LOCALIZED EIGENFUNCTIONS OF THE LAPLACIAN ON p.c.f. SELF-SIMILAR SETS

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ABSTRACT

In this paper we consider the form of the eigenvalue counting function ρ for Laplacians on p.c.f. self-similar sets, a class of self-similar fractal spaces. It is known that on a p.c.f. self-similar set K the function $\rho(x) = O(x^{d_s/2})$ as $x \to \infty$, for some $d_s > 0$. We show that if there exist localized eigenfunctions (that is, a non-zero eigenfunction which vanishes on some open subset of the space) and K satisfies some additional conditions ('the lattice case') then $\rho(x)x^{-d_s/2}$ does not converge as $x \to \infty$. We next establish a number of sufficient conditions for the existence of a localized eigenfunction in terms of the symmetries of the space K. In particular, we show that any nested fractal with more than two essential fixed points has localized eigenfunctions.

1. *Introduction*

Since the construction of Brownian motion on the Sierpinski gasket by Kusuoka [**9**] and Goldstein [**3**], analysis on fractals has been developed from both probabilistic and analytical points of view. This work has focused on constructing analytical structures such as diffusion processes, Laplacians and Dirichlet forms of self-similar sets. In particular, for finitely ramified self-similar sets, there are now numerous results – see, for example, [**1, 4, 11, 10, 8**].

In [**5**], it was shown how one can define natural Laplacians on post critically finite self-similar sets (for short, p.c.f. self-similar sets), which are abstract 'finitely ramified fractals'. We review the results in [**5**] in §2. The essential idea is that the Laplacian can be defined as the scaled limit of discrete Laplacians on a sequence of finite graphs which approximate the given p.c.f. self-similar set. If the discrete Laplacians are invariant under a certain kind of renormalization, we can construct a natural Dirichlet form and an associated Laplacian, which is denoted by Δ in this section, on the p.c.f. self-similar set. We shall study eigenvalues and eigenfunctions of $-\Delta$ with Dirichlet or Neumann boundary conditions. (See §3 for the precise definitions.)

In [**7**] it is proved that there exists $d_s > 0$ such that

$$0 < \liminf_{x\to\infty} \rho(x)\, x^{-d_s/2} \leqslant \limsup_{x\to\infty} \rho(x)\, x^{-d_s/2} < \infty, \tag{1.1}$$

where $\rho(x) = \#\{\text{eigenvalues of } -\Delta \text{ which are at most } x\}$ is called the eigenvalue counting function, or integrated density of states. This result had already been proved for various more restricted classes of fractals in [**14, 13**] etc. In [**15**] and [**2**] it was shown by Fukushima and Shima that for the Sierpinski gasket, a strict inequality holds in (1.1), so that one has

$$\liminf_{x\to\infty} \rho(x)\, x^{-d_s/2} < \limsup_{x\to\infty} \rho(x)\, x^{-d_s/2}. \tag{1.2}$$

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Their proof uses a particular property of the Sierpinski gasket, that of *spectral decimation*. More precisely, if V_n is in the nth approximation to the Sierpinski gasket, and H_n is the discrete Laplacian on V_n, then if f_n is an eigenfunction of H_n with eigenvalue k, so that

$$H_n f_n = k f_n,$$

then $f_{n-1} = f_n|_{V_{n-1}}$ satisfies $H_{n-1} f_{n-1} = \Psi(k) f_{n-1}$, where $\Psi(t) = t(5-t)$. Thus f_{n-1} is an eigenfunction of H_{n-1}. Under these circumstances, it is possible to obtain quite detailed information on the behaviour of ρ.

More recently, Shima [**16**] has extended these results to more general p.c.f. self-similar sets which admit spectral decimation. However problems with this approach are, first, that it is hard to check spectral decimation in all but very simple cases, and second, that it seems likely that spectral decimation holds only under rather exceptional circumstances.

In this paper we give a number of sufficient conditions under which (1.2) holds. In particular we prove that (1.2) holds for all nested fractals; see [**11**]. Our approach also provides an intuitive explanation for the oscillation phenomenon (1.2). A localized eigenfunction is one which is zero outside some open subset $O \subset K$, with $O \cap V_0 = \emptyset$. Given a localized eigenfunction ψ we can use scaling to define other localized eigenfunctions, and in particular we prove that in any open set U there exists an eigenfunction which is zero outside U. Under additional conditions on the p.c.f. self-similar set, and the measure μ (the 'lattice case') we find that there exists $\varepsilon > 0$ and eigenvalues $k_n \to \infty$ with multiplicity greater than $\varepsilon k_n^{d_s/2}$ for all $n \geqslant 1$. Hence $x^{-d_s/2}\rho(x)$ cannot converge as $x \to \infty$.

In Sections 5 and 6 we then investigate sufficient conditions for a p.c.f. self-similar set to have localized eigenfunctions. We introduce the group $\mathcal{G}$ of symmetries of a p.c.f. self-similar set (which preserve the measure μ and the harmonic structure), and prove that, if $\mathcal{G}$ is large enough, then K has localized eigenfunctions. Nested fractals have a very large symmetry group $\mathcal{G}$, and in Section 6 we show that any nested fractal with more than 2 boundary points has localized eigenfunctions.

2. *The* p.c.f. *self-similar sets*

In this section, we shall briefly recall results from [**5, 7**] on the construction and properties of Dirichlet forms and Laplacians on post critically finite self-similar sets. A more detailed summary is found in [**7**].

First, we introduce the notion of a self-similar structure, which provides a purely topological description of self-similarity.

DEFINITION 2.1. Let K be a compact metrizable topological space, let S be a finite set, and let F_i for $i \in S$ be continuous injections from K to itself. Then, $(K, S, \{F_i\}_{i\in S})$ is called a *self-similar structure* if there exists a continuous surjection $\pi: \Sigma \to K$ such that $F_i \circ \pi = \pi \circ i$ for every $i \in S$, where $\Sigma = S^{\mathbb{N}}$ is the one-sided shift space and $i: \Sigma \to \Sigma$ is defined by $i(w_1 w_2 w_3 \cdots) = i w_1 w_2 w_3 \cdots$ for each $w_1 w_2 w_3 \cdots \in \Sigma$.

NOTATION. Let $W_m = S^m$ be the collection of words with length m. For $w = w_1 w_2 \cdots w_m \in W_m$, we define $F_w: K \to K$ by $F_w = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m}$, and set $K_w = F_w(K)$. Also we define

$$W_* = \bigcup_{m \geqslant 0} W_m.$$

Post critically finite self-similar sets are finitely ramified self-similar sets.

DEFINITION 2.2. Let $(K, S, \{F_i\}_{i\in S})$ be a self-similar structure. We define the *critical set* $\mathscr{C} \subset \Sigma$ and the *post critical set* $\mathscr{P} \subset \Sigma$ by

$$\mathscr{C} = \pi^{-1}\left(\bigcup_{i\neq j}(K_i \cap K_j)\right) \quad \text{and} \quad \mathscr{P} = \bigcup_{n\geqslant 1} \sigma^n(\mathscr{C}),$$

where σ is the shift map from Σ to itself defined by $\sigma(w_1 w_2 w_3 \cdots) = w_2 w_3 w_4 \cdots$. A self-similar structure is called *post critically finite* (p.c.f. for short) if and only if $\#(\mathscr{P})$ is finite; we then say that K is a *post critically finite self-similar set*.

From now on, we shall fix a p.c.f. self-similar structure $(K, S, \{F_i\}_{i\in S})$ with $S = \{1, 2, \cdots, N\}$.

NOTATION. Let $V_0 = \pi(\mathscr{P})$. For $m \geqslant 1$, set

$$V_m = \bigcup_{w\in W_m} F_m(\pi(\mathscr{P})) \quad \text{and} \quad V_* = \bigcup_{m\geqslant 0} V_m.$$

It is easy to see that $V_m \subset V_{m+1}$ and that K is the closure of V_*. In particular, V_0 is thought of as the 'boundary' of K.

To construct the theory of harmonic calculus on p.c.f. self-similar sets, we shall use some concepts from the theory of electrical networks.

DEFINITION 2.3. For a finite set V, we denote the collection of real-valued functions on V by $l(V)$. For a symmetric linear map $H: l(V) \to l(V)$, we define a symmetric bilinear form $\mathscr{E}_H$ by $\mathscr{E}_H(u, v) = -{}^t uHv$ for $u, v \in l(V)$. Then (V, H) is called a *resistance network* (r-network for short) if $H_{pq} \leqslant 0$ for $p \neq q$, $\mathscr{E}_H(u, u) \geqslant 0$, and equality holds if and only if u is constant on V.

DEFINITION 2.4. If (V, H) and (V', H') are r-networks, then write $(V, H) \leqslant (V', H')$ if and only if $V \subset V'$ and, for every $v \in l(V)$, we have

$$\mathscr{E}_H(v, v) = \min\{\mathscr{E}_{H'}(u, u): u \in l(V'), u|_V = v\}.$$

REMARK. This electrical network formulation was not used in [**5**], but was introduced subsequently in [**6**].

Given an r-network (V_0, D) we define a sequence of r-networks $\{(V_m, H_m)\}_{m\geqslant 0}$.

DEFINITION 2.5. Let (V_0, D) be an r-network and let $r = (r_1, r_2, \ldots, r_N)$, where $r_i > 0$ for $i = 1, 2, \cdots, N$. We define $H_m: l(V_m) \to l(V_m)$ by $H_m = \Sigma_{w\in W_m} r_w^{-1}\, {}^t R_w D R_w$, where $r_w = r_{w_1} r_{w_2} \cdots r_{w_m}$ for $w = w_1 w_2 \ldots w_m \in W_m$, and $R_w: l(V_m) \to l(V_0)$ is defined by $R_w f = f \circ F_w$.

It is shown in [**5**, Proposition 4.3] that (V_m, H_m) is an r-network.

DEFINITION 2.6. The pair (D, r) is called a *harmonic structure* if and only if $(V_0, D) \leqslant (V_1, \lambda H_1)$ for some $\lambda > 0$. Moreover, if $r_i < \lambda$ for all $i = 1, 2, \cdots, N$, then (D, r) is called a *regular harmonic structure*.

Replacing $r = (r_1, r_2, \cdots, r_N)$ by $(r_1/\lambda, r_2/\lambda, \cdots, r_N/\lambda)$ for a harmonic structure (D, r), we have $(V_0, D) \leqslant (V_1, H_1)$. Thus we can always renormalize r so that $\lambda = 1$. Note that we then have $(V_m, H_m) \leqslant (V_{m+1}, H_{m+1})$ for all $m \geqslant 0$.

THEOREM 2.7 [**5**, Theorem 7.4]. *Let (D, r) be a regular harmonic structure with $\lambda = 1$ and write $\mathscr{E}_m(u, v) = \mathscr{E}_{H_m}(u, v)$. We define*

$$\mathscr{F} = \{u \in l(V_*) : \lim_{m\to\infty} \mathscr{E}_m(u|_{V_m}, u|_{V_m}) < \infty\} \quad \text{and} \quad \mathscr{E}(u, v) = \lim_{m\to\infty} \mathscr{E}_m(u|_{V_m}, v|_{V_m})$$

for $u, v \in \mathscr{F}$. Then,

(1) *$\mathscr{F}$ is naturally embedded in $C(K)$, which is the collection of all continuous functions on K;*

(2) *let μ be a finite Borel measure on K which satisfies $\mu(O) > 0$ for every non-empty open subset $O \subset K$, then $(\mathscr{E}, \mathscr{F})$ is a local regular Dirichlet form on $L^2(K, \mu)$.*

Furthermore $(\mathscr{E}, \mathscr{F})$ has the following scaling property.

LEMMA 2.8 [**7**, Lemma 6.1]. *For any $u, v \in \mathscr{F}$ and all $i = 1, 2, \cdots, N$ we have $u \circ F_i$, $v \circ F_i \in \mathscr{F}$, and*

$$\mathscr{E}(u, v) = \sum_{i=1}^{N} r_i^{-1} \mathscr{E}(u \circ F_i, v \circ F_i).$$

Let μ be a measure on K satisfying property (2) of Theorem 2.7. We now give a direct definition of the Laplacian associated with $(\mathscr{E}, \mathscr{F}, \mu)$, as a scaling limit of the discrete Laplacians H_m on V_m.

DEFINITION 2.9. For $p \in V_m$, let $\psi_{m,p}$ be the unique function in $\mathscr{F}$ that attains the following minimum: $\min\{\mathscr{E}(u, u) : u \in \mathscr{F}, u|_{V_m} = 1_{\{p\}}\}$. For $u \in C(K)$, if there exists $f \in C(K)$ such that

$$\lim_{m\to\infty} \max_{p \in V_m \setminus V_0} |\mu_{m,p}^{-1}(H_m u)(p) - f(p)| = 0,$$

where $\mu_{m,p} = \int_K \psi_{m,p}\, d\mu$, then we define the μ-Laplacian Δ_μ by $\Delta_\mu u = f$. The domain of Δ_μ is denoted by $\mathscr{D}_\mu$.

THEOREM 2.10 (Gauss–Green formula). (a) *The domain $\mathscr{D}_\mu \subset \mathscr{F}$, and the Neumann derivative on the boundary, defined by $(dv)_p = \lim_{m\to\infty} -(H_m v)(p)$, exists for $v \in \mathscr{D}_\mu, p \in V_0$.*

(b) *For $u \in \mathscr{F}$ and $v \in \mathscr{D}_\mu$, we have*

$$\mathscr{E}(u, v) = \sum_{p \in V_0} u(p)\,(dv)_p - \int_K u \Delta_\mu v\, d\mu.$$

We may also define the Green's function $g(x, y)$ associated with $(\mathscr{E}, \mathscr{F})$.

THEOREM 2.11. *There exists a non-negative continuous function $g : K \times K \to \mathbb{R}$, with $g(x, y) = g(y, x)$ for all $x, y \in K$, that satisfies $\mathscr{E}(g^x, f) = f(x)$ for all $f \in \mathscr{F}$ with $f|_{V_0} = 0$, where $g^x(y) = g(x, y)$. Also for given $\phi \in C(K)$, there exists a unique $f \in \mathscr{D}_\mu$ which satisfies*

$$\begin{cases} \Delta_\mu f = \phi, \\ f|_{V_0} = 0. \end{cases}$$

Furthermore, f is given by $f(x) = -\int_K g(x, y)\phi(y)\mu(dy)$.

3. *Eigenfunctions*

DEFINITION 3.1. For $u \in \mathscr{D}_\mu$ and $k \in \mathbb{R}$, if

$$\begin{cases} \Delta_\mu u = -ku, \\ u|_{V_0} = 0, \end{cases}$$

then k is called a *Dirichlet eigenvalue* (D-*eigenvalue* for short) of $-\Delta_\mu$ and u is said to be a *Dirichlet eigenfunction* (D-*eigenfunction* for short) belonging to the D-eigenvalue k. Also, if

$$\begin{cases} \Delta_\mu u = -ku, \\ (du)_p = 0 \quad \text{for all } p \in V_0, \end{cases}$$

then k is called a *Neumann eigenvalue* (N-*eigenvalue* for short) of $-\Delta_\mu$ and u is said to be a *Neumann eigenfunction* (N-*eigenfunction* for short) belonging to the N-eigenvalue k.

The eigenvalue problems above are equivalent to the following variational problems involving the Dirichlet form $(\mathscr{E}, \mathscr{F})$. Write $(u, v)_\mu$ for the inner product on $L^2(K, \mu)$.

PROPOSITION 3.2 [**7**, Proposition 5.1 and 5.2]. *With the above notation k is a* D-*eigenvalue of $-\Delta_\mu$ and $u \in \mathscr{F}_0 = \{u \in \mathscr{F} : u|_{V_0} = 0\}$ is an associated* D-*eigenfunction if and only if $\mathscr{E}(u, v) = k(u, v)_\mu$ for all $v \in \mathscr{F}_0$. Also k is an* N-*eigenvalue of $-\Delta_\mu$ and $u \in \mathscr{F}$ is an associated* N-*eigenfunction if and only if $\mathscr{E}(u, v) = k(u, v)_\mu$ for all $v \in \mathscr{F}$.*

It is known that the D-eigenvalues (and also the N-eigenvalues) are non-negative, of finite multiplicity and that their only accumulation point is $+\infty$. See [**7**, §4 and §5].

DEFINITION 3.3. For $* = \mathrm{D}, \mathrm{N}$, let $\{k_i^*(\mu)\}_{i=1,2,\dots}$, where $k_i^*(\mu) \leqslant k_{i+1}^*(\mu)$ for all $i = 1, 2, \cdots$, be the set of $*$-eigenvalues of $-\Delta_\mu$, taking the multiplicity into account. The *eigenvalue counting function* $\rho_*(x, \mu)$ is defined by $\rho_*(x, \mu) = \#\{i : k_i^*(\mu) \leqslant x\}$. Note that ρ_* is right-continuous and non-decreasing, and set $\rho_*(x-, \mu) = \lim_{y \to z, y < x} \rho_*(y, \mu)$.

Combining Theorem 2.11 and the Proposition 3.2, a D-eigenvalue k and a D-eigenfunction u are characterized by

$$u(x) = k \int_K g(x, y) u(y) \mu(dy).$$

Hence, applying the classical theory of integral operators, we have the following.

PROPOSITION 3.4. *If ϕ_i is the normalized* D-*eigenfunction* (*that is*, $(\phi_i, \phi_i)_\mu = 1$) *belonging to the* D-*eigenvalue $k_i^D(\mu)$, then $\{\phi_i\}_{i=1,2,\cdots}$ is a complete orthonormal basis of $L^2(K, \mu)$.*

Kigami and Lapidus [**7**] studied the behaviour of $\rho_*(x, \mu)$ as $x \to \infty$ when μ is a Bernoulli measure. (A Bernoulli measure μ is characterized by $\mu(K_w) = \mu_{w_1} \mu_{w_2} \cdots \mu_{w_m}$ for all $w = w_1 w_2 \cdots w_m \in W_*$, where $\mu_i = \mu(K_i)$ for $i = 1, 2, \cdots, N$).

THEOREM 3.5 [7, Theorem 2.4]. *Let d_s be the unique real number d that satisfies $\Sigma_{i=1}^N \gamma_i^d = 1$, where $\gamma_i = (r_i \mu_i)^{1/2}$. Then*

$$0 < \liminf_{x\to\infty} \rho_*(x,\mu)/x^{d_s/2} \leqslant \limsup_{x\to\infty} \rho_*(x,\mu)/x^{d_s/2} < \infty$$

for $* = \mathrm{D}, \mathrm{N}$. Here d_s is called the spectral exponent of $(\mathscr{E}, \mathscr{F}, \mu)$. Moreover, we have the following.

(1) *Non-lattice case: if $\Sigma_{i=1}^N \mathbb{Z}\log\gamma_i$ is a dense subgroup of $\mathbb{R}$, then the limit $\lim_{x\to\infty} \rho_*(x,\mu)/x^{d_s/2}$ exists.*

(2) *Lattice case: if $\Sigma_{i=1}^N \mathbb{Z}\log\gamma_i$ is a discrete subgroup of $\mathbb{R}$, let $T > 0$ be its generator. Then, $\rho_*(x,\mu) = (G(\log x/2) + o(1))x^{d_s/2}$, where G is a (right-continuous) T-periodic function with $0 < \inf G(x) \leqslant \sup G(x) < \infty$ and $o(1)$ is a term which vanishes as $x \to \infty$.*

Remark. More concrete expressions for the value of the limit in the non-lattice case and the function G in the lattice case are obtained in [7]. In particular, these limits are independent of $* = \mathrm{D}$ or N.

4. *Localized eigenfunctions and oscillations*

In the following sections, we shall assume that μ is a Bernoulli measure on K.

DEFINITION 4.1. The function $u \in \mathscr{D}_\mu$ is called a *pre-localized eigenfunction* of $-\Delta_\mu$ if u is both a Neumann and a Dirichlet eigenfunction for a (Neumann and Dirichlet) eigenvalue k.

REMARK. By using the variational expression in Proposition 3.2, it is easily seen that u is a pre-localized eigenfunction if and only if $u \in \mathscr{F}_0$ and $\mathscr{E}(u,v) = k(u,v)_\mu$ for all $v \in \mathscr{F}$.

LEMMA 4.2. *Let u be a pre-localized eigenfunction. For $w \in W_*$ define u_w by*

$$u_w(x) = \begin{cases} x(F_w^{-1}(x)) & \text{if } x \in K_w, \\ 0 & \text{otherwise.} \end{cases}$$

Then u_w is a pre-localized eigenfunction belonging to the eigenvalue $k/r_w\mu_w$, where $\mu_w = \mu(K_w) = \mu_{w_1}\mu_{w_2}\cdots\mu_{w_m}$ for $w = w_1 w_2 \cdots w_m$.

Proof. Applying Lemma 2.8 and Proposition 3.2, we have

$$\mathscr{E}(u_w, v) = \sum_{w' \in W_m} (r_{w'})^{-1} \mathscr{E}(u_w \circ F_{w'}, v \circ F_{w'}) = (r_w)^{-1} \mathscr{E}(u, v \circ F_w) = \frac{k}{r_w}(u, v \circ F_w)_\mu.$$

for any $v \in \mathscr{F}$. On the other hand,

$$(u_w, v)_\mu = \sum_{w' \in W_m} \mu_{w'} (u_w \circ F_{w'}, v \circ F_{w'})_\mu = \mu_w (u, v \circ F_w)_\mu.$$

Hence for all $v \in \mathscr{F}$, we have $\mathscr{E}(u_w, v) = (k/(r_w \mu_w))(u_w, v)_\mu$.

REMARK. Note also that the function u_w is a localized eigenfunction in the sense that $u = 0$ outside the set K_w.

The following proposition is an immediate consequence of Lemma 4.2.

PROPOSITION 4.3. *There exists a pre-localized eigenfunction of* $-\Delta_\mu$ *if and only if for any non-empty open subset* $O \subset K$, *there exists a pre-localized eigenfunction* u *such that* $\operatorname{supp} u \subset O$.

Proof. For any non-empty open subset $O \subset K$, there exists $w \in W_*$ such that $K_w \subset O$. If there exists a pre-localized eigenfunction u, then u_w is a pre-localized eigenfunction with $\operatorname{supp} u_w \subset K_w \subset O$. The converse direction is obvious.

We now consider the lattice case in Theorem 3.5. Our main result is the following.

THEOREM 4.4. *For the lattice case, the following four conditions are equivalent:*
(1) *there exists a pre-localized eigenfunction of* $-\Delta_\mu$;
(2) G *has a discontinuous point;*
(3) *for any* $M \in \mathbb{N}$, *there exists a (Neumann or Dirichlet) eigenvalue of* $-\Delta_\mu$ *whose multiplicity is greater than* M;
(4) *there exists a (Neumann or Dirichlet) eigenvalue of* $-\Delta_\mu$ *whose multiplicity is greater than* $\#(V_0)$.

COROLLARY 4.5. *For the lattice case, if there does exist a pre-localized eigenfunction, then* $\rho_*(x,\mu)/x^{d_s/2}$ *does not converge as* $x \to \infty$.

The rest of this section is devoted to proving the above theorem. First we recall and introduce some notation for the lattice case. As T is the positive generator of $\Sigma_{i=1}^N \mathbb{Z} \log \gamma_i$, where $\gamma_i = (r_i \mu_i)^{1/2}$, we can write $-\log \gamma_i = m_i T$ for $i = 1, 2, \cdots, N$, where the m_i are positive integers whose largest common divisor is 1. Note that $k/r_w \mu_w = k/\gamma_w^2 = k \exp(2T\, \Sigma_{i=1}^l m_{w_i})$ for any $w = w_1 w_2 \cdots w_l \in W_*$, where $\gamma_w = \gamma_{w_1}\gamma_{w_2}\cdots\gamma_{w_l}$. Hence if

$$M(n) = \#\left\{ w = w_1 w_2 \cdots w_l \in W_* : \frac{k}{r_w \mu_w} = k\, e^{2nT} \right\},$$

for $n \in \mathbb{N}$, then $M(n) = \#\{w = w_1 w_2 \cdots w_l \in W_* : \Sigma_{i=1}^l m_{w_i} = n\}$. We also define $M(0) = 1$ and $M(n) = 0$ for any negative integer n.

LEMMA 4.6. *If* $p = e^{Td}$, *then* $\lim_{n\to\infty} M(n)/p^n = (\Sigma_{i=1}^N m_i p^{-m_i})^{-1}$.

Proof. For $n \in \mathbb{N}$, we have $M(n) = \Sigma_{i=1}^N M(n - m_i)$. Hence defining $Z(x)$ for $x \in \mathbb{R}$ by $Z(x) = M(x)/p^x$ for $x \in \mathbb{Z}$, and $Z(x) = 0$ otherwise, we have $Z(x) - \Sigma_{i=1}^N Z(x - m_i)/p^{m_i} = U(x)$, where $U(x) = 0$ if $x \neq 0$ and $U(0) = 1$. Therefore we have the following renewal equation

$$Z(x) = U(x) + \int_0^\infty Z(x-t)\, \nu(dt) \quad \text{for all } x \in \mathbb{R},$$

where $\nu = \Sigma_{i=1}^N p^{-m_i} \delta_{m_i}$ and δ_x is the Dirac point mass at x. Note that $\Sigma_{i=1}^N p^{-m_i} = \Sigma_{i=1}^N \gamma_i^{d_s} = 1$. By the renewal theorem we have

$$\lim_{n\to\infty} Z(n) = \left(\int_0^\infty x \nu(dx) \right)^{-1} \sum_{n=-\infty}^{+\infty} U(n) = \left(\sum_{i=1}^N m_i p^{-m_i} \right)^{-1}.$$

Proof of Theorem 4.4. (1) implies (2). Let u be a pre-localized eigenfunction belonging to the eigenvalue k. Then by Lemma 4.6, $k_n = k\,e^{2nT}$ is a (Dirichlet and Neumann) eigenvalue with multiplicity at least $M(n)$. Hence $\rho_*(k_n,\mu)-\rho_*(k_n-,\mu) \geqslant M(n)$. As $\lim_{x\to\infty}|\rho_*(x,\mu)/x^{d_s/2}-G(\log x/2)| = 0$ and $\log k_n = (\log k/2)+nT$, by Lemma 4.6, we have that

$$\lim_{t\to\alpha+0} G(t)-\lim_{t\to\alpha-0} G(t) \geqslant \lim_{n\to\infty}\frac{M(n)}{k^{d_s/2}\,e^{nTd_s}} = \left(\sum_{i=1}^{N}\frac{m_i}{p^{m_i}}\right)^{-1} k^{-d_2/2} > 0,$$

where $\alpha = \log k/2$. Hence G is discontinuous at $\alpha = \log k/2$.

(2) implies (3). Let $G_n(t) = \rho_*(e^{2(t+nT)},\mu)/e^{t+nTd_s}$, and $\varepsilon_n = \sup_{0\leqslant t\leqslant 1}|G_n(t)-G(t)|$. Then, by Theorem 3.4, $\varepsilon_n \to 0$ as $n\to\infty$. Now if G is discontinuous at α, then we can choose $a_m > 0$ and $b_m > 0$ so that $\lim_{m\to\infty} a_m = \lim_{m\to\infty} b_m = 0$ and $L = \liminf_{m\to\infty}|G(\alpha+a_m)-G(\alpha-b_m)| > 0$. It follows that

$$\liminf_{m\to\infty}|G_n(\alpha+a_m)-G_n(\alpha-b_m)| \geqslant L-2\varepsilon_n.$$

This implies that $e^{2(\alpha+nT)}$ is an eigenvalue whose multiplicity is no less than $e^{\alpha+nTd_s}$. This implies (3).

(3) implies (4). This is obvious.

(4) implies (1). Let k be an N-eigenvalue whose multiplicity is greater than $\#(V_0)$ and let E_k be the collection of N-eigenfunctions belonging to k. Define a linear map $\tau: E_k \to l(V_0)$ by $\tau(u)(x) = u(x)$ for $x\in V_0$. As $\dim E_k > \dim l(V_0) = \#(V_0)$, the kernel of τ is not trivial. Hence there exists non-trivial $u\in E_k$ that satisfies $u|_{V_0} = 0$, and so u is a pre-localized eigenfunction.

The same argument works for the case of D-eigenvalues.

5. *Symmetry and the existence of pre-localized eigenfunctions*

We now give some sufficient conditions, in terms of the geometry of K, for the existence of localized eigenfunctions.

Let $(K, S, \{F_i\}_{i\in S})$ be a p.c.f. self-similar set, and μ be a Bernoulli measure on K. If $g: K\to K$ is a bijection, and $f: K\to\mathbb{R}$, define $T_g f: K\to\mathbb{R}$ by $T_g f(x) = f(g^{-1}(x))$.

Definition 5.1. A function $g: K\to K$ is a p.c.f. *morphism* if
(i) g is bijective,
(ii) g is a homeomorphism of K,
(iii) $g: V_0\to V_0$,
(iv) $\mu\circ g^{-1} = \mu$,
(v) if $\phi\in\mathscr{F}$ then $T_g\phi\in\mathscr{F}$, and $\mathscr{E}(\phi,\psi) = \mathscr{E}(T_g\phi, T_g\psi)$ for all $\psi\in\mathscr{F}$.

Let $\mathscr{G}$ be the group of p.c.f. morphisms, and write ι for the identity element of $\mathscr{G}$. Let E be the collection of Dirichlet eigenfunctions of $-\Delta_\mu$, and for $\phi\in E$ write $\varepsilon(\phi)$ for the associated eigenvalue. Set $\Lambda = \{\Sigma_{i-1}^n \alpha_i T_{g_i}, \alpha_i\in\mathbb{R}, g_i\in\mathscr{G}\}$.

Lemma 5.2. *Let* $R = \Sigma_{i=1}^n \alpha_i T_{g_i}\in\Lambda$. *If* $Ru\neq 0$ *for some* $u\in L^2(K,\mu)$ *and* $R^*v\in\mathscr{F}_0$ *for all* $v\in\mathscr{F}$, *where* $R^* = \Sigma_{i=1}^n \alpha_i T_{g_i^{-1}}$, *then there exists a pre-localized eigenfunction.*

Proof. By Proposition 3.4, the set of normalized D-eigenfunctions $\{\phi:\phi\in E\}$ is a complete orthonormal base of $L^2(K,\mu)$. So if $R\phi = 0$ for all $\phi\in E$, then $Ru = 0$ for any $u \in L^2(K, \mu)$. Hence there exists $\phi \in E$ such that $R\phi \neq 0$. Let $\psi = R\phi$, and note that $\psi \in \mathscr{F}_0$. Now for all $v \in \mathscr{F}$, as $R^* v \in \mathscr{F}_0$, using Proposition 3.2,

$$\mathscr{E}(\psi, v) = \mathscr{E}(\phi, R^* v) = \varepsilon(\phi)(\phi, R^* v)_\mu = \varepsilon(\phi)(\psi, v)_\mu.$$

Hence by the variational formulation of pre-localized eigenfunctions (see the remark after Definition 4.1) ψ is a pre-localized eigenfunction.

For $g \in \mathscr{G}$ set $\mathscr{S}(g) = \{x \in K : g(x) = x\}$.

PROPOSITION 5.3. (a) *If there exists $h \in \mathscr{G} \backslash \{\iota\}$ such that $V_0 \subset \mathscr{S}(h)$, then there exist pre-localized eigenfunctions.*

(b) *If $\mathscr{G}$ is infinite then there exist pre-localized eigenfunctions.*

Proof. (a) Let $R = I - T_h \in \Lambda$. As $h \neq \iota$, there exists $x \in K$ such that $h(x) \neq x$. Since h is continuous there exists a neighbourhood A of x such that $h(A) \cap A = \varnothing$. Set $u = 1_A$: we have $Ru \neq 0$.

As $V_0 \subset \mathscr{S}(h)$, it follows that $h^{-1}(x) = x$ for all $x \in V_0$. Hence $v - T_{h^{-1}} v = R^* v \in \mathscr{F}_0$ for all $v \in \mathscr{F}$. Now using Lemma 5.2, we can complete the proof.

(b) If $\mathscr{G}$ is infinite then, since V_0 is finite, a counting argument shows that there exist distinct elements g_1, g_2 of $\mathscr{G}$ with the same action on V_0. Hence $V_0 \subset \mathscr{S}(g_1^{-1} g_2)$, and the result is immediate from (a).

There do exist p.c.f. self-similar sets for which $\mathscr{G}$ is infinite (the Vicsek set is one), but this is a little exceptional. We now turn to the more complicated situation when $\mathscr{G}$ is finite.

THEOREM 5.4. *Suppose that G is a finite subgroup of $\mathscr{G}$ which is vertex transitive on V_0, and that there exists $h \in \mathscr{G}$ with $h \notin G$, such that*

$$\mathscr{S}_G(h) = \bigcup_{g \in G} \mathscr{S}(h^{-1} g) \neq K. \tag{5.1}$$

Then there exist pre-localized eigenfunctions.

Proof. Set $R_G = \Sigma_{g \in G} T_g = \Sigma_{g \in G} T_{g^{-1}}$, and $R = R_G(T_{h^{-1}} - I) \in \Lambda$. Let $x \in K \backslash \mathscr{S}_G(h)$. Then $\{g(x) : g \in G\}$ is finite and does not contain $h(x)$. Hence there exists a neighbourhood A of x such that $h(A) \cap g(A) = \varnothing$ for all $g \in G$. Set $u = 1_{h(A)}$. If $y \in A$, then $u(g(y)) = 0$ for $g \in G$, and so

$$Ru(y) = \sum_{g \in G} u(h(g(y))) - \sum_{g \in G} u(g(y)) \geqslant u(h(y)) - \sum_{g \in G} u(g(y)) = u(h(y)) = 1,$$

proving that $Ru \neq 0$.

Let $v \in \mathscr{F}$. As G is vertex transitive, if $y \in V_0$ then

$$R_G v(y) = \sum_{g \in G} v(g(y)) = \frac{\#(G)}{\#(V_0)} \sum_{p \in V_0} v(p),$$

which is independent of y. If $x \in V_0$ then $h^{-1}(x) \in V_0$, and therefore $R^* v = (T_h - I) R_G v \in \mathscr{F}_0$.

Now using Lemma 5.2, we can complete the proof.

REMARK 1. As the condition (5.1) is a little troublesome to verify, one might hope that this symmetry argument would work under the weaker condition that

$$\mathscr{G} \text{ contains a vertex transitive subgroup } G \text{ and there exists } h \in \mathscr{G} \backslash G. \tag{5.2}$$

However, it is easy to see that (5.1) is equivalent to the statement that $R_G(T_{h^{-1}} - I) \neq 0$, and in Section 6 we give an example of a p.c.f. self-similar set K which satisfies (5.1) but not (5.2).

On the other hand, note that if (5.1) fails, then $\mathscr{S}(h^{-1}g)$ has an interior point for some $g \in G$. Hence (5.2) along with

$$\mathscr{S}(h^{-1}g) \text{ has no interior point for any } g \in G \tag{5.3}$$

implies (5.1). In fact we shall see below (Lemma 6.3) that for nested fractals (5.3) always holds, so that (5.2) is all that needs to be verified.

REMARK 2. While the two results in this section prove the existence of pre-localized eigenfunctions for a variety of p.c.f. self-similar sets, there are many cases they do not cover. It is possible that every p.c.f. self-similar set with $\#(V_0) > 2$ has pre-localized eigenfunctions. (See the remark following Theorem 6.6 for the case $\#(V_0) = 2$.)

6. *Nested fractals and other examples*

In this section we shall discuss the case where K is a subset of $\mathbb{R}^d$ for some d and the F_i are restrictions of similitudes of $\mathbb{R}^d$; the map $F: \mathbb{R}^d \to \mathbb{R}^d$ is called a *similitude* if $F(x) = \alpha Tx + b$, where $\alpha \in (0,1)$, $T \in O(d)$ and $b \in \mathbb{R}^d$. In such a case, we can assume, without loss of generality, that

$$\sum_{i=1}^{M} p_i = 0, \tag{6.1}$$

where $V_0 = \{p_1, p_2, \cdots, p_M\}$ and that

$$\{x - y : x, y \in K\} \quad \text{spans} \quad \mathbb{R}^d. \tag{6.2}$$

Under these assumptions we have the following lemma.

LEMMA 6.1. (1) *If f is an affine map from $\mathbb{R}^d$ to itself with $f(V_0) = V_0$, then $f(0) = 0$.*

(2) *Let $f_i: \mathbb{R}^d \to \mathbb{R}^d$ be linear for $i = 1, 2$. If $f_1|_K = f_2|_K$, then $f_1(x) = f_2(x)$ for all $x \in \mathbb{R}^d$.*

(3) *If $f: \mathbb{R}^d \to \mathbb{R}^d$ is a linear map with $f(K) = K$, then $f \in O(d)$. Moreover, if f is not the identity map, then $\mathscr{S}(f) = \{x \in K: f(x) = x\}$ contains no interior point in the intrinsic topology K.*

Proof. (1) Let $f(x) = Ax + b$ where A is a $d \times d$-matrix and $b \in \mathbb{R}^d$. As $\Sigma_{i=1}^M f(p_i) = \Sigma_{i=1}^M p_i = 0$, we have $A(\Sigma_{i=1}^M p_i) + Mb = 0$. Hence $b = 0$.

(2) This is immediate from (6.2).

(3) As $K \subset \operatorname{Im} f$ and (6.2) holds, it follows that f is invertible. Note that K is bounded. We can easily see that $f^n(x)$ and $f^{-n}(x)$ remain bounded as $n \to \infty$ for any $x \in \mathbb{R}^d$. Hence if we extend f to a map from $\mathbb{C}^d$ to itself, f is semisimple and the absolute values of its eigenvalues are all equal to 1. Therefore $f \in O(d)$.

Next suppose there exists a non-empty open subset O of K such that $f(x) = x$ for any $x \in O$. By (6.2), there exist $x_i, y_i \in K$ for $i = 1, 2, \cdots, d$ such that $(x_1 - y_1, x_2 - y_2, \cdots, x_d - y_d)$ is a base of $\mathbb{R}^d$. Now choose $w \in W_*$ so that $F_w(K) \subset O$, and write $z_i =$

$F_w(x_i) - F_w(y_i)$; then $(z_1, z_2, \cdots, z_d)$ is a base of $\mathbb{R}^d$ and $f(z_i) = z_i$ for $i = 1, 2, \cdots, d$. Thus f is the identity map.

Hereafter, we consider a special subgroup of the group of p.c.f. morphisms $\mathscr{G}$.

THEOREM 6.2. *Define* $\mathscr{G}_0 = \{g \in \mathscr{G}; g = f|_K$ *for some linear map* $f: \mathbb{R}^d \to \mathbb{R}^d\}$. *If there exists a proper subgroup G of* $\mathscr{G}_0$ *which is vertex transitive on* V_0, *then there exists a pre-localized eigenfunction.*

REMARK. By Lemma 6.1, we can consider $\mathscr{G}_0$ to be a subgroup of $O(d)$.

Proof. As G is a proper subgroup of $\mathscr{G}_0$, there exists $h \in \mathscr{G}_0 \backslash G$. By Lemma 6.1 (3), we see that $\mathscr{S}(h^{-1}g)$ has no interior point for any $g \in G$. Hence by Remark 1 following Theorem 5.4, we deduce that there exists a pre-localized eigenfunction.

The corollary below is sometimes easier to apply to examples than Theorem 6.2. Let P_{V_0} be the group of permutations of V_0. We can define a natural map $\chi: \mathscr{G} \to P_{V_0}$ by $\chi(g) = g|_{V_0}$.

COROLLARY 6.3. (1) *If* χ *is not injective then there exists a pre-localized eigenfunction.*

(2) *Set* $G_0 = \chi(\mathscr{G}_0)$. *If there exists a proper subgroup of* G_0 *which is vertex transitive on* V_0, *then there exists a pre-localized eigenfunction.*

Proof. (1) If χ is not injective then there exists $g \neq \iota \in \mathscr{G}$ with $g(x) = x$ for all $x \in V_0$. The result is now immediate from Proposition 5.3 (a).

(2) We can find a proper subgroup of $\mathscr{G}_0$ which is vertex transitive on V_0 by using χ^{-1}. Then use Theorem 6.2.

Here are some cases where we can apply Corollary 6.3 (2).

EXAMPLE 6.4. (1) If $G_0 = P_{V_0}$ and $\#(V_0) \geqslant 3$, then the group of even permutations is a proper subgroup of P_{V_0} which is vertex transitive on V_0.

(2) Let V_0 be a regular n-sided polygon for $n > 2$ and suppose that G_0 contains D_n, where D_n is the symmetry group of the regular n-sided polygon. We may write $V_0 = \{(\cos(2\pi j/n), \sin(2\pi j/n)) : j = 1, 2, \cdots, n\}$. Let g be the rotation by $2\pi/n$ around $(0, 0)$. Then $\{g^j$ for $j = 1, 2, \cdots, n\}$ is a proper subgroup of D_n and is vertex transitive on V_0.

(3) Let V_0 be a cube. In this case, we may write

$$V_0 = \{((-1)^i, (-1)^j, (-1)^k) : i, j, k \in \{0, 1\}\}.$$

Suppose G_0 contains the symmetry group of the cube. Let g_1 be rotation by $\pi/2$ around the z-axis and g_2 be reflection in the xy-plane. The group generated by g_1 and g_2 is a proper subgroup of G_0 which is vertex transitive on V_0.

Next we give an example where we can apply Corollary 6.3 (1).

EXAMPLE 6.5. Set $F_1(z) = \frac{1}{2}(z+1), F_2(z) = \frac{1}{2}(z-1), F_3(z) = \frac{1}{4}(-1)^{1/2}(z+1)$ and $F_4(z) = \frac{1}{4}(-1)^{1/2}(z-1)$ for $z \in \mathbb{C}$. Let K be the unique non-empty compact subset of $\mathbb{C}$ that satisfies $K = \bigcup_{i=1,2,3,4} F_i(K)$. It is easy to see that $(K, S, \{F_i\}_{i \in S})$ where $S = \{1, 2, 3, 4\}$ is a p.c.f. self-similar structure. In fact, $\bigcup_{i \neq j}(K_i \cap K_j) = \{0\}$, $\mathscr{C} = \pi^{-1}(0) = \{2\dot{1}, 1\dot{2},$

$3\dot{2}, 4\dot{1}\}$ and $\mathscr{P} = \{\dot{1}, \dot{2}\}$, where $\dot{k} = kkk\cdots$. As $\pi(\dot{1}) = 1$ and $\pi(\dot{2}) = -1$, it follows that $V_0 = \{-1, 1\}$.

Let $t \in (0, 1)$, and set

$$D = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad r = (\tfrac{1}{2}, \tfrac{1}{2}, t, t).$$

Then (D, r) is a regular harmonic structure with $\lambda = 1$. Also let μ be a Bernoulli measure on K that satisfies $\mu_1 = \mu_2$ and $\mu_3 = \mu_4$.

The reflections in the real axis and the imaginary axis, denoted by g_1 and g_2 respectively, are p.c.f.-morphisms with respect to (D, r) and μ, and $\mathscr{G}_0$ is the group generated by $\{g_1, g_2\}$. Obviously $\chi(g_2)$ is the identity map on V_0. Hence by Corollary 6.3 (1), there exists a pre-localized eigenfunction of $-\Delta_\mu$. (Note also that $\mathscr{G}$ contains infinitely many elements, and so the existence of a pre-localized eigenfunction also follows from Proposition 5.3 (b).)

Now let $G = \{\iota, g_1\}$, and let $h: K \to K$ be defined by $h(x) = x$ for $x \in K_1 \cup K_2$ and $h(x) = g_2(x)$ for $x \in K_3 \cup K_4$. It is not hard to check that h is a p.c.f.-morphism. Then G and h satisfy (5.2) but $R_G(T_{h^{-1}} - I) = 0$. So this example shows that (5.1) and (5.2) are not equivalent; recall Remark 1 of Theorem 5.4.

In the rest of this section, we shall consider the case of nested fractals.

NOTATION. For $x, y \in \mathbb{R}^2$ with $x \neq y$, let H_{xy} be the hyperplane bisecting the line segment $[x, y]$, and let $g[xy]: \mathbb{R}^d \to \mathbb{R}^d$ be reflection in H_{xy}.

Nested fractals, defined in [**11**], are a subset of the class of p.c.f. self-similar sets, and may be described by saying that a nested fractal is a p.c.f. self-similar set $(K, S, \{F_i\}_{i=1}^N)$ such that for some $\alpha \in (0, 1)$, we have

(N1) $K \subset \mathbb{R}^d$,
(N2) each F_i is a similitude with a contraction factor α,
(N3) $g[xy](V_n) \subset V_n$ for all $x, y \in V_0$.

We may also assume that (6.1) and (6.2) hold. It is shown in [**11**] that a nested fractal has a (non-degenerate) harmonic structure (D, r), with $r_i = 1$ for $i = 1, 2, \cdots, N$. Let μ be the Bernoulli measure on K obtained by setting $\mu_i = 1/N$ for $i = 1, 2, \cdots, N$, and let $\lambda > 1$ be the scaling factor given by Definition 2.6. After normalization (see the discussion following Definition 2.6), we may write $r_i = 1/\lambda$ for $i = 1, 2, \cdots, N$.

The description of the harmonic structure (D, r) given in [**11**] shows that $g[xy]|_K$ satisfies property (v) of Definition 5.1 for each $x, y \in V_0$. Since the other properties are evident, $g[xy]|_K$ is a p.c.f. morphism belonging to $\mathscr{G}_0$. Let $\mathscr{G}_1$ be the subgroup of $\mathscr{G}_0$ generated by $\{g[xy]|_K : x, y \in V_0, x \neq y\}$.

THEOREM 6.6. *Let K be a nested fractal with $\#(V_0) \geqslant 3$. Then K has a pre-localized eigenfunction.*

Proof In view of Lemma 6.1 (2) we may identify $g[xy]$ and $g[xy]|_K$, and so regard $\mathscr{G}_1$ as a subgroup of $O(d)$. Note that, as $g[xy]$ is a reflection, $\det(g[xy]) = -1$. Let $\mathscr{G}_2$ be the set of $g \in \mathscr{G}_1$ which are the product of an even number of the $g[xy]$. Then every element g of $\mathscr{G}_2$ has $\det(g) = 1$, and so $\mathscr{G}_2$ is a proper subgroup of $\mathscr{G}_1$ Furthermore, if $\#(V_0) \geqslant 3$ then $\mathscr{G}_2$ is vertex transitive. For, if $x, y \in V_0$, let $z \in V_0 \backslash \{x, y\}$: then $g[yz]g[xz](x) = y$. The result now follows from Theorem 6.2.

REMARK. If $\#(V_0) = 2$ then examples show that both possibilities (that is, existence or non-existence of localized eigenfunctions) can arise. The unit interval $[0, 1]$ is a nested fractal, with $V_0 = \{0, 1\}$, and of course has no localized eigenfunctions. On the other hand, it is shown in [**12**] that the modified Koch graph does have localized eigenfunctions.

For a nested fractal, we have $\gamma_i = (r_i \mu_i)^{1/2} = (\lambda N)^{-1/2}$ for $i = 1, 2, \cdots, N$. So the lattice case of Theorem 3.5 holds and, using Theorem 4.4, we deduce the following corollary.

COROLLARY 6.7. *Let K be a nested fractal with $\#(V_0) \geqslant 3$. Then $\rho_*(x, \mu)/x^{d_s/2}$ does not converge as $x \to \infty$.*

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