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Self-Similarity of Volume Measures for Laplacians on P. C. F. Self-Similar Fractals

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Abstract: Our main goal in this paper is to obtain a precise analogue of Weyl's asymptotic formula for the eigenvalue distribution of Laplacians on a certain class of "finitely ramified" (or p.c.f.) self-similar fractals, building, in particular, on the work of [7, 9, 22, 24]. Our main result consists in precisely identifying (for the class of "decimable fractals") the volume measures constructed by the second author in [24] for general p.c.f. fractals and showing that they are self-similar.

From a physical point of view, our results should be relevant to the study of the density of states for diffusions and wave propagation in fractal media.

1. Introduction

In this paper, we will obtain a refined version of Weyl's formula for the eigenvalue distribution of Laplacians on certain self-similar fractals. There is now a well-developed theory of Laplacians and diffusions on "finitely ramified" self-similar sets. (See, for example, Kusuoka [23], Goldstein [14], Barlow and Perkins [4].) Before discussing our results, we first recall Weyl's classical formula for Laplacians on Riemannian manifolds. (See, for example, Hörmander [17] and in the Euclidean case, Reed and Simon [31].)

Let $-\Delta$ be the positive Laplacian (or Laplace–Beltrami operator) on a closed, compact d-dimensional smooth (connected) Riemannian manifold M. Then it is well-known that $-\Delta$ has a discrete spectrum $\{\lambda_j\}_{j=1}^{\infty}$ which can be written in non-decreasing order according to multiplicity as follows:

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots \to \infty.$$

For $x > 0$, let $\rho(x) = \#\{j \ge 1 : \lambda_j \le x\}$ denote the eigenvalue counting function of $-\Delta$. Then Weyl's asymptotic formula in this context states that

$$\rho(x) = c_d \mathrm{Vol}(M) x^{d/2}(1 + o(1)) \tag{1.1}$$

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as $x \to \infty$, where c_d is a positive constant depending only on d and where $\mathrm{Vol}(M)$ denotes the Riemannian volume of M. Henceforth, $o(1)$ stands for a function that tends to zero as $x \to \infty$. We note that if M is a compact manifold with smooth boundary, an entirely analogous formula holds for the Dirichlet Laplacian on M. (See also, for example, [2 and 31] for various physical applications of Weyl's formula in the case where M is a bounded smooth domain in Euclidean space.)

If, in addition, M is a (closed) spin manifold, Connes ([7, 8, §VI.1]) has used the notion of Dixmier trace (a suitable scale-invariant trace which is well-suited for dealing with logarithmic divergences) to reconstruct the Riemannian volume measure of M and hence to reinterpret Weyl's formula within the framework of noncommutative geometry.

In the case of a "finitely ramified" (that is, p. c. f.) self-similar fractal K instead of a smooth manifold, Kigami and Lapidus [22] have obtained a partial analogue of Weyl's formula for the Dirichlet Laplacian on K. When K is in "general position" (the "non-lattice case"), the counterpart of (1.1) is then given by

$$\rho(x) = Cx^{d_S/2}(1 + o(1)) \tag{1.2}$$

as $x \to \infty$, where C is a positive constant depending on K and $d_S > 0$ is a suitable "spectral exponent" defined in Theorem 3.2 below. On the other hand, in the "lattice case" (also called the "arithmetic case" in probability theory), the analogue of (1.1) is given by

$$\rho(x) = (G(\log x/2) + o(1))x^{d_S/2} \tag{1.3}$$

as $x \to \infty$, where G is a positive periodic function that is bounded away from zero and infinity; so that $\rho(x) \asymp x^{d_S/2}$. (See Theorem 3.2 below.)

Motivated by the above mentioned work of Connes [7] and using, in particular, the results of [22], Lapidus [24] has constructed a *"volume measure"* ν on the p. c. f. self-similar set K, associated with the Dirichlet Laplacian on K. (See Theorem 4.1 below.) Moreover, he has shown that the total mass of ν, namely, $\nu(K)$, is given by the constant C appearing in (1.2) in the non-lattice case, and by the mean-value of the periodic function G occurring in (1.3) in the lattice case.

In part by analogy with the work of Connes and Sullivan on the "quantized calculus" on limit sets of quasi-Fuchsian groups ([9, 8, §IV.3]), such as certain hyperbolic Julia sets, the second author has also conjectured that this volume measure (or rather, the associated probability measure $\nu/\nu(K)$) is "approximately self-similar". (See [24, §5.1 and 25, §6.)

In the present paper, under a certain hypothesis, we will identify the volume measure ν constructed in [24] and show that it is equal to a constant multiple of a self-similar measure on K. (See Theorem 4.7 in conjunction with Hypothesis 4.6.) Moreover, we will verify that this hypothesis holds for the class of p. c. f. self-similar sets satisfying the eigenvalue decimation property, which was first introduced by the physicists Rammal and Toulouse [30] and Rammal [29] for the case of the Sierpinski gasket. Several examples of such "decimable fractals" are provided in Sect. 5 below.

A sample of physics papers studying finitely ramified fractals includes Dhar [11], Alexander and Orbach [1], Berry [5, 6], Hattori et al. [15], along with the survey articles by Liu [26], Havlin and Bunde [16] and by Nakayama et al. [28]. We note that in the mathematics literature, the eigenvalue decimation method – which provides an explicit algorithm to compute the eigenvalues and the eigenfunctions of the Laplacian – has been justified rigorously by Fukushima and Shima 13] for the Sierpinski gasket and later on, by Shima [33] for the more general class of p.c.f. self-similar sets considered here. (See also the recent work by Teplyaev [34].) We believe that Hypothesis 4.6 under which our main result is established should hold more generally than for "decimable fractals", but unfortunately, we cannot prove it at this point. We also remark that under our hypothesis, the normalized volume measure $\nu/\nu(K)$ coincides with the original self-similar measure defining the mass distribution of K if (and only if) d_S coincides with the spectral dimension of K, as defined in [22]. In that case, ν was proposed in [24, 25] to be thought of as an analogue of Riemannian volume on K.

As an immediate consequence of our results (combined with the earlier works in [22] and [24]), one obtains a more precise version of Weyl's classical formula in the present context of Laplacians on (certain) self-similar fractals.

Part of our present joint results was announced in Sect. 6 of [25]. The interested reader can find in [24, 25] further discussion of the possible connections between aspects of noncommutative geometry [8] and of spectral and fractal geometry.

The rest of this paper is organized as follows. In Sect. 2, we briefly review the analytic definition of Laplacians on p. c. f. self-similar fractals. In Sect. 3, we recall the main result of [22] concerning the eigenvalue distribution of Laplacians on p.c.f. fractals and provide some preparatory lemmas and definitions. In Sect. 4, we recall the main result of [24] concerning the construction of volume measures on fractals. We also briefly discuss the notion of Dixmier trace and introduce Hypothesis 4.6 as well as derive its main consequence, Theorem 4.7, which proves the self-similarity of $\nu/\nu(K)$. Finally, in Sect. 5, we establish a sufficient condition for the self-similarity of volume measures (that is, for Hypothesis 4.6 to be satisfied); see Theorem 5.2. We also provide several examples illustrating our results.

2. Laplacians on P. C. F. Self-Similar Sets

In this section, we will define post critically finite self-similar sets and construct Laplacians on them. See [18, 19] for details.

Definition 2.1. *Let K be a compact metrizable topological space and let $\mathcal{S}$ be a finite set. In this paper, $\mathcal{S} = \{1, 2, \cdots, N\}$. Also, let F_i, for $i \in \mathcal{S}$, be a continuous injection from K to itself. Then, $(K, \mathcal{S}, \{F_i\}_{i\in\mathcal{S}})$ is called a self-similar structure if there exists a continuous surjection $\pi : \Sigma \to K$ such that $F_i \circ \pi = \pi \circ i$ for every $i \in \mathcal{S}$, where $\Sigma = \mathcal{S}^{\mathbb{N}}$ is the one-sided shift space and $i : \Sigma \to \Sigma$ is defined by $i(w_1w_2w_3\cdots) = iw_1w_2w_3\cdots$ for each $w_1w_2w_3\cdots \in \Sigma$.*

Note that if $(K, \mathcal{S}, \{F_i\}_{i\in\mathcal{S}})$ is a self-similar structure, then K is self-similar in the following sense:

$$K = \bigcup_{i\in\mathcal{S}} F_i(K). \tag{2.1}$$

Notation. $W_m = \mathcal{S}^m$ is the collection of words with length m. For $w = w_1\cdots w_m \in W_m$, we define $F_w : K \to K$ by $F_w = F_{w_1} \circ \cdots \circ F_{w_m}$ and $K_w = F_w(K)$. In particular, $W_0 = \{\emptyset\}$ and $F_\emptyset$ is the identity map. Also we define $W_* = \cup_{m\geq 0} W_m$.

Definition 2.2. *Let $(K, \mathcal{S}, \{F_i\}_{i\in\mathcal{S}})$ be a self-similar structure. We define the critical set $\mathcal{C} \subset \Sigma$ and the post critical set $\mathcal{P} \subset \Sigma$ by*

$$\mathcal{C} = \pi^{-1}(\bigcup_{i\neq j}(K_i \cap K_j)) \quad \text{and} \quad \mathcal{P} = \bigcup_{n\geq 1} \sigma^n(\mathcal{C}),$$

where σ is the shift map from Σ to itself defined by $\sigma(\omega_1\omega_2\cdots) = \omega_2\omega_3\cdots$. A self-similar structure is called post critically finite (p. c. f. for short) if and only if $\#(\mathcal{P})$ is finite.

Now, we fix a p. c. f. self-similar structure $(K, \mathcal{S}, \{F_i\}_{i\in\mathcal{S}})$.

Definition 2.3. *Let $V_0 = \pi(\mathcal{P})$. For $m \geq 1$. Also set*

$$V_m = \bigcup_{w\in W_m} F_w(\pi(\mathcal{P})) \quad \text{and} \quad V_* = \bigcup_{m\geq 0} V_m.$$

It is easy to see that $V_m \subset V_{m+1}$ and that K is the closure of V_*. In particular, V_0 is thought of as the "boundary" of K. Next we explain how to construct Laplacians on a p. c. f. self-similar set. First we define a Laplacian on a finite set.

Definition 2.4. *Let V be a finite set. We denote the collection of real-valued functions on V by $\ell(V)$. The space $\ell(V)$ is equipped with the standard inner product $(u, v) = \sum_{p\in V} u(p)v(p)$ for $u, v \in \ell(V)$. A symmetric linear operator $H : \ell(V) \to \ell(V)$ is called a Laplacian on V if it satisfies*

(L1) *H is non-positive definite,*
(L2) *$Hu = 0$ if and only if u is a constant on V, and*
(L3) *$H_{pq} \geq 0$ for all $p \neq q \in V$.*

We use $\mathcal{L}(V)$ to denote the collection of Laplacians on V. For $H \in \mathcal{L}(V)$, $\mathcal{E}_H(\cdot,\cdot)$ is a non-negative symmetric bilinear form defined by $\mathcal{E}_H(u, v) = -(Hu, v)$ for $u, v \in \ell(V)$.

Proposition 2.5. *Let $D \in \mathcal{L}(V_0)$ and let $\mathbf{r} = (r_1, \cdots, r_N)$, where $r_i > 0$ for $i \in \mathcal{S}$. Define a symmetric bilinear form $\mathcal{E}^{(m)}$ on $\ell(V_m)$ by $\mathcal{E}^{(m)}(u, v) = \sum_{w\in W_m} {r_w}^{-1}\mathcal{E}_D(u \circ F_w, v \circ F_w)$, where $r_w = r_{w_1}\cdots r_{w_m}$ for $w = w_1\cdots w_m \in W_m$. Then there exists $H_m \in \mathcal{L}(V_m)$ that satisfies $\mathcal{E}^{(m)} = \mathcal{E}_{H_m}$.*

Definition 2.6. *$(D, \mathbf{r})$ is said to be a harmonic structure if and only if*

$$\mathcal{E}^{(m)}(u, u) = \min\{\mathcal{E}^{(m+1)}(v, v) : v \in \ell(V_{m+1}), v|_{V_m} = u\} \tag{2.2}$$

for all $m \geq 0$ and for any $u \in \ell(V_m)$.

It is known that (2.2) holds for all $m \geq 0$ if and only if it holds for $m = 0$.

Definition 2.7. *If $(D, \mathbf{r})$ is a harmonic structure, then we define*

$$\mathcal{F} = \{u : u \in \ell(V_*), \lim_{m\to\infty} \mathcal{E}^{(m)}(u|_{V_m}, u|_{V_m}) < \infty\}$$

and $\mathcal{E}(u, v) = \lim_{m\to\infty} \mathcal{E}^{(m)}(u|_{V_m}, v|_{V_m})$ for $u, v \in \mathcal{F}$. Also $\mathcal{F}_0 = \{u \in \mathcal{F} : u|_{V_0} = 0\}$.

Since $\mathcal{E}^{(m)}$ is defined in a self-similar fashion, $\mathcal{E}$ naturally satisfies the following self-similarity property.

Proposition 2.8. *$u \in \mathcal{F}$ if and only if $u \circ F_i \in \mathcal{F}$ for all $i \in \mathcal{S}$. Also*

$$\mathcal{E}(u, v) = \sum_{i \in \mathcal{S}} {r_i}^{-1} \mathcal{E}(u \circ F_i, v \circ F_i)$$

for any $u, v \in \mathcal{F}$.

Proposition 2.9 (Self-similar measure). *If $\mu_i > 0$ for each $i \in \mathcal{S}$ and $\sum_{i \in \mathcal{S}} \mu_i = 1$, then there exists a unique Borel regular probability measure μ on K such that*

$$\int_K f d\mu = \sum_{i \in \mathcal{S}} \mu_i \int_K f \circ F_i d\mu$$

for any continuous function on K. μ is called a self-similar measure on K with weight $(\mu_1, \cdots, \mu_N)$.

If μ is a self-similar measure, then $\mu(K_w) = \mu_w$, where $\mu_w = \mu_{w_1} \cdots \mu_{w_m}$ for $w = w_1 \cdots w_m \in W_m$. Now we give a direct definition of the Laplacian associated with $(\mathcal{E}, \mathcal{F})$ and a measure μ. Let $C(K)$ be the collection of all real-valued continuous functions on K.

Definition 2.10. *For $p \in V_m$, let ψ_p^m be the unique function in $\mathcal{F}$ that attains the following minimum:*

$$\min\{\mathcal{E}(u, u) : u \in \mathcal{F}, u(p) = 1, u(q) = 0 \text{ for } q \in V_m \backslash \{p\}\}.$$

For $u \in C(K)$, if there exists $f \in C(K)$ such that

$$\lim_{m \to \infty} \max_{p \in V_m \backslash V_0} |\mu_{m,p}^{-1} (H_m u)(p) - f(p)| = 0,$$

where $\mu_{m,p} = \int_K \psi_p^m d\mu$, then we define the μ-Laplacian Δ_μ by $\Delta_\mu u = f$. The domain of Δ_μ is denoted by $\mathcal{D}_\mu$.

Proposition 2.11. *For $u \in \mathcal{D}_\mu$ and $p \in V_0$,*

$$-\lim_{m \to \infty} (H_m u)(p) = -(Du)(p) + \int_K \psi_p^0 \Delta_\mu u d\mu.$$

The above limit is denoted by $(du)_p$ and is called the Neumann derivative at p.

There is a natural relation between Δ_μ, $(\mathcal{E}, \mathcal{F})$ and Neumann derivatives.

Proposition 2.12 (Gauss-Green's formula). *For $u \in \mathcal{F}$ and $v \in \mathcal{D}_\mu$,*

$$\mathcal{E}(u, v) = \sum_{p \in V_0} u(p)(dv)_p - \int_K u \Delta_\mu v d\mu.$$

Theorem 2.13. *Let $(D, \mathbf{r})$ be a harmonic structure on a p.c.f. self-similar structure $(K, \mathcal{S}, \{F_i\}_{i \in \mathcal{S}})$. Also let μ be a self-similar measure on K with weight $(\mu_1, \cdots, \mu_N)$. If $\mu_i r_i < 1$ for all $i \in \mathcal{S}$, then $\mathcal{F}$ is naturally embedded in $L^2(K, \mu)$. $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}, \mathcal{F}_0)$ are local regular Dirichlet forms on $L^2(K, \mu)$. Moreover, let H_N and H_D be non-negative self-adjoint operators on $L^2(K, \mu)$ associated with $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}, \mathcal{F}_0)$ respectively, then both H_N and H_D have compact resolvent.*

The operators H_N and H_D are defined through the abstract theory of closed quadratic forms on a Hilbert space. See [10, 32] for the general theory. For example, let u and f be in $L^2(K, \mu)$, $u \in \mathrm{Dom}(H_N)$ and $H_N u = f$ if and only if $u \in \mathcal{F}$ and $\mathcal{E}(v, u) = (v, f)_\mu$ for all $v \in \mathcal{F}$, where $(u, v)_\mu$ is the inner product of $L^2(K, \mu)$.

The operator $-H_N$ is thought to be a Laplacian on K with Neumann boundary conditions while $-H_D$ is thought to be a Laplacian on K with Dirichlet boundary conditions. In fact, if

$$\mathcal{D}_N = \{u \in \mathcal{D}_\mu : (du)_p = 0 \quad \text{for any } p \in V_0\},$$

then the above characterization of H_N along with Proposition 2.12 implies that $\mathcal{D}_N \subset \mathrm{Dom}(H_N)$ and $\Delta_\mu = -H_N$ on $\mathcal{D}_N$. Similarly, if

$$\mathcal{D}_D = \{u \in D_\mu : u|_{V_0} = 0\},$$

then $\mathcal{D}_D \subset \mathrm{Dom}(H_D)$ and $\Delta_\mu = -H_D$ on $\mathcal{D}_D$. Moreover we can verify the following theorem.

Theorem 2.14. *The operators $-H_N$ and $-H_D$ are the Friedrichs extensions of $\Delta_\mu|_{\mathcal{D}_N}$ and $\Delta_\mu|_{\mathcal{D}_D}$, respectively.*

3. Eigenvalue Distribution of Laplacians

In this section, we will discuss results concerning the eigenvalue distributions of Laplacians on p. c. f. self-similar sets. Throughout the rest of this paper, $(K, \mathcal{S}, \{F_i\}_{i\in\mathcal{S}})$ is a p. c. f. self-similar structure with $\mathcal{S} = \{1, 2, \cdots, N\}$ and $(D, \mathbf{r})$ is a harmonic structure, where $\mathbf{r} = (r_1, \cdots, r_N)$. Further, μ is a self-similar measure on K with weight $(\mu_1, \cdots, \mu_N)$ that satisfies $r_i\mu_i < 1$ for all $i \in \mathcal{S}$. In the following, the symbol $*$ always represents D or N.

Definition 3.1 (Eigenvalues and Eigenfunctions). *For $k \in \mathbb{R}$, we define*

$$E_*(k) = \{u : u \in \mathrm{Dom}(H_*), H_* u = ku\}.$$

If $\dim E_*(k) \geq 1$, *then k is called a $*$-eigenvalue and $u \in E_*(k)$ is said to be a $*$-eigenfunction belonging to the $*$-eigenvalue k.*

It is known that if $u \in E_*(k)$, then $u \in \mathcal{D}_*$ and $\Delta_\mu u = -ku$. See [22, 28]. Since H_* has compact resolvent, the $*$-eigenvalues are non-negative, of finite multiplicity and the only accumulation point is ∞. Precisely, there exist a complete orthonormal system of $L^2(K, \mu)$, $\{\varphi_j^*\}_{j\geq 1} \subset \mathcal{D}_*$ and $\{k_j^*\}_{j\geq 1}$ such that $H_*\varphi_j^* = k_j^*\varphi_j^*$ and $k_j^* \leq k_{j+1}^*$ for all $j \geq 1$. Hence if we let

$$\rho_*(x, \mu) = \sum_{k\leq x} \dim E_*(k) = \#\{j : k_j^* \leq x\},$$

$\rho_*(x, \mu)$ is well-defined and $\rho_*(x, \mu) \to \infty$ as $x \to \infty$. We call $\rho_*(x, \mu)$ the eigenvalue counting function. The following theorem gives an analogue of Weyl's asymptotic formula for the eigenvalue counting functions.

Theorem 3.2 ([22]). *Let d_S be the unique positive number d that satisfies*

$$\sum_{i \in \mathcal{S}} \gamma_i^d = 1,$$

where $\gamma_i = \sqrt{r_i \mu_i}$ for $i \in \mathcal{S}$. Then

$$0 < \liminf_{x \to \infty} \rho_*(x, \mu)/x^{d_S/2} \leq \limsup_{x \to \infty} \rho_*(x, \mu)/x^{d_S/2} < \infty$$

for $* = D, N$. The positive number d_S is called the spectral exponent of $(\mathcal{E}, \mathcal{F}, \mu)$. Moreover, we have the following dichotomy:

(1) *Non-lattice case: If $\sum_{i \in \mathcal{S}} \mathbb{Z} \log \gamma_i$ is a dense subgroup of $\mathbb{R}$, then the limit $\lim_{x \to \infty} \rho_*(x, \mu)/x^{d_S/2}$ exists.*
(2) *Lattice case: If $\sum_{i \in \mathcal{S}} \mathbb{Z} \log \gamma_i$ is a discrete subgroup of $\mathbb{R}$, let $T > 0$ be its generator. Then*

$$\rho_*(x, \mu) = (G(\log x/2) + o(1))x^{d_S/2},$$

where G is a right-continuous T-periodic function such that $0 < \inf G(x) \leq \sup G(x) < \infty$ and $o(1)$ denotes a term which vanishes as $x \to \infty$.

It is known that $0 \leq \rho_N(x, \mu) - \rho_D(x, \mu) \leq \#(V_0)$. See [22, 18]. Hence the limit $\lim_{x \to \infty} \rho_*(x, \mu)/x^{d_S/2}$ (or the periodic function G) is independent of the boundary conditions. In fact, if $R(x) = \rho_D(x, \mu) - \sum_{i \in \mathcal{S}} \rho_D(\gamma_i^2 x, \mu)$, then

$$\lim_{x \to \infty} \rho_*(x, \mu)/x^{d_S/2} = \Big(-\sum_{i \in \mathcal{S}} \nu_i \log \nu_i\Big)^{-1} d_S \int_{-\infty}^{\infty} U(t)dt \tag{3.1}$$

in the non-lattice case and

$$G(t) = \Big(-\sum_{i \in \mathcal{S}} \nu_i \log \nu_i\Big)^{-1} d_S T \sum_{j=-\infty}^{\infty} U(t + jT) \tag{3.2}$$

in the lattice case, where $\nu_i = \gamma_i^{d_S}$ for $i \in \mathcal{S}$ and $U(t) = e^{-d_S t} R(e^{2t})$. In light of (3.2), we immediately deduce the following lemma.

Lemma 3.3. *In the lattice case, we have*

$$\frac{1}{T} \int_0^T G(t)dt = \Big(-\sum_{i \in S} \nu_i \log \nu_i\Big)^{-1} d_S \int_{-\infty}^{\infty} U(t)dt. \tag{3.3}$$

By analogy with Weyl's classical theorem (see (1.1) or [22, Theorem 0.1] for example), the limit (3.1) may represent a kind of volume of the space in the non-lattice case. Even in the lattice case, we may use the integral average (3.3) as a substitute for the value of the limit.

Definition 3.4 (Spectral Volume). *The spectral volume* $\mathrm{vol}(K, \mu)$ *is defined by*

$$\mathrm{vol}(K, \mu) = \Big(-\sum_{i \in \mathcal{S}} \nu_i \log \nu_i\Big)^{-1} d_S \int_{-\infty}^{\infty} U(t)dt. \tag{3.4}$$

Note that $0 < \mathrm{vol}(K,\mu) < \infty$ by (3.1) and (3.3). To justify this analogy, we need some kind of natural measure ν defined on K that satisfies $\nu(K) = \mathrm{vol}(K,\mu)$. Such a measure was in fact defined by Lapidus in [24]. We will introduce it in the next section. In the meantime, we derive a formula for the spectral volume. Let k_j denote the j^{th} Dirichlet eigenvalue k_j^D for $j \geq 1$.

Proposition 3.5.

$$\begin{aligned}\mathrm{vol}(K,\mu) &= \Big(-\sum_{i\in\mathcal{S}} \nu_i \log \nu_i\Big)^{-1} \lim_{x\to\infty}(q(x) - \sum_{i\in\mathcal{S}} \nu_i q(\gamma_i^2 x)) \\ &= \Big(-\sum_{i\in\mathcal{S}} \nu_i \log \nu_i\Big)^{-1} \lim_{t\to 0}(\tilde{q}(t) - \sum_{i\in\mathcal{S}} \nu_i \tilde{q}(t/\nu_i)),\end{aligned}$$

where $q(x) = \sum_{k_j\leq x} k_j^{-d_S/2}$ *and* $\tilde{q}(t) = \sum_{t\leq k_j^{-d_S/2}} k_j^{-d_S/2}$.

Proof. We need to show that

$$d_S \int_{-\infty}^{\infty} e^{-d_S t} R(e^{2t})dt = \lim_{x\to\infty}(q(x) - \sum_{i\in\mathcal{S}} \nu_i q(\gamma_i^2 x)).$$

Although $R(x) = \rho_D(x,\mu) - \sum_{i\in\mathcal{S}} \rho_D(\gamma_i^2 x,\mu)$ is a step function, we can still use the formula of integration by parts. Then

$$d_S \int_{-\infty}^{\infty} e^{-d_S t} R(e^{2t})dt = \int_{-\infty}^{\infty} e^{-d_S t} (R(e^{2t}))' dt.$$

Now $\rho_D(e^{2t},\mu)' = \sum_j \delta_{t_j}$, where $t_j = \log k_j/2$ and δ_x is the Dirac point mass at x. Hence we have

$$\int_{-\infty}^{t} e^{-d_S t} (\rho_D(e^{2t},\mu))' dt = \sum_{t_j\leq t} k_j^{-d_S/2}.$$

Therefore it follows that

$$\int_{-\infty}^{t} e^{-d_S t} (R(e^{2t}))' dt = q(e^{2t}) - \sum_{i\in\mathcal{S}} \nu_i q(\gamma_i^2 e^{2t}).$$

By letting $t \to \infty$, we deduce the proposition. □

4. Volume Measures

First we will recall the notion of volume measures introduced by Lapidus in [24]. Combining [24, Theorem 4.41] and [24, Corollary 4.45], we obtain the following result.

Theorem 4.1. *There exists a unique positive Borel regular measure* ν *on* K *such that*

$$\int_K f d\nu = \mathrm{Tr}_{\mathrm{w}}(M_f \circ H_D^{-d_S/2})$$

for any $f \in C(K)$*, where* $\mathrm{Tr}_{\mathrm{w}}(\cdot)$ *is the Dixmier trace of operators* (*as explained just below*) *and* M_f *is the multiplication operator on* $L^2(K,\mu)$ *defined by* $M_f(u) = fu$. *Moreover, the total mass of* K *with respect to* ν *is equal to the spectral volume. In other words,* $\mathrm{vol}(K,\mu) = \nu(K)$.

The Borel regular measure ν in the above theorem is called the *volume measure* associated with $(\mathcal{E}, \mathcal{F}, \mu)$ and is denoted by ν_μ.

Next, we briefly recall the notion of Dixmier trace ([12, 8, §IV.2]), which is a very useful tool in Connes' noncommutative geometry and quantized calculus. (See, for example, [8, Chapters IV and VI].) Given a compact (nonnegative and self-adjoint) operator R on a Hilbert space $\mathcal{H}$, with eigenvalues $\{\kappa_j(R)\}_{j=1}^{\infty} \downarrow 0$, we say that $R \in \mathcal{L}^{1+}$ (the "Matsaev ideal" [8]) if the sequence $(\ln J)^{-1} \sum_{j=1}^{J} \kappa_j(R)$ is bounded. (In Theorem 4.1, the Hilbert space $\mathcal{H}$ is equal to $L^2(K, \mu)$.) Then, roughly speaking, the *Dixmier trace* of R is defined by

$$\mathrm{Tr}_{\mathrm{w}}(R) = \operatorname*{Lim}_{\mathrm{w}} (\ln J)^{-1} \sum_{j=1}^{J} \kappa_j(R), \tag{4.1}$$

where "$\mathrm{Lim}_{\mathrm{w}}$" is a suitable notion of limit of (bounded sequences) with nice scale-invariance (i. e., renormalization) properties. See, e.g., [7, 8, §IV.2] and [24, §4.1] for more details and additional relevant references. (Intuitively, $\mathrm{Tr}_{\mathrm{w}}(R)$ captures the "semi-classical information" contained in R.) Further, Tr_w extends to a finite, positive (non-normal and unitary) trace on $\mathcal{L}^{1+}$. The following proposition summarizes some of the basic properties of Tr_w.

Proposition 4.2. *Let A and B belong to $\mathcal{L}^{1+}$.*

(1) $\mathrm{Tr}_{\mathrm{w}}(A \circ B) = \mathrm{Tr}_{\mathrm{w}}(B \circ A)$.
(2) *If A belongs to the trace class, then* $\mathrm{Tr}_{\mathrm{w}}(A) = 0$.
(3) *If A is non-negative, then* $\mathrm{Tr}_{\mathrm{w}}(A) \geq 0$.

Our main interest in this paper is to determine the nature of the volume measure. In particular, we conjecture that the normalized volume measure $\nu_\mu / \nu_\mu(K)$ is the self-similar measure with weight $(\nu_1, \cdots, \nu_N)$. Recall that $\nu_i = {\gamma_i}^{d_S}$ for $i \in \mathcal{S}$. In the next section, we will prove this conjecture for a class including the standard Laplacians on the Sierpinski gaskets.

Set $\widetilde{\mathcal{F}}_0 = \{u \in \mathcal{F} : u|_{V_1} = 0\}$. It is easy to see that $(\mathcal{E}, \widetilde{\mathcal{F}}_0)$ becomes a local regular Dirichlet form on $L^2(K, \mu)$. Let $\widetilde{H}_D$ be a non-negative self-adjoint operator associated with $(\mathcal{E}, \widetilde{\mathcal{F}}_0)$. Note that $\mathcal{E}(u, v) = (u, \widetilde{H}_D v)_\mu$ for all $v \in \widetilde{\mathcal{F}}$. Then Proposition 2.8 implies the following lemma.

Lemma 4.3. *Let φ_j denote the j^{th} Dirichlet eigenfunction φ_j^D for all $j \geq 1$. Set $\varphi_{j,i} = (\mu_i)^{-1/2} S_i \varphi_j$, where*

$$S_i(f)(x) = \begin{cases} f(F_i^{-1}(x)) & \text{if } x \in K_w, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{\varphi_{j,i}\}_{j \geq 1, i \in \mathcal{S}}$ is a complete orthonormal system of $L^2(K, \mu)$. Moreover, $\widetilde{H}_D \varphi_{j,i} = \frac{k_j}{r_i \mu_i} \varphi_{j,i}$.

Lemma 4.4. *For all $f \in C(K)$,*

$$M_f \circ \widetilde{H}_D^{-d_S/2} = \sum_{i \in \mathcal{S}} \nu_i S_i \circ M_{f \circ F_i} \circ {H_D}^{-d_S/2} \circ R_i,$$

where $R_i(u) = u \circ F_i$.

Remark. For all $i \in \mathcal{S}$, $R_i \circ S_i$ is the identity and $S_i \circ R_i u = \chi_{K_i} u$, where χ_{K_i} is the characteristic function of K_i.

Proof. Let $u = \sum_{j,i} \alpha_{j,i}\varphi_{j,i}$, then

$$\widetilde{H}_D{}^{-d_S/2}u = \sum_{i\in\mathcal{S}} \nu_i \sum_{j\geq 1} \alpha_{j,i} k_j^{-d_S/2}\varphi_{j,i}.$$

This implies $\widetilde{H}_D{}^{-d_S/2} = \sum_{i\in\mathcal{S}} \nu_i S_i \circ H_D{}^{-d_S/2} \circ R_i$. Now we can easily obtain the proposed equality. □

Proposition 4.5. *For all $f \in C(K)$,*

$$\nu_\mu(f) - \sum_{i\in\mathcal{S}} \nu_i \nu_\mu(f \circ F_i) = \mathrm{Tr}_{\mathrm{w}}(M_f \circ (H_D{}^{-d_S/2} - \widetilde{H}_D{}^{-d_S/2})).$$

Proof. By Lemma 4.4,

$$\begin{aligned}\mathrm{Tr}_{\mathrm{w}}(M_f \circ \widetilde{H}_D^{-d_S/2}) &= \sum_{i\in\mathcal{S}} \nu_i \mathrm{Tr}_{\mathrm{w}}(S_i \circ M_{f\circ F_i} \circ H_D{}^{-d_S/2} \circ R_i)\\ &= \sum_{i\in\mathcal{S}} \nu_i \mathrm{Tr}_{\mathrm{w}}(M_{f\circ F_i} \circ H_D{}^{-d_S/2}),\end{aligned}$$

where we also use Proposition 4.2 (1). This immediately implies the proposition. □

The following hypothesis is a key to show self-similarity of volume measures in the present approach. We believe that it is always satisfied but unfortunately, so far, we do not know how to verify it in general.

Hypothesis 4.6. *The operator $H_D{}^{-d_S/2} - \widetilde{H}_D{}^{-d_S/2}$ belongs to the trace class and*

$$\mathrm{vol}(K,\mu) = \Big(-\sum_{i\in\mathcal{S}} \nu_i \log \nu_i\Big)^{-1} \mathrm{tr}(H_D{}^{-d_S/2} - \widetilde{H}_D{}^{-d_S/2}). \tag{4.2}$$

In the next section, we will show that the above hypothesis holds for the Laplacians associated with strong harmonic structures in the sense of Shima [33], where the eigenvalue decimation method can be applied. This class includes the standard Laplacians on the Sierpinski gaskets. We give several examples in the next section.

Theorem 4.7. *Define the normalized volume measure $\tilde{\nu}_\mu$ by $\tilde{\nu}_\mu = \nu_\mu/\nu_\mu(K)$. If Hypothesis 4.6 is true, then the normalized volume measure $\tilde{\nu}_\mu$ is the self-similar measure with weight $(\nu_1, \cdots, \nu_N)$.*

Proof. Assume $H_D{}^{-d_S/2} - \widetilde{H}_D{}^{-d_S/2}$ belongs to the trace class. Then, since the trace class is an ideal in the algebra of all bounded linear operators (see Reed & Simon [32] for example), $M_f \circ (H_D{}^{-d_S/2} - \widetilde{H}_D{}^{-d_S/2})$ also belongs to the trace class. Hence, by (2) of Proposition 4.2, $\mathrm{Tr}_{\mathrm{w}}(M_f \circ (H_D{}^{-d_S/2} - \widetilde{H}_D{}^{-d_S/2})) = 0$. So Proposition 4.5 implies $\nu_\mu(f) = \sum_{i\in\mathcal{S}} \nu_i \nu_\mu(f \circ F_i)$ for any $f \in C(K)$. Using Proposition 2.9, we see that $\tilde{\nu}_\mu$ is the self-similar measure with weight $(\nu_1, \cdots, \nu_N)$. □

Remark. If Hypothesis 4.6 is true, then

$$\begin{aligned}\mathrm{vol}(K,\mu) = \nu_\mu(K) &= \mathrm{Tr}_{\mathrm{w}}({H_D}^{-d_S/2}) \\ &= \Big(-\sum_{i\in\mathcal{S}} \nu_i \log \nu_i\Big)^{-1} \mathrm{tr}({H_D}^{-d_S/2} - {\widetilde{H}_D}{}^{-d_S/2}) \\ &= \Big(-\sum_{i\in\mathcal{S}} \nu_i \log \nu_i\Big)^{-1} \lim_{x\to\infty} (q(x) - \sum_{i\in\mathcal{S}} \nu_i q(\gamma_i^2 x)).\end{aligned}$$

In the rest of this section, we discuss properties of volume measures assuming Hypothesis 4.6. Note that in general the self-similar measure $\tilde{\nu}_\mu$ has a different weight from that of the original self-similar measure μ. More precisely, $\mu = \tilde{\nu}_\mu$ if and only if the harmonic structure $(D, \mathbf{r})$ is regular (i. e., $0 < r_i < 1$ for all $i \in \mathcal{S}$) and $\mu_i = r_i^{d_H}$ for all $i \in \mathcal{S}$, where d_H is defined as the unique $d > 0$ that satisfies $\sum_{i\in\mathcal{S}} r_i^d = 1$. Assume that the harmonic structure (D, r) is regular. Let μ^* be the self-similar measure which satisfies $\mu^* = \tilde{\nu}_{\mu^*}$. Then by the appendix of Kigami–Lapidus [22], μ^* is the unique self-similar measure that attains the following maximum

$$\max\{d_S : \mu \text{ is a self-similar measure on } K\}$$

and $d_S = \frac{2d_H}{d_H+1}$. Also, Kigami [20] has shown that d_H is equal to the Hausdorff dimension of K with respect to the effective resistance metric. If $\mu \neq \mu^*$, ν_μ and μ are mutually singular.

In [24], the measure $\nu_{\mu^*} = \mathrm{vol}(K, \mu^*)\mu^*$ is called the "natural volume measure" on K (associated with the harmonic structure $(D, \mathbf{r})$) and is suggested to be a counterpart of the usual Riemannian volume measure for this class of self-similar fractals, by analogy with the work of Connes in [7] for smooth Riemannian (spin) manifolds. In general, the value of the Dixmier trace may depend on the choice of the mean w used to define Tr_{w} in (4.1); see [8, §IV.2.β]. It follows from [24] that the total mass of ν, namely, $\nu(K) = \mathrm{vol}(K, \mu)$, is always independent of w. (See Theorem 4.1 above.) Moreover, Theorem 4.7 implies that the measure ν itself is independent of the choice of w under Hypothesis 4.6.

5. A Sufficient Condition for Self-Similarity and Examples

In this section, we will give a sufficient condition related to localized eigenfunctions for Hypothesis 4.6 to be satisfied. To state our sufficient condition, we need to recall some notions about localized (and non-localized) eigenfunctions and corresponding eigenvalue counting functions.

Definition 5.1. *We define $E^W(k) = E_D(k) \cap E_N(k)$ and $E^F(k) = E_D(k) \cap E^W(k)^\perp$. We also define corresponding eigenvalue counting functions as follows:*

$$\rho^W(x,\mu) = \sum_{k\le x} \dim E^W(k) \quad \textit{and} \quad \rho^F(x,\mu) = \sum_{k\le x} \dim E^F(k).$$

Obviously, $\rho_D(x,\mu) = \rho^W(x,\mu) + \rho^F(x,\mu)$. If $u \in E^W(k)$ for some $k > 0$, then u is called a pre-localized eigenfunction.

Theorem 5.2. *Suppose that there exists a pre-localized eigenfunction. If*

$$\kappa_F = \limsup_{x\to\infty} \frac{\log \rho^F(x,\mu)}{\log x} < \frac{d_S}{2}, \tag{5.1}$$

then Hypothesis 4.6 is satisfied.

Recall Theorem 3.2, where we obtain that $\rho_D(x,\mu) \asymp x^{d_S/2}$ as $x \to \infty$. Hence the above condition requires that the counting function of non-localized eigenfunctions $\rho^F(x,\mu)$ is asymptotically much smaller than that of localized eigenfunctions $\rho^W(x,\mu)$. In [21], (5.1) is conjectured to be true whenever there exists a pre-localized eigenfunction. In particular, it was shown in [21, Theorem 4.5] that (5.1) is true if the harmonic structure is a strong harmonic structure in the sense of Shima [33]. In this paper, we will not go into the details. Instead, we will give examples where (5.1) has been verified in [21].

Example 5.3 (Sierpinski gasket). Let $\{p_1, p_2, p_3\} \subset \mathbb{C}$ satisfy $|p_i - p_j| = 1$ for any $i \neq j$. Define $F_i : \mathbb{C} \to \mathbb{C}$ by $F_i(z) = (z - p_i)/2 + p_i$ for $i \in \mathcal{S}$, where $\mathcal{S} = \{1, 2, 3\}$. The Sierpinski gasket is the unique non-empty compact set K that satisfies (2.1). Clearly $(K, \mathcal{S}, \{F_i\}_{i\in\mathcal{S}})$ is a p. c. f. self-similar structure and $V_0 = \{p_1, p_2, p_3\}$. Now if

$$D = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = (\frac{3}{5}, \frac{3}{5}, \frac{3}{5}),$$

then $(D, \mathbf{r})$ is a harmonic structure. Also let μ be the self-similar measure on K with weight $(1/3, 1/3, 1/3)$. The Laplacian associated with $(D, \mathbf{r})$ and μ is called the standard Laplacian on the Sierpinski gasket K. By Theorem 4.4 of [21], we can verify (5.1). In fact, $\kappa_F = \log 2/\log 5 < d_S/2 = \log 3/\log 5$. Hence Hypothesis 4.6 is true. So the normalized volume measure $\tilde{\nu}_\mu$ is a self-similar measure. Since $\mu_i r_i = 1/5$ for all $i \in S$, it follows that $\tilde{\nu}_\mu$ is the self-similar measure with weight $(1/3, 1/3, 1/3)$ and hence it coincides with μ. Analogous results are also valid for the higher-dimensional Sierpinski gaskets. We have discussed only the above case for simplicity.

Example 5.4 (Vicsek set, [21, Example 4.6]). For $1 \leq j \leq 5$, define $F_j : \mathbb{C} \to \mathbb{C}$ by $F_j = (z - p_j)/3 + p_j$, where $p_1 = 1$, $p_2 = \sqrt{-1}$, $p_3 = -1$, $p_4 = -\sqrt{-1}$ and $p_5 = 0$. The Vicsek set K is the unique non-empty compact set that satisfies (2.2), where $\mathcal{S} = \{1, 2, 3, 4, 5\}$. $(K, \mathcal{S}, \{F_i\}_{i\in\mathcal{S}})$ is a p. c. f. self-similar structure and $V_0 = \{p_1, p_2, p_3, p_4\}$. Define $D \in \mathcal{L}(V_0)$ by $D_{p_j p_k} = 1$ for $1 \leq j \neq k \leq 4$ and $D_{p_j p_j} = -3$ for all j and let $r = (s, s, s, s, t)$, where $t > 0, s > 0$ and $2s + t = 1$. Then $(D, \mathbf{r})$ is a regular harmonic structure. Moreover, set $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \frac{t}{4t+s}$ and $\mu_5 = \frac{s}{4t+s}$. Then in [21], it was shown that $d_S/2 = \frac{\log 5}{\log n_0}$ and $\kappa_F = \frac{\log 3}{\log n_0}$, where $n_0 = \frac{4t+s}{st}$. So by Theorem 5.2 and Theorem 4.1, the normalized volume measure $\tilde{\nu}_\mu$ is a self-similar measure. As $\mu_i r_i = n_0^{-1}$ for all $i \in \mathcal{S}$, $\nu_i = 1/5$ for all $i \in \mathcal{S}$. Therefore, $\mu = \tilde{\nu}_\mu$ if and only if $s = t = 1/3$.

Example 5.5 (modified Koch curve, [2], [21, Example 4.7]). Let $f_{p,q}(z) = (q-p)z + p$ for $p, q \in \mathbb{C}$. Define $F_1 = f_{0,1/3}$, $F_2 = f_{2/3,1}$, $F_3 = f_{1/3,2/3}$, $F_4 = f_{1/3,c}$ and $F_5 = f_{c,2/3}$, where $c = \frac{1}{2} + \frac{\sqrt{-1}}{2\sqrt{3}}$. The modified Koch curve is the unique compact set K that satisfies (2.1), where $\mathcal{S} = \{1, 2, 3, 4, 5\}$. Obviously, $(K, \mathcal{S}, \{F_i\}_{i\in\mathcal{S}})$ is a

p. c. f. self-similar structure and $V_0 = \{0, 1\}$. Set $D = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ and $\mathbf{r} = (s, s, t, h, h)$ with $2s + \frac{2ht}{t+2h} = 1$ for $s, t, h > 0$. Then $(D, \mathbf{r})$ is a harmonic structure. Note that one of the numbers t or h can be arbitrarily large. In such a case, $(D, \mathbf{r})$ is not a regular harmonic structure. Now set $\mu_1 = \mu_2 = (n_0 s)^{-1}$, $\mu_3 = (n_0 t)^{-1}$ and $\mu_4 = \mu_5 = (n_0 h)^{-1}$, where $n_0 = 2s^{-1} + t^{-1} + 2h^{-1}$. Then it was shown in [21] that $d_S/2 = \frac{\log 5}{\log n_0}$ and $\kappa_F = \frac{\log 4}{\log n_0}$. So by Theorem 5.2 and Theorem 4.1, the normalized volume measure $\tilde{\nu}_\mu$ is a self-similar measure. As $\mu_i r_i = n_0^{-1}$ for all $i \in \mathcal{S}$, $\nu_i = 1/5$ for all $i \in \mathcal{S}$. Hence $\mu = \tilde{\nu}_\mu$ if and only if $s = t = h = 3/8$.

In the rest of this section, we will prove Theorem 5.2. First we will introduce some properties of pre-localized eigenfunctions. A pre-localized eigenfunction can generate a sequence of infinitely many pre-localized eigenfunctions as follows.

Proposition 5.6 ([3, Lemma 4.2]). *Let u be a pre-localized eigenfunction with $u \in E^W(k)$. Define $u_w = S_{w_1} \circ \cdots \circ S_{w_m}(u)$ for any $w = w_1 \cdots w_m \in W_*$. Then u_w is also a pre-localized eigenfunction belonging to the eigenvalue $\frac{k}{r_w \mu_w}$.*

Note that $S_j(E^W(\mu_j r_j k)) \subset E^W(k)$.

Naturally, the eigenfunctions in $S_j(E^W(\mu_j r_j k))$ are thought to be offsprings of the preceding eigenfunctions in $E^W(\mu_j r_j k)$. From such an observation, we can divide $E^W(k)$ into offsprings $E_2^W(k)$ and generators $E_1^W(k)$.

Definition 5.7.

$$E_2^W(k) = \bigoplus_{i \in \mathcal{S}} S_i(E^W(k\mu_i r_i)) \qquad \textit{and} \qquad E_1^W(k) = (E_2^W(k))^\perp \cap E^W(k).$$

Now we can choose k_j^W and $\phi_j \in E_1^W(k_j^W)$ for $j \geq 1$ so that $k_j^W \leq k_{j+1}^W$ and $\{\phi_j\}_{j=1}^\infty$ is a complete orthonormal system of $E_1^W = \overline{\oplus_k E_1^W(k)}$. Then $\{\phi_{j,w} | j \geq 1, w \in W_*\}$ is a complete orthonormal system of $E^W = \overline{\oplus_k E^W(k)}$, where $\phi_{j,w} = (\mu_w)^{-1/2} S_{w_1} \circ \cdots \circ S_{w_m}(\phi_j)$ for $w = w_1 \cdots w_m \in W_*$. Note that $\phi_{j,w} \in E_2^W(k_i^W/(\mu_w r_w))$ if $w \notin W_0$ and $\{\phi_{j,w}\}_{j \geq 1, w \in W_* \setminus W_0}$ is a complete orthonormal system of $E_2^W = \overline{\oplus_k E_2^W(k)}$. The following proposition was obtained in [21].

Proposition 5.8 ([21, Theorem 3.5]). *Suppose that there exists a pre-localized eigenfunction.*

(1) *In the lattice case, $\rho^W(x, \mu) = (G^W(\log x/2) + o(1))x^{d_S/2}$ as $x \to \infty$, where G^W is a discontinuous T-periodic function with $0 < \inf G^W \leq \sup G^W < \infty$.*
(2) *In the non-lattice case, the limit $\lim_{x\to\infty} \rho^W(x, \mu)/x^{d_S/2}$ exists and is positive.*
(3) *$\sum_{j \geq 1}(k_j^W)^{-d_S/2} < \infty$ and*

$$c_W = \Big(-\sum_{i \in \mathcal{S}} \nu_i \log \nu_i\Big)^{-1} \sum_{j \geq 1} (k_j^W)^{-d_S/2}, \tag{5.2}$$

where

$$c_W = \begin{cases} \frac{1}{T}\int_0^T G^W(t)dt & \textit{in the lattice case,} \\ \lim_{x\to\infty} \rho^W(x, \mu)/x^{d_S/2} & \textit{in the non-lattice case.} \end{cases}$$

By the above proposition, we have the following lemma.

Lemma 5.9. *If* (5.1) *is satisfied, then*

$$\mathrm{vol}(K,\mu) = c_W = \Big(-\sum_{i\in\mathcal{S}} \nu_i \log \nu_i\Big)^{-1} \sum_{j\geq 1} (k_j^W)^{-d_S/2}.$$

Proof. If (5.1) is satisfied, then we see that $G = G^W$ in the lattice case and $\lim_{x\to\infty} \rho^F(x,\mu)/x^{d_S/2} = 0$ in the non-lattice case. Hence comparing the definitions of $\mathrm{vol}(K,\mu)$ and c_W, we obtain $\mathrm{vol}(K,\mu) = c_W$. □

Next we choose $k_j^F > 0$ and $\xi_j \in E^F(k_j^F)$ for $j \geq 1$ so that $k_j^F \leq k_{j+1}^F$ and $\{\xi_j\}_{j\geq 1}$ is a complete orthonormal system of $E^F = \overline{\oplus_k E^F(k)}$. It follows immediately that $L^2(K,\mu) = E^F \oplus E_1^W \oplus E_2^W$ and $\{\xi_j, \phi_{j,w}\}_{j\geq 1, w\in W_*}$ is a complete orthonormal system of $L^2(K,\mu)$.

Lemma 5.10. *If* (5.1) *is satisfied, then* $\sum_{j\geq 1}(k_j^F)^{-d_S/2} < \infty$.

Proof. Choose α so that $\kappa_F < \alpha < d_S/2$. Note that $\rho^F(x,\mu) = \#\{j : k_j^F \leq x\}$. So by (5.1), we obtain that there exists $c > 0$ such that $cj^{1/\alpha} \leq k_j^F$ for any $j \geq 1$. Therefore $(k_j^F)^{-d_S/2} \leq cj^{-d_S/(2\alpha)}$. Now as $1 < d_S/(2\alpha)$, $\sum_{j\geq 1} j^{-d_S/(2\alpha)} < \infty$. □

Lemma 5.11. *Let* $\xi_{j,i} = S_i(\xi_j)$ *for any* $j \geq 1$ *and* $i \in \mathcal{S}$. *Then* $\{\xi_{j,i}\}_{j\geq 1, i\in\mathcal{S}}$ *is a complete orthonormal system of* $E^F \oplus E_1^W$.

Proof. Applying the same argument as in Lemma 4.3 to $\{\xi_j, \phi_{j,w}\}_{j\geq 1, w\in W_*}$, we see that $\{\xi_{j,i}, \phi_{j,w}\}_{j\geq 1, i\in\mathcal{S}, w\in W_*\backslash W_0}$ is a complete orthonormal system of $L^2(K,\mu)$. Recall that $\{\phi_{j,w}\}_{j\geq 1, w\in W_*\backslash W_0}$ is a complete orthonormal system of E_2^W. Hence $\{\xi_{j,i}\}_{j\geq 1, i\in\mathcal{S}}$ is a complete orthonormal system of the orthogonal complement of E_2^W, which is $E^F \oplus E_1^W$. □

Proof of Theorem 5.2. Let P_F, P_1 and P_2 be the orthogonal projection of $L^2(K,\mu)$ onto E^F, E_1^W and E_2^W, respectively. Also let $A = H_D{}^{-d_S/2}$ and $B = \widetilde{H}_D{}^{-d_S/2}$. By Proposition 5.6 and Lemma 4.3, $A\phi_{j,w} = B\phi_{j,w} = (\mu_w r_w)^{d_S/2}(k_j^W)^{-d_S/2}\phi_{j,w}$ for $j \geq 1$ and $w \in W_*\backslash W_0$. Hence $A \circ P_2 = B \circ P_2$. Therefore,

$$A - B = A_1 + A_F - B_{F1}$$

where $A_F = A \circ P_F$, $A_1 = A \circ P_1$ and $B_{F1} = B \circ (P_F + P_1)$. Note that $A_F\xi_j = (k_j^F)^{-d_S/2}\xi_j$, $A_1\phi_j = (k_j^W)^{-d_S/2}\phi_j$ and $B_{F1}\xi_{j,i} = \nu_i(k_j^F)^{-d_S/2}\xi_{j,i}$. So it is easy to see that A_F, A_1 and B_{F1} are bounded non-negative self-adjoint operators. Now by Lemma 5.9 and Lemma 5.10, it follows that $\mathrm{tr}(A_F) = \sum_{j\geq 1}(k_j^F)^{-d_S/2} < \infty$, $\mathrm{tr}(A_1) = \sum_{j\geq 1}(k_j^W)^{-d_S/2} < \infty$ and $\mathrm{tr}(B_{F1}) = \sum_{j\geq 1, i\in\mathcal{S}} \nu_i (k_j^F)^{-d_S/2} = \sum_{j\geq 1}(k_j^F)^{-d_S/2} < \infty$. Hence A_F, A_1 and B_{F1} belong to the trace class. Therefore $A - B$ belongs to the trace class. Moreover,

$$\begin{aligned}\mathrm{tr}(A-B) &= \mathrm{tr}(A_F) + \mathrm{tr}(A_1) - \mathrm{tr}(B_{F1}) = \mathrm{tr}(A_1)\\ &= \sum_{j\geq 1}(k_j^W)^{-d_S/2}.\end{aligned}$$

This along with Lemma 5.9 implies (4.2). □

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