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Taylor Approximations on Sierpinski Gasket Type Fractals

Robert S. Strichartz[1]

Department of Mathematics, Cornell University, Malott Hall, Ithaca, New York 14853
E-mail: str@math.cornell.edu

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For a class of fractals that includes the familiar Sierpinski gasket, there is now a theory involving Laplacians, Dirichlet forms, normal derivatives, Green's functions, and the Gauss–Green integration formula, analogous to the theory of analysis on manifolds. This theory was originally developed as a by-product of the construction of stochastic processes analogous to Brownian motion, but has been given by a direct analytic construction in the work of Kigami. Until now, this theory has not provided anything analogous to the gradient of a function, or a local Taylor approximation. In this paper we construct a family of derivatives, which includes the known normal derivative, at vertex points in the graphs that approximate the fractal, and obtain Taylor approximations at these points. We show that a function in the domain of Δ^n can be locally well approximated by an n-harmonic function (solution of $\Delta^n u = 0$). One novel feature of this result is that it requires several different estimates to describe the optimal rate of approximation.

1. INTRODUCTION

A theory of analysis on fractals, analogous to the theory of analysis on manifolds, is under construction. We will follow the approach of Kigami [Ki1–Ki8] in which a Dirichlet form $\mathscr{E}$ (analogous to $\mathscr{E}(u, v) = \int \nabla u \cdot \nabla v \, dx$ in manifold theory) is constructed as a renormalized limit of Dirichlet forms on a sequence of graphs approximating the fractal. The Laplacian Δ_μ is then defined by

$$\int v \, \Delta_\mu u \, d\mu = -\mathscr{E}(u, v) \tag{1.1}$$

for a suitable class of test functions v, for a given measure μ. One can also define $\Delta_\mu u$ directly as a renormalized limit of graph Laplacians. It is also possible to define a normal derivative $\partial_n u$ at boundary points of the fractal K (in this theory the boundary is always a finite set, so K is more like an

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interval than a higher dimensional manifold) in such a way that the Gauss–Green integration formula holds,

$$\int_K (u\,\Delta v - v\,\Delta u)\,d\mu = \sum_{\partial K} u\,\partial_n v - v\,\partial_n u. \tag{1.2}$$

There is a substantial literature concerning this theory. Many of these works are included in the references. Despite the accomplishments, there are two obvious weaknesses in the theory developed so far: (1) the class of fractals to which it applies is still rather limited, and (2) until now, there has been no satisfactory theory of gradients and Taylor approximations. In this paper we attempt to remedy the second weakness, although in the process we will retreat a little regarding the first weakness, by considering a still narrower class of fractals. It is an article of faith among mathematicians that if we can obtain a deep understanding of a few key elementary mathematical models, we can use this understanding to tackle problems involving more realistic models. It is our hope that the Sierpinski gasket, like the circle and the triangle, will prove to be one of these key examples.

One possible approach to finding gradients in this theory is to look to the Dirichlet form as an integral of the inner product of gradients. Some work on this approach has been done by Kusuoka [Ku2] and Kigami [Ki3], but it does not appear to give much hope for a pointwise gradient. For example, one result in [Ku2] is that the measure involved in the Dirichlet form is different from the measure one would like to use in defining the Laplacian.

Our approach is to build upon the normal derivatives that are already defined. It is not difficult to localize the definition so that it applies to all vertices of the graphs approximating the fractal, and in the process the Gauss–Green formula (1.2) is also extended to integrations over simple sets. These vertex points form a countable dense subset of the fractal, but the set has measure zero. For the most part this paper deals only with these points, except for a brief discussion in Section 7 of the local behavior of a function near a generic point. We will introduce other derivatives at a vertex point x, so that together with the normal derivatives we can put them together to make up a gradient $df(x)$. We then define the tangent of order one at x, $T_1(f)$, to be the harmonic function with the same value and the same gradient as f at x. (For simplicity of notation we write just $T_1(f)$ and suppress mention of the vertex point x.) The space of harmonic functions is finite dimensional, and reduces to the space of linear functions when the fractal is just an interval. Thus $T_1(f)$ is the correct analog of the tangent function to a differentiable function on the line. We then obtain a first order Taylor theorem describing the local approximation of $T_1(f)$ to f in a neighborhood of x. It turns out that we require several different

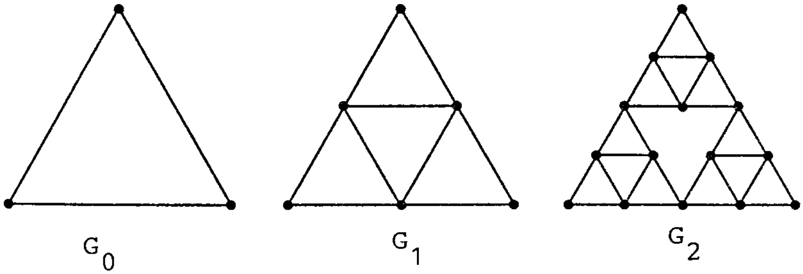

FIG. 1.1. The first 3 graphs, G_0, G_1, G_2 in the approximation to the Sierpinski gasket.

estimates to describe the approximation that are optimal in the sense that they hold for a reasonable class of functions and they uniquely determine the tangent $T_1(f)$.

We now describe the situation in more detail in the case of the usual Sierpinski gasket, which is the fractal generated by the iterated function system (i.f.s) in the plane consisting of 3 similarities F_j with contraction ratio 1/2 and fixed points at the vertices v_1, v_2, v_3 of a triangle. The initial graph G_0 consists of just these vertices and the edges of the triangle, and each subsequent graph G_k is obtained from the previous one G_{k-1} as the union $F_1 G_{k-1} \cup F_2 G_{k-1} \cup F_3 G_{k-1}$, identifying the common images $F_j v_k = F_k v_j$, $j \neq k$. See Fig. 1.1. The initial vertices v_1, v_2, v_3 are defined to be the boundary of each graph G_k. The Sierpinski gasket SG is the limit of G_k as $k \to \infty$, and its boundary is defined to be the initial vertices as well. The natural energy form $\mathscr{E}_k$ defined on functions on G_k is

$$\mathscr{E}_k(u, v) = \sum_{x \sim_k y} (u(x) - u(y))(v(x) - v(y)), \tag{1.3}$$

where $x \sim_k y$ means there is an edge in G_k joining x to y. It is necessary to multiply $\mathscr{E}_k$ by the renormalization factor $(5/3)^k$ in order to make the following consistency property hold [Ki1]:

Lemma 1.1. *For every function f on G_k there exists a unique extension $\tilde{f}$ to G_{k+1} minimizing $\mathscr{E}_{k+1}(\tilde{f}, \tilde{f})$. Then*

$$(5/3)^k \, \mathscr{E}_k(f, f) = (5/3)^{k+1} \, \mathscr{E}_{k+1}(\tilde{f}, \tilde{f}). \tag{1.4}$$

In view of this, we define

$$\mathscr{E}(f, f) = \lim_{k \to \infty} (5/3)^k \, \mathscr{E}_k(f, f) \qquad (\text{in } [0, \infty]) \tag{1.5}$$

for any function f on SG (the restriction of f to G_k is still denoted f) and we say f belongs to dom($\mathscr{E}$) when this limit is finite. All functions in dom($\mathscr{E}$) turn out to be continuous. For $f, g \in \text{dom}(\mathscr{E})$,

$$\mathscr{E}(f, g) = \lim_{k \to \infty} (5/3)^k \mathscr{E}_k(f, g) \tag{1.6}$$

exists and is finite, and $\mathscr{E}$ is a regular Dirichlet form with respect to any reasonable measure. In particular, we choose the normalized Hausdorff measure μ on SG, which is the unique probability measure μ satisfying the self-similar identity

$$\mu = \tfrac{1}{3} \mu \circ F_1^{-1} + \tfrac{1}{3} \mu \circ F_2^{-1} + \tfrac{1}{3} \mu \circ F_3^{-1}. \tag{1.7}$$

We then define the Laplacian Δ_μ and its domain dom(Δ_μ) as follows: $u \in \text{dom}(\Delta_\mu)$ if $u \in \text{dom}(\mathscr{E})$ and there exists a continuous function, denoted $\Delta_\mu u$, such that (1.1) holds for every $v \in \text{dom}(\mathscr{E})$ vanishing on the boundary. Note that we are using the symbol "dom" to refer to an L^2 domain for the Dirichlet form but a continuous domain for the Laplacian. It is the fact that points have positive capacity, which also holds for the interval but not in higher dimensional manifolds, that forces functions in dom($\mathscr{E}$) to be continuous, and also makes possible the success of this pointwise approach.

The direct definition of the Laplacian at a nonboundary vertex point x is

$$\Delta_\mu f(x) = \lim_{x \to \infty} 5^k \tfrac{3}{2} \Big(-4f(x) + \sum_{y \sim_k x} f(y) \Big). \tag{1.8}$$

Note that each nonboundary vertex has exactly 4 neighbors in each graph. The renormalization factor 5 is in fact the product of the 5/3 factor from the energy form and 1/(1/3) from the self-similar identity (1.7) to scale the measure. (The 3/2 factor is less important, and is inadvertently omitted in some references, but is needed to make (1.2) valid.)

It is not necessary to invoke the measure to define harmonic functions, although it is true that these are just the solutions of $\Delta_\mu h = 0$. The more direct definition is that

$$h(x) = \tfrac{1}{4} \sum_{y \sim_k x} h(y) \tag{1.9}$$

for every nonboundary vertex and every k. The Dirichlet principle holds: harmonic functions minimize energy subject to the boundary conditions. In particular, the space of harmonic functions is 3-dimensional, and the values at the 3 boundary points may be freely assigned. Moreover, there is a simple efficient algorithm for computing the values of a harmonic function exactly at all vertex points in terms of the boundary values. In every way, these harmonic functions are the analogs of linear functions on an interval.

The normal derivative is also defined without reference to the measure,

$$\partial_n f(v_1) = \lim_{k\to\infty} (5/3)^k \left(2f(v_1) - f(F_1^k v_2) - f(F_1^k v_3)\right) \tag{1.10}$$

and similarly for the other boundary points. The renormalization factor 5/3 is the same as for the Dirichlet form in (1.6), and not the same as for the Laplacian in (1.8). Strictly speaking, the Laplacian is not defined at boundary points, although $\Delta_\mu f = g$ requires that f and g be continuous up to the boundary, so it could be defined by continuity, but (1.8) does not make sense. As in the manifold theory, Neumann boundary conditions arise "naturally" when no controls are imposed at boundary points. Normal derivatives exist for functions in $\mathrm{dom}(\Delta_\mu)$.

We can localize the definition of normal derivative as follows. We let w denote a word $w = (w_1, ..., w_N)$ from the alphabet $(1, 2, 3)$ and write $F_w = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_N}$. We call $F_w K$ a cell of level N, and consider the vertex $x = F_w v_1$ in G_N. If $w \neq (1, ..., 1)$ then x is not a boundary point in K, but is a boundary point of $F_w K$ and one other level N cell. We call x a *junction vertex*. (The Sierpinski gasket has the property that every non-boundary vertex is a junction vertex, but this is not true of all the fractals we consider.) We define the normal derivative at x with respect to $F_w K$ by

$$\lim_{k\to\infty} (5/3)^{N+k} \left(2f(F_w v_1) - f(F_w F_1^k v_2) - f(F_w F_1^k v_3)\right) \tag{1.11}$$

if the limit exists. The exponent is $N+k$ because the points $F_w v_1$, $F_w F_1^k v_2$, $F_w F_1^k v_3$ are the boundary points of the level $N+k$ cell $F_w F_1^k K$. There is another normal derivative at x with respect to the other cell containing x. For a general function these may be unrelated, but for $f \in \mathrm{dom}(\Delta_\mu)$ the sum of the two normal derivatives must vanish. We call this the *compatibility condition*. In general, this compatibility condition is necessary and sufficient for patching together two functions whose Laplacian is defined on neighboring cells.

In addition to the normal derivatives, we define transverse derivatives as follows. First, at a boundary point, say v_1, we take

$$\lim_{k\to\infty} 5^k \left(f(F_1^k v_2) - f(F_1^k v_3)\right). \tag{1.12}$$

More generally, at $x = F_w v_1$, there is a transverse derivative

$$\lim_{k\to\infty} 5^{k+N} \left(f(F_w F_1^k v_2) - f(F_w F_1^k v_3)\right) \tag{1.13}$$

with respect to $F_w K$, and another one with respect to the other cell. We will show that these derivatives exist if $f \in \mathrm{dom}(\Delta_\mu)$. The factor 5 in this case is just coincidentally the same as the factor 5 in the definition of the Laplacian (1.8). The explanation of the factor comes from the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 2/5 & 2/5 & 1/5 \\ 2/5 & 1/5 & 2/5 \end{pmatrix} \tag{1.14}$$

which describes the algorithm for extending a harmonic function

$$\begin{pmatrix} h(F_1 v_1) \\ h(F_1 v_2) \\ h(F_1 v_3) \end{pmatrix} = M \begin{pmatrix} h(v_1) \\ h(v_2) \\ h(v_3) \end{pmatrix} \tag{1.15}$$

from the boundary of K to the boundary of $F_1 K$. The eigenvalues of M are 1, 3/5, 1/5, (the same is true for the matrices giving the extensions to $F_2 K$ and $F_3(K)$), and the factors 5/3 and 5 in (1.11) and (1.13) are the reciprocals of the nontrivial eigenvalues (the eigenvalue 1 corresponds to extending a constant function). The two transverse derivatives at x are not related in any way. We combine the 2 normal derivatives and 2 transverse derivatives of f at x into a gradient $df(x)$, which lies in a 3-dimensional subspace of $\mathbb{R}^4$, assuming the compatibility condition.

The values of $f(x)$ and $df(x)$ provide us with 4 local coordinates out of which we construct a tangent of order 1. Since the space of global harmonic functions on K has dimension 3, this is not the correct space to look for the tangent. Instead, we look in a space of harmonic functions defined on a certain neighborhood of x we call $U_0(x)$. If x first appears in graph G_N, then $U_0(x)$ consists of the 2 cells of level N containing x. More generally, we denote by $U_m(x)$ the union of the 2 cells of level $N+m$ containing x, forming a canonical system of neighborhoods of x. $U_0(x)$ has a boundary consisting of 4 points, namely the other boundary points of each of the cells (see Fig. 1.2), so the space of harmonic functions on $U_0(x)$ is 4-dimensional and it turns out that such harmonic functions are uniquely determined by the values $h(x)$ and $dh(x)$. Thus it makes sense to define the tangent $T_1(f)$ to be the harmonic function on $U_0(x)$ with the same value and gradient as f at x. In fact it is possible to extend every harmonic function on $U_0(x)$ to a function harmonic on $K \backslash x^*$ (see Fig. 1.2 for the location of x^*) with a possible pole singularity at x^*. We will not be concerned with this extension further, since its significance is unclear, and it does not generalize to other fractals. What is significant is that the geometry of the fractal sets an upper bound for the neighborhood in which the Taylor approximation can be accurate.

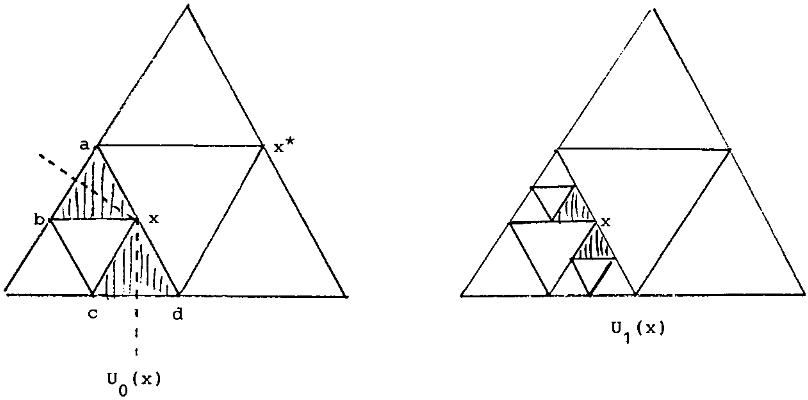

FIG. 1.2. The neighborhoods $U_0(x)$ (left) and $U_1(x)$ (right) of a point x that first appears in graph G_2. The 4 boundary points of $U_0(x)$ are labeled a, b, c, d. Also shown is the point x^*. The two dotted lines are reflection axes for the local symmetry g_x.

We now describe the approximation of f by $T_1(f)$ near x. For this we require the local symmetry g_x on $U_0(x)$ which simply reflects each cell in the bisector through x (the dotted lines in Fig. 1.2). Then the two estimates we want are

$$(f - T_1(f))|_{U_m(x)} = o((3/5)^m) \tag{1.16}$$

and

$$(f - T_1(f) - (f - T_1(f)) \circ g_x)|_{U_m(x)} = o(1/5^m). \tag{1.17}$$

In effect this sets a faster rate of decay for the odd part of the remainder $f - T_1(f)$ than for the even part. We will show that these conditions uniquely determine the tangent. More precisely, if h is a harmonic function on $U_0(x)$ satisfying

$$h|_{U_m(x)} = o((3/5)^m) \tag{1.18}$$

and

$$(h - h \circ g_x)|_{U_m(x)} = o((1/5)^m) \tag{1.19}$$

then h vanishes. These estimates are sharp, since there exist nonvanishing harmonic functions satisfying the same conditions with little oh replaced by big oh. One of our main theorems states that if $f \in \operatorname{dom}(\Delta_\mu)$ and $\Delta_\mu f$ satisfies a Hölder condition of any positive order, then (1.16) and (1.17) do hold. (The requirement of the Hölder condition is related to the coincidence of the two 5's.)

We also develop a theory of higher order tangents and Taylor approximations. This does not require introducing any new derivatives, since we use the normal and transverse derivatives of powers of the Laplacian of the function. For the order n approximation, $T_n(f)$ will denote the n-harmonic function (solution of $\Delta_\mu^n h=0$) with the same values as $\Delta_\mu^k f$ and $d\Delta_\mu^k f$ at x for all $k<n$. In place of (1.16) and (1.17) we want the estimates

$$(f-T_n(f))|_{U_m(x)}=o(((1/5)^{n-1}(3/5))^m) \tag{1.20}$$

and

$$(f-T_n(f)-(f-T_n(f))\circ g_x)|_{U_m(x)}=o(((1/5)^{n-1}(1/5))^m), \tag{1.21}$$

where the new factor $(1/5)^{n-1}$ is related to the 5 that appears in the definition of the Laplacian (1.8). Again we show that such estimates uniquely determine an n-harmonic function $T_n(f)$, and the order of approximation is sharp. Furthermore, if $f\in\text{dom}(\Delta_\mu^n)$ and $\Delta_\mu^n f$ satisfies a Hölder condition, then (1.20) and (1.21) hold.

We may regard $\text{dom}(\Delta_\mu^\infty)=\bigcap_{n=1}^\infty \text{dom}(\Delta_\mu^n)$ as the analog of the space of C^∞ functions on a manifold. Of course there are some significant differences. For example, it is shown in [BST] that $\text{dom}(\Delta_\mu)$ and also $\text{dom}(\Delta_\mu^\infty)$ is not closed under multiplication (in fact $f^2\notin\text{dom}(\Delta_\mu)$ if $f\in\text{dom}(\Delta_\mu)$ is nonconstant). For functions in $\text{dom}(\Delta_\mu^\infty)$ the results of this paper provide local approximations $T_n(f)$ at a vertex point x of increasing accuracy. It is tempting to believe that there is a class of functions, analogous to the analytic functions on manifolds, for which the Taylor approximations $T_n(f)$ converge as $n\to\infty$ in some neighborhood of x. It is not difficult to construct artificial examples that are not n-harmonic for any n, but it is not known whether there are any natural examples, such as eigenfunctions of the Laplacian. A theory of infinite Taylor expansions awaits development.

We now describe the organization of this paper. We begin with a digression. Section 2 describes the theory of Taylor approximations on the unit interval using the standard Dirichlet form $\int_0^1 u'(x)\,v'(x)\,dx$ but using a singular self-similar measure μ in place of Lebesgue measure in (1.1) to define a different Laplacian. Here the gradient is the ordinary derivative and $T_1(f)$ is the ordinary tangent, but the higher order Taylor approximations are different. Because the situation is so simple, we are easily able to work out all the details, and certain features of the general theory become apparent. The results of this section are not used in the sequel, however. We begin the general theory in Section 3, dealing with a wide class of harmonic structures on post-critically finite (p.c.f.) fractals, and defining gradients and first order tangents (only in a weak sense, however), using only the harmonic structure and not the measure. In Section 4 we continue the development, bringing into play the measure and the Laplacian. In this general context we are only able to obtain rather weak theorems, so in Section 5 we drastically reduce generality and consider only structures with dihedral 3 symmetry (in particular, this excludes the asymmetric Laplacians of Section 2). We present 3 nontrivial examples, the Sierpinski gasket being one of them, and derive the first order Taylor theorem as stated above. In Section 6 we derive the higher order Taylor approximations for the same class of structures. In Section 7 we briefly discuss the possibility of Taylor approximations about a generic point. In Section 8, as an appendix to the paper, we establish a Hölder estimate for functions in the domain of the Laplacian.

After the completion of this paper, Teplyaev [T2] proposed another approach to defining gradients. His paper explains the relationship between the two approaches and also relates them to other gradient concepts in [Ku2, Ki3]. He also obtains a proof of a slightly weaker version of our Hypothesis 8.1.

2. THE UNIT INTERVAL

In this section we derive the complete theory for the simplest example, the unit interval with the usual harmonic structure, but with a general self-similar measure μ. We let $F_0 x = \frac{1}{2}x$ and $F_1 x = \frac{1}{2}x + \frac{1}{2}$ be the two contractions that define the interval, and let μ be the unique probability measure satisfying

$$\mu(A) = p\mu(F_0^{-1}A) + (1-p)\,\mu(F_1^{-1}A), \tag{2.1}$$

or equivalently

$$\int_0^1 f(x)\,d\mu(x) = p\int_0^1 f(F_0 x)\,d\mu(x) + (1-p)\int_0^1 f(F_1 x)\,d\mu(x). \tag{2.2}$$

Without loss of generality we may take $\frac{1}{2} \leqslant p < 1$, with $p = \frac{1}{2}$ corresponding to Lebesgue measure. We will see that when $p = \frac{1}{2}$ our theory reduces to ordinary calculus.

We can describe the basic notions in traditional terms as follows. We start with the standard Dirichlet form

$$\mathscr{E}(f, g) = \int_0^1 f'(x)\,g'(x)\,dx \tag{2.3}$$

with domain dom($\mathscr{E}$) the Sobolev space of functions f whose distributional derivative f' belongs to $L^2(dx)$. Such functions must be continuous and are representable as

$$f(x) = a + \int_0^x g(y)\,dy \tag{2.4}$$

for some $g \in L^2(dx)$, with $f' = g$. The Laplacian Δ_μ with domain dom(Δ_μ) is defined by

$$f \in \mathrm{dom}(\Delta_\mu) \qquad \text{and} \qquad \Delta_\mu f = g \tag{2.5}$$

if there exists a continuous g such that $f'' = g\,d\mu$ in the distribution sense. Note that we are using the word "domain" in a different sense for the Dirichlet form (L^2 sense) and the Laplacian (continuous sense). Also note that $f \in \mathrm{dom}(\Delta_\mu)$ implies that f is C^1, but in general ($p \neq \frac{1}{2}$) it will contain no C^2 function other than the linear ones.

It is also convenient to have an integral representation analogous to (2.4). For this we use the Green's function

$$G(x, y) = \begin{cases} y(1-x) & \text{if } \; y \leqslant x \\ x(1-y) & \text{if } \; x \leqslant y \end{cases} \tag{2.6}$$

for the Dirichlet boundary conditions. Note that the Green's function depends only on the harmonic structure, not the measure. We then have

$$f(x) = ax + b + \int_0^1 G(x, y)\, g(y)\, d\mu(y) \tag{2.7}$$

if and only if $-\Delta_\mu f = g$. Note that the only harmonic functions (solutions of $\Delta_\mu h = 0$) are the linear functions $ax + b$. The representation (2.7) is suited to the solution of the boundary value problem $-\Delta_\mu f = g$ with $f(0)$ and $f(1)$ specified, since the Green's function vanishes at $x = 0$ or 1. We will also need a local Green's function $G_z(x, y)$ for solving the initial value problem at the point z. This is just

$$G_z(x, y) = \begin{cases} y - x & \text{if } \; z \leqslant y \leqslant x \\ x - y & \text{if } \; x \leqslant y \leqslant z \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

Then

$$\begin{aligned} f(x) &= \int_0^1 G_z(x, y)\, g(y)\, d\mu(y) \\ &= \int_z^x (y - x)\, g(y)\, d\mu(y) \end{aligned} \tag{2.9}$$

(the second integral in (2.9) must be understood as an oriented integral) gives a solution of $-\Delta_\mu f = g$ which satisfies $f(z) = f'(z) = 0$. We thus have $f(x) = o(|z - x|)$ as $x \to z$. One of our tasks is to make a more precise estimate of the rate of vanishing as $x \to z$. This will depend on the nature of the point z.

The Laplacian Δ_μ can also be defined as a limit of difference quotients following Definition 6.1 of [Ki2]. If x is an interior dyadic point, say $x = j2^{-m}$ ($j \neq 0$ or 2^m), and $k \geqslant m$, let $\psi_{x,k}$ be the continuous piecewise linear function which takes the value 1 at x and 0 at all other dyadic points of the form $\ell 2^{-k}$. For

$$\varepsilon_k(x) = \int \psi_{x,k}(y)\, d\mu(y) \tag{2.10}$$

we form the difference quotient

$$\frac{f(x + 2^{-k}) + f(x - 2^{-k}) - 2f(x)}{2^{-k}\varepsilon_k(x)} \tag{2.11}$$

and define $\Delta_\mu f = g$ (for f and g continuous) provided that (2.11) converges to g uniformly as $k \to \infty$ (in other words, the maximum of the difference between (2.11) and $g(x)$ over all interior dyadic points $x = \ell 2^{-k}$ goes to zero as $k \to \infty$). The uniformity of the convergence turns out to be essential, as we will soon see.

The equivalence of the difference quotient definition and the Green's function representation (2.7) is proved in [Ki2]. The equivalence of (2.5) and (2.7) is routine. A key observation is that if f has the representation (2.7) then the difference quotient (2.11) is equal to

$$\frac{\int \psi_{x,k}(y)\, g(y)\, d\mu(y)}{\int \psi_{x,k}(y)\, d\mu(y)} \tag{2.12}$$

which makes the uniform convergence to $g(x)$ obvious.

The value of $\varepsilon_k(x)$ may be computed exactly using the self-similarity property (2.2) of the measure. Suppose the finite binary expansion of x has n_1 ones and n_0 zeroes before the last one (in place $n_0 + n_1$). For $k \geqslant n_0 + n_1$, we find $\varepsilon_k(x) = p^{n_0}(1-p)^{n_1 - 1}\, \varepsilon_{k - n_0 - n_1 + 1}(1/2)$ by repeated use of (2.2). Also $\varepsilon_m(1/2) = p^m(1-p) + p(1-p)^m$, which uses the fact that $\int_0^1 y\, d\mu(y) = 1 - p$, also obtained from (2.2). All together

$$\varepsilon_k(x) = p^{k - n_1 + 1}(1-p)^{n_1} + p^{n_0 + 1}(1-p)^{k - n_0}. \tag{2.13}$$

When $p > \frac{1}{2}$ we may drop the second term in (2.13) in the limit as $k \to \infty$, so

$$\Delta_\mu f(x) = p^{1-n_1}(1-p)^{n_1} \lim_{K \to \infty} \left(\frac{2}{p}\right)^K (f(x+2^{-K}) + f(x-2^{-K}) - 2f(\gamma)). \tag{2.14}$$

However, the existence of the limit in (2.14) alone does not imply that $f \in \mathrm{dom}(\Delta_\mu)$. In fact if f is any C^2 function, the limit exists and is zero because $2/p < 4$, and the limit exists with $2/p$ replaced by 4. Nevertheless, the only C^2 functions in $\mathrm{dom}(\Delta_\mu)$ are the linear functions. The explanation for this paradox is that the limit will not be uniform: for fixed k, we can choose x with n_1 close to k and n_0 close to zero, so $\varepsilon_k(x) \approx (1-p)^k$ and $2/(1-p) > 4$.

Now we consider Taylor approximations for $f \in \mathrm{dom}(\Delta_\mu)$ about a point z. We write

$$f(x) = T_1(f)(x, z) + R_1(x, z), \tag{2.15}$$

where

$$T_1(f)(z, x) = f(z) + f'(z)(x-z) \tag{2.16}$$

is the usual Taylor approximation to a C^1 function. The estimates on the remainder $R_1(x, z)$ will be quite different form the usual ones for C^2 functions, and will be quite different for different z. We consider first the case when z is dyadic.

Theorem 2.1. *Let $f \in \mathrm{dom}(\Delta_\mu)$ and let z be dyadic. Then*

$$\begin{cases} R_1(x, z) = O(|x-z|^{\alpha_-}) & \text{for} \quad x < z \\ R_1(x, z) = O(|x-z|^{\alpha_+}) & \text{for} \quad x > z, \end{cases} \tag{2.17}$$

where

$$\begin{cases} \alpha_+ = 1 + \dfrac{\log 1/p}{\log 2} \\ \alpha_- = 1 + \dfrac{\log 1/(1-p)}{\log 2}. \end{cases} \tag{2.18}$$

Remarks. When $p > 1/2$, we have $\alpha_+ < 2$ and $\alpha_- > 2$. The constants in (2.17) depend on z.

Proof. Let $g = \Delta_\mu f$. Then $R_1(x, z)$ has the representation (2.9), and so we have the trivial estimates

$$|R_1(x, z)| \leqslant |x-z| \, \|g\|_\infty \, \mu([x, z])$$

for $x<z$, and similarly with $\mu([z, x])$ for $x>z$. If $x<z$ there exists k such that $z-2^{-k} \leqslant x \leqslant z-2^{-k-1}$. Then $\mu([x, z]) \leqslant \mu([z-2^{-k}, z])$ and $|x-z| \geqslant 2^{-k-1}$. For k sufficiently large we have $\mu([z-2^{-k}, z]) \leqslant c(z)(1-p)^k$, and for a suitable choice of constant this holds for all k. Thus

$$|R_1(x, z)| \leqslant |x-z| \, \|g\|_\infty \, c(z)(1-p)^k \leqslant \|g\|_\infty \, c(z) \, |x-z|^{1+\alpha_-}$$

with α_- given by (2.18). Similarly $\mu([z, z+2^{-k}]) \leqslant c(z) \, p^k$ and this gives (2.17) for $x>z$. Q.E.D

We consider next the estimate for the remainder at a generic point z. By "generic point" we mean that z belongs to a certain set of full μ measure.

THEOREM 2.2. *There exists a set of full μ measure such that for all z in this set and all $f \in \operatorname{dom}(\Delta_\mu)$ we have*

$$R_1(x, z) = O(|x-z|^{\alpha-\varepsilon}) \qquad \textit{for any} \quad \varepsilon>0, \tag{2.19}$$

where

$$\alpha = 1 + \frac{p \log 1/p + (1-p) \log 1/(1-p)}{\log 2}. \tag{2.20}$$

Remark. For $p>\frac{1}{2}$ we have $\alpha_+ < \alpha < 2$.

Proof. As before we need to estimate $\mu([x, z])$ or $\mu([z, x])$. Let I_k denote the dyadic interval of length 2^{-k} containing z, and let I_k^+ and I_k^- denote the neighboring dyadic intervals of the same length. If $2^{-k-1} \leqslant |x-z| \leqslant 2^{-k}$ then $[x, z]$ or $[z, x]$ is contained in $I_k \cup I_k^+ \cup I_k^-$, so $\mu([x, z])$ or $\mu([z, x])$ is bounded above by $\mu(I_k)+\mu(I_k^+)+\mu(I_k^-)$. By the law of large numbers we can estimate these measures from above by $c(p^p(1-p)^{1-p})^{(1-\varepsilon)k}$ for any $\varepsilon>0$ for a generic set of z (the constant depending on z and ε), and this easily yields (2.19). Q.E.D

By using more precise theorems in probability (law of the iterated logarithm, or central limit theorem) we can obtain slightly better estimates than (2.19), but we will not give the details.

We show next that we can use the methods of [BST] to prove that f^2 is never in $\operatorname{dom}(\Delta_\mu)$ if f is a nonconstant function in $\operatorname{dom}(\Delta_\mu)$ for $p>\frac{1}{2}$.

COROLLARY 2.3. *Let $p>1/2$. If $f \in \operatorname{dom}(\Delta_\mu)$ is nonconstant, then $f^2 \notin \operatorname{dom}(\Delta_\mu)$.*

Proof. It suffices to prove that $f, f^2 \in \operatorname{dom}(\Delta_\mu)$ implies $f'(z)=0$ for every dyadic z, since f is C^1. Note that $f_1(x)=(f(x)-f(z))^2$ is also in

$\operatorname{dom}(\Delta_\mu)$ under our assumptions. Since $f_1(z)=f_1'(z)=0$ we have $T_1(f_1)(x,z)=0$, so the estimate (2.17) holds for $f_1(x)$. Choosing the $x<z$ case with $\alpha_->2$ we have $|f(x)-f(z)|^2 \leqslant c\,|x-z|^{\alpha_-}$ for $x<z$ which implies $f'(z)=0$. Q.E.D

We define powers Δ_μ^n of the Laplacian and their domains $\operatorname{dom}(\Delta_\mu^n)$ for positive integers n in the obvious way. For $f\in\operatorname{dom}(\Delta_\mu^n)$ and any fixed z, we define $T_n(f)(x,z)$ to be the unique solution to $\Delta_\mu^n h(x)=0$ satisfying

$$\begin{cases} \Delta_\mu^k h(z)=\Delta_\mu^k f(z), & 0\leqslant k<n \\ (\Delta_\mu^k h)'(z)=(\Delta_\mu^k f)'(z), & 0\leqslant k<n. \end{cases} \tag{2.21}$$

For example,

$$T_2(f)(x,z)=f(z)+f'(z)(x-z) + \int_z^x (x-y)(\Delta_\mu f(z)+(\Delta_\mu f)'(z)(y-z))\,d\mu(y) \tag{2.22}$$

and it is possible to give explicit formulas for all $T_n(f)$ involving iterated integrals. We write

$$f(x)=T_n(f)(x,z)+R_n(x,z) \tag{2.23}$$

and seek estimates for the remainder. By induction it is easy to establish a representation analogous to (2.9) for $R_n(x,z)$, namely

$$R_n(x,z)=\int_{S_n}(x-x_1)(x_1-x_2)\cdots(x_{n-1}-x_n)\,g_n(x_n)\,d\mu(x_1)\cdots d\mu(x_n), \tag{2.24}$$

where $g_n=\Delta_\mu^n f$ and S_n denotes the oriented simplex

$$S_n=\{z\leqslant x_n\leqslant x_{n-1}\leqslant\cdots\leqslant x_1\leqslant x\} \tag{2.25}$$

if $z<x$, with the reverse inequalities if $x<z$.

THEOREM 2.4. (a) *Let $f\in\operatorname{dom}(\Delta_\mu^n)$ and let z be dyadic. Then*

$$\begin{cases} R_n(x,z)=O(|x-z|^{n\alpha_-}) & \text{for } x<z \\ R_n(x,z)=O(|x-z|^{n\alpha_+}) & \text{for } x>z \end{cases} \tag{2.26}$$

with α_+ and α_- given by (2.18).

(b) *There exists a set of full μ measure such that for all z in this set and all $f \in \operatorname{dom}(\Delta_\mu^n)$ we have*

$$R_n(x, z) = O(|x - z|^{n\alpha - \varepsilon}) \qquad \text{for any} \quad \varepsilon > 0 \tag{2.27}$$

with α given by (2.20).

Proof. The proof is the same as that of Theorems 2.1 and 2.2, using (2.24), estimating the integrand by $2^{-nk} \|g\|_\infty$ for $2^{-k-1} \leqslant |x-z| \leqslant 2^{-k}$ and estimating the measure of S_n by $\mu([x, z])^n$ or $\mu([z, x])^n$. Q.E.D

It is also possible to obtain slightly weaker approximation estimates under weaker hypotheses.

THEOREM 2.5. (a) *Let $f \in \operatorname{dom}(\Delta_\mu^{n-1})$ and assume $\Delta_\mu^{n-1} f$ is C^1. Let z be dyadic. Then*

$$\begin{cases} R_n(x, z) = o(|x-z|^{1+(n-1)\alpha_-}) & \text{for} \quad x < z \\ R_n(x, z) = o(|x-z|^{1+(n-1)\alpha_+}) & \text{for} \quad x > z. \end{cases} \tag{2.28}$$

Furthermore, if $(\Delta_\mu^{n-1} f)'$ satisfies a Hölder condition of order β then

$$\begin{cases} R_n(x, z) = o(|x-z|^{\beta+1+(n-1)\alpha_-}) & \text{for} \quad x < z \\ R_n(x, z) = o(|x-z|^{\beta+1+(n-1)\alpha_+}) & \text{for} \quad x > z. \end{cases} \tag{2.29}$$

(b) *There exists a set of full μ measure such that for z in this set and all $f \in \operatorname{dom}(\Delta_\mu^{n-1})$ for which $\Delta_\mu^{n-1} f \in C^{1+\beta}$ for some $\beta > 0$, we have*

$$R_n(x, z) = O(|x-z|^{\beta+1-\varepsilon+(n-1)\alpha}) \qquad \text{for any} \quad \varepsilon > 0. \tag{2.30}$$

Proof. We need a different expression for R_n, namely

$$\begin{aligned} R_n(x, z) = \int_{S_{n-1}} & (x - x_1)(x_1 - x_2) \cdots (x_{n-2} - x_{n-1}) \\ & \times (g_{n-1}(x_{n-1}) - g_{n-1}(z) - g'_{n-1}(z)(x_{n-1} - z)) \\ & \times d\mu(x_1) \cdots d\mu(x_{n-1}). \end{aligned} \tag{2.31}$$

It is easy to establish (2.31) by induction, and the rest of the proof is as before, using (2.31) in place of (2.24). Q.E.D

The estimates (2.28) and (2.30) are the weakest that will imply the uniqueness of $T_n(f)$. That is the reason we did not allow just $\Delta_\mu^{n-1} f \in C^1$ in part (b). In fact we should think of $T_n(f)$ as a linear combination of the $2n$ functions φ_j and ψ_j, $0 \leqslant j \leqslant n-1$, characterized by the conditions $\Delta_\mu^{j+1} \varphi_j = \Delta_\mu^{j+1} \psi_j = 0$ and

$$\begin{cases} \Delta_\mu^k \varphi_j(z) = \delta_{jk} & (\Delta_\mu^k \varphi_j)'(z) = 0 \\ \Delta_\mu^k \psi_j(z) = 0 & (\Delta_\mu^k \psi_j)'(z) = \delta_{jk} \end{cases} \tag{2.32}$$

for $k \leqslant j$. It is easy to see that $\varphi_0(x) = 1$, $\psi_0(x) = x - z$, and

$$\begin{cases} \varphi_j(x) = \int_{S_j} (x - x_1)(x_1 - x_2) \cdots (x_{j-1} - x_j) \, d\mu(x_1) \cdots d\mu(x_j) \\ \psi_j(x) = \int_{S_j} (x - x_1)(x_1 - x_2) \cdots (x_{j-1} - x_j)(x_j - z) \, d\mu(x_1) \cdots d\mu(x_j) \end{cases} \tag{2.33}$$

for $j \geqslant 1$. At a dyadic point z we have

$$\varphi_j(x) = \begin{cases} O(|x - z|^{j\alpha_-}) & \text{for} \quad x < z \\ O(|x - z|^{j\alpha_+}) & \text{for} \quad x > z \end{cases} \tag{2.34}$$

$$\psi_j(x) = \begin{cases} O(|x - z|^{1 + j\alpha_-}) & \text{for} \quad x < z \\ O(|x - z|^{1 + j\alpha_+}) & \text{for} \quad x > z, \end{cases} \tag{2.35}$$

while at a generic point z we have

$$\varphi_j(x) = O(|x - z|^{j\alpha - \varepsilon}) \qquad \text{for any} \quad \varepsilon > 0 \tag{2.36}$$

$$\psi_j(x) = O(|x - z|^{1 + j\alpha - \varepsilon}) \qquad \text{for any} \quad \varepsilon > 0, \tag{2.37}$$

and these estimates are sharp.

$T_n(f)$ is in fact the analog of the usual Taylor polynomial of degree $2n-1$, with $\varphi_j(x)$ and $\psi_j(x)$ the analogs of $(x-z)^{2j}$ and $(x-z)^{2j+1}$, respectively. The notation may seem a bit perverse, but it is motivated by the fact that we only have the analogs of these Taylor approximations for general fractals. In the present setting it is straightforward to define also $T_{n+1/2}(f)$, the analog of the usual Taylor polynomial of degree $2n$, to be the unique solution of $(\Delta_\mu^n h)' = 0$ satisfying

$$\begin{cases} \Delta_\mu^k h(z) = \Delta_\mu^k f(z), & 0 \leqslant k \leqslant n \\ (\Delta_\mu^k h)'(z) = (\Delta_\mu^k f)'(z), & 0 \leqslant k < n. \end{cases} \tag{2.38}$$

Clearly $T_{n+1/2}(f)$ is a linear combination of the $2n+1$ functions φ_j, $0 \leqslant j \leqslant n$ and ψ_j, $0 \leqslant j \leqslant n-1$. We have the following expression for the remainder $R_{n+1/2}(x, z) = f - T_{n+1/2}(f)$,

$$R_{n+1/2}(x, z) = \int_{S_n} (x - x_1)(x_1 - x_2) \cdots (x_{n-1} - x_n)(g_n(x_n) - g_n(z)) \times d\mu(x_1) \cdots d\mu(x_n). \tag{2.39}$$

THEOREM 2.6. (a) *Let $f \in \mathrm{dom}(\Delta_\mu^n)$ and let z be dyadic. Then*

$$\begin{cases} R_{n+1/2}(x, z) = o(|x-z|^{n\alpha_-}) & \text{for} \quad x < z \\ R_{n+1/2}(x, z) = o(|x-z|^{n\alpha_+}) & \text{for} \quad x > z. \end{cases} \tag{2.40}$$

Furthermore, if $\Delta_\mu^n f \in C^\beta$ for some β with $0 < \beta \leqslant 1$, then

$$\begin{cases} R_{n+1/2}(x, z) = O(|x-z|^{\beta + n\alpha_-}) & \text{for} \quad x < z \\ R_{n+1/2}(x, z) = O(|x-z|^{\beta + n\alpha_+}) & \text{for} \quad x > z. \end{cases} \tag{2.41}$$

(b) *Let $f \in \mathrm{dom}(\Delta_\mu^n)$ and $\Delta_\mu^n f \in C^\beta$ for some β with $0 < \beta \leqslant 1$. For generic z,*

$$R_{n+1/2}(x, z) = O(|x-z|^{\beta - \varepsilon + n\alpha}) \tag{2.42}$$

for any $\varepsilon > 0$.

Proof. This is as before, using (2.39). Q.E.D

3. FIRST ORDER TANGENTS

We assume that a regular harmonic structure is given on a p.c.f. self-similar fractal. The reader is referred to [Ki2] for exact definitions, and any unexplained notation. We will make two additional assumptions. The first is geometric in nature, depending only on the self-similar structure, while the second is analytic, depending on the harmonic structure. These assumptions are essential to our approach. The second assumption, which requires that certain matrices be diagonalizable over the reals, is valid in all examples we have looked at.

HYPOTHESES 3.1. (a) *Each point v_j, $j = 1, ..., N_0$ in the post critical set V_0 is the fixed point of a unique mapping in the i.f.s., which we will denote F_j. Also we assume that for any F_j and F_ℓ in the i.f.s., $j \neq \ell$, the intersection $F_j K \cap F_\ell K$ consists of at most one point x with $x = F_j v_m = F_\ell v_n$ for some* *points v_m and v_n in V_0.* (b) *For each v_j in V_0, let M_j denote the $N_0 \times N_0$ matrix that transforms the values $u|_{V_0}$ to $u|_{F_j V_0}$ for harmonic functions u; i.e.,*

$$u(F_j v_k) = \sum_{\ell=1}^{N_0} (M_j)_{k\ell}\, u(v_\ell). \tag{3.1}$$

We assume that each M_j has a complete set of real left eigenvectors β_{jk} with real nonzero eigenvalues λ_{jk},

$$\beta_{jk} M_j = \lambda_{jk} \beta_{jk}. \tag{3.2}$$

We will assume that for each j the eigenvalues λ_{jk} are labeled in decreasing order of absolute value. We write $\lambda_{j2} = r_j$.

Since $F_j v_j = v_j$, the jth row of M_j is δ_{kj}. Other than that row, all the entries of M_j are strictly positive, and all row sums are one. Thus the largest eigenvalue is $\lambda_{j1} = 1$ with right eigenvector constant (the exact form of β_{j1} is not of interest). If we let $\tilde{M}_j$ denote the matrix obtained from M_j by deleting the jth row and column, then the largest eigenvalue of $\tilde{M}_j$ becomes the second largest eigenvalue λ_{j2} of M_j, and the eigenvector β_{j2} can be chosen to have -1 in the jth place and all other entries positive. In fact, if $\tilde{\beta}_{j2}$ denotes the positive left eigenvector of $\tilde{M}_j$ normalized to have sum one, then β_{j2} is obtained from $\tilde{\beta}_{j2}$ by inserting -1 in the jth place. We have $\lambda_{j2} < 1$ because the row sums of $\tilde{M}_j$ are strictly less than one, and $|\lambda_{jk}| < \lambda_{j2}$ for $k \geqslant 3$ because $\tilde{M}_j$ is strictly positive. The derivative associated with β_{j2} will just be a multiple of the normal derivative at v_j. We will also define derivatives associated to all β_{jk} with $k \geqslant 3$, but since the absolute values of the eigenvalues $|\lambda_{jk}|$ are strictly less than λ_{j2}, these will be derivatives of a somewhat higher "order."

DEFINITION 3.2. Let f be a continuous function defined in a neighborhood of v_j. Then the derivatives $d_{jk} f(v_j)$ for $2 \leqslant k \leqslant N_0$ are defined by the following limits, if they exist,

$$d_{jk} f(v_j) = \lim_{m \to \infty} \lambda_{jk}^{-m} \beta_{jk} f|_{F_j^m V_0}, \tag{3.3}$$

where $\beta_{jk} f|_{F_j^m V_0}$ means

$$\sum_{\ell=1}^{N_0} (\beta_{jk})_\ell\, f(F_j^m v_\ell). \tag{3.4}$$

LEMMA 3.3. *If h is harmonic in a neighborhood of v_j then all the derivatives $d_{jk} h(v_j)$ exist and may be evaluated without taking the limit in* (3.3).

Proof. Suppose first that h is a harmonic function on K. Then the definition of β_{jk} yields the equality of the $m=0$ and $m=1$ terms on the right side of (3.3). By applying the same argument to $h \circ F_j^m$ we obtain that all terms on the right side of (3.3) are equal. If h is only harmonic in a neighborhood of v_j we begin the argument with $h \circ F_j^m$ for m sufficiently large so that $F_j^m K$ is contained in that neighborhood. Q.E.D

LEMMA 3.4. *Fix v_j.*

(a) *A harmonic function h is uniquely determined by the values of $h(v_j)$ and $d_{jk}h(v_j)$, $2 \leqslant k \leqslant N_0$, and any values may be assigned.*

(b) *Let h be a harmonic function satisfying*

$$\beta_{jk} h|_{F_j^m V_0} = o((\lambda_{jk})^m) \qquad \text{as} \quad m \to \infty \tag{3.5}$$

for $2 \leqslant k \leqslant N_0$ and $h(v_j)=0$. Then h is identically zero.

Proof. (a) A harmonic function h is uniquely determined by the values $h|_{V_0}$. By assumption, the vectors β_{jk} for $2 \leqslant k \leqslant N_0$ span the space of vectors orthogonal to the constant vector. By Lemma 3.3, $d_{jk}h(v_j) = \beta_{jk} h|_{V_0}$ so the values of $d_{jk}h(v_j)$ determine h up to a constant, and $h(v_j)$ determines the constant.

(b) The assumption (3.5) implies $d_{jk}h(v_j)=0$ for $2 \leqslant k \leqslant N_0$. Since $h(v_j)=0$ we may apply (a) to conclude that h vanishes identically. Q.E.D

We now want to extend these definitions and observations to all points in V_n, n arbitrary. Specifically, suppose n is the first value for which $x \in V_n$. We call x a *nonjunction vertex* if there is a unique word w of length $|w|=n$ and j such that $x = F_w v_j$. We say that x is a *junction vertex of order $m \geqslant 2$* if there are exactly m solutions of $x = F_w v_j$ with $|w|=n$. By our first hypothesis, all the different types of junction vertices show up already in V_1. All the other junction vertices are simply images (under an iterated map F_w) of a junction vertex in V_1. We could, of course, consider a nonjunction vertex as the case $m=1$ of a junction vertex, but in fact we will treat the two types of vertices in a somewhat different fashion. The local harmonic functions at a nonjunction vertex will not be required to satisfy $\Delta h(x)=0$ at the vertex x, but merely be continuous there. This is consistent with the definition of global harmonic function at a boundary point, and we want the theory at a nonjunction vertex $x = F_w v_j$ to be exactly the same as at the corresponding boundary point v_j. But at a junction point x it is natural to impose the harmonic equation $\Delta h(x)=0$, and this gives rise to a compatibility condition on the different normal derivatives of h at x.

Let x be a junction vertex in V_1. Let $J(x)$ denote the set of indices j such that there exists j' $(1 \leqslant j' \leqslant N_0)$ with $x = F_j v_{j'}$ (a priori the different values

j' do not have to be distinct, whereas the values of j must be distinct; this explains the awkward notation). A neighborhood of x contains portions of $F_j K$ for all $j \in J(x)$, and each portion is an image under F_j of a neighborhood of $v_{j'}$ in K. Also $\# J(x) = m$, the order of the junction vertex. More generally, if x is a junction vertex in V_n, then $x = F_w x'$ for x' a junction vertex in V_1 and $|w| = n-1$, and we set $J(x) = J(x')$. Then $x = F_w F_j v_{j'}$ for all $j \in J(x)$.

DEFINITION 3.5. Let f be a continuous function defined in a neighborhood of a vertex $x \in V_n$ (but $x \notin V_{n-1}$).

(a) Let $x = F_w v_j$ be a nonjunction vertex. Then the derivatives $d_{jk} f(x)$ for $2 \leqslant k \leqslant N_0$ are defined by the following limits, if they exist,

$$d_{jk} f(x) = \lim_{m \to \infty} r_w^{-1} \lambda_{jk}^{-m} \beta_{jk} f|_{F_w F_j^m V_0}. \tag{3.6}$$

(b) Let x be a junction vertex. Then the derivatives $d_{j'k} f(x)$ for $j \in J(x)$ and $2 \leqslant k \leqslant N_0$ are defined by the following limits, if they exist,

$$d_{j'k} f(x) = \lim_{m \to \infty} r_w^{-1} r_j^{-1} \lambda_{j'k}^{-m} \beta_{j'k} f|_{F_w F_j F_{j'}^m V_0}. \tag{3.7}$$

Furthermore, the normal derivatives $d_{j'2} f(x)$ are said to satisfy the compatibility condition if

$$\sum_{j \in J(x)} d_{j'2} f(x) = 0. \tag{3.8}$$

We write $df(x)$ for the collection of all derivatives defined here, and refer to it as the gradient of f at x.

Remarks. The factors r_w^{-1} and $r_w^{-1} r_j^{-1}$ in (3.6) and (3.7) are chosen to make the normal derivatives comparable at different points x. It is not clear that they are the best choices when $k \geqslant 3$.

LEMMA 3.6. *If h is harmonic in a neighborhood of a vertex x, then all the derivatives $d_{jk} h(x)$ or $d_{j'k} h(x)$ exist, and may be evaluated without taking the limit in* (3.6) *or* (3.7). *Furthermore, if x is a junction vertex, then the compatibility condition* (3.8) *for the normal derivatives holds.*

Proof. The existence follows by Lemma 3.3 applied to $h \circ F_w$ or $h \circ F_w \circ F_j$. If x is a junction vertex then the condition $\Delta h(x) = 0$ is equivalent to the compatibility condition (3.8). Q.E.D

Definition 3.7. For a vertex x we define a standard system of neighborhoods $U_m(x)$ by

$$U_m(x) = F_w F_j^m K \qquad \text{for} \quad x = F_w v_j \text{ nonjunction} \tag{3.9}$$

$$U_m(x) = \bigcup_{j \in J(x)} F_w F_j F_{j'}^m K \qquad \text{for } x \text{ a junction point.} \tag{3.10}$$

The boundary of $U_m(x)$ is taken to be $\{F_w F_j^m v_k\}$ in the first case (including x), and $\{F_w F_j F_{j'}^m v_k\}$ with x deleted in the second case. A function harmonic in $U_m(x)$ must be continuous and satisfy the harmonic condition at all points except the boundary points.

Lemma 3.8. *Fix a vertex x.*

(a) *A harmonic function h on $U_m(x)$ is uniquely determined by the values of $h(x)$ and the gradient $dh(x)$, and any values satisfying the compatibility condition* (3.8) *(x a junction vertex) may be freely assigned.*

(b) *Let h be a harmonic function on some $U_{m_0}(x)$ satisfying $h(x) = 0$ and*

$$\beta_{jk} h|_{F_w F_j^m V_0} = o(\lambda_{jk}^m) \qquad \textit{as} \quad m \to \infty \tag{3.11}$$

for $2 \leqslant k \leqslant N_0$ ($x = F_w v_j$ nonjunction) or

$$\beta_{j'k} h|_{F_w F_j F_{j'}^m V_0} = o(\lambda_{j'k}^m) \qquad \textit{as} \quad m \to \infty \tag{3.12}$$

for all $j \in J(x)$ and $2 \leqslant k \leqslant N_0$ (x a junction vertex). Then h is identically zero on $U_{m_0}(x)$.

Proof. Just apply Lemma 3.4 to $h \circ F_w$ or $h \circ F_w \circ F_j$. The compatibility condition (3.8) is equivalent to the requirement that h be harmonic at x in the junction case. Q.E.D

Definition 3.9. Let f be a continuous function defined in a neighborhood of a vertex x. A harmonic function h on $U_0(x)$ is said to be a weak tangent of order one to f at x if

$$\beta_{jk}(f - h)|_{F_w F_j^m V_0} = o(\lambda_{jk}^m) \qquad \text{as} \quad m \to \infty \tag{3.13}$$

for $2 \leqslant k \leqslant N_0$, and $f(x) = h(x)$, if $x = F_w v_j$ is nonjunction, or

$$\beta_{j'k}(f - h)|_{F_w F_j F_{j'}^m V_0} = o(\lambda_{j'k}^m) \qquad \text{as} \quad m \to \infty \tag{3.14}$$

for $2 \leqslant k \leqslant N_0$ and all $j \in J(x)$, and $f(x) = h(x)$, if x is a junction vertex. Weak tangents, if they exist, are unique by Lemma 3.8(b), and will be denote by $T_1(f)$.

Remark. Harmonic functions on any neighborhood $U_m(x)$ can be uniquely extended to harmonic functions on $U_0(x)$. This is a consequence of Hypothesis 3.1(b) which implies that the matrices M_j are invertible. For a junction vertex we do the extension separately from $F_w F_j F_{j'}^m K$ to $F_w F_j K$ for each j. There is of course no expectation that a harmonic function on $U_0(x)$ can be extended to all of K. The extent to which this is possible will depend on both the geometry of K and the specific harmonic function. Note that as x varies, the size of the neighborhood $U_0(x)$ will vary in a discontinuous manner. Thus the geometry of the vertex x will put inherent bounds on the size of neighborhood of approximation by a tangent, regardless of the analytic properties of the function f. This is a strong contrast with the situation in ordinary calculus.

THEOREM 3.10. *Let f be a continuous function defined in a neighborhood of a vertex x. The following are equivalent*:

(i) *f has a weak tangent $T_1(f)$ of order one at x.*

(ii) *All the gradient $df(x)$ of Definition* 3.5 *exist*, *and the compatibility condition* (3.8) *holds if x is a junction vertex.*

Furthermore, *the function $T_1(f)$ in* (i) *is equal to the harmonic function whose gradient is equal to the gradient of f at x in* (ii).

A function satisfying one of the equivalent conditions above will be called differentiable at *x*.

Proof. (i) $\Rightarrow$ (ii). If the weak tangent $h = T_1(f)$ exists, then the gradient of h exist by Lemma 3.6. It follows easily from (3.13) or (3.14) that f and h have the same gradient at x. Also the compatibility condition holds for h hence for f, if x is a junction vertex.

(ii) $\Rightarrow$ (i). If the gradient exist for f at x, we can construct a harmonic function with the same gradient by Lemma 3.8. Then it is easy to verify that (3.13) or (3.14) holds, so we have the weak tangent. Q.E.D

THEOREM 3.11. *Let f be a continuous function defined in a neighborhood of a vertex x and differentiable as in Theorem* 3.10. *Let h_m denote the harmonic function that assumes the same values as f at the boundary points of $U_m(x)$ as in Definition* 3.7, *extended to be harmonic on $U_0(x)$. Then h_m converges uniformly to $T_1(f)$ on $U_0(x)$ as $m \to \infty$.*

Proof. First assume x is a nonjunction vertex. Note that on the right side of (3.6) we may replace f by h_m since they are equal on $F_w F_j^m V_0 = \partial U_m(x)$. By Lemma 3.6 we then have

$$d_{jk} f(x) = \lim_{m \to \infty} d_{jk} h_m(x); \tag{3.15}$$

in particular, this shows the limit exists. We have $h_m(x) = f(x)$ for all m since x is a boundary point of $U_m(x)$. Since the space of harmonic functions on $U_0(x)$ is finite dimensional, Lemma 3.8 implies that there is an estimate $|h(y)| \leqslant c(|h(x)| + \|dh(x)\|)$ uniformly for $y \in U_0(x)$ for such functions. Using this estimate for $h_m - T_1(f)$ shows that h_m converges uniformly on $U_0(x)$ to $T_1(f)$.

If x is a junction point we no longer have x as a boundary point of $U_m(x)$, so the argument for the analog of (3.15) is only valid for $k \geqslant 3$. It is easy to see that $\lim_{m \to \infty} h_m(x) = f(x)$, but to handle the normal derivatives we need a slightly stronger statement, namely

$$h_m(x) - f(x) = o(\lambda_{j'2}^m) \qquad \text{for all} \quad j \in J(x). \tag{3.16}$$

To prove (3.16) we use the compatibility condition (3.8), which says

$$\begin{aligned} 0 &= \lim_{m \to \infty} \sum_{j \in J(x)} r_w^{-1} r_j^{-1} \lambda_{j'2}^{-m} \beta_{j'2} f|_{F_w F_j F_{j'}^m V_0} \\ &= \lim_{m \to \infty} \sum_{j \in J(x)} r_w^{-1} r_j^{-1} \lambda_{j'2}^{-m} \beta_{j'2} h_m|_{F_w F_j F_j^m V_0} \\ &\quad + \lim_{m \to \infty} \sum_{j \in J(x)} r_w^{-1} r_j^{-1} \lambda_{j'2}^{-m} (f(x) - h_m(x)) \end{aligned}$$

because h_m and f agree at all the other vertices of $F_w F_j F_{j'}^m V_0$. But

$$\sum_{j \in J(x)} r_w^{-1} r_j^{-1} \lambda_{j'2}^{-m} \beta_{j'2} h_m|_{F_w F_j F_{j'}^m V_0} = 0$$

because h_m is harmonic, and this establishes (3.16).

Using (3.16) we can replace f by h_m on the right side of (3.7) for $k = 2$ and control the error at x,

$$\begin{aligned} d_{j'2} f(x) &= \lim_{m \to \infty} r_w^{-1} r_j^{-1} \lambda_{j'2}^{-m} \beta_{j'2} h_m|_{F_w F_j F_j^m V_0} \\ &\quad + \lim_{m \to \infty} r_w^{-1} r_j^{-1} \lambda_{j'2}^{-m} (f(x) - h_m(x)) \\ &= \lim_{m \to \infty} d_{j'2} h_m(x). \end{aligned}$$

We can now complete the proof as before. Q.E.D

It is not possible to conclude from the convergence of h_m anything about the differentiability of f. This is even the case for the unit interval as the example $f(x) = |x - \frac{1}{2}|$ shows.

4. DIFFERENTIABILITY OF FUNCTIONS IN DOM(Δ_μ)

We now consider a self-similar measure μ on K. Given probability weights $\mu_1, ..., \mu_N$ we let μ be the unique probability measure satisfying

$$\mu = \sum_{j=1}^{N} \mu_j \mu \circ F_j^{-1}. \tag{4.1}$$

It follows that

$$\mu(F_w K) = \prod_{j=1}^{m} \mu_{w_j} \qquad \text{for} \quad w = (w_1, ..., w_m). \tag{4.2}$$

We let Δ_μ be the Laplacian associated to the harmonic structure and the measure μ, with domain dom(Δ_μ). We recall the Gauss–Green formula

$$\int_K (f\, \Delta_\mu g - g\, \Delta_\mu f)\, d\mu = \sum_{j=1}^{N_0} f(v_j)\, \partial_n g(v_j) - g(v_j)\, \partial_n f(v_j) \tag{4.3}$$

for f and g in dom(Δ_μ), where the normal derivative $\partial_n f(v_j)$ is equal (up to a normalization constant) to the derivative $d_{j2} f(v_j)$. There is also a local version,

$$\int_\Omega (f\, \Delta_\mu g - g\, \Delta_\mu f)\, d\mu = \sum_{\partial\Omega} f\, \partial_n g - g\, \partial_n f \tag{4.4}$$

for sets Ω that can be written as finite unions of images $F_w K$, provided the boundary is suitably interpreted. In particular, it holds for the sets $U_m(x)$ in Definition 3.7.

THEOREM 4.1. *Suppose*

$$r_j \mu_j < \lambda_{jN_0} \qquad \textit{for every } j. \tag{4.5}$$

Then for every $f \in \text{dom}(\Delta_\mu)$ *and every vertex* x, f *is differentiable at* x (*it satisfies the equivalent conditions of Theorem* 3.10).

Remark. There are some important examples (including SG) for which the hypothesis (4.5) does not hold, but for which the conclusion of the theorem is still valid, provided we assume in addition that the function $\Delta_\mu f$ satisfies an appropriate Hölder condition. We will discuss these examples in Section 5.

Proof. Assume first that $x = v_j$ is a boundary point. Let h be a continuous function that is piecewise harmonic on each $F_m K$, $1 \leqslant m \leqslant N$. We will use the local Gauss–Green formula (4.4) for f and h on each $F_m K$, and add up. We make h harmonic at all vertices in V_1 except those in V_0 and $F_j V_0$. Furthermore we set h equal to zero on V_0. Thus the only boundary terms remaining are at vertices in V_0 and $F_j V_0$, and none of them involve $h\,\partial_n f$, since at V_0 vertices h vanishes, while at $F_j V_0$ vertices the sum of the normal derivatives of f will vanish. We will be left with the identity

$$\int_K h\, \Delta_\mu f\, d\mu = - \sum_{V_0 \cup F_j V_0} f\, \partial_n h. \tag{4.6}$$

At $F_j V_0$ vertices the sum will include the values of $\partial_n h$ for all the sets $F_\ell K$ that meet there. These need not sum to zero because h is not assumed harmonic at these points.

Now the function h is uniquely specified by the values $h(F_j v_\ell) = a_\ell$ for $\ell \neq j$. This forms a vector space of dimension $N_0 - 1$. For each such function the identity (4.6) can be written

$$\int h\, \Delta_\mu f\, d\mu = \sum b_\ell f(v_\ell) + \sum_{k \neq j} c_k f(F_j v_k) \tag{4.7}$$

for certain coefficients $\{b_\ell\}$ and $\{c_k\}$ that depend linearly on $\{a_\ell\}$. If f is harmonic then the left side of (4.7) is zero, and also (3.1) holds. Thus

$$b_\ell = - \sum_{k \neq j} c_k (M_j)_{k\ell} \tag{4.8}$$

since we can assign any values to $f(v_\ell)$. We claim that the mapping from $\{a_\ell\}$ to $\{c_k\}$ is invertible. Since both vector spaces have the same dimension $N_0 - 1$, it suffices to show that there is no nontrivial kernel. But if a nonzero vector $\{a_\ell\}$ gave rise to $\{c_k\}$ zero, then the whole right side of (4.7) would vanish in view of (4.8). But we an easily construct functions f for which the left side of (4.7) is nonzero, simply by solving $\Delta_\mu f$ equal to a function supported in a small neighborhood of one of the v_ℓ where h is either strictly positive or negative.

Thus it is possible to choose h so that (4.7) holds for any choice of $\{c_k\}$, where $\{b_\ell\}$ is given by (4.8). For each m with $2 \leqslant m \leqslant N_0$, we let h_m denote the function corresponding to $c_k = (\beta_{jm})_k$ for $k \neq j$. Since $(M_j)_{j\ell} = \delta_{j\ell}$ we find from (4.8) and (3.2) that

$$b_\ell = -\lambda_{jm}(\beta_{jm})_\ell \qquad \text{for} \quad \ell \neq j$$

and

$$b_j = (1-\lambda_{jm})(\beta_{jm})_j.$$

Thus we have

$$\lambda_{jm}^{-1} \int h_m \,\Delta_\mu f \, d\mu = \lambda_{jm}^{-1} \beta_{jm} f|_{F_j V_0} - \beta_{jm} f|_{V_0}. \tag{4.9}$$

Next we want to scale the identity (4.9) down to the neighborhoods $U_n(v_j)$. For this we define

$$h_{mn} = \begin{cases} h_m \circ F_j^{-n} & \text{on } F_j^n K \\ 0 & \text{otherwise.} \end{cases} \tag{4.10}$$

The normal derivatives of h_{mn} are the same as h_m at corresponding points except for the scaling factor r_j^{-n}. Thus the scaled version of (4.9) is

$$\int_{F_j^n K} r_j^n \lambda_{jm}^{-n-1} h_{mn} \,\Delta_\mu f \, d\mu$$
$$= \lambda_{jm}^{-n-1} \beta_{jm} f|_{F_j^{-n-1} V_0} - \lambda_{jm}^{-n} \beta_{jm} f|_{F_j^n V_0}. \tag{4.11}$$

Thus, to prove the existence of $d_{jm} f(v_j)$ it suffices to show that

$$\sum_n \left| \int_{F_j^n K} r_j^n \lambda_{jm}^{-n-1} h_{mn} \,\Delta_\mu f \, d\mu \right| \tag{4.12}$$

converges. To see this we take the absolute value inside the integral and use the boundedness of h_{mn} and $\Delta_\mu f$. Each term in (4.12) is majorized by a multiple of $(r_j \lambda_{jm}^{-1} \mu_j)^n$, and by (4.5) this is a convergent geometric series. This completes the proof of condition (ii) of Theorem 3.10 when $x = v_j$.

The proof of the existence of derivatives of general vertex points x is essentially the same. It is only necessary to adapt the functions h_{mn} to the point x. It is also necessary to verify the compatibility condition (3.8) when x is a junction point, but this is an easy consequence of the existence of $\Delta_\mu f(x)$.

Q.E.D

Now fix a vertex z, and consider the "initial value problem"

$$\begin{cases} \Delta_\mu u = f & \text{in } U_0(z) \\ u(z) = 0 & \\ d_{jk} u(z) = 0, & 2 \leqslant k \leqslant N_0, \quad z = F_w x_j \text{ nonjunction or} \\ d_{j'k} u(z) = 0, & j \in J(z), \quad 2 \leqslant k \leqslant N_0, \quad z \text{ junction} \end{cases} \tag{4.13}$$

for f continuous on the closure of $U_0(z)$. By Lemma 3.8 we have existence and uniqueness (for existence we first extend f to be continuous on K, then solve $\Delta_\mu v = f$ on K, and then modify v by a harmonic function on $U_0(z)$ to satisfy the initial conditions). At least formally, the solution is given by

$$u(x) = \int_{U_0(z)} G_z(x, y) f(y) \, d\mu(y) \tag{4.14}$$

for a local Green's function $G_z(x, y)$. If $G(x, y)$ denotes the Green's function for the Dirichlet problem on K, then

$$G_z(x, y) = G(x, y) + H_z(x, y), \tag{4.15}$$

where for each fixed z and y $(z \neq y)$, $H_z(\cdot, y)$ is a harmonic function in $U_0(z)$. Indeed, $H_z(\cdot, y)$ is the unique harmonic function satisfying

$$\begin{cases} H_z(z, y) = -G(z, y) & \text{and} \\ d_{jk} H_z(\cdot, y)|_z = -d_{jk} G(\cdot, y)|_z & \text{or} \\ d_{j'k} H_z(\cdot, y)|_z = -d_{j'k} G(\cdot, y)|_z. & \end{cases} \tag{4.16}$$

THEOREM 4.2. *For fixed z, the local Green's function $G_z(x, y)$ is continuous for $y \neq z$. It is localized to the region where y lies roughly "between" x and z in the following precise sense*: $G_z(x, y) = 0$ *if* $x \in U_m(z)$ *and* $y \notin U_m(z)$, *and, in addition, in the junction vertex case that* $z = F_w F_j v_{j'}$ *for* $j \in J(z)$, *if* $x \in F_w F_{j_1} K$ *and* $y \in F_w F_{j_2} K$ *for* $j_1 \neq j_2$.

Proof. Since we are assuming that the harmonic structure regular, we know that $G(x, y)$ is continuous on $K \times K$. For fixed $y \neq z$, the function $G(\cdot, y)$ is harmonic in a neighborhood of z, so the derivatives on the right side of (4.16) may be evaluated according to Lemma 3.6 without taking the limit in (3.6) or (3.7). Thus $H_z(x, y)$ is continuous as well.

For the localization property, observe that if f vanishes in $U_m(z)$, then the solution of (4.13) must also vanish in $U_m(z)$ by the uniqueness part of Lemma 3.8. In view of (4.14) this means $G_z(x, y) = 0$ for $x \in U_m(z)$ and $y \notin U_m(z)$. Similarly, for z a junction vertex, if f vanishes in $F_w F_{j_1} K$ then so does u, hence $G_z(x, y) = 0$ for $x \in F_w F_{j_1} K$ and $y \in F_w F_{j_2} K$ for $j_1 \neq j_2$. Q.E.D

The local Green's function may not be continuous, or even bounded, as $y \to z$. This is shown in [KSS]. That paper also gives conditions for the convergence of the integral in (4.14).

We will show next that the normal derivatives of a function in $\mathrm{dom}(\Delta_\mu)$ are uniformly bounded. In contrast, the other derivatives need not be uniformly bounded, even for a harmonic function. For a simple example, consider the harmonic function h on the Sierpinski gasket with boundary values $h(v_1)=0$, $h(v_2)=h(v_3)=1$. Then $h(F_1^m v_2)=h(F_1^m v_3)=(3/5)^m$, hence

$$d_{13}h(F_1^m v_2)=5^m(h(F_1^m v_3)-h(F_1^m v_1))=5^m(3/5)^m=3^m.$$

THEOREM 4.3. *Assume Hypothesis* 8.1. *Let* $f \in \mathrm{dom}(\Delta_\mu)$.

(a) *The normal derivatives* $d_{j2}(x)$ *and* $d_{j'2}(x)$ *are uniformly bounded as* x *varies over all vertices.*

(b) *The sum of normal derivatives over the boundary vertices of* $F_w K$ *goes to zero as* $|w| \to \infty$. *More precisely*

$$\left| \sum_{x \in \partial F_w K} d_{j2} f(x) \right| \leqslant c\mu_w. \tag{4.17}$$

(c) *Fix a vertex* x, *and let* h *denote any harmonic function on* $U_0(x)$ *with zero normal derivative*(*s*) *at* x. *Then for any* $\varepsilon > 0$ *there exists* m *such that*

$$\begin{cases} |d_{j2}h(y)| \leqslant \varepsilon, & \text{or} \\ |d_{j'2}h(y)| \leqslant \varepsilon \end{cases} \tag{4.18}$$

for all vertices $y \in U_m(x)$.

Proof. (a) Let g be the solution of $\Delta_\mu^2 g = 0$ on K with vanishing normal derivatives and $g(v_j)=\delta_{j\ell}$. Note that

$$\int_K \Delta_\mu g \, d\mu = 0 \tag{4.19}$$

by the Gauss–Green formula for g and 1. Now we apply the Gauss–Green formula for f and $g \circ F_w$ on $F_w K$, for any w, to obtain

$$\partial_n f(F_w v_\ell) = \int_{F_w K} (g \circ F_w \, \Delta_\mu f - f \, \Delta_\mu g \circ F_w) \, d\mu. \tag{4.20}$$

Thus we need a uniform bound for the integrals in (4.20). There is no difficulty with $\int_{F_w K} g \circ F_w \, \Delta f \, d\mu$ since the integrand is uniformly bounded. Then $\Delta_\mu g \circ F_w = (r_w \mu_w)^{-1} (\Delta_\mu g) \circ F_w$ and $\mu(F_w K) = \mu_w$, so a direct estimate of $\int_{F_w K} f \Delta_\mu g \circ F_w \, d\mu$ is not sufficient. However, because of (4.19) we can write this as

$$\int_{F_w K} (f(y) - f(F_w v_\ell)) \, \Delta_\mu g \circ F_w(y) \, d\mu(y). \tag{4.21}$$

We then pick up the required factor of r_w from the estimate

$$|f(x) - f(y)| \leqslant c r_w \qquad \text{for} \quad x, y \in F_w K \tag{4.22}$$

which will be proved in Section 8.

(b) We use the Gauss–Green formula for f and 1 over $F_w K$ to obtain

$$\int_{F_w K} \Delta_\mu f \, d\mu = \sum_{\partial F_w K} \partial_n f, \tag{4.23}$$

and (4.17) follows from the obvious estimates on the integral in (4.23).

(c) For simplicity we give the proof in the case of a nonjunction vertex, $x = F_w v_j$. It suffices to prove the result for h varying over a basis for the space of harmonic functions with $d_{j2} f(x) = 0$. Let h_k denote the harmonic function on K with $\{h_k(v_\ell)\}$ equal to the eigenvector of M_j with eigenvalue λ_{jk} for $3 \leqslant k \leqslant N_0$. Then $h_k \circ F_j^m = \lambda_{jk}^m h_k$, and $h_k \circ F_w^{-1}$ is the desired basis. Let c denote an upper bound for all normal derivatives of h_k, as guaranteed by part (a). Then for any $y \in U_m(x)$,

$$|d_{j2} h_k \circ F_w^{-1}(y)| \leqslant c r_w^{-1} r_j^{-m} \lambda_{jk}^m.$$

Since $\lambda_{jk} < r_j = \lambda_{j2}$ for all $k \geqslant 3$, we obtain the desired estimate (4.18) by taking m large enough. Q.E.D

Part (c) may be interpreted as a weak continuity for normal derivatives for harmonic functions. The result one would really like is the following: if $f \in \operatorname{dom}(\Delta_\mu)$ and x is any fixed vertex, then the normal derivatives of f at x determine the normal derivatives of f at all points $y \in U_m(x)$ up to an error of at most ε (where m depends on ε). It is not clear whether or not this is true in general, but part (c) shows it is true for harmonic functions.

5. STRUCTURES WITH DIHEDRAL-3 SYMMETRY

We assume now that $\# V_0 = 3$ and all structures possess full D_3 symmetry. This means there exists a group $\mathcal{G}$ of homeomorphisms of K isomorphic to D_3 that acts as permutations on V_0, and $\mathcal{G}$ preserves the self-similar and harmonic structures and the self-similar measure. In particular, this forces the difference matrix defining the harmonic structures to be a multiple of

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \tag{5.1}$$

and without loss of generality we will take the multiple to be one. We must have $r_1 = r_2 = r_3$ and $\mu_1 = \mu_2 = \mu_3$, but in general it is not necessary that all r's and all μ's be the same. We denote by ρ the value of $r_j \mu_j$ for $j = 1, 2, 3$.

For each vertex v_j in V_0, let g_j denote the symmetry in $\mathcal{G}$ that fixes v_j and interchanges the other 2 vertices. We will call g_j the *point symmetry* at v_j. More generally, we will define a local point symmetry g_x at every vertex x in V_m, acting on $U_0(x)$ as follows. If $x = F_w v_j$ is a nonjunction vertex, let

$$g_x = F_w g_j F_w^{-1}; \tag{5.2}$$

while if $x = F_w F_j v_{j'}$ for $j \in J(x)$ is a junction vertex then

$$g_x = F_w F_j g_{j'} (F_w F_j)^{-1} \qquad \text{on } F_w F_j K \tag{5.3}$$

for all $j \in J(x)$.

In other words, g_x flips points in $F_w F_j K$, keeping x fixed and interchanging the other 2 vertices. Note that g_x preserves all structures in $U_0(x)$. In particular, $h \circ g_x$ is harmonic on $U_0(x)$ if h is.

The symmetry assumption implies that all matrices M_j are permutations of M_1, which has the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 1-a-b & a & b \\ 1-a-b & b & a \end{pmatrix}$$

for some constants satisfying $a > 0$, $b > 0$, $a + b < 1$. Then Hypothesis 3.1(b) is simply the condition $a \neq b$. The left eigenvectors are $\beta_{j2} = (2 \ \ -1 \ \ -1)$ with eigenvalue $a + b$ corresponding to the normal derivative, and $\beta_{j3} = (0 \ \ 1 \ \ -1)$ with eigenvalue $a - b$ corresponding to what we will call a *transverse derivative*. We observe easily that for any vertex x, functions which are even with respect to g_x have zero transverse derivatives, and functions which are odd have zero normal derivatives.

EXAMPLES. (i) The Sierpinski gasket SG has $a=2/5$ and $b=1/5$, hence $\lambda_{j2}=3/5$ and $\lambda_{23}=1/5$. In this case all $r_j=3/5$ and all $\mu_j=1/3$, so (4.5) just fails (with equality). Here $\rho=1/5$.

(ii) The hexagasket, or fractal Star of David, can be generated by 6 mappings with simultaneously rotate and contract by a factor of 1/3 in the plane. Thus V_0 consists of 3 points of an equilateral triangle, and V_1 consists of the vertices of the Star of David, as shown in Fig. 5.1. Although the same geometric fractal can be constructed by using contractions which do not rotate, this gives rise to a different self-similar structure (in particular with $\# V_0=6$). Our choice of self-similar structure destroys the D_6 symmetry of the geometric fractal, but it has the advantage of easier computation. We show in Fig. 5.1 the values on V_1 of one harmonic function. A basis is easily obtained by rotating this function by $2\pi/3$ and $4\pi/3$. From this data we easily find $a=2/7$ and $b=1/7$, hence $\lambda_{j2}=3/7$ and $\lambda_{j3}=1/7$. Since all $r_j=3/7$ and all $\mu_j=1/6$, condition (4.5) holds. Here $\rho=1/14$.

(iii) The level 3 Sierpinski gasket, obtained by taking 6 contractions of ratio 1/3 as shown in Fig. 5.2. Here we have $a=4/15$ and $b=3/15$, so $\lambda_{j2}=7/15$ and $\lambda_{j3}=1/15$. Since all $r_j=7/15$ and $\mu_j=1/6$, condition (4.5) fails with $\rho=7/90$ and $\lambda_{j3}=1/15$. Note that in this example we have two different types of junction points, so that $J(x)$ may have 2 or 3 elements.

We now consider a condition which is stronger than the notion of weak tangent given in Definition 3.9.

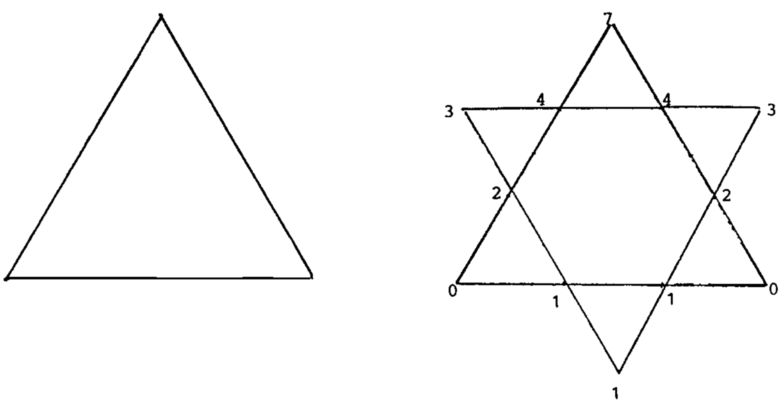

FIG. 5.1. The graphs of V_0 and V_1 for the fractal Star of David. The mappings into the 3 points of the Star not in V_0 involve a rotation. The values marked at the vertices of V_1 are the values of the harmonic function taking values (7, 0, 0) at the vertices of V_0.

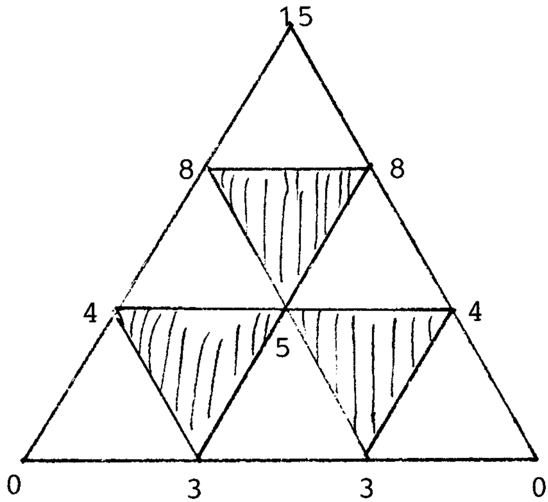

FIG. 5.2. The graph of the V_1 vertices of the level 3 Sierpinski gasket, with the values of the harmonic function that takes the values (15, 0, 0) at the vertices of V_0.

DEFINITION 5.1. Let f be a continuous function defined in a neighborhood of a vertex x. A harmonic function h on $U_0(x)$ is said to be a *tangent of order one* to f at x if

$$(f-h)|_{U_m(x)} = o((a+b)^m) \tag{5.4}$$

and

$$[(f-h)-(f-h)\circ g_x]|_{U_m(x)} = o((a-b)^m).$$

It is easy to see that a tangent is automatically a weak tangent. Indeed, since the definition implies $h(x)=f(x)$, we have (5.4) implies (3.13) or (3.14) for $k=2$, while (5.5) implies the same for $k=3$.

THEOREM 5.2. (a) *If $\rho < a-b$, then for every $f \in \mathrm{dom}(\Delta_\mu)$ and every vertex x, f has a tangent of order one at x.*

(b) *Even if $\rho \geqslant a-b$, we obtain the same conclusion if $g=\Delta_\mu f$ satisfies the following Hölder condition at x,*

$$|g(y)-g(x)| \leqslant c\gamma^m \qquad \textit{for all } y \in U_m(x) \tag{5.6}$$

for some γ satisfying

$$\rho\gamma < a-b.$$

Proof. (a) Note that $\rho < a-b$ is exactly (4.5). By Theorem 4.1 there exists a weak tangent, so (5.4) and (5.5) hold at the boundary points of $U_m(x)$. We need to extend these estimates to all of $U_m(x)$. Now $\Delta_\mu(f-h) = \Delta_\mu f = g$ is a bounded continuous function. Thus we can write $f-h = f_1 + f_2$ on $U_m(x)$ where f_1 is the harmonic function taking the same values as $f-h$ on the boundary of $U_m(x)$, and f_2 is the solution of $\Delta_\mu f_2 = g$ on $U_m(x)$ with zero boundary values. Now f_1 satisfies (5.4) on all of $U_m(x)$ because it is harmonic. On the other hand, f_2 is given by the integral of g on $U_m(x)$ against a scaled Green's function. Since the scaling factor is ρ^m, we have

$$f_2|_{U_m(x)} = O(\rho^m) = o((a-b)^m) \tag{5.8}$$

by (4.5), overkill! To extend (5.5) we use a similar argument for the function $(f-h)-(f-h)\circ g_x$. Here we use the structure symmetry to obtain

$$\Delta_\mu[(f-h)-(f-h)\circ g_x] = g - g\circ g_x.$$

Note that this time (5.8) is exactly what is needed for (5.5).

(b) First we need to modify the proof of Theorem 4.1 to obtain (5.4) and (5.5) on the boundary of $U_m(x)$. The argument up to (4.12) is the same. For the normal derivative ($m=2$ in (4.12)), the same estimate $(r_j \lambda_{j2}^{-1} \mu_j)^n$ is sufficient, since $\lambda_{j2} = r_j$ and $\mu_j < 1$. (The existence of normal derivatives was already known, of course!) We need something better when $m=3$, and this is provided by the Hölder estimate (5.6). The key observation is that h_{3n} has integral zero, since it is odd under the symmetry g_x. Thus

$$\int_{F_j^n K} r_j^n \lambda_{j2}^{-n-1} h_{3n}\, \Delta_\mu f\, d\mu = \int_{F_j^n K} r_j^n \lambda_{j2}^{-n-1} h_{3n}(y)(g(y)-g(x))\, d\mu(y)$$

and this is estimated by a multiple of $(r_j \lambda_{j3}^{-1} \mu_j \gamma)^n$. Since $\lambda_{j2} = a-b$, (5.7) guarantees geometric convergence of (4.12).

Next we need to modify the argument in part (a) to obtain (5.4) and (5.5) in general. For (5.4) no change is required, since $O(\rho^m) = o((a+b)^m)$ (remember that only $j = 1, 2, 3$ are involved in these estimates). For (5.5) we observe that $g - g\circ g_x$ is an odd function under the symmetry g_x, so (5.6) implies

$$|g(y) - g\circ g_x(y)| \leqslant c\gamma^m \qquad \text{for all} \quad y \in U_m(x). \tag{5.9}$$

Thus we obtain

$$f_2|_{U_m(x)} = O((\gamma\rho)^m)$$

and so (5.7) allows us to complete the proof as before. Q.E.D

How reasonable is it to expect a Hölder condition (5.6) to hold? If g is harmonic, or even just in $\mathrm{dom}(\Delta_\mu)$ then (5.6) holds with $\gamma = a+b$, but no smaller value. (This will be proved in Section 8.) Thus part (b) of the theorem is interesting provided

$$\rho(a+b) < (a-b). \tag{5.10}$$

This holds in examples (i) and (iii) above. Even if (5.10) does not hold, it might be possible to replace the hypothesis (5.6) with (5.9) and obtain a useful result, but we will not pursue this here. Note that for the Sierpinski gasket, the condition (5.7) just requires $\gamma < 1$, or in other words a Hölder condition of any order.

We consider next an example of a computation of a tangent to a function. This is interesting because it shows that the analog of Fermat's theorem does not hold: at a local maximum, the tangent does not have to have a local maximum. On the Sierpinski gasket we consider the ground state eigenfunction. The values of this function have been computed explicitly in [DSV] (see Fig. 4.9 there). It assumes its maximum (taken to be 1) on the boundary of largest deleted triangle. On any triangle of level m bordering on this boundary, function takes on the values 1, 1, $1-\lambda_m/2$ at the vertices, where $\lambda_0 = 2$ and $\lambda_m = (5-\sqrt{25-4\lambda_{m-1}})/2$. In particular $\lambda = \lim_{m\to\infty} 5^m \lambda_m$ exists and is nonzero ($\lambda \approx 2.2421385630294296...$). Thus if we fix a vertex on the boundary, the normal derivative is $\lim_{m\to 0} (5/3)^m \lambda_m/2 = 0$, and the transverse derivative is $\lim_{m\to\infty} 5^m\lambda_m/2 = \lambda/2$.

6. HIGHER ORDER TANGENTS

We continue the assumptions of Section 5, and in addition we will assume (5.10). We call a function n-harmonic if it satisfies $\Delta_\mu^n f = 0$, it being understood that this means $\Delta_\mu^k f \in \mathrm{dom}(\Delta_\mu)$ for $0 \leqslant k \leqslant n-1$. Tangents of order n will be n-harmonic functions on $U_0(x)$ that approximate a given function to higher order, extending (5.4) and (5.5), essential by inserting a factor $\rho^{(n-1)m}$ on the right side. The condition (5.10) is necessary if we want the approximation of order 2 to be better than the odd part approximation of order 1. It will not be necessary to define any new derivatives, as we will be able to use $d\Delta_\mu^k f$.

We need to extend Lemma 3.8 to n-harmonic functions. We begin with the easy part (a).

LEMMA 6.1. *Fix a vertex x. An n-harmonic function h on $U_m(x)$ is uniquely determined by the values of $\Delta_\mu^k h(x)$ and the gradients $d\Delta_\mu^k h(x)$ for $0 \leqslant k \leqslant n-1$, and any values satisfying the compatibility conditions*

$$\sum_{j \in J(x)} d_{j'2} \, \Delta_\mu^k h(x) = 0 \qquad \text{for} \quad 0 \leqslant k \leqslant n-1 \tag{6.1}$$

(*when x is a junction vertex*) *may be freely assigned.*

Proof. The existence of the gradients is guaranteed by Theorem 5.2(b) and condition (5.10). The compatibility conditions (6.1) follow from Theorem 3.10. Existence and uniqueness are equivalent because the dimension of the space of n-harmonic functions on $U_m(x)$ is equal to the number of derivatives (modulo the compatibility conditions).

We prove uniqueness by induction. The case $n=1$ is Theorem 3.8(a). Assume uniqueness holds for $n-1$. If h is n-harmonic and $\Delta_\mu^k h(x)=0$ and $d\Delta_\mu^k h(x)=0$ for $0 \leqslant k \leqslant n-1$, then the hypotheses of the induction step hold for $\Delta_\mu h$. Thus by the induction assumption we have $\Delta_\mu h=0$. So h is harmonic and we can apply Theorem 3.8(a) again to conclude $h \equiv 0$. Q.E.D

To obtain the analog of Theorem 3.8(b) we first need to discuss the fine properties of n-harmonic functions. These questions will be discussed in greater detail in [SU].

The existence and uniqueness for the Dirichlet problem in [Ki2] implies that an n-harmonic function h on K is uniquely determined by the values of $\Delta_\mu^k h|_{V_0}$ for $0 \leqslant k \leqslant n-1$. For simplicity of notation fix $j=1$, 2, or 3 and write F for F_j, and M for the matrix M_j in Hypothesis 3.1(b). Since the scaling factor for Δ_μ under F is ρ, there exist matrices $\tilde{M}_1, ..., \tilde{M}_{n-1}$ such that

$$h|_{F^m V_0} = Mh|_{F^{m-1} V_0} + \sum_{k=1}^{n-1} \rho^{km} \tilde{M}_k \, \Delta_\mu^k h|_{F^{m-1} V_0}. \tag{6.2}$$

The exact formula for $\tilde{M}_k$ will not be needed here, but symmetry considerations dictate the general form. Of course the jth row of $\tilde{M}_k$ must be identically zero, since $h(F^m v_j) = h(v_j)$ is given by the first term on the right in (6.2). Also (0, 1, 1) and (0, 1, -1) are left eigenvectors (for $j=1$), with nonzero eigenvalues (this by uniqueness of the Dirichlet problem).

LEMMA 6.2. *Let h be an n-harmonic function on some $U_{m_0}(x)$ satisfying $h(x)=0$ and*

$$\beta_{jk}h|_{F_wF_j^m V_0} = o((\rho^{n-1}\lambda_{jk})^m) \qquad as \quad m\to\infty \tag{6.3}$$

for $k=2, 3$ ($x=F_w v_j$ nonjunction), or

$$\beta_{j'k}h|_{F_wF_jF_{j'}^m V_0} = o((\rho^{n-1}\lambda_{j'k})^m) \qquad as \quad m\to\infty \tag{6.4}$$

for all $j\in J(x)$ and $k=2, 3$ (x a junction vertex). Then h is identically zero on $U_{m_0}(x)$.

Remark. We have written (6.3) and (6.4) to be the exact analogs of (3.11) and (3.12), but it is possible to rephrase them more in line with (5.4) and (5.5),

$$h|_{\partial U_m(x)} = o((\rho^{n-1}(a+b))^m) \tag{6.5}$$

and

$$(h-h\circ g_x)|_{\partial U_m(x)} = o((\rho^{n-1}(a-b))^m). \tag{6.6}$$

Here (6.6) corresponds to $k=3$, and (6.5) follows by combining the weaker estimates for $k=2$ and 3.

Proof. We give the proof in the nonjunction case with $w=\phi$ and $j=1$, for simplicity. For the sake of clarity we explain the argument first in the case $n=2$. We split h into its even and odd part under g_x, and show that each vanishes. For the odd part we note that (6.3) for $k=3$ reads

$$h(F^m v_2) - h(F^m v_3) = o((\rho(a-b))^m). \tag{6.7}$$

Now from (6.2) we obtain

$$\begin{aligned} h(F^m v_2) - h(F^m v_3) &= (a-b)(h(F^{m-1}v_2) - h(F^{m-1}v_3)) \\ &\quad + c\rho^m(\Delta_\mu h(F^{m-1}v_2) - \Delta_\mu h(F^{m-1}v_3)) \end{aligned} \tag{6.8}$$

for some nonzero c depending on $\tilde{M}_1$. There is a 2-dimensional space of odd 2-harmonic functions, and corresponding solutions of (6.8). A basis may be given by h_1, the odd harmonic function, and h_2 satisfying $\Delta_\mu h_2 = h_1$. Then (6.8) gives

$$h_1(F^m v_2) - h_1(F^m v_3) = c_1(a-b)^m, \qquad c_1 \neq 0, \tag{6.9}$$

as we already know, and then substituting (6.9) back in (6.8) for h_2,

$$h_2(F^m v_2) - h_2(F^m v_3) = c_2(a-b)^m + c_3(\rho(a-b))^m, \qquad c_3\neq 0. \tag{6.10}$$

But now it is clear that no nontrivial linear combination of (6.9) and (6.10) can satisfy (6.7). Thus the odd part vanishes.

For the even part, since we are assuming $h(v_1)=0$, (6.3) for $k=2$ reads

$$h(F^m v_2)+h(F^m v_3)=o((\rho(a+b))^m). \tag{6.11}$$

In place of (6.8) we have

$$h(F^m v_2)+h(F^m v_3)=(a+b)(h(F^{m-1}v_2)+h(F^{m-1}v_3)) \\ +c'\rho^m(\Delta_\mu h(F^{m-1}v_2)+\Delta_\mu h(F^{m-1}v_3)) \tag{6.12}$$

for $c'\neq 0$. One new observation we need is that $\Delta_\mu h(v_1)=0$, which follows directly from the definition and the estimates (6.7) and (6.11). Thus we again have a 2-dimensional space of even 2-harmonic functions with $h(v_1) =\Delta_\mu h(v_1)=0$, and a corresponding space of solutions of (6.12). We then obtain a basis h_3 and h_4 with $\Delta_\mu h_3=0$ and $\Delta_\mu h_4=h_3$ satisfying

$$h_3(F^m v_2)+h_3(F^m v_3)=c_1'(a+b)^m, \qquad c_1'\neq 0 \tag{6.13}$$

$$h_4(F^m v_2)+h_4(F^m v_3)=c_2'(a+b)^m+c_3'(\rho(a+b))^m, \qquad c_3'\neq 0 \tag{6.14}$$

and the argument is the same as before.

It is also possible to prove the vanishing of the even part without using the observation $\Delta_\mu h(v_1)=0$. Then the space of solutions is 3-dimensional and we need to add to the basis a function h_5 satisfying $\Delta_\mu h_5=1$. Then the solution of (6.12) satisfies

$$h_5(F^m v_2)+h_5(F^m v_3)=c_4'(a+b)^m+c_5'\rho^m, \qquad c_5'\neq 0. \tag{6.15}$$

Note that $\rho<a+b$ because $\rho=r_j\mu_j$ and $r_1=a+b$. Once again we can argue that no nontrivial linear combination of (6.13), (6.14), and (6.15) can satisfy (6.11).

We now discuss the extension to general n. For the odd part, we have in place of (6.8)

$$h(F^m v_2)-h(F^m v_3) \\ =(a-b)(h(F^{m-1}v_2)-h(F^{m-1}v_3)) \\ +\sum_{k=1}^{n-1} c_k\rho^{km}(\Delta_\mu^k h(F^{m-1}v_2)-\Delta_\mu^k h(F^{m-1}v_3)) \tag{6.16}$$

for nonzero coefficients c_k. We show by induction that any solution of (6.16) must be a linear combination of $(\rho^k(a-b))^m$ for $0 \leqslant k \leqslant n-1$. But in place of (6.7) we have

$$h(F^m v_2) - h(F^m v_3) = o((\rho^{n-1}(a-b))^m) \tag{6.17}$$

and no nontrivial linear combination can satisfy this. Thus the odd part vanishes. For the even part we extend the second argument given for $n=2$. In place of (6.16) we have

$$\begin{aligned} h(F^m v_2) &+ h(F^m v_3) \\ &= (a+b)(h(F^{m-1}v_2) + h(F^{m-1}v_3)) \\ &\quad + \sum_{k=1}^{n-1} c'_k \rho^{km}(\Delta_\mu^k h(F^{m-1}v_2) + \Delta_\mu^k h(F^{m-1}v_3)). \end{aligned} \tag{6.18}$$

Working in the $(2n-1)$-dimensional space of even n-harmonic functions with $h(v_1)=0$ we show by induction that the solutions to (6.18) are the linear combinations of $(\rho^k(a+b))^m$ for $0 \leqslant k \leqslant n-1$ and ρ^{km} for $1 \leqslant k \leqslant n-1$. Since we are assuming

$$h(F^m v_2) - h(F^m v_3) = o((\rho^{n-1}(a+b))^m) \tag{6.19}$$

we conclude that the even part also vanishes. Q.E.D

We note that the Lemma is sharp, as the proof shows that there do exist nonzero n-harmonic functions satisfying big-Oh rather than little-oh estimates in (6.3) or (6.4).

Definition 6.3. Let x be a vertex and f a function defined in a neighborhood of x. We say that an n-harmonic function h is a weak tangent of order n to f at x if

$$(f-h)|_{\partial U_m(x)} = o((\rho^{n-1}(a+b))^m) \tag{6.20}$$

and

$$(f-h-(f-h)\circ g_x)|_{\partial U_m(x)} = o((\rho^{n-1}(a-b))^m). \tag{6.21}$$

In that case h is unique by Lemma 6.2, and we write $h = T_n(f)$. We say that $T_n(f)$ is a tangent of order n to f at x if

$$(f - T_n(f))|_{U_m(x)} = o((\rho^{n-1}(a+b))^m) \tag{6.22}$$

and

$$(f - T_n(f) - (f - T_n(f)) \circ g_x)|_{U_m(x)} = o((\rho^{n-1}(a-b))^m). \qquad (6.23)$$

THEOREM 6.4. *Let $f \in \mathrm{dom}(\Delta_\mu^n)$ for some n, and let x be a vertex. Assume either $\rho < a-b$, or that $\Delta_\mu^n f$ satisfies the Hölder condition* (5.6) *for some γ satisfying* (5.7). *Then $d\Delta_\mu^k f(x)$ exists for all $k < n$. Let h be the n-harmonic function on $U_m(x)$ satisfying $\Delta_\mu^k h(x) = \Delta_\mu^k f(x)$ and $d\Delta_\mu^k h(x) = d\Delta_\mu^k f(x)$ for all $k < n$. Then $h = T_n(f)$ is a tangent of order n to f at x.*

Proof. The existence of the gradients $d\Delta_\mu^k f(x)$ follows by repeated application of Theorem 5.2. The Hölder condition is automatic for $\Delta_\mu^k f$ for $k < n$ because we are assuming (5.10). We can then construct h uniquely by Lemma 6.1.

We prove the estimates (6.22) and (6.23) by induction on n. The case $n = 1$ is just Theorem 5.2. We assume the result is true for all smaller values of n for the induction hypothesis. For simplicity of notation we give the proof for the point $x = v_1$. Without loss of generality we may assume $T_n(f) = 0$. Now $\Delta_\mu f$ satisfies the hypotheses of the theorem for $n-1$, and also $T_{n-1}(\Delta_\mu f) = \Delta_\mu T_n(f) = 0$ by construction. Thus (6.22) and (6.23) take the form

$$\Delta_\mu f|_{U_m(x)} = o((\rho^{n-2}(a+b))^m) \qquad (6.24)$$

and

$$(\Delta_\mu f - (\Delta_\mu f) \circ g_x)|_{U_m(x)} = o((\rho^{n-2}(a-b))^m). \qquad (6.25)$$

We will use these estimates in the identity (4.11), which in this case says

$$(a+b)^{-1} \int_{F_1^m K} h_{2m} \Delta_\mu f \, d\mu = (a+b)^{-m-1} (f(F_1^{m+1} v_2) + f(F_1^{m+1} v_3)) - (a+b)^{-m} (f(F_1^m v_2) + f(F_1^m v_3)) \qquad (6.26)$$

and

$$(a-b)^{-m-1} r^m \int_{F_1^m K} h_{3m} \Delta_\mu f \, d\mu = (a-b)^{-m-1} (f(F_1^{m+1} v_2) - f(F_1^{m+1} v_3)) - (a-b)^{-m} (f(F_1^m v_2) - f(F_1^m v_3)), \qquad (6.27)$$

where we have used the assumption $f(v_1) = 0$ in simplifying (6.26). We also observe that since h_{3m} is odd with respect to g_x, we can replace $\Delta_\mu f$ by its odd part in the integral in (6.27). Now we sum these identities from $m=k$ to ∞, making use of the assumption $df(v_1)=0$, to obtain

$$f(F_1^k v_2)+f(F_1^k v_3)=-(a+b)^{k-1}\sum_{m=k}^{\infty}\int_{F_1^m K} h_{2m}\,\Delta_\mu f\,d\mu \tag{6.28}$$

and

$$f(F_1^k v_2)-f(F_1^k v_3) = -\tfrac{1}{2}(a-b)^k \sum_{m=k}^{\infty} r^m (a-b)^{-m-1}\, h_{3n}(\Delta_\mu f-(\Delta_\mu f)\circ g_x)\,d\mu. \tag{6.29}$$

We substitute estimate (6.24) in (6.28) and (6.25) in (6.29) and use the uniform boundedness of h_{2m} and h_{3m} and the measure identity $\mu(F_1^m K)=\mu^m$ to obtain

$$\begin{aligned} f(F_1^k v_2)+f(F_1^k v_3) &= (a+b)^k \sum_{m=k}^{\infty} o(\rho^{(n-1)m}) \\ &= o((\rho^{n-1}(a+b))^k) \end{aligned} \tag{6.30}$$

and

$$\begin{aligned} f(F_1^k v_2)-f(F_1^k v_s) &= (a-b)^k \sum_{m=k}^{\infty} o((r(a-b)^{-1}\mu\rho^{(n-2)}(a-b))^m) \\ &= (a-b)^k \sum_{m=k}^{\infty} o(\rho^{(n-1)m}) = o((\rho^{n-1}(a-b))^k). \end{aligned} \tag{6.31}$$

This is exactly (6.20) and (6.21), showing that $T_n(f)$ is a weak tangent.

The argument to show $T_n(f)$ is actually a tangent is very similar to the argument in the proof of Theorem 5.2. Again assuming without loss of generality that $T_n(f)=0$, we write $f=f_1+f_2$ where $\Delta_\mu^n f_1=0$ and $\Delta_\mu^k f|_{\partial U_m} =\Delta_\mu^k f_1|_{\partial U_m}$ for all $k<n$. Now f_2 is given by the n-fold product of the Green's operator on $\Delta_\mu^n f$, so we obtain in place of (5.8)

$$f_2|_{U_m(x)}=O(\rho^{nm}) \tag{6.32}$$

by using a uniform bound for $\Delta_\mu^n f$, which suffices for both estimates for f_2 under the assumption $\rho<a-b$. In the other case, we still get (6.22) out of (6.32), and for the odd part of f_2 we can squeeze out an extra factor of γ^m from the Hölder condition on $\Delta_\mu^n f$, giving (6.23). Thus f_2 satisfies the required estimates.

Now for f_1 we have not only the estimates (6.20) and (6.21), but also the analogous estimates for $\Delta_\mu^k f_1$ for $k<n$ by the induction hypothesis, so

$$\Delta_\mu^k f_1|_{\partial U_m(x)} = o((\rho^{n-k-1}(a+b))^m) \tag{6.33}$$

and

$$\Delta_\mu^k (f_1 - f_1 \circ g_x)|_{\partial U_m(x)} = o((\rho^{n-k-1}(a-b))^m). \tag{6.34}$$

Thus to complete the proof we need the following a priori estimate for solutions of $\Delta_\mu^n u = 0$,

$$\| u|_{U_m(x)} \|_\infty \leqslant c \sum_{k=0}^{n-1} \rho^{km} \| \Delta_\mu^k u|_{\partial U_m} \|_\infty . \tag{6.35}$$

But (6.35) is immediate for $m=0$ because the space of solutions is finite dimensional and $u|_{U_m(x)}$ is uniquely determined by $\Delta_\mu^k u|_{\partial U_m(x)}$ for $k<n$. It then follows for general m by a scaling argument. Q.E.D

An important observation made in the proof is that

$$\Delta_\mu T_n(f) = T_{n-1}(\Delta_\mu f), \tag{6.36}$$

or more generally,

$$\Delta_\mu^k T_n(f) = T_{n-k}(\Delta_\mu^k f) \qquad \text{for} \quad k<n. \tag{6.37}$$

In other words, we can apply the Laplacian to a Taylor approximation. This is the analog of being able to differentiate a standard Taylor approximation.

A simple corollary of the theorem is that we can improve the approximation to the rate of decay of the next term $(T_{n+1}(f) - T_n(f))$ by assuming greater smoothness. The proof of Lemma 6.2 shows that this yields a rate of $O((\rho^n(a-b))^m)$ for the odd part, but only $O(\rho^{nm})$ for the even part.

Definition 6.5. We say that $T_n(f)$ is a strong tangent of order n to f at x if we have

$$(f - T_n(f))|_{U_m(x)} = O(\rho^{nm}) \tag{6.38}$$

and

$$f - T_n(f) - (f - T_n(f)) \circ g_x|_{U_m(x)} = O((\rho^n(a-b))^m). \tag{6.39}$$

Corollary 6.6. *Assume the hypotheses of Theorem 6.4 hold for $n+1$. Then $T_n(f)$ is a strong tangent to f at x.*

Proof. It is clear that (6.38) and (6.39) hold for $T_{n+1}(f)$ in place of $T_n(f)$, because (6.22) and (6.23) for $n+1$ are stronger estimates. Thus it suffices to show that

$$(T_{n+1}(f) - T_n(f))|_{U_m(x)} = O(\rho^{nm}) \tag{6.40}$$

and

$$(T_{n+1}(f) - T_n(f) - (T_{n+1}(f) - T_n(f)) \circ g_x)|_{U_m(x)} = O((\rho^n(a+b))^m). \tag{6.41}$$

Now $u = T_{n+1}(f) - T_n(f)$ is an $(n+1)$-harmonic function satisfying $\Delta_\mu^k u(x) = 0$ and $d\Delta_\mu^k u(x) = 0$ for $k < n$. Also $u - u \circ g_x$ has the same properties and in addition is odd. The proof of Lemma 6.2 gives (6.40) and (6.41). Q.E.D

There is one more type of result that should be part of the story, characterizing weak tangents in terms of existence of derivatives, and generalizing Theorem 3.10. We state one possible version as a conjecture.

Conjecture 6.7. Let $f \in \operatorname{dom}(\Delta_\mu^{n-1})$. Then f has a weak tangent of order n at x if and only if $d\Delta_\mu^{n-1} f(x)$ exists.

It is easy to rewrite the Taylor approximations as sums of terms of different orders. We say that f vanishes to order k at x if $\Delta_\mu^j f(x) = 0$ and $d\Delta_\mu^j f(x) = 0$ for all $j < k$. Then $T_n(f) = f_0 + f_1 + \cdots + f_n$ where f_0 is constant $(= f(x))$ and f_k is k-harmonic and vanishes to order k at x. In fact $f_1 = T_1(f) - f(x)$ and $f_k = T_k(f) - T_{k-1}(f)$ for $k \geqslant 2$. We can then ask for $f \in \operatorname{dom}(\Delta_\mu^\infty)$ whether the infinite series $\sum_0^\infty f_n$ converges to f on some neighborhood of x. Such functions could be called *analytic*. Of course this will be true if f is n-harmonic for some n, for then the series terminates. Given any sequence $\{f_n\}$ of n-harmonic functions vanishing to order n at x, we can form the function $f = \sum_{n=0}^\infty \varepsilon_n f_n$ for a sequence ε_n going to zero rapidly enough to make the series converge and so that $T_n(f) = \sum_{k=0}^n \varepsilon_k f_k$. This gives a rather artificial example of an analytic function. It is not known even if an eigenfunction of Δ_μ must be analytic. Also, it is not clear that analytic functions enjoy any of the properties we would expect of them!

We conclude this section with a list of problems of a somewhat more technical nature that should be solved to complete the story of Taylor approximations.

(1) It should be possible to calculate $\Delta_\mu^k f(x)$ and all the derivatives in $d\Delta_\mu^k f(x)$ in terms of limits of linear combinations of the values of f at points approaching x.

(2) It should also be possible to calculate the same quantities as limits of integrals involving f over $U_m(x)$, as in the proof of Theorem 4.1.

(3) The tangents $T_n(f)$ should be expressible as limits of n-harmonic functions that agree with f in a suitable sense on $U_m(x)$. The model for this is Theorem 3.11. There are several ways one might do this. The simplest is to require the functions and all powers of Δ_μ up to $n-1$ to be equal on $\partial U_m(x)$. Another possibility is to make just the functions equal on $\partial U_{m+k}(x)$ for $0 \leqslant k \leqslant n-1$.

(4) It may be possible to obtain the conclusions of Theorem 6.4 and Corollary 6.6 under weaker hypotheses.

7. TAYLOR APPROXIMATION AT GENERIC POINTS

The results in this section deal with the Sierpinski gasket only. We would like to understand the approximation of functions in a neighborhood of a "generic" point. We thus exclude vertices, and as needed we will assume the point x satisfies conditions that hold almost everywhere with respect to the measure μ. There is a canonical system of neighborhoods of x that is quite different from the system $U_m(x)$ at vertices. For generic x there is a unique infinite word $w = (w_1, w_2, ...)$ such that $x = \lim_{m \to 0} F_{W_m} K$ where $W_m = (w_1, ..., w_m)$. We then take $F_{W_m} K$ as our neighborhood basis.

If h is any harmonic function on K we know

$$h|_{\partial F_{W_m} K} = M_{w_m} \cdots M_{w_1} h|_{\partial K}, \tag{7.1}$$

where

$$M_1 = \frac{1}{5}\begin{pmatrix} 5 & 0 & 0 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}, \quad M_2 = \frac{1}{5}\begin{pmatrix} 2 & 2 & 1 \\ 0 & 5 & 0 \\ 1 & 2 & 2 \end{pmatrix}, \quad M_3 = \frac{1}{5}\begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 5 \end{pmatrix}. \tag{7.2}$$

Inverting (7.1), we obtain

$$h|_{\partial K} = M_{w_1}^{-1} \cdots M_{w_m}^{-1} h|_{\partial F_{W_m} K}. \tag{7.3}$$

Thus we can use (7.1) to "zoom in" and (7.3) to "zoom out" on harmonic functions. The form of (7.1) fits exactly the theory of products of random matrices, while the multiplications in (7.3) are in the reverse order. Before using this theory, however, we want to factor out by the constant functions. We choose a 2-dimensional space of harmonic functions (for example, functions with mean value zero, which is equivalent to $h(v_1) + h(v_2) + h(v_3) = 0$) complementary to the constants. There is no way to do this so that the mappings $h \to (h|_{F_j K}) \circ F_j^{-1}$ (represented by the matrices M_j) preserve this subspace, nor can we preserve the symmetry under permutations. Since each matrix M_j has eigenvalues 1, 3/5, 1/5, with the eigenvalue 1 corresponding to the constant functions, we are left after reduction by three 2×2 matrices $\tilde{M}_j$ with eigenvalues 3/5, 1/5, and the set of these matrices has no nontrivial invariant subspaces. The specific choice of basis $(1, -1, 0)$ and $(0, 1, -1)$ leads to the specific matrices

$$\tilde{M}_1 = \tfrac{1}{5}\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \tilde{M}_2 = \tfrac{1}{5}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \qquad \tilde{M}_3 = \tfrac{1}{5}\begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix}. \tag{7.4}$$

Now given a continuous function defined in a neighborhood of a generic point x, we define a first order tangent at x as the harmonic function h on K, if it exists, that is the limit of harmonic functions h_m defined by

$$h_m|_{\partial F_{w_m} K} = f|_{\partial F_{w_m} K}. \tag{7.5}$$

Of course the convergence of h_m to h only depends on the convergence of $h_m|_{\partial K}$. The continuity of f and (7.5) imply that $\lim_{m \to \infty} h_m(x) = f(x)$. This means that the tangent h, if it exists, must lie in the 2-dimensional affine subspace of harmonic functions satisfying $h(x) = f(x)$.

Now assume that $f \in \operatorname{dom}(\Delta_\mu)$. Then from (7.5) we know

$$(h_m - f)|_{F_{w_m} K} = O(5^{-m}). \tag{7.6}$$

By substituting (7.6) for m and $m-1$ we obtain

$$(h_m - h_{m-1})|_{F_{w_m} K} = O(5^{-m}). \tag{7.7}$$

Since $h_m - h_{m-1}$ is harmonic, we have (7.1) holding for $h = h_m - h_{m-1}$. Now the theory of products of random matrices tells us that there are 2 indices, which we denote α_+, α_-, associated with independent random choice of $\tilde{M}_1$, $\tilde{M}_2$, $\tilde{M}_3$ with equal probability. Since these matrices have determinant 3/25, the indices satisfy

$$\alpha_+ \alpha_- = 3/25 \tag{7.8}$$

and

$$\alpha_- < \sqrt{3}/5 < \alpha_+. \tag{7.9}$$

According to theorems of Furstenberg [Fu] and Oseledec [CKM, O], with probability one there exists a nonzero vector u_- such that

$$\lim_{m \to \infty} m^{-1} \log \|\tilde{M}_{w_m} \cdots \tilde{M}_{w_1} u\| = \begin{cases} \log \alpha_- & \text{if} \quad u = c u_- \\ \log \alpha_+ & \text{otherwise} \end{cases} \tag{7.10}$$

(the values $\log \alpha_+$ and $\log \alpha_-$ are called Lyapunov indices). The vector u_- depends on x but the values of α_+, α_- do not.

We are grateful to Divakar Viswanath for the computer assisted proof of the next lemma.

LEMMA 7.1. *We have the bounds*

$$0.67 - \log 5 \leqslant \log \alpha_+ \leqslant 0.725 - \log 5. \tag{7.11}$$

Proof. A well-known formula of Furstenberg [Fu] gives

$$\log \alpha_+ = \mathrm{Exp}_u \left(\tfrac{1}{3} \sum_{j=1}^{3} \log \|\tilde{M}_j u\| \right), \tag{7.12}$$

where the expectation is taken with respect to an invariant measure for the random matrices acting on unit 2-vectors. While it is difficult to compute this measure exactly, we can obtain upper and lower bounds for $\log \alpha_+$ simply by taking upper and lower bounds of the integrand. We actually need to do this for the set of 3^k matrices consisting of all k-fold products of $\tilde{M}_1$, $\tilde{M}_2$, $\tilde{M}_3$ with equal probabilities. This system has the same Lyapunov exponents and invariant measure, but the integrand

$$\frac{1}{3^k} \sum_{j_1=1}^{3} \cdots \sum_{j_k=1}^{3} \log \|\tilde{M}_{j_k} \cdots \tilde{M}_{j_1} u\| \tag{7.13}$$

has less variability (see [BL, V]). The estimate (7.11) was obtained by a computer calculation of upper and lower bounds for (7.13) with $k = 10$. Q.E.D

THEOREM 7.2. *For a generic point x, and any $f \in \mathrm{dom}(\Delta_\mu)$, the tangent $T_1(f)$ at x exists, and in fact*

$$\|h_m - T_1(f)\|_\infty = O((5(\alpha_- - \varepsilon))^{-m}) \tag{7.14}$$

for any $\varepsilon > 0$.

Proof. For a generic point x (independent of the choice of f), we may deduce from (7.7) that

$$\|h_m - h_{m-1}\|_\infty = O((5(\alpha_- - \varepsilon))^{-m}) \tag{7.15}$$

for any $\varepsilon > 0$. From (7.8) and the upper bound in (7.11) we have $\alpha_- > 1/5$, hence (7.15) is a geometric convergence rate and so the tangent h exists, and (7.14) holds. Q.E.D

Next we consider the approximation properties of the tangent.

THEOREM 7.3. *At a generic point* x, *for any* $f \in \mathrm{dom}(\Delta_\mu)$, *we have*

$$(f - T_1(f))|_{F_{W_m}K} = O(\beta^m) \tag{7.16}$$

provided

$$\beta > \alpha_+/5\alpha_-. \tag{7.17}$$

Proof. Write $T_1(f) = h$. In view of (7.6), it suffices to estimate $(h - h_m)|_{F_{W_m}K}$. We again use (7.10) to obtain this information from the global estimate (7.14), and from the behavior at the point x, where $h(x) = f(x)$ and so, by (7.6)

$$h(x) - h_m(x) = O(5^{-m}). \tag{7.18}$$

Of course $h(x) - h_m(x)$ is just a convex combination of the values $h(F_{W_m}v_k) - h_m(F_{W_m}v_k)$, $k = 1, 2, 3$ on the boundary of $\partial F_{W_m}K$. From (7.14) and (7.10) we obtain

$$h(F_{W_m}v_k) - h(F_{W_m}v_\ell) - (h_m(F_{W_m}v_k) - h_m(F_{W_m}v_\ell)) = O(\beta^m) \tag{7.19}$$

provided (7.17) holds. This implies (7.16). Q.E.D

LEMMA 7.4. *For a generic point* x, *any harmonic function* h *satisfying*

$$h|_{F_{W_m}K} = O(\beta^m) \qquad \text{for some } \beta < \alpha_- \tag{7.20}$$

must vanish identically.

Proof. This follows immediately from (7.10). Q.E.D

COROLLARY 7.5. *For a generic point* x *and* $f \in \mathrm{dom}(\Delta_\mu)$, *the tangent* $T_1(f)$ *is the unique harmonic function satisfying the estimate* (7.16) *if we take* β *close enough to* $\alpha_+/5\alpha_-$.

Proof. If $\alpha_+/5\alpha_- < \alpha_-$ then we can choose β in between to satisfy both $\beta < \alpha_-$ and (7.17). But by (7.8) this is equivalent to the estimate

$$\alpha_+ < 3^{2/3}5, \tag{7.21}$$

and the upper bound in (7.11) is better ($\frac{2}{3}\log 3 \approx 0.7324081 > 0.725$). Thus the difference between two harmonic functions satisfying (7.16) for such β would satisfy (7.20) and hence be zero. Q.E.D

It should also be possible to define tangents of order k for functions in $\mathrm{dom}(\Delta_\mu^k)$ and obtain improved decay in the estimate (7.16). One way to do this would be to take the limit as $m \to \infty$ of k-harmonic functions $h_{m,k}$ which satisfy

$$\Delta_{\mu}^{j} h_{m,k}|_{\partial F_{W_m} K} = \Delta_{\mu}^{j} f|_{\partial F_{W_m} K} \tag{7.22}$$

for all $j < k$.

On the other hand, it is not clear how to extend the ideas presented in this section to very many other fractals. One of the difficulties is that we often don't have the analog of (7.3), because the matrices M_j that arise are not all invertible (the ones corresponding to mappings F_j that fix a boundary point are always invertible, however). This is already the case for the hexagasket.

8. HÖLDER ESTIMATES

As an appendix to this paper, we establish the Hölder estimates for functions in the domain of the Laplacian. We first prove the estimates for harmonic functions, and then show that functions in the domain of the Laplacian enjoy the same property. For the first part we will need an additional assumption on the matrices M_j that represent the map $h \to (h|_{F_j K}) \circ F_j^{-1}$ for all F_j in the i.f.s., not just the ones that fix a boundary vertex. The identity

$$h|_{\partial F_j K} = M_j h|_{\partial K} \tag{8.1}$$

defines M_j. Write

$$\mathscr{E}_0(u) = \sum_{k,\ell} D_{k\ell}(u_\ell - u_k)^2 \tag{8.2}$$

for the energy form on V_0 that defines the harmonic structure. The self-similar identity for $\mathscr{E}_0$ is

$$\mathscr{E}_0(u) = \sum_j r_j^{-1} \mathscr{E}_0(M_j u). \tag{8.3}$$

HYPOTHESIS 8.1. *Assume*

$$\mathscr{E}_0(M_j u) \leqslant r_j^2 \mathscr{E}_0(u). \tag{8.4}$$

Remark. A slightly weaker statement is shown to be true in general in [T2].

In terms of energy of harmonic functions, (8.3) becomes

$$\mathscr{E}(h, h) = \sum_j r_j^{-1} \mathscr{E}(h|_{F_jK} \circ F_j^{-1}, h|_{F_jK} \circ F_j^{-1}) \tag{8.5}$$

and (8.4) becomes

$$\mathscr{E}(h|_{F_jK} \circ F_j^{-1}, h|_{F_jK} \circ F_j^{-1}) \leqslant r_j^2 \mathscr{E}(h, h) \tag{8.6}$$

(in fact (8.5) holds for all functions). For $1 \leqslant j \leqslant N_0$, when F_j fixes the vertex v_j, then r_j is the second eigenvalue of M_j which shows that (8.4) is sharp, becoming an equality if u is an eigenvector ($M_j u = r_j u$).

Lemma 8.2. *Under the dihedral-*3 *symmetry assumption of Section* 6, (8.4) *holds for* $j = 1, 2, 3$.

Proof. In this case

$$\mathscr{E}_0(u) = \sum_{j,k} (u_j - u_k)^2$$

and

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1-a-b & a & b \\ 1-a-b & b & a \end{pmatrix}$$

with $a + b < 1$, and $r_j = a + b$. Thus

$$\begin{aligned} \mathscr{E}_0(M_1 u) &= (a(u_1 - u_2) + b(u_1 - u_3))^2 + (a-b)^2 (u_2 - u_3)^2 \\ &\quad + (a(u_3 - u_1) + b(u_2 - u_1))^2 \\ &= (a^2 + b^2)(u_1 - u_2)^2 + (u_1 - u_3)^2 + (a-b)^2 (u_2 - u_3)^2 \\ &\quad + 4ab(u_1 - u_2)(u_2 - u_3) \end{aligned}$$

and the desired estimate follows from

$$4ab(u_1 - u_2)(u_1 - u_3) \leqslant 2ab((u_1 - u_2)^2 + (u_1 - u_3)^2).$$

The same argument works for M_2 and M_3. Q.E.D

The significance of Hypothesis 8.1 is for the other matrices M_j. It is easy to verify that it holds for Examples (ii) and (iii) in Section 6. In fact, we do not know any examples where Hypothesis 8.1 does not hold.

THEOREM 8.3. *Assume Hypothesis* 8.1. *Then*

$$|h(x)-h(y)| \leqslant cr_w \qquad \text{if} \quad x, y \in F_w K \tag{8.7}$$

for any harmonic function h and any word w, where the constant c may be taken to be a multiple of $\|h\|_\infty$.

Proof. By the maximum principle it suffices to prove (8.7) for x and y on the boundary of $F_w K$. Now the energy form $\mathscr{E}_0$ annihilates only the constant vectors, so there is an estimate

$$|u_k - u_\ell| \leqslant c\mathscr{E}_0(u)^{1/2} \tag{8.8}$$

for some constant c. So

$$|h(F_w v_k) - h(F_w v_\ell)| \leqslant c\mathscr{E}_0(h|_{\partial F_w K}) \leqslant cr_w \mathscr{E}_0(h|_{\partial K})$$

by repeated use of (8.4), and of course $\mathscr{E}_0(h|_{\partial K})$ can be bounded by a multiple of $\|h\|_\infty$. Q.E.D

THEOREM 8.4. *Let* μ *be a self-similar measure. If* (8.7) *holds for every harmonic function, then it holds for every function in* $\text{dom}(\Delta_\mu)$.

Proof. If $f \in \text{dom}(\Delta_\mu)$ and h is the harmonic function taking the same boundary values as f, then

$$f(x) - h(x) = \int G(x, z)\, g(z)\, d\mu(z) \tag{8.9}$$

for $g = \Delta_\mu f$. Since h satisfies (8.7) by assumption, it suffices to verify it for $f-h$. Since g is bounded, this amounts to showing

$$\int |G(x, z) - G(y, z)|\, d\mu(z) \leqslant cr_w \tag{8.10}$$

for $x, y \in F_w K$. By the form of (8.10) it suffices to show this for x and y on the boundary of $F_w K$.

Now we use a formula of Kigami for the Green's function [Ki2]. For $x \in \partial F_w K$,

$$G(x, z) = \sum_{n=0}^{m} (r_{w_1} \cdots r_{w_n})\, \Phi(F_{w_n}^{-1} \cdots F_{w_1}^{-1} x, F_{w_n}^{-1} \cdots F_{w_1}^{-1} z), \tag{8.11}$$

where Φ is a continuous piecewise harmonic function (harmonic on each cell $F_j K$), and the summand in zero unless $z \in F_{w_1} \cdots F_{w_n} K$. More precisely,

Φ is a linear combination of products $\psi_p(x)\,\psi_q(z)$ where ψ_p are piecewise harmonic, and hence satisfy (8.7). So, for $x,\ y \in F_{w_1} \cdots F_{w_m} K$,

$$|\Phi(x, z) - \Phi(y, z)| \leqslant c r_{w_1} \cdots r_{w_m}$$

and also

$$\Phi(F_{w_n}^{-1} \cdots F_{w_1}^{-1}, x, F_{w_n}^{-1} \cdots F_{w_1}^{-1} z) \\ - \Phi(F_{w_n}^{-1} \cdots F_{w_1}^{-1} y, F_{w_n}^{-1} \cdots F_{w_1}^{-1} z)| \leqslant c r_{w_{n+1}} \cdots r_{w_m}.$$

When we substitute this estimate to estimate $|G(x, z) - G(y, z)|$ using (8.11), each summand is of the order r_w. This is not quite enough, since it produces an estimate $m r_w$ with the extraneous factor of m. We can eliminate this, however, when we estimate the integral in (8.10), since for each fixed z not all summands will be present. More precisely, if $n(z)$ is the largest value of n for which $z \in F_{w_1} \cdots F_{w_n} K$, then the sum in (8.11) stops at $n = n(z)$. Thus we have

$$\int |G(x, z) - G(y, z)|\, d\mu(z) \leqslant c r_w \int n(z)\, d\mu(z).$$

But it is easy to see that

$$\int n(z)\, d\mu(z) \leqslant \sum_{n=0}^{\infty} \mu(F_{w_1} \cdots F_{w_n} K)$$

and this is uniformly bounded for any self-similar measure. This proves (8.10).
Q.E.D

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