 such that for $s \geqslant 0$ small enough,

$$\frac{d\sigma^2(s)}{ds} \leqslant c_3 \frac{\sigma^2(s)}{s} \quad \text{and} \quad \Big|\frac{d^2\sigma^2(s)}{ds^2}\Big| \leqslant c_4 \frac{\sigma^2(s)}{s^2}. \tag{5.2}$$

Remark 5.1 It follows from Theorem 1.8.2 in Bingham et al (1987) that, without loss of generality, we may and will assume that $L(s)$ varies smoothly near the origin with index 0. Hence,

$$\frac{s^n L^{(n)}(s)}{L(s)} \to 0 \quad \text{as} \quad s \to 0 \ \text{ for } n \geqslant 1, \tag{5.3}$$

where $L^{(n)}(s)$ is the n-th derivative of $L(s)$.

It is known (cf. Bingham et al (1987)) that the slowly varying function $L(s)$ can be represented by

$$L(s) = \exp\Big(\theta(s) + \int_s^1 \frac{\varepsilon(t)}{t}\ dt\Big),$$

where $\theta(s)$ and $\varepsilon(s)$ are bounded measurable functions and

$$\lim_{s\to 0} \theta(s) = c\ ,\ \ |c| < \infty; \quad \lim_{s \to 0} \varepsilon(s) = 0.$$

It is clear that

$$\tilde{L}(s) = \exp\Big(\int_s^1 \frac{|\varepsilon(t)|}{t}\ dt\Big)$$

is also slowing varying at 0.

Theorem 5.2 *Let G be a mean zero Gaussian process with stationary increments that satisfies Condition 6. Then, for all analytic sets $E \subset [0,1]$,*

$$\sup_{t\in E} \limsup_{h \to 0+} \frac{|G(t+h)-G(t)|}{\sqrt{2\sigma^2(h)|\log h|}} = \big(\dim_{\mathrm{P}}(E)\big)^{1/2}, \qquad \text{a.s.}$$

Theorem 5.2 refines earlier work of Marcus (1968).

Proof. As in our proof of Theorem 1.1, we let

$$Y(t,h) := \frac{|G(t+h)-G(t)|^2}{2\sigma^2(h)|\log h|}.$$

Standard estimates on the tails of Gaussian distributions reveal that Condition 1 holds with $y_0 = 0$ and any y_1. To check Condition 2, we use the same argument as that in Section 4 of Khoshnevisan and Shi (1998) to see that

$$\begin{aligned}&\mathbb{P}\Big(Y(t,h) > \gamma \mid Y(s,h) > \gamma\Big)\\ &\leqslant \mathbb{P}\Big(Y(t,h) > \gamma\Big)\frac{(1-\rho^2)}{(1-\rho^+)^{3/2}}(1-a^{-2})\exp\Big(\frac{a^2}{2}\frac{\rho^+-\rho^2}{1-\rho^2}\Big),\end{aligned}$$

where $\rho^+ := \max\{\rho, 0\}$, $a := \sqrt{2\gamma|\log h|}$ and

$$\begin{aligned}\rho &= \frac{1}{2\sigma^2(h)}E\Big[(G(t+h)-G(t))(G(s+h)-G(s))\Big]\\ &= \frac{1}{2\sigma^2(h)}\Big[\sigma^2(|t+h-s|)+\sigma^2(|t-h-s|)-2\sigma^2(|t-s|)\Big].\end{aligned}$$

Clearly, it suffices to show the existence of a function $h \mapsto \psi(h)$ which is regularly varying at zero of order 1, such that uniformly for all $t, s \in [0,1]$ which satisfy $|t-s| \geqslant \psi(h)$, we have $a^2\rho \to 0$, as $h \to 0+$.

Let $K(h) := \sigma^2(h)$ $(h \geqslant 0)$ and use Taylor expansion about $|t-s|$ to see that

$$\rho = \frac{h^2(K''(|t-s-\theta_1 h|)+K''(|t-s+\theta_2 h|))}{4K(h)},$$

where $0 \leqslant \theta_1, \theta_2 \leqslant 1$. By (5.3) it is easy to see that there exists a constant $\delta > 0$ such that the function $s^{2\alpha-2}L(s)$ is decreasing on $[0, 2\delta]$. Hence it follows from (5.1) and (5.2) that there exists a positive and finite constant c_5 such that for all $t, s \in [0,1]$ satisfying $|t-s| \leqslant \delta$ and all $h \in [0, \delta/2]$

$$|\rho| \leqslant c_5 \Big|\frac{h}{|t-s|-h}\Big|^{2-2\alpha}\frac{L(||t-s|-h|)}{L(h)}.$$

If $|t-s| \geqslant 2h$, then

$$\frac{L(||t-s|-h|)}{L(h)} \leqslant c_6 \tilde{L}(h)$$

for some finite constant c_6 depending on θ only. On the other hand, by the continuity of $K''(s)$ there exists a positive and finite constant c_7 (which may depend on δ) such that for all $t, s \in [0,1]$ satisfying $|t-s| \geqslant \delta$ and all $h \in [0, \delta/2]$ we have

$$|\rho| \leqslant c_7 \frac{h^{2-2\alpha}}{L(h)}.$$

Define

$$\psi(h) \triangleq h(|\log h|)^{2/(2-2\alpha)}\tilde{L}(h)^{1/(2-2\alpha)}.$$

We conclude that $a^2\rho \to 0$, as $h \to 0+$ uniformly for all $s, t \in [0,1]$ with $|t-s| \geqslant \psi(h)$. This implies Condition 2. Finally, by Theorem 2.1, it suffices to check Condition 3. By standard entropy methods (see Adler 1990), there exists a constant c_8 such that

$$\limsup_{h\to 0+} \sup_{0\leqslant t\leqslant 1} \frac{|G(t+h)-G(t)|}{\sigma(h)\sqrt{|\log h|}} \leqslant c_8, \qquad \text{a.s.}$$

This fact, the triangle inequality and the first inequality in (5.2) together imply that $Y(t,h)$ satisfies Condition 3. The theorem now follows from Theorem 2.1. $\square$

As a canonical example, consider fractional Brownian motion β of index α, where $\alpha \in]0,1[$ is fixed. In other words, $(\beta(t);\ t \geqslant 0)$ is a mean zero Gaussian process with $\beta(0) = 0$ and $\mathbb{E}[(\beta(t+h)-\beta(t))^2] = h^{2\alpha}$. Clearly, this is a Gaussian process which satisfies the conditions of the above Theorem. In fact, similar considerations yield the following which completes Theorem 1.1 of Khoshnevisan and Shi (1998).

Theorem 5.3 *Suppose $(\beta(t);\ t\geqslant 0)$ is fractional Brownian motion of index α. For any analytic set $E \subset [0,1]$,*

$$\sup_{t\in E} \limsup_{h\to 0+} \frac{|\beta(t+h)-\beta(t)|}{h^\alpha\sqrt{2|\log h|}} = (\dim_{\mathrm{P}}(E))^{1/2}, \qquad \text{a.s.}$$

Moreover,

$$\begin{aligned}
\limsup_{h\to 0+} \sup_{t\in E} \frac{|\beta(t+h)-\beta(t)|}{h^\alpha\sqrt{2|\log h|}} &= (\overline{\dim}_{\mathrm{M}}(E))^{1/2}, \qquad \text{a.s.}\\
\liminf_{h\to 0+} \sup_{t\in E} \frac{|\beta(t+h)-\beta(t)|}{h^\alpha\sqrt{2|\log h|}} &= (\underline{\dim}_{\mathrm{M}}(E))^{1/2}. \qquad \text{a.s.}
\end{aligned}$$

6 Rate of convergence in the functional LIL

Theorem 1.7 ensures that when $\dim_{\mathrm{P}}(E) > \|f\|_{\mathbb{H}}^2$, there a.s. exists some $t \in E$ so that f can be uniformly approximated on $[0,1]$ by normalized Brownian increments $\Delta_h[t]$, where $\Delta_h[t](s) = [X(t+sh)-X(t)]/\sqrt{2h|\log h|}$.

The following theorem shows that the rate of convergence of these approximations is completely determined by the gap $\dim_{\mathrm{P}}(E) - \|f\|_{\mathbb{H}}^2$. This theorem extends the results of Deheuvels and Mason (1998) (who considered $E = [0,1]$); it also refines the one-dimensional case of Theorem 1.7.

For any $f \in \mathbb{H}$ with $\|f\|_{\mathbb{H}} < 1$ and $c \geqslant 1$, let

$$\mathfrak{S}(f,c) = \Big\{ t \in [0,1] :\ \liminf_{h\to 0+} |\log h| \|\Delta_h[t] - f\|_\infty \leqslant c\omega_f \Big\},$$

where $\omega_f = \frac{\pi}{4}(1-\|f\|_{\mathbb{H}}^2)^{-1/2}$. Deheuvels and Mason (1998) proved that almost surely

$$\lim_{h\to 0+} \inf_{0\leqslant t\leqslant 1} |\log h| \cdot \|\Delta_h[t] - f\|_\infty = \omega_f$$

and

$$\dim_{\mathrm{H}} \mathfrak{S}(f,c) = (1 - \|f\|_{\mathbb{H}}^2)(1-c^{-2}) := 1 - \lambda(f,c),$$

where $\lambda(f,c) \triangleq c^{-2}(1-\|f\|_{\mathbb{H}}^2) + \|f\|_{\mathbb{H}}^2$.

Theorem 6.1 *Let X denote linear Brownian motion. Then, for any analytic set $E \subset [0,1]$ and any $f \in \mathbb{H}$ with $\|f\|_{\mathbb{H}} < \dim_{\mathrm{P}}(E)$,*

$$\inf_{t\in E} \liminf_{h\to 0+} |\log h| \|\Delta_h[t] - f\|_\infty = \frac{\pi}{4}\Big(\dim_{\mathrm{P}}(E) - \|f\|_{\mathbb{H}}^2\Big)^{-1/2},$$

and for any compact set $E \subset [0,1]$,

$$\mathbb{P}(\mathfrak{S}(f,c) \cap E \neq \varnothing) = \begin{cases} 1, & \text{if } E \notin \{\dim_{\mathrm{P}} < \lambda(f,c)\}_\sigma \\ 0, & \text{otherwise} \end{cases}.$$

In case $E \notin \{\dim_{\mathrm{P}} < \lambda(f,c)\}_\sigma$, then for all compact sets $E \subset [0,1]$, we have $\dim_{\mathrm{P}}(\mathfrak{S}(f,c) \cap E) = \dim_{\mathrm{P}}(E)$*, a.s., while*

$$\dim_{\mathrm{H}}(E) - \lambda(f,c) \leqslant \dim_{\mathrm{H}}(\mathfrak{S}(f,c)\cap E) \leqslant \dim_{\mathrm{P}}(E) - \lambda(f,c), \ \ a.s.$$

Proof. Let

$$Y(t,h) = \|f\|_{\mathbb{H}}^2 + \frac{1 - \|f\|_{\mathbb{H}}^2}{\Big(\omega_f^{-1}|\log h| \ \|\Delta_h[t] - f\|_\infty\Big)^2}.$$

Then, for $t \in \mathbb{R}^1_+$, $h > 0$ and $\gamma > \|f\|_{\mathbb{H}}^2$,

$$\mathbb{P}\Big(Y(t,h) > \gamma\Big) = \mathbb{P}\Big(\|\Delta_h[t] - f\|_\infty < \frac{\pi}{4(\gamma - \|f\|_{\mathbb{H}}^2)^{1/2}} \ \frac{1}{|\log h|}\Big).$$

Theorem 3.3 of de Acosta (1983) states that: *For every $\|f\|_{\mathbb{H}} < 1$ and $r > 0$,*

$$\lim_{\lambda\to\infty} \lambda^{-2} \log \mathbb{P}\Big(\|\lambda^{-1}W - f\|_\infty \leqslant \lambda^{-2} r\Big) = -\frac{\pi^2}{8} \ r^{-2} - \frac{1}{2}\|f\|_{\mathbb{H}}^2.$$

By taking $\lambda = \sqrt{2|\log h|}$ in de Acosta's theorem, we see that Condition 1 is satisfied with $y_0 = \|f\|_{\mathbb{H}}^2$ and any y_1. Conditions 2 and 3 are also satisfied due to the independence of increments and Lévy's modulus of continuity. Since $\mathfrak{S}(f,c) = \Gamma(\lambda(f,c))$, the assertions of the theorem follow from Theorem 2.1. □

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