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Large deviations for Brownian motion on the Sierpinski gasket

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Abstract

We study large deviations for Brownian motion on the Sierpinski gasket in the short time limit. Because of the subtle oscillation of hitting times of the process, no large deviation principle can hold. In fact, our result shows that there is an infinity of different large deviation principles for different subsequences, with different (good) rate functions. Thus, instead of taking the time scaling $\varepsilon \to 0$, we prove that the large deviations hold for $\varepsilon_n^z \equiv (\frac{2}{5})^n z$ as $n \to \infty$ using one parameter family of rate functions I^z ($z \in [\frac{2}{5}, 1)$). As a corollary, we obtain Strassen-type laws of the iterated logarithm. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper, we obtain some large deviation results in the short-time regime (Schilder-type large deviations) for Brownian motion on the Sierpinski gasket. Let E be the Sierpinski gasket on $\mathbb{R}^2$ with an intrinsic geodesic metric d, called a shortest path metric, and let $X^x(t)$ be Brownian motion on E starting at $x \in E$. This process exhibits sub-diffusive behavior in the sense that $E^x[d(X(t),x)] \asymp t^{1/d_w}$ for small $t > 0$ where $d_w = \log 5/\log 2 > 2$ and $f(t) \asymp g(t)$ means $f(t)/g(t)$ is bounded from above and below by some positive constants. For fixed $T > 0$, let $\Omega_x \equiv C_x([0,T] \to E) = \{\phi \in C([0,T] \to E)\colon \phi(0) = x\}$ with uniformly continuous topology. Let P_ε^x be the law for $X^x(\varepsilon t)$. Then, our main theorem is the following.

Theorem 1.1. *For each* $z \in [\frac{2}{5}, 1), A \subset \Omega_x$,

$$-\inf_{\phi \in \mathrm{Int}\, A} I_x^z(\phi) \leqslant \liminf_{n\to\infty} ((\tfrac{2}{5})^n z)^{1/(d_w-1)} \log P^x_{(2/5)^n z}(A)$$
$$\leqslant \limsup_{n\to\infty} ((\tfrac{2}{5})^n z)^{1/(d_w-1)} \log P^x_{(2/5)^n z}(A) \leqslant -\inf_{\phi \in \mathrm{Cl}\, A} I_x^z(\phi).$$

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Here $\{I_x^z\}_{z\in[\frac{2}{5},1)}$ is a sequence of rate functions defined as follows for each $\phi\in\Omega_x$,

$$I_x^z(\phi)=\begin{cases}\displaystyle\int_0^T(\dot\phi(t))^{d_w/(d_w-1)}F(z/\dot\phi(t))\,\mathrm{d}t & \phi \text{ is absolutely continuous,}\\ \infty & \text{otherwise,}\end{cases}\tag{1.1}$$

where F is a periodic non-constant positive continuous function with period $\frac{5}{2}$ and $\dot\phi(t)\equiv\lim_{s\to t}d(\phi(s),\phi(t))/|s-t|$ for $t\in[0,T]$. This result tells us that the large deviation result does not hold when one takes the time scaling ε to 0. Instead, for each fixed z, it holds via the sequence $\varepsilon_n^z\equiv(\frac{2}{5})^nz$ as $n\to\infty$. This result is closely related to the following estimate for the transition probability density $p_t(x,y)$ of X obtained in Kumagai (1997):

$$-\lim_{n\to\infty}((\tfrac{2}{5})^nz)^{1/(d_w-1)}\log p_{(\frac{2}{5})^nz}(x,y)=d(x,y)^{d_w/(d_w-1)}F\left(\frac{z}{d(x,y)}\right),$$

for $x,y\in E$ where F is the same periodic function as above. This F is defined as a Legendre transform of some limiting function of a Laplace transform of some hitting time of X and a "tiny" oscillation of the hitting times makes F non-constant. When $A=\{f\in\Omega_x: f(T)=y\}$, $\inf\{I_x^z(\phi):\phi\in A\}$ is attained independently of z by the path(s) which moves on the geodesic(s) between x and y homogeneously. Thus "the most probable path" should be this path, but the energy (action functional) of the path depends on time sequences determined by z.

For Brownian motion on a smooth Riemannian manifold, it is well known (see Varadhan, 1967) that Schilder-type large deviations hold with $I_x(\phi)=\frac{1}{2}\int_0^T|\dot\phi(t)|^2\,\mathrm{d}t$ (this rate function is recovered from (1.1) by taking $d_w=2$, $F\equiv$ constant). Our result shows an interesting contrast to this fact.

In Section 2, we will briefly present Brownian motion on the Sierpinski gasket and the apriori estimates on its hitting times and transition probabilities. In Section 3, we will show some properties of our rate functions and sketch the proof of our main theorem. The corresponding large deviation results for pinned Brownian motion is also introduced. In Section 4 we will obtain a Strassen-type law of the iterated logarithm as an application of our main theorem.

2. Brownian motion on the Sierpinski gasket

2.1. The Sierpinski gasket and its Dirichlet form

Let $\{\Psi_i\}_{i=1}^3$ be a family of affine maps on $\mathbb{R}^2$ with contraction rate $\frac{1}{2}$ where the fixed point of Ψ_i $(1\leqslant i\leqslant 3)$ is $(0,0),(1,0)$ and $(\frac{1}{2},\sqrt{\frac{3}{2}})$, respectively. Then, there exists unique non-void compact set E such that $E=\bigcup_{i=1}^3\Psi_i(E)$ which is called the (two-dimensional) Sierpinski gasket. Let $F^{(0)}=\{(0,0),(1,0),(\frac{1}{2},\sqrt{\frac{3}{2}})\}$. For $A\subset\mathbb{R}^2$, define $\Psi_{i_1,\dots,i_n}(A)=\Psi_{i_1}\circ\cdots\circ\Psi_{i_n}(A)$. We will call the set $\Psi_{i_1,\dots,i_n}(F^{(0)})$ an n-cell and $\Psi_{i_1,\dots,i_n}(E)$ an n-complex. Set

$$F^{(n)}=\bigcup_{i_1,\dots,i_n=1}^N\Psi_{i_1,\dots,i_n}(F^{(0)}),\quad F^{(\infty)}=\bigcup_{n=0}^\infty F^{(n)}.$$

Taking closure, E can be recovered: $E=\mathrm{Cl}(F^{(\infty)})$.

We next introduce an intrinsic metric on the gasket which we call a shortest path metric. For $x, y \in F^{(m)}$, let

$$\pi_m(x,y) = \{\pi_m\colon \pi_m \text{ is an } m\text{-walk in } E \text{ from } x \text{ to } y \text{ which does not contain multiple points}\}. \tag{2.1}$$

Here $\pi_m = \{p_k, p_{k+1}\}_{k=1}^{l}$ is an *m-walk* if $l \in \mathbb{N}$, $p_k \in F^{(m)}$ for $1 \leqslant k \leqslant l$ and p_k and p_{k+1} are in the same m-cell for $1 \leqslant k \leqslant l-1$. For $\pi_m = \{p_k, p_{k+1}\}_{k=1}^{l} \in \pi_m(x,y)$, we say the length of π_m is l and denotes it by $|\pi_m| = l$. Now we define the distance on $F^{(m)}$ as follows:

$$d_{F^{(m)}}(x,y) = \min_{\pi \in \pi_m(x,y),\, \pi = \{(p_k, p_{k+1})\}_{k=1}^{|\pi|}} \sum_{k=1}^{|\pi|} 2^{-m}.$$

Then, we can easily prove the following (see Fitzsimmons et al., 1994 for the proof under more general situations).

Lemma 2.1. (1) $d_{F^{(m)}}(x,y) = d_{F^{(m+1)}}(x,y)$ if $x, y \in F^{(m)}$.

(2) *For any choice of* $p, q \in E$, *define* $d(p,q)$ *by* $d(p,q) = \lim_{n\to\infty} d(p_n, q_n)$, *where* $p_n, q_n \in F^{(\infty)}$ *and* $p_n \to p, q_n \to q$ *as* $n \to \infty$. *Then* d *is well defined and* d *is a metric on* E.

We now define the Dirichlet form on the gasket. Let $f, g \in l(F^{(\infty)}) = \{f : F^{(\infty)} \to \mathbb{R}\}$ and define

$$\mathscr{E}_n(f,g) = \frac{1}{2} \sum_{1 \leqslant k_1,\dots,k_n \leqslant 3} \sum_{x,y \in F^{(0)}} \left(\frac{5}{3}\right)^n (f(\Psi_{k_1,\dots,k_n}(x)) - f(\Psi_{k_1,\dots,k_n}(y))) \times (g(\Psi_{k_1,\dots,k_n}(x)) - g(\Psi_{k_1,\dots,k_n}(y))). \tag{2.2}$$

This is the energy of the network on $F^{(n)}$ with conductance $(\frac{5}{3})^n$ on each n-complex. The sequence of quadratic forms has the following consistency:

$$\inf\{\mathscr{E}_n(f,f)\colon f|_{F^{(n-1)}} = v\} = \mathscr{E}_{n-1}(v,v) \quad \text{for all } v \in l(F^{(n-1)}).$$

By this,

$$\mathscr{E}_n(f|_{F^{(n)}}, f|_{F^{(n)}}) \leqslant \mathscr{E}_{n+1}(f|_{F^{(n+1)}}, f|_{F^{(n+1)}}) \quad \text{for all } f \in l(F^{(\infty)}).$$

For $f \in l(F^{(\infty)})$, define $\mathscr{F} = \{f\colon \sup_n \mathscr{E}_n(f,f) < \infty\}$ and $\mathscr{E}(f,f) = \lim_{n\to\infty} \mathscr{E}_n(f,f)$ for $f \in \mathscr{F}$. Then, from Fukushima (1992), Kigami (1993), Kusuoka (1993), we have the following.

Theorem 2.2. (1) *Any function in* $\mathscr{F}$ *can be extended uniquely to a continuous function on* E (*thus we can consider* $\mathscr{F} \subset C(E) = \{f\colon f$ *is a continuous function on* $E\}$). *Further*, $(\mathscr{E}, \mathscr{F})$ *is a local regular Dirichlet form on* $\mathbb{L}^2(E,\mu)$ (μ *is a normalized*

Hausdorff measure) *which has the following properties*:

$$\mathscr{E}(f,g)=\sum_{i=1}^{3}\tfrac{5}{3}\mathscr{E}(f\circ\Psi_i,g\circ\Psi_i)\quad \textit{for all } f,g\in\mathscr{F}, \tag{2.3}$$

$$\sup_{x\in E}|f(x)|\leqslant c_{2.1}\cdot\mathscr{E}_{(1)}(f,f)\quad \textit{for all } f\in\mathscr{F}, \tag{2.4}$$

where $\mathscr{E}_{(\beta)}(\cdot,\cdot)\equiv\mathscr{E}(\cdot,\cdot)+\beta(\cdot,\cdot)_{\mathbb{L}^2(E,\mu)}$ *for* $\beta>0$ *and* $c_{2.1}>0$ *is a constant.*

(2) $\mathscr{E}_{(\beta)}$ *admits a positive symmetric continuous reproducing kernel* $g_\beta(\cdot,\cdot)$ *which is bounded and equi-uniform continuous w.r.t.* $\beta>0$ *on E. The corresponding diffusion is point recurrent.*

As we have a local regular Dirichlet form, there is a corresponding diffusion process $X(t)$. Because of the geometric characterization of this process in Section 8 of Barlow and Perkins (1988), we will call this process *Brownian motion* on the gasket.

2.2. A priori estimates

In the following, we will explain estimates for Brownian motion on E which are already obtained in Barlow and Perkins (1988), Kumagai (1997).

2.2.1. Hitting time estimates

Let $X(t)$ be Brownian motion on E and let

$$W_n=\inf\{t\geqslant 0\colon X(t)\in F^{(n)}\setminus\{X(0)\}\}.$$

In this case, W_0 is a limit random variable of a supercritical branching process divided by its mean. Setting $\phi(s)=E^0[\exp(-sW_0)]$, the following equation holds (see Barlow and Perkins, 1988, Section 2):

$$\phi(5z)=\frac{\phi(z)^2}{4-3\phi(z)}. \tag{2.5}$$

Using this fact, we have the following:

Proposition 2.3 (Barlow and Perkins, 1988, Section 3)**.** *There exist positive constants* $c_{2.2}\sim c_{2.9}$ *such that*

$$c_{2.2}\exp(-c_{2.3}s^{1/d_w})\leqslant\phi(s)\leqslant c_{2.4}\exp(-c_{2.5}s^{1/d_w}) \tag{2.6}$$

$$c_{2.6}\exp(-c_{2.7}s^{-1/(d_w-1)})\leqslant P^0(W_0\leqslant s)\leqslant c_{2.8}\exp(-c_{2.9}s^{-1/(d_w-1)}) \tag{2.7}$$

for all $s>0$.

Let $L(s)=-s^{1/d_w}\log\phi(s)$. Then, we further have the following.

Proposition 2.4 (Barlow and Perkins, 1988, Section 3; Kumagai, 1997, Section 4)**.** $k(s)\equiv\lim_{n\to\infty}L(s\cdot 5^n)>0$ *exists and it is not a constant. Moreover,* $s^{1/d_w}k(s)$ *is strictly concave and real analytic on* $\mathbb{R}_+$.

This proposition means that there is an oscillation of hitting times so that one cannot take $c_{2.3} = c_{2.5}$ in (2.6).

2.2.2. Heat kernel estimates

Let $\bar{F}(y) = \sup_s\{k(s)s^{1/d_w} - ys\}$ and $F(y) = y^{1/(d_w-1)}\bar{F}(y)$. Remark that by Proposition 2.4 and by the properties of the Legendre transform, we see that $\bar{F}(y)$ is monotone decreasing and strictly convex. Also $F(\frac{5}{2}y) = F(y)$ and F is not constant.

Theorem 2.5 (Barlow and Perkins, 1988, Theorem 1.5; Kumagai, 1997, Theorem 1.2). (1) *There exists a jointly continuous transition density* $p_t(x,y)$ *for* $X(t)$ *w.r.t. the Hausdorff measure* μ *on* E *which satisfies the following*:

$$c_{2.10}t^{-d_s/2}\exp(-c_{2.11}\Psi(d(x,y),t)) \leqslant p_t(x,y) \leqslant c_{2.12}t^{-d_s/2}\exp(-c_{2.13}\Psi(d(x,y),t))$$

for all $0 < t < 1$, $x, y \in E$, *where* $\Psi(z,t) = (z^{d_w}t^{-1})^{1/(d_w-1)}$, $d_s = 2\log 3/\log 5$ *and* $c_{2.10} \sim c_{2.13}$ *are positive constants.*

(2) *The following holds for all* $z > 0$, $x, y \in E$. (*The right-hand side is* 0 *when* $d(x,y) = 0$.):

$$-\lim_{n\to\infty}((2/5)^n z)^{1/(d_w-1)}\log p_{(2/5)^n z}(x,y) = d(x,y)^{d_w/(d_w-1)}F\left(\frac{z}{d(x,y)}\right). \tag{2.8}$$

For each fixed z, *this convergence is compact uniform for* x, y *w.r.t.* d.

As a corollary to this theorem, we see that Varadhan-type estimates do not hold for the heat kernel of Brownian motion on the gasket.

Corollary 2.6 (Kumagai, 1997, Corollary 4.1). *There is no function* $f : E \times E \to \mathbb{R}_+$ *which satisfies the following for some bounded function* G:

$$-\lim_{t\to 0} G(t)t^{1/(d_w-1)}\log p_t(x,y) = f(x,y), \quad \forall x, y \in E. \tag{2.9}$$

3. Large deviations for Brownian motion on the Sierpinski gasket

For $\phi \in \Omega_x$, we say ϕ is absolutely continuous if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\sum_{i=1}^n d(\phi(t_i), \phi(t_{i-1})) < \varepsilon$ for any n and any disjoint collection of intervals $\{(t_{i-1}, t_i)\}_{i=1}^n$ in $[0,T]$ whose lengths satisfy $\sum_i(t_i - t_{i-1}) < \delta$. It can be proved by routine arguments that if ϕ is absolutely continuous, then $\dot{\phi}(t) \equiv \lim_{s\to t} d(\phi(s),\phi(t))/|s-t|$ exists for a.e. $t \in [0,T]$, $\dot{\phi} \in \mathbb{L}^1([0,T], \mathrm{d}t)$ and $\int_0^T \dot{\phi}(t)\,\mathrm{d}t$ is a length of the path $\{\phi(s)\colon 0 \leqslant s \leqslant T\}$. Now, for $z > 0$ and $\phi \in \Omega_x$, define $I_x^z(\phi)$ as (1.1) where we consider $0^{1/(d_w-1)}F(\infty) = 0$. When $\phi \in C([0,T] \to E)$ (no restriction for $\phi(0)$), we denote the corresponding rate function as $I^z(\phi)$.

Remark 3.1. The definition of the rate function in p. 226 of Kumagai (1997) was wrong unless E is a line. But the results in the paper hold using the rate function introduced here.

We next give some notation. For $\Delta : 0=t_0 < t_1 < t_2 < \cdots < t_m=T$ and $\phi \in \Omega_x$, we set $\Pi_\Delta \phi = \{\phi(t_1), \ldots, \phi(t_m)\}$. Also, define $\phi_\Delta \in \Omega_x$ by taking points $\{\phi(t_j)\}$ and joining the successive ones by geodesics with natural parametrization. If there are more than one geodesics between two such points, it is immaterial which one is chosen. Thus, ϕ_Δ is piecewise geodesic and $\phi_\Delta(t_j) = \phi(t_j)$ $(0 \leqslant j \leqslant m)$. We then have the following:

Lemma 3.2. (a) *On* $C([0,T] \to E)$, *it holds that*

$$\inf_{\substack{\phi(\alpha)=a,\\ \phi(\beta)=b}} I^z(\phi) = \left(\frac{d(a,b)^{d_w}}{\beta-\alpha}\right)^{1/(d_w-1)} F\left(\frac{z(\beta-\alpha)}{d(a,b)}\right),$$

where the infimum is attained by the geodesic on the gasket.

(b) *On* $C_x([0,T] \to E)$, *it holds that*

$$\inf_{\substack{\phi(t_i)=x_i,\\ i=1,\ldots,m}} I_x^z(\phi) = I_x^z(\phi_\Delta) = \sum_{i=1}^m \left(\frac{d(x_i,x_{i-1})^{d_w}}{t_i-t_{i-1}}\right)^{1/(d_w-1)} F\left(\frac{z(t_i-t_{i-1})}{d(x_i,x_{i-1})}\right),$$

where $\Delta : 0=t_0 \leqslant t_1 \leqslant \cdots \leqslant t_m \leqslant T$, $x_0=x, x_1, \ldots, x_m \in E$ *and* ϕ_Δ *is piecewise geodesic with* $\phi_\Delta(t_j)=x_j (0 \leqslant j \leqslant m)$.

Proof. Note that (b) is an obvious extension of (a). For (a), it is enough to prove for the case $\alpha=0, \beta=T$ as otherwise the infimum is attained by the path which does not move in the intervals $[0,\alpha]$ and $[\beta,T]$. Now, for each $\phi \in \Omega_x$ which is absolutely continuous, set $D(\phi)=\int_0^T \dot\phi(t)\,dt$. Then,

$$\begin{aligned}
\int_0^T (\dot\phi(t))^{d_w/(d_w-1)} F\left(\frac{z}{\dot\phi(t)}\right) dt &= \int_0^T (\dot\phi(t))^{d_w/(d_w-1)} \left(\frac{z}{\dot\phi(t)}\right)^{1/(d_w-1)} \bar F\left(\frac{z}{\dot\phi(t)}\right) dt \\
&= z^{1/(d_w-1)} D(\phi) \int_0^T \bar F\left(\frac{z}{\dot\phi(t)}\right) \dot\phi(t) \frac{dt}{D(\phi)} \\
&\geqslant z^{1/(d_w-1)} D(\phi) \bar F\left(\frac{1}{D(\phi)} \int_0^T \frac{z}{\dot\phi(t)} \dot\phi(t)\,dt\right) \\
&= z^{1/(d_w-1)} D(\phi) \bar F\left(\frac{zT}{D(\phi)}\right) \\
&\geqslant z^{1/(d_w-1)} d(a,b) \bar F(zT/d(a,b)) \\
&= \left(\frac{d(a,b)^{d_w}}{T}\right)^{1/(d_w-1)} F(zT/d(a,b)).
\end{aligned}$$

Here we use Jensen's inequality in the first inequality and the second inequality is because $D(\phi) \geqslant d(a,b)$ and $\bar F$ is monotone decreasing. As $\bar F$ is strictly convex, the equalities hold if and only if $\dot\phi(t)=\text{constant}$ and $D(\phi)=d(a,b)$. That is the geodesic with natural parametrization. We thus obtain the result. □

Using the results, we see for $\phi \in \Omega_x$ and $0 \leqslant \alpha \leqslant \beta \leqslant T$

$$I_x^z(\phi) \geqslant \left(\frac{d(\phi(\alpha),x)^{d_w}}{\alpha}\right)^{1/(d_w-1)} F\left(\frac{z\alpha}{d(\phi(\alpha),x)}\right)$$

$$+\left(\frac{d(\phi(\beta),\phi(\alpha))^{d_w}}{\beta-\alpha}\right)^{1/(d_w-1)} F\left(\frac{z(\beta-\alpha)}{d(\phi(\beta),\phi(\alpha))}\right)$$

$$\geqslant M\left(\frac{d(\phi(\beta),\phi(\alpha))^{d_w}}{\beta-\alpha}\right)^{1/(d_w-1)},$$

where $M \equiv \min_{2/5 \leqslant x \leqslant 1} F(x) > 0$. Thus,

$$d(\phi(\alpha),\phi(\beta)) \leqslant M' I_x^z(\phi)^{(d_w-1)/d_w}(\beta-\alpha)^{1/d_w} \tag{3.1}$$

for some $M' > 0$.

For $\psi, \phi \in \Omega_x$, define $||\psi - \phi|| = \sup_{0 \leqslant t \leqslant T} d(\psi(t), \phi(t))$.

Lemma 3.3. (1) *The function $I_x^z(\phi)$ is lower semi-continuous. Further, for every $N > 0$, $\{\phi\colon I_x^z(\phi) \leqslant N\}$ is compact.*
(2) *If $C \subset \Omega_x$ is closed in Ω_x, then*

$$\lim_{\delta\to 0}\inf_{\phi\in C_\delta} I_x^z(\phi) = \inf_{\phi\in C} I_x^z(\phi),$$

where $C_\delta = \{\phi \in \Omega_x\colon ||\phi - \psi|| < \delta$ for some $\psi \in C\}$.

Proof. For the lower semi-continuity, it is enough to show that if $I_x^z(\phi_n) \leqslant N$ and $||\phi_n - \phi|| \to 0$, then $I_x^z(\phi) \leqslant N$. But this can be proved similarly to Varadhan (1980, p. 157), noting that $x \mapsto x^{d_w/(d_w-1)}F(z/x) = z^{1/(d_w-1)} x\bar{F}(z/x)$ is monotone increasing and strictly convex. Next, (3.1) shows that $\{\phi : I_x^z(\phi) \leqslant N\}$ is uniformly bounded and equi-continuous. As it is closed by the lower semi-continuity of I_x^z, (1) follows from Ascoli–Arzéla's theorem. (2) can be proved in the same way as in Varadhan (1980, p. 159), using (3.1) and (1). □

We are now ready to prove Theorem 1.1. In fact, it can be proved following the argument of the corresponding proof in Varadhan (1967). In the following, we state key lemmas of the proof for readers' convenience. Set $\varepsilon_n^z = (2/5)^n z$.

Lemma 3.4. *Let $C \subset \Omega_x$ be a closed set of the form $\Pi_\Delta^{-1}A$, where $A \subset E^m$ is closed. Then*

$$\limsup_{n\to\infty} (\varepsilon_n^z)^{1/(d_w-1)} \log P_{\varepsilon_n^z}^x(C) \leqslant -\inf_{\phi\in C} I_x^z(\phi).$$

Proof. Using Theorem 2.5 and Lemma 3.2, this can be proved in the same way as Lemma 3.1 of Varadhan (1967). □

For $m \in \mathbb{N}$, let Δ_m: $0 = t_0 < t_1 < t_2 < \cdots < t_m = T$ be an equally spaced partition, i.e. $t_j = jT/m$ $(0 \leqslant j \leqslant m)$.

Lemma 3.5. *For every* $\delta > 0$,

$$\limsup_{m\to\infty}\limsup_{n\to\infty}(\varepsilon_n^z)^{1/(d_w-1)}\log P_{\varepsilon_n^z}^x(||\phi-\phi_{\Delta_m}||\geqslant\delta))=-\infty.$$

Proof. Remark that $W_n = W_0/5^n$, where W_n, W_0 are hitting times appeared in Section 2. By this, one has the corresponding estimates of (2.6), (2.7) for W_n. Using this, the proof is the same as Lemma 3.2 of Varadhan (1967). □

By these lemmas, we can prove the third inequality of Theorem 1.1. Indeed, it is enough to prove the inequality when A is closed. Let $I_x^{z,\delta}(\omega)=\inf_{\omega':||\omega'-\omega||<\delta}I_x^z(\omega')$ and define $T_\delta=\inf_{\omega\in C_\delta}I_x^z(\omega)$. If $\omega\in C$ then $I_x^{z,\delta}(\omega)\geqslant T_\delta$ and therefore

$$P_{\varepsilon_n^z}^x[C]\leqslant P_{\varepsilon_n^z}^x[I_x^{z,\delta}(\omega)\geqslant T_\delta]\leqslant P_{\varepsilon_n^z}^x[||\omega-\omega_{\Delta_m}||\geqslant\delta]+P_{\varepsilon_n^z}^x[I_x^z(\omega_{\Delta_m})\geqslant T_\delta].$$

From Lemma 3.5,

$$\limsup_{m\to\infty}\limsup_{n\to\infty}(\varepsilon_n^z)^{1/(d_w-1)}\log P_{\varepsilon_n^z}^x[||\omega-\omega_{\Delta_m}||\geqslant\delta]=-\infty.$$

As the set $(C\subset)\{I_x^z(\omega_{\Delta_m})\geqslant T_\delta\}$ is equal to

$$\left\{\omega:\sum_{i=1}^m\left(\frac{d(\omega(t_i),\omega(t_{i-1}))^{d_w}}{t_i-t_{i-1}}\right)^{1/(d_w-1)}F\left(\frac{z(t_i-t_{i-1})}{d(\omega(t_i),\omega(t_{i-1}))}\right)\geqslant T_\delta\right\},$$

we see from Lemma 3.4 that

$$\limsup_{m\to\infty}\limsup_{n\to\infty}(\varepsilon_n^z)^{1/(d_w-1)}\log P_{\varepsilon_n^z}^x[I_x^z(\omega_{\Delta_m})\geqslant T_\delta]\leqslant -T_\delta.$$

Combined these facts with $\lim_{\delta\to0}T_\delta=\inf_{\phi\in A}I_x^z(\phi)$, which comes from Lemma 3.3(2), we obtain the third inequality of Theorem 1.1.

Next, comes the lemma for the lower bound.

Lemma 3.6. *Let* $f\in\Omega_x$, $V=\{\phi\in\Omega_x:||\phi-f||<\delta\}$, *where* $\delta>0$. *Then*

$$\liminf_{n\to\infty}(\varepsilon_n^z)^{1/(d_w-1)}\log P_{\varepsilon_n^z}^x(V)\geqslant -I_x^z(f).$$

Proof. Using Theorem 2.5, Lemma 3.2 and the third inequality of Theorem 1.1, this can be proved in the same way as Lemma 3.4 of Varadhan (1967). □

By this lemma, we can prove the first inequality of Theorem 1.1. Indeed, it is enough to prove the inequality when A is open. For $f\in A$, take a sphere V around f contained in A. Then, by the above lemma,

$$\liminf_{n\to\infty}(\varepsilon_n^z)^{1/(d_w-1)}\log P_{\varepsilon_n^z}^x(A)\geqslant\liminf_{n\to\infty}(\varepsilon_n^z)^{1/(d_w-1)}\log P_{\varepsilon_n^z}^x(V)\geqslant -I_x^z(f).$$

As this is true for all $f\in A$, we have the result. This concludes the proof of Theorem 1.1.

Finally in this section, we mention the large deviations for pinned Brownian motion (cf. Hsu, 1990; Fujita and Watanabe, 1989). For $x\neq y\in E$, let $P_\varepsilon^{x,y}$ be the pinned measure $P_\varepsilon^x[\,\cdot\,|X^x(\varepsilon T)=y]$. We set $\varepsilon_n^z=(2/5)^n z$ as before and define $f_T^z(x,y)=(d(x,y)^{d_w}/T)^{1/(d_w-1)}F(zT/d(x,y))$.

Theorem 3.7. *For each* $z \in [\frac{2}{5}, 1)$, $A \subset \Omega_x$,

$$-\inf_{\substack{\phi\in \mathrm{Int}\, A\\ \phi(T)=y}} \{I_x^z(\phi) - f_T^z(x,y)\} \leqslant \liminf_{n\to\infty}(\varepsilon_n^z)^{1/(d_w-1)}\log P_{\varepsilon_n^z}^{x,y}(A)$$

$$\leqslant \limsup_{n\to\infty}(\varepsilon_n^z)^{1/(d_w-1)}\log P_{\varepsilon_n^z}^{x}(A) \leqslant -\inf_{\substack{\phi\in \mathrm{Cl}\, A\\ \phi(T)=y}} \{I_x^z(\phi) - f_T^z(x,y)\}.$$

Let $\hat{p}(u,v,z_1,z_2)$ be the probability density of the pinned process under $P^{\cdot}(\ |X(T)=y) = P^{\cdot,y}$, i.e.

$$\hat{p}(u,v,z_1,z_2) = P^{\cdot,y}(X(v) \in dz_2 | X(u) = z_1)/\mu(dz_2)$$

for $0 \leqslant u < v \leqslant T$, $z_1, z_2 \in E$. Then

$$\hat{p}(u,v,z_1,z_2) = p_{v-u}(z_1,z_2)p_{T-v}(z_2,y)/p_{T-u}(z_1,y).$$

Replacing $p_t(x,y)$ by $\hat{p}(u,v,z_1,z_2)$ in the proof of Theorem 1.1 with suitable modifications of the lemmas, the proof of Theorem 3.7 can be given in the same way. We thus omit it.

4. Law of the iterated logarithm

In this section, we will mention the Strassen-type law of the iterated logarithm. Define $\hat{E} = \bigcup_{n=1}^{\infty} 2^n E$ and call it the unbounded Sierpinski gasket. As is mentioned in Fukushima (1992), Fitzsimmons et al. (1994), we can construct Brownian motion on $\hat{E}$ via Dirichlet forms. By the same argument mentioned in the last section, we can show that Theorem 1.1 holds for Brownian motion on $\hat{E}$. For Brownian motion starting at 0, set

$$\xi_n(t,\omega) = 2^{-n}X(5^n(\log n)^{1-d_w}t,\omega) \overset{\mathscr{L}}{\sim} X((\log n)^{1-d_w}t,\omega).$$

Then, we can prove the following theorem using Theorem 1.1 by a simple modification of the proof of Theorem 1.17 in Stroock (1984).

Theorem 4.1. *For P^0-a.a. ω, the sequence $\{\xi_n(\cdot,\omega)\}_2^\infty$ has the following properties*:

(1) $\{\xi_n(\cdot,\omega)\}_2^\infty$ *is precompact in* $C_0([0,T] \to \hat{E})$.
(2) *If* $\{\xi_{n'}(\cdot,\omega)\}_2^\infty$ *is a convergent subsequence of* $\{\xi_n(\cdot,\omega)\}_2^\infty$ *and* ψ *is its limit, then* $\min_z I_0^z(\psi) \leqslant 1$.
(3) *If* $\psi \in C_0([0,T] \to \hat{E})$ *with* $\max_z I_0^z(\psi) \leqslant 1$, *then there is a subsequence of* $\{\xi_n(\cdot,\omega)\}_2^\infty$ *which converges to* ψ.
In particular, if $\Phi : C_0([0,T) \to \hat{E}) \to \mathbf{R}$ *is a continuous function, then*

$$P^0\left(\sup_{\psi\in K_*}\Phi(\psi) \leqslant \limsup_{n\to\infty}\Phi(\xi_n(\cdot)) \leqslant \sup_{\psi\in K^*}\Phi(\psi)\right) = 1, \tag{4.1}$$

where $K_* = \{\phi \in C_0([0,T] \to \hat{E})\colon \max_z I_0^z(\phi) \leqslant 1\}$ *and* $K^* = \{\phi \in C_0([0,T] \to \hat{E})\colon \min_z I_0^z(\phi) \leqslant 1\}$.

Taking $\Phi(\xi) = \sup_{0 \leqslant s \leqslant 1} d(\xi(s), 0)$ in (4.1), we can easily see $0 < \sup_{\psi \in K_*} \Phi(\psi) \leqslant \sup_{\psi \in K^*} \Phi(\psi) < \infty$ using (3.1). Thus, we have for some $c_{4.1}, c_{4.2} > 0$,

$$P^0\left(c_{4.1} \leqslant \limsup_{n\to\infty} \sup_{0\leqslant s\leqslant 1} d(\xi_n(s,\omega),0) \leqslant c_{4.2}\right) = 1. \tag{4.2}$$

Now we quote a powerful $0-1$ law due to Barlow and Bass (1999, Theorem 8.4).

Theorem 4.2 (Barlow and Bass, 1999). *Suppose Γ is a tail event: $\Gamma \in \bigcap_t \sigma\{X_u : u \geqslant t\}$. Then, either $P^x(\Gamma)$ is 0 for all x or else it is 1 for all x.*

Using this, we have the classical law of the iterated logarithm.

Theorem 4.3. *There exists $C > 0$ such that*

$$\limsup_{t\to\infty} \frac{\sup_{0\leqslant s\leqslant t} d(X_s, X_0)}{t^{1/d_w}(\log\log t)^{1-1/d_w}} = C, \quad P^x\text{-a.s., } x \in \hat{E}.$$

Proof. Note that $d(\xi_n(s,\omega),0) = d(X(5^n(\log n)^{1-d_w}s),0)/2^n$ and $t \asymp 5^n(\log n)^{1-d_w}$ if and only if $2^n \asymp t^{1/d_w}(\log\log t)^{1-1/d_w}$. Then the result is an immediate consequence of (4.2) and Theorem 4.2. □

Remark 4.4. (1) Theorem 4.3 can be obtained more directly and easily using hitting time estimates in Barlow and Perkins (1988) and Theorem 4.2. In fact, the following was already mentioned by Fukushima and Shima (unpublished).

$$c_{4.3} \leqslant \limsup_{t\to\infty} \frac{\sup_{0\leqslant s\leqslant t}|X_s - X_0|}{t^{1/d_w}(\log\log t)^{1-1/d_w}} \leqslant c_{4.4}, \quad P^x\text{-a.s., } x \in \hat{E}$$

for some $c_{4.3}, c_{4.4} > 0$.

(2) In Bass and Kumagai (1998), the result of this section is generalized to diffusion processes whose heat kernels have Aronson-type estimates. In Fukushima et al. (1998), and Bass and Kumagai (1998), several laws of the iterated logarithm for diffusion processes on fractals are studied.

5. For Further Reading

The following reference is also of interest to the reader: Schilder, 1966.

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