Math. Proc. Camb. Phil. Soc. (2000), **129**, 331

Printed in the United Kingdom

Splines on fractals

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(*Received* 18 *March* 1999)

Abstract

A general theory of piecewise multiharmonic splines is constructed for a class of fractals (post-critically finite) that includes the familiar Sierpinski gasket, based on Kigami's theory of Laplacians on these fractals. The spline spaces are the analogues of the spaces of piecewise C^j polynomials of degree $2j + 1$ on an interval, with nodes at dyadic rational points. We give explicit algorithms for effectively computing multiharmonic functions (solutions of $\Delta^{j+1}u = 0$) and for constructing bases for the spline spaces (for general fractals we need to assume that j is odd), and also for computing inner products of these functions. This enables us to give a finite element method for the approximate solution of fractal differential equations. We give the analogue of Simpson's method for numerical integration on the Sierpinski gasket. We use splines to approximate functions vanishing on the boundary by functions vanishing in a neighbourhood of the boundary.

1. *Introduction*

For a large class of fractals, called *post-critically finite* (p.c.f.), that includes the familiar Sierpinski gasket, Kigami [**Ki1**–**Ki8**] has constructed a theory of Laplacians based on the renormalized limits of graph Laplacians. This allows a theory of *fractal differential equations*, although strictly speaking these are not differential equations. While there are no specific applications of this theory at the moment, it has the potential to be used as a model for various physical processes on fractal objects. It is therefore desirable to develop numerical analysis methods to approximate solutions to these equations. In [**DSV**] the analogue of finite difference methods were used. Here we will develop the analogue of the finite element method using spline spaces.

† Research supported in part by the National Science Foundation, Grant DMS-9623205.
‡ Research supported by the National Science Foundation through the Research Experiences for Undergraduates (REU) Program.

The p.c.f. fractals are in many ways more closely related to the unit interval than domains in higher dimensional spaces, so we should look to the theory of piecewise polynomial splines on an interval for inspiration. The space of polynomials of degree at most $2j + 1$ is identical to the space of solutions of $\Delta^{j+1}u = 0$ on the line, so we will use the analogous spaces of multiharmonic functions on our fractals as the model spaces in our construction of splines. These spaces are finite dimensional and as j increases the approximation power increases. In [**S2**] these spaces were used to establish the analogue of Taylor approximations.

The first goal of this paper is to give an effective algorithm for the computation of multiharmonic functions. A solution of $\Delta^{j+1}u = 0$ is uniquely determined by the values of $\Delta^{\ell}u$ at boundary points for all $\ell \leqslant j$. In Section 2 we find a recursive local algorithm to determine the solution. The fractal is a limit of finite graphs Γ_m with vertices V_m, the boundary being exactly V_0. The algorithm successively computes the values of $\Delta^{\ell}u(x)$ for $x \in V_m$ and $\ell \leqslant j$ in terms of the values of $\Delta^{\ell}u(y)$ for $y \in V_{m-1}$ and $\ell \leqslant j$, but for each x it is only necessary to consider those vertices y in a neighbourhood of x. The algorithm for harmonic functions ($j = 0$) was given in [**Ki2**] and for biharmonic function ($j = 1$) on the Sierpinski gasket an ad hoc method was used in [**DSV**] to find the algorithm. The approach in [**DSV**] will not work in general, so we use a different method. We also compute the inner products of multiharmonic functions, and in fact it turns out that the two problems are linked. In order to obtain the computation algorithm and the inner products for one value of j, it is necessary to have both for the value $j - 1$. Since the results are known for $j = 0$, we have an inductive solution to both problems.

The results of Section 2 yield an easy basis for the space $\mathscr{H}_j$ of solutions of $\Delta^{j+1}u = 0$, but this basis is not well adapted to construct splines; on the unit interval the analogous construction would give a polynomial of degree at most $2j + 1$ in terms of the values of $\Delta^{\ell}f$ on the boundary for $\ell \leqslant j$, in other words just even order derivatives. To get C^j splines on the line we need to control all derivatives of order $\leqslant j$ at nodes. In the case of fractals this involves a mixture of normal derivatives and Laplacians. In Section 3 we consider the global problem of finding a better basis for $\mathscr{H}_j$ where we control the values of $\Delta^{\ell}u$ on the boundary for $\ell \leqslant j/2$ and the values of the normal derivatives $\partial_n\Delta^{\ell}u$ on the boundary for $\ell < j/2$. We are able to give a general solution only under the assumption that j is odd, but this hypothesis is unnecessary for the Sierpinski gasket (of course for $j = 0$ there is no problem since the basis of Section 2 is the solution).

We localize the construction in Section 4 to obtain the splines spaces. Our fractals K are given by an iterated function system (i.f.s.) of mappings F_i, $1 \leqslant i \leqslant N$, with intersections $F_iK \cap F_{i'}K$ consisting of points in V_1. More generally, for any word $w = (w_1, \ldots, w_m)$ on N letters of length m we write $F_w = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m}$ and we have the decomposition $K = \bigcup_{|w|=m} F_wK$ of the fractal into cells F_wK which intersect at points in V_m. We call these intersection points *junction points* and each junction point is contained in more than one cell F_wK (in the case of the Sierpinski gasket each junction point belongs to exactly 2 cells). There are also points in V_m that are not junction points (the Sierpinski gasket is rather atypical in that the only nonjunction points in V_m are boundary points). We define the spline space $S(\mathscr{H}_j, V_m)$ to be, roughly speaking, the space of functions that belong to $\mathscr{H}_j$ on each cell F_wK for $|w| = m$ and which satisfy certain matching conditions at junction points, namely that $\Delta^\ell u(x)$ should be well defined for $\ell \leqslant j/2$ and the sum of the normal derivatives of $\Delta^\ell u$ over all the cells containing x should vanish for $\ell < j/2$. Because the analogue of the Gauss–Green formula holds, the matching conditions are exactly what is needed to have $\Delta^\ell u$ defined globally as an L^∞ function for $\ell \leqslant (j+1)/2$. There is a small technical problem here: we allow functions to be locally in $\mathscr{H}_j$ that are not restrictions of global functions in $\mathscr{H}_j$ since we do not require the equation $\Delta^{j+1}u = 0$ to hold at points in V_m that are not either junction points or boundary points. This problem does not arise in the case of the Sierpinski gasket. Once we have the spline spaces defined, we prove some approximation results, both in energy and sup norms, that say that for functions in the domain of a power of the Laplacian we can increase the rate of approximation by increasing j.

In Section 5 we specialize to the case of the Sierpinski gasket with its standard Laplacian. We present all the algorithms of the previous sections in explicit and simplified form. We omit the proofs, since they are just routine but lengthy calculations. We then derive the analog of Simpson's method for numerical integration.

In Section 6 we describe the finite element method using the spline spaces for general fractals and prove some rate of convergence results. These are of the expected form, with some restrictions that may or may not be really necessary. A full implementation and test of the method on the Sierpinski gasket may be found at http://mathlab.cit.cornell.edu/~gibbons. See also [**GRS**].

In Section 7 we use splines to show that a function (on the Sierpinski gasket) vanishing on the boundary in an appropriate sense may be approximated by functions vanishing in a neighbourhood of the boundary, with the type of approximation linked to the vanishing condition. The simpler approach based on multiplication by cut-off functions is not always available in this context because of negative results in [**BST**]. As an application we give an improved version of the 'weak = strong' result for solutions of $\Delta u = f$ from [**S1**]. This technique is expected to have many other applications.

We now give a brief summary of Kigami's theory of Laplacians on p.c.f. fractals. The reader should consult [**Ki2**] or [**Ki8**] for more details. We will make a few simplifying assumptions that are not strictly speaking necessary, but they do not seem to rule out any interesting examples.

We assume the fractal K is a compact subset of a Euclidean space defined by the self-similar identity

$$K = \bigcup_{i=1}^{N} F_i K, \tag{1·1}$$

where F_i are contractive similarities. In fact the metric and Euclidean structures are not used in the theory and only the combinatorial properties of the local connectivity of K are important. We assume K is connected, but just barely, in that the nontrivial intersections of the cells $F_i K$ are just finite sets of points. We assume that a finite set V_0 (with $\#V_0 = N_0$), the *boundary* of K, is given. We form the sequence $V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots$ by setting $V_m = V_{m-1} \cup (\bigcup_{i=1}^{N} F_i V_{m-1})$. The basic assumption is that the nonempty intersections of two cells $F_w K$ and $F_{w'} K$ for $|w| = |w'| = m$ must consist only of points in V_m. We form a graph Γ_m with vertices V_m by joining those pairs of points $x, y \in V_m$ for which there exists a cell $F_w K$ with $|w| = m$ containing both of them. The graph Γ_m encodes the combinatorics of the local connectivity of K at the resolution level m; in other words, if we are nearsighted enough to perceive the cells F_wK as connected blobs, we need only pay attention to the points in V_m. In essence, the Laplacians we consider on K are just limits of graph Laplacians on Γ_m, as $m \to \infty$.

The unit interval is an example of such a fractal, where $F_1x = \frac{1}{2}x, F_2x = \frac{1}{2}x+\frac{1}{2}$ and the boundary V_0 is the usual boundary $\{0, 1\}$. The points in V_m are just the dyadic rationals $k2^{-m}$ and the cells F_wK are just the dyadic intervals $[k2^{-m}, (k+1)2^{-m}]$. The simplest nontrivial example is the Sierpinski gasket. Here $K \subseteq \mathbb{R}^2$ is generated by 3 contractions with fixed points (V_0, V_1, V_2) the vertices of a triangle and contraction ratio $\frac{1}{2}$. These 3 vertices form the boundary V_0. In this example every point in V_m is either a boundary point or is the intersection point of exactly 2 distinct cells.

In order to construct a Laplacian, we first construct a Dirichlet form, which is the analogue of the standard energy form

$$\mathscr{E}(u, v) = \int_0^1 u'(x)v'(x)dx \tag{1·2}$$

on the unit interval. We begin with a Dirichlet form on V_0

$$\left.\begin{aligned}\mathscr{E}(u, v) &= \sum_{j<k} D_{jk}(u(x_j) - u(x_k))(v(x_j) - v(x_k)) \\ &= -\sum_{j=1}^{N_0} \sum_{j=1}^{N_0} D_{jk}u(x_j)v(x_k).\end{aligned}\right\} \tag{1·3}$$

Here D_{jk} is a symmetric matrix of coefficients, with $D_{jj} = \sum_{k\neq j} D_{jk}$ for the consistency of the two expressions. We assume that $D_{jk} \geqslant 0$ for $j \neq k$ and that the matrix is irreducible. We next require a vector $r = (r_1, \dots, r_N)$ of scaling factors with $0 < r_j < 1$ for all j and we use them to extend the Dirichlet form to V_1 by self-similarity:

$$\mathscr{E}_1(u, v) = \sum_{i=1} r_i^{-1} \mathscr{E}_0(u \circ F_i, v \circ F_i). \tag{1·4}$$

More generally we define a sequence of Dirichlet forms $\mathscr{E}_m$ on V_m by

$$\mathscr{E}_m(u, v) = \sum_{|w|=m} r_w^{-1} \mathscr{E}_0(u \circ F_w, v \circ F_w), \tag{1·5}$$

where $r_w = r_{w_1} r_{w_2} \dots r_{w_m}$. The final assumption, which is the most delicate, is that the sequence of Dirichlet forms be consistent. Given a function u on V_{m-1}, consider all extensions $\tilde{u}$ to V_m and minimize the energy $\mathscr{E}_m(\tilde{u}, \tilde{u})$. The extension realizing the minimum is called the *harmonic* extension. The consistency assumption is that

$$\mathscr{E}_{m-1}(u, u) = \mathscr{E}_m(\tilde{u}, \tilde{u}) \tag{1·6}$$

for the harmonic extension. It suffices to verify the condition for $m = 1$ and then it follows for all m. However, the consistency condition places severe restrictions on the choice of the matrix D_{jk} and the r vector. For the Sierpinski gasket it is known that the choice $D_{jk} = 1$ if $j \neq k$, $D_{jj} = -2$ and $r = (\frac{3}{5}, \frac{3}{5}, \frac{3}{5})$ satisfies the consistency condition. See [**Sa**] for the full story of all consistent choices for this example. For the general p.c.f. fractal, it is still an open question whether or not there exist consistent choices.

Any consistent choice is called a *harmonic structure*. We can then define a Dirichlet form on continuous functions $u(x)$ on K by

$$\mathscr{E}(u,u) = \lim_{m\to\infty} \mathscr{E}_m(u,u), \tag{1·7}$$

where we denote the restriction of u to V_m by the same letter u. The limit always exists as an extended real because the sequence is monotone increasing and we define the domain of $\mathscr{E}$ to be those functions for which the limit is finite. It can be shown that $\mathscr{E}$ is a local regular Dirichlet form with respect to any reasonable measure and points have positive capacity (this explains why it is no loss of generality to restrict attention to continuous functions). We will only consider self-similar measures, which are probability measures satisfying the self-similar identity

$$\left.\begin{aligned} \mu &= \sum_{i=1}^{N} \mu_j \mu \circ F_i^{-1}, \quad \text{or equivalently} \\ \int f d\mu &= \sum_{i=1}^{N} \mu_i \int f \circ F_i \, d\mu, \end{aligned}\right\} \tag{1·8}$$

for some finite non-zero probabilities $\{\mu_i\}$. It is important to understand that the measure μ has nothing to do with the definition of $\mathscr{E}$ and it is decidedly not true (see [**Ku**]) that

$$\mathscr{E}(u,u) = \int |\nabla u|^2 d\mu.$$

The harmonic structure alone gives rise to the class of harmonic functions, the minimizers of $\mathscr{E}(u,u)$ subject to the boundary values $u|_{V_0}$. It also gives a definition of normal derivative $\partial_n u(x)$ at boundary points x for any $u \in \operatorname{dom}(\mathscr{E})$. We combine the harmonic structure and the measure μ to define a Laplacian Δ_μ by

$$\mathscr{E}(u,v) = -\int v \Delta_\mu u d\mu \tag{1·9}$$

for all $v \in \operatorname{dom}(\mathscr{E})$ vanishing at the boundary. More precisely, u is in the domain of Δ_μ if $u \in \operatorname{dom}(\mathscr{E})$ and there exists a continuous function $\Delta_\mu u$ that makes (1·9) valid. For simplicity of notation we will drop the subscript μ on the Laplacian. It is possible to give a pointwise definition of $\Delta u(x)$ for points $x \in V_m$ and it is true that u is harmonic as defined above if and only if $\Delta u = 0$. A very important property of the Laplacian is the Gauss–Green formula

$$\int (u\Delta v - v\Delta u) d\mu = \sum_{x\in V_0} u(x)\partial_n v(x) - v(x)\partial_n u(x). \tag{1·10}$$

We will also use the scaling identities

$$\mathscr{E}(u,v) = \sum_{i=1}^{N} r_i^{-1} \mathscr{E}(u\circ F_i, v\circ F_i) \tag{1·11}$$

and

$$\Delta(u\circ F_i) = r_i \mu_i (\Delta u)\circ F_i \tag{1·12}$$

for the Dirichlet form and Laplacian.

When K is the unit interval, the choice

$$D = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad r = \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

yields the standard Dirichlet form (1·2). With $\mu_1 = \mu_2 = \frac{1}{2}$ the measure μ is just Lebesgue measure and Δ is just the usual second derivative. For the example of the Sierpinski gasket, we define the *standard Laplacian* by taking the harmonic structure described above and the self-similar measure with $\mu_1 = \mu_2 = \mu_3 = \frac{1}{3}$. This measure is just normalized Hausdorff measure in dimension $\log 3/\log 2$. In this case harmonic functions are characterized by the property that for non-boundary points x in V_m, $u(x)$ is just the average of $u(y)$ over the 4 neighbouring points y joined to x in Γ_m (we write $y \sim_m x$). The pointwise formula for the Laplacian is

$$\Delta u(x) = \tfrac{3}{2} \lim_{m\to\infty} 5^m \left(\sum_{y \sim_m x} u(y) - 4u(x) \right) \tag{1·13}$$

for any point x in one of the V_m. Note that we have complete dihedral-3 symmetry for this example.

Another important ingredient in the theory is an explicit formula for the Green's function. The Dirichlet problem

$$-\Delta u = f \quad \text{with } u|_{V_0} = 0 \tag{1·14}$$

has a unique solution for each continuous function f and the solution is expressible as

$$u(x) = \int G(x, y) f(y) d\mu(y) \tag{1·15}$$

for a specific continuous function $G(x, y)$ called the Green's function (the continuity of G is related to the fact that points have positive capacity). In fact G depends only on the harmonic structure, not the measure, and there is an explicit formula that will be described in more detail in Section 2.

2. *Multiharmonic functions*

We consider the general setting of a p.c.f. self-similar fractal K generated by contractions $F_i, i = 1, \ldots, N$, with boundary V_0, $\#V_0 = N_0$, a regular harmonic structure on K with Dirichlet form $\mathscr{E}$ satisfying

$$\mathscr{E}(u, v) = \sum_{i=1}^{N} r_i^{-1} \mathscr{E}(u \circ F_i, v \circ F_i) \tag{2·1}$$

and a self-similar probability measure μ satisfying

$$\mu = \sum_{i=1}^{N} \mu_j \mu \circ F_i^{-1} \tag{2·2}$$

for a set of discrete probability weights $\{\mu_i\}$. We denote the associated Laplacian simply as Δ and for each $j \geqslant 0$ we let $\mathscr{H}_j = \{f : \Delta^{j+1} f = 0\}$ be the space of $(j+1)$-harmonic functions. Then $\dim \mathscr{H}_j = (j+1)N_0$ and we describe next the 'easy' basis for $\mathscr{H}_j$:

Definition 2·1. For $j \geqslant 0$ and $1 \leqslant k \leqslant k \leqslant N_0$ define f_{jk} to be the solution of $\Delta^{j+1} f_{jk} = 0$ satisfying the boundary conditions

$$\Delta^m f_{jk}(v_n) = \delta_{mj}\delta_{kn} \quad \text{for} \quad 1 \leqslant n \leqslant N_0 \quad \text{and} \quad m \geqslant 0. \tag{2·3}$$

Of course $\Delta^m f_{jk} = 0$ for $m > j$ so we could restrict $m \leqslant j$ in (2·3). Note that

$$f_{jk}(x) = -\int G(x,y) f_{(j-1)k}(y)\, d\mu(y) \quad \text{for} \quad j \geqslant 1 \tag{2·4}$$

since $\Delta f_{jk} = f_{(j-1)k}$ and f_{jk} vanishes on V_0. This will be the key to computing f_{jk} algorithmically. We will also need the scaling identity

$$\Delta(\varphi \circ F_i) = r_i \mu_i (\Delta\varphi) \circ F_i, \tag{2·5}$$

which follows from (2·1), (2·2) and the definition of the Laplacian, and its iteration

$$\Delta^m(\varphi \circ F_i) = (r_i\mu_i)^m (\Delta^m \varphi) \circ F_i. \tag{2·6}$$

Lemma 2·2. *$\{f_{mk}\}_{0 \leqslant m \leqslant j, 1 \leqslant k \leqslant N_0}$ is a basis for $\mathscr{H}_j$ and any $f \in \mathscr{H}_j$ has the explicit representation*

$$f = \sum_{m=0}^{j} \sum_{k=1}^{N_0} (\Delta^m f(v_k)) f_{mk}. \tag{2·7}$$

Proof. Both sides of (2·7) belong to $\mathscr{H}_j$ and give the same value for $\Delta^\ell f(v_n)$ for $0 \leqslant \ell \leqslant j, \ell \leqslant n \leqslant N_0$.

Lemma 2·3. *For all j, k, i, we have*

$$f_{jk} \circ F_i = \sum_{\ell=0}^{j} \sum_{n=1}^{N_0} (r_i\mu_i)^\ell f_{(j-\ell)k}(F_i v_n) f_{\ell n}. \tag{2·8}$$

Proof. By (2·6) we have $f_{jk} \circ F_i \in \mathscr{H}_j$, so we apply (2·7) to $f = f_{jk} \circ F_i$. To compute $\Delta^\ell f(v_n)$ we use (2·6) and the result is (2·8).

We need to be able to compute the values of $f_{jk}(F_i v_n)$, for then (2·8) will be an explicit scaling identity for the functions f_{jk}. Note that we cannot use (2·8) directly to find the required values, since evaluating (2·8) at v_n just yields a tautology. We will succeed in finding a recursive formula for these values and at the same time find a recursive solution to another important problem, that of computing inner products of the basis functions. It is important to note that these recursions must proceed simultaneously. Thus we let

$$I(jk, j'k') = \int f_{jk}\, f_{j'k'}\, d\mu. \tag{2·9}$$

We begin with the recursion formula for the Is. Although it is linear in I, it involves the values of $f_{jk}(F_i v_n)$ quadratically. Also, terms of the highest order appear on both sides of the identity.

Lemma 2·4. *For all j, k, j', k', we have*

$$I(jk, j'k') = \sum_{i=1}^{N} \sum_{\ell=0}^{j} \sum_{n=1}^{N_0} \sum_{\ell'=0}^{j'} \sum_{n'=1}^{N_0} \mu_i (r_i\mu_i)^{\ell+\ell'} f_{(j-\ell)k}(F_i v_n) f_{(j'-\ell')k}(F_i v_{n'}) I(\ell n, \ell' n'). \tag{2·10}$$

Proof. We use (2·2) on the right-hand side of (2·9) to obtain

$$I(jk, j'k') = \sum_{i=1}^{N} \mu_i \int (f_{jk} \circ F_i)(f_{j'k'} \circ F_i) d\mu.$$

We then substitute (2·8) and the result is (2·10).

When $j = j' = 0$ (2·10) takes the form

$$I(0k, 0k') = \sum_{n=1}^{N_0} \sum_{n'=1}^{N_0} A(kk', nn') I(0n, 0n') \tag{2·11}$$

for

$$A(kk', nn') = \sum_{i=1}^{N} \mu_i f_{0k}(F_i v_n) f_{0k'}(F_i v_{n'}). \tag{2·12}$$

Since the harmonic functions f_{0k} are all non-negative (2·11) says that $I(0k, 0k')$ is a non-negative eigenvector (with eigenvalue 1) for the non-negative matrix $A(kk', nn')$. *We will assume that this matrix is irreducible.* This is true under very weak assumptions on the harmonic structure and is probably true for any nondegenerate structure. With this assumption (2·11) determines $I(0n, 0n')$ up to a normalization factor and that factor is determined by the condition

$$\sum_{k=1}^{N_0} \sum_{k'=1}^{N_0} I(0k, 0k') = 1, \tag{2·13}$$

which is just the statement that a constant function integrates to the constant.

Theorem 2·5. *Assume the matrix $A(kk', nn')$ is irreducible. If the values of $f_{ml}(F_i v_n)$ are known for $m \leqslant \max(j, j')$, then* (2·10) *and* (2·13) *uniquely determine $I(jk, j'k')$.*

Proof. We have already seen how $I(0k, 0k')$ is determined. So assume $I(\ell k, \ell' k')$ is known if $\ell \leqslant j$ and $\ell' \leqslant j'$ and both inequalities are not strict. Then transporting all the highest order terms in (2·10) to the left side, we have

$$I(jk, j'k') - \sum_{n=1}^{N_0} \sum_{n'=1}^{N_0} \tilde{A}(kk', nn') I(jn, j'n')$$

expressed in terms of known values, where

$$\tilde{A}(kk', n'n') = \sum_{i=1}^{N} \mu_i (\mu_i r_i)^{j+j'} f_{0k}(F_i v_n) f_{0k'}(F_i v_{n'}).$$

Comparing this expression with (2·12), we see that the spectral radius is strictly less than 1 since $\mu_i r_i < 1$ and so the identity minus $\tilde{A}$ is an invertible matrix, hence (2·10) is solvable.

To get a recursion formula for the values of $f_{jk}(F_i v_n)$ we use (2·4) and an explicit formula for the Green's function, which we write as follows:

$$G(F_i v_n, F_{i'} y) = \sum_{n'=1}^{N_0} \gamma(i, i', n, n') f_{0n'}(y). \tag{2·14}$$

It is clear that there must be such an identity for some constants $\gamma(i,i',n,n')$ because $G(F_iv_n, F_{i'}y)$ is a harmonic function of y since the only singularity of the Green's function is on the diagonal and $F_iv_n = F_{i'}y$ would imply $y \in V_0$. But we can identify the constants exactly, namely

$$\gamma(i,i',n,n') = G_{pq} \quad \text{for} \quad p = F_iv_n, \quad q = F_{i'}v_{n'} \tag{2·15}$$

and the matrix G_{pq} is given by $G = -X^{-1}$, where X denotes the restriction to $(V_1\backslash V_0) \times (V_1\backslash V_0)$ of the Dirichlet form $\mathscr{E}_1$ on $V_1 \times V_1$. Indeed, since $F_iv_n \in V_1$, definition 5·3 in [**Ki2**] yields

$$G(F_iv_n, F_{i'}y) = \sum G_{pq}\psi_p(F_iv_n)\psi_q(F_{i'}y),$$

where ψ_p denotes the continuous piecewise harmonic function satisfying $\psi_p(q) = \delta_{pq}$ for $q \in V_1$. Thus $\psi_p(F_iv_n)$ vanishes unless $p = F_iv_n$. Similarly, $\psi_q(F_{i'}y) = 0$ unless $q = F_{i'}v_{n'}$ for some n'. But in that case $\psi_q(F_{i'}y) = f_{0n'}(y)$, and this proves (2·14) and (2·15).

In any particular example it is easy to compute the coefficients G_{pq}. It is also easy to compute the values $f_{0k}(F_iv_n)$, since f_{0k} is harmonic. Again there are explicit matrices A_i such that

$$h(F_iv_n) = \sum_{k=1}^{N_0}(A_i)_{nk}h(v_k) \tag{2·16}$$

for any harmonic function and so

$$f_{0k}(F_iv_n) = (A_i)_{nk}. \tag{2·17}$$

This is the initial step and then we use the following recursion formula:

LEMMA 2·6. *For any j,k,i,n, we have*

$$f_{jk}(F_iv_n) = -\sum_{i'=1}^{N}\sum_{\ell=0}^{j-1}\sum_{n'=1}^{N_0}\sum_{k'=1}^{N_0}\mu_{i'}(r_{i'}\mu_{i'})^{\ell}\gamma(i,i',n,n')I(\ell k',0n')f_{(j-1-\ell)k}(F_{i'}v_{k'}). \tag{2·18}$$

Proof. We use (2·4) with $x = F_iv_n$ and (2·2) to obtain

$$f_{jk}(F_iv_n) = -\sum_{i'=1}^{N}\mu_{i'}\int G(F_iv_n, F_{i'}y)f_{(j-1)k}\circ F_{i'}(y)\,d\mu(y).$$

We substitute (2·8) for $f_{(j-1)k}\circ F_{i'}$ and (2·14) for $G(F_iv_n, F_{i'}y)$ and then do the integrals to obtain (2·18).

THEOREM 2·7. *The values of $f_{jk}(F_iv_n)$ and $I(jk,0k')$ for any j may be successively computed* (*based on the values for lower j*) *by first using* (2·18) *and then* (2·10).

Proof. The values of ℓ in $I(\ell k',0n')$ and $j-1-\ell$ in $f_{(j-1-\ell)k}(F_{i'}v_{k'})$ that appear on the right side of (2·18) are less than j, so these terms will have already been computed. Thus (2·18) is an explicit formula for $f_{jk}(F_iv_n)$. Then we have all the information required in Theorem 2·5 to compute $I(jk,0k')$ using (2·10).

There is a simple identity for computing all the inner products in terms of inner products with harmonic functions.

Theorem 2·8. *For all j, k, j', k', we have*

$$I(jk, j'k') = I((j + j')k, 0k'). \tag{2·19}$$

Proof. We apply the Gauss–Green formula to the functions $f_{(j+1)k}$ and $f_{j'k'}$ for $j' > 0$:

$$\int (f_{j'k'}\Delta f_{(j+1)k} - f_{(j+1)k}\Delta f_{j'k'})\, d\mu = 0$$

since $f_{(j+1)k}$ and $f_{j'k'}$ vanish on the boundary. But $\Delta f_{(j+1)k} = f_{jk}$ and $\Delta f_{j'k'} = f_{(j'-1)k'}$. Thus $I(jk, j'k') = I((j+1)k, (j'-1)k')$. Then (2·19) follows by induction.

The inner products are also useful for computing normal derivatives. We are grateful to Teplyaev for the following result.

Theorem 2·9. *For every j, k, ℓ, m, with $\ell < j$ we have*

$$\partial_n(\Delta^\ell f_{jk})(v_m) = I((j - 1 - \ell)k, 0m). \tag{2·20}$$

Proof. We apply the Gauss–Green formula to the functions $\Delta^\ell f_{jk}$ and f_{0m} to obtain

$$\int (f_{0m}\Delta^{\ell+1} f_{jk} - (\Delta f_{0m})\Delta^\ell f_{jk}) d\mu = \sum_{m'=1}^{N_0} (f_{0m}(v_{m'})\partial_n \Delta^\ell f_{jk}(v_{m'}) - \partial_n f_{0m}(v_{m'})\Delta^\ell f_{jk}(v_{m'})). \tag{2·21}$$

However $\Delta f_{0m} = 0$ and $\Delta^{\ell+1} f_{jk} = f_{(j-1-\ell)k}$ so the left-hand side of (2·21) is just $I((j-1-\ell)k, 0m)$. On the other hand, $\Delta^\ell f_{jk}(v_{m'}) = 0$ since $\ell < j$, and $f_{0m}(v_{m'}) = \delta_{m,m'}$, so the right-hand side of (2·21) is just $\partial_n \Delta^\ell f_{jk}(v_m)$.

Now that we know how to compute the values of the basis functions on points of V_1, we can use (2·8) inductively to compute the values on V_m and by (2·7) to values of any function in $\mathscr{H}_j$ on V_m. Note that this is a local computation: to get the values at $F_w V_0$ for any word $w = (w_1, \ldots, w_m)$ we only need to compute values at $F_{w_1} V_0, F_{w_1} F_{w_2} V_0, \ldots, F_{w_1} \cdots F_{w_{m-1}} V_0$.

3. *A better basis*

In order to combine piecewise multiharmonic functions into splines we need to be able to match normal derivatives at nodes. The easy basis for $\mathscr{H}_j$ will not allow us to control normal derivatives and so is inadequate for the purpose. We are thus led to the problem of constructing a better basis for $\mathscr{H}_j$ involving normal derivatives. The method we use will work for any odd value of j. To avoid cumbersome notation we just present the cases $j = 1$ and $j = 3$.

The easy basis for $\mathscr{H}_1$ consists of the $2N_0$ functions f_{0k} and f_{1k}, which satisfy the boundary conditions

$$f_{0k}(v_m) = \delta_{k,m}, \quad f_{1k}(v_m) = 0, \tag{3·1}$$

$$\Delta f_{0k}(v_m) = 0, \quad \Delta f_{1k}(v_m) = \delta_{k,m}. \tag{3·2}$$

The better basis we will construct will consist of the $2N_0$ functions $f_{0k}^{(1)}$ and $g_{0k}^{(1)}$ and they will satisfy the boundary conditions

$$f_{0k}^{(1)}(v_m) = \delta_{k,m}, \quad g_{0k}^{(1)} v_m = 0, \tag{3·3}$$

$$\partial_n f_{0k}^{(1)}(v_m) = 0, \quad \partial_n g_{0k}^{(1)}(v_m) = \delta_{k,m}. \tag{3·4}$$

Each function in the new basis must be a linear combination of functions in the old basis and in view of (3·1) and (3·3) we must have

$$\left.\begin{aligned} f_{0k}^{(1)} &= f_{0k} + \sum_{\ell=1}^{N_0} b_{k\ell} f_{1\ell} \\ g_{0k}^{(1)} &= \sum_{\ell=1}^{N} d_{k\ell} f_{1\ell}. \end{aligned}\right\} \tag{3·5}$$

This gives (3·3) and to verify (3·4) we need, using Theorem 2·9,

$$\left.\begin{aligned} &\partial_n f_{0k}(v_m) + \sum_{\ell=1}^{N_0} b_{k\ell} I(0\ell, 0m) = 0 \\ &\sum_{\ell=1}^{N_0} d_{k\ell} I(0\ell, 0m) = \delta_{k,m}. \end{aligned}\right\} \tag{3·6}$$

Now the $N_0 \times N_0$ matrix $I(0\ell, 0m)$ is invertible, since it is the matrix of inner products of the easy basis of $\mathscr{H}_0$. Let J denote its inverse. Then

$$\left.\begin{aligned} d_{k\ell} &= J_{k\ell} \\ b_{k\ell} &= -\sum_{m=1}^{N_0} J_{m\ell} \partial_n f_{0k}(v_m) \end{aligned}\right\} \tag{3·7}$$

solves (3·6) and completes the construction of the new basis. The values of the normal derivatives of f_{0k} are easily computed, since these are harmonic functions so it is not necessary to take the limit in the definition. In fact

$$\partial_n f_{0k}(v_m) = -H_{mk}, \tag{3·8}$$

where $\{H_{mk}\}$ denotes the Dirichlet form $\mathscr{E}_0$ on $V_0 \times V_0$.

Next we consider the case $j = 3$. The easy basis of $\mathscr{H}_3$ consists of the $4N_0$ functions $f_{0k}, f_{1k}, f_{2k}, f_{3k}$ satisfying

$$\Delta^\ell f_{nk}(v_m) = \delta_{\ell,n}\delta_{k,m} \quad \text{for} \quad 0 \leqslant \ell \leqslant 3. \tag{3·9}$$

The better basis will consist of the $4N_0$ functions $f_{0k}^{(3)}, f_{1k}^{(3)}, g_{0k}^{(3)}, g_{1k}^{(3)}$ satisfying

$$\Delta^\ell f_{nk}^{(3)}(v_m) = \delta_{\ell,n}\delta_{k,m}, \quad \Delta^\ell g_{nk}^{(3)}(v_m) = 0, \tag{3·10}$$

$$\partial_n \Delta^\ell f_{nk}^{(3)}(v_m) = 0, \quad \partial_n \Delta^\ell g_{nk}^{(3)}(v_m) = \delta_{l,n}\,\delta_{k,m}, \tag{3·11}$$

for $\ell = 0, 1$. Expressing the new basis in terms of the old basis, and taking into account just (3·10), we find

$$\left.\begin{aligned} f^{(3)}_{0k} &= f_{0k} + \sum_{\ell=1}^{N_0}(a_{k\ell}f_{2\ell} + b_{k\ell}f_{3\ell}) \\ f^{(3)}_{1k} &= f_{1k} + \sum_{\ell=1}^{N_0}(c_{k\ell}f_{2\ell} + d_{k\ell}f_{3\ell}) \\ g^{(3)}_{0k} &= \sum_{\ell=1}^{N_0}(a'_{k\ell}f_{2\ell} + b'_{k\ell}f_{3\ell}) \\ g^{(3)}_{1k} &= \sum_{\ell=1}^{N_0}(c'_{k\ell}f_{2\ell} + d'_{k\ell}f_{3\ell}) \end{aligned}\right\} \tag{3·12}$$

for certain coefficients to be determined. Now we impose the conditions (3·11), using Theorems 2·8 and 2·9 to obtain

$$\left.\begin{aligned} &\sum_{\ell=1}^{N_0}(a_{k\ell}I(0\ell,0m) + b_{k\ell}I(1\ell,0m)) = 0 \\ &\sum_{\ell=1}^{N_0}(a_{k\ell}I(1\ell,0m) + b_{k\ell}I(1\ell,1m)) = -\partial_n f_{0k}(v_m), \end{aligned}\right\} \tag{3·13}$$

$$\left.\begin{aligned} &\sum_{\ell=1}^{N_0}(c_{k\ell}I(0\ell,0m) + d_{k\ell}I(1\ell,0m)) = -\partial_n f_{0k}(v_m) \\ &\sum_{\ell=1}^{N_0}(c_{k\ell}I(1\ell,0m) + d_{k\ell}I(1\ell,1m)) = -I(0k,0m), \end{aligned}\right\} \tag{3·14}$$

$$\left.\begin{aligned} &\sum_{\ell=1}^{N_0}(a'_{k\ell}I(0\ell,0m) + b'_{k\ell}I(1\ell,0m)) = 0 \\ &\sum_{\ell=1}^{N_0}(a'_{k\ell}I(1\ell,0m) + b'_{k\ell}I(1\ell,1m)) = \delta_{k,m}, \end{aligned}\right\} \tag{3·15}$$

$$\left.\begin{aligned} &\sum_{\ell=1}^{N_0}(c'_{k\ell}I(0\ell,0m) + d'_{k\ell}I(1\ell,0m)) = \delta_{k,m} \\ &\sum_{\ell=1}^{N_0}(c'_{k\ell}I(1\ell,0m) + d'_{k\ell}I(1\ell,1m)) = 0. \end{aligned}\right\} \tag{3·16}$$

Note that each of these 4 systems of $2N_0$ equations in $2N_0$ unknowns involves the $2N_0 \times 2N_0$ matrix of inner products for the easy basis for $\mathscr{H}_1$. Thus the systems are uniquely solvable and this gives the new basis.

In general we can find a basis for $\mathscr{H}_j$ for j odd consisting of functions $f^{(j)}_{nk}$ and $g^{(j)}_{nk}$ for $n \leqslant (j-1)/2$ satisfying

$$\Delta^\ell f^{(j)}_{nk}(v_m) = \delta_{\ell,n}\delta_{k,m}, \quad \Delta^\ell g^{(j)}_{nk}(v_m) = 0, \tag{3·17}$$

$$\partial_n\Delta^\ell f^{(j)}_{nk}(v_m) = 0, \quad \partial_n\Delta^\ell g^{(j)}_{nk}(v_m) = \delta_{\ell,n}\delta_{k,m} \tag{3·18}$$

for $\ell \leqslant (j-1)/2$. This leads to systems of equations that are solvable because the matrix involved is the matrix of inner products for the easy basis for $\mathscr{H}_{(j-1)/2}$. We omit the details. The situation for j even is less clear. We want a basis with $f_{nk}^{(j)}$ for $n \leqslant j/2$ and $g_{nk}^{(j)}$ for $n < j/2$, with (3·17) holding for $n \leqslant j/2$ and (3·18) holding for $n < j/2$. We consider the case $j = 2$. Then

$$\left.\begin{aligned} f_{0k}^{(2)} &= f_{0k} + \sum_{\ell=1}^{N_0} a_{k\ell} f_{2\ell} \\ f_{1k}^{(2)} &= f_{1k} + \sum_{\ell=1}^{N_0} b_{k\ell} f_{2\ell} \\ g_{0k}^{(2)} &= \sum_{\ell=1}^{N_0} c_{k\ell} f_{2\ell} \end{aligned}\right\} \tag{3·19}$$

from (3·17) and

$$\left.\begin{aligned} \sum_{\ell+1}^{N_0} a_{k\ell} I(1\ell, 0m) &= -\partial_n f_{0k}(v_m) \\ \sum_{\ell=1}^{N_0} b_{k\ell} I(1\ell, 0m) &= -I(0k, 0m) \\ \sum_{\ell=1}^{N_0} c_{k\ell} I(1\ell, 0m) &= \delta_{k,m} \end{aligned}\right\} \tag{3·20}$$

from (3·18). We can solve these systems provided the $N_0 \times N_0$ matrix $I(1\ell, 0m)$ is invertible. We do not know if this is true in general.

4. *Splines*

We will be dealing with weak solutions of $\Delta u = f$ and we now give precise definitions. The domain of the Dirichlet form, $\operatorname{dom}\mathscr{E}$, is defined in the usual way based on L^2. The functions in $\operatorname{dom}\mathscr{E}$ are nevertheless continuous. In [**Ki2**] the domain of the Laplacian is defined based on C, the space of continuous functions. We will write $\operatorname{dom}_C(\Delta)$ to emphasize this.

Definition 4·1. We say $u \in \operatorname{dom}_C(\Delta)$ and $\Delta u = f$ if $u \in \operatorname{dom}\mathscr{E}$, $f \in C(K)$ and

$$\mathscr{E}(u, v) = -\int f v \, d\mu \tag{4·1}$$

for every $v \in \operatorname{dom}\mathscr{E}$ vanishing on the boundary. We say $u \in \operatorname{dom}_{\mathscr{M}}(\Delta)$ and $\Delta u = \nu$ if $u \in \operatorname{dom}(\mathscr{E})$, $\nu \in \mathscr{M}(K)$, the space of finite measures on K, and

$$\mathscr{E}(u, v) = -\int v \, d\nu \tag{4·2}$$

for every $v \in \operatorname{dom}\mathscr{E}$ vanishing on the boudary. Similarly, $\operatorname{dom}_B(\Delta)$ is defined for any Banach space B intermediate between C and $\mathscr{M}$. For $j > 1$ we define $f \in \operatorname{dom}_B(\Delta^j)$ inductively by $f \in \operatorname{dom}_B(\Delta)$ and $\Delta f \in \operatorname{dom}_B(\Delta^{j-1})$.

In [**Ki2**] the domain $\mathrm{dom}_C(\Delta)$ is first defined by the uniform convergence of a difference quotient and the above definition appears as a theorem. It should be a relatively routine matter to establish equivalent difference quotient characterizations of the domains $\mathrm{dom}_B(\Delta)$ as well, but we will not do this here. A more subtle question is the extent that the Gauss–Green formula holds for these weak domains. We also note that (4·2) does not determine the measure ν uniquely, since atoms on the boundary will not make a difference. We do not want to rule out atoms altogether since $G(\cdot, y)$ will belong to $\mathrm{dom}_{\mathscr{M}}(\Delta)$ with $\Delta G(\cdot, y) = \delta_y$.

Next we consider the localization of the Laplacian to a cell $K_w = F_w K$ for $w = (w_1, \ldots, w_m)$ any word. We do this simply by taking the composition with F_w and referring all questions back to the global setting, scaling using (2·5).

Definition 4·2. We say $u \in \mathrm{Dom}_B(\Delta\big|_{F_w K})$ and $\Delta u = f$ on $F_w K$ if $u \circ F_w \in \mathrm{Dom}_B(\Delta)$ and

$$\Delta(u \circ F_w) = r_w \mu_w f \circ F_w. \tag{4·3}$$

This definition is not as straightforward as it seems since it treats all the points in $F_w V_0$ as boundary points of $F_w K$. Thus there may be functions which are harmonic on $F_w K$ which do not satisfy the pointwise condition for being harmonic at points in $F_w V_0$ which are neither boundary points of K nor junction points (for the Sierpinski gasket such points are nonexistent).

We also need to localize the notion of normal derivative to $F_w K$. For each boundary point $F_w v_k$ we define

$$\partial_n u(F_w v_k) = r_w^{-1} \partial_n (u \circ F_w)(v_k). \tag{4·4}$$

The same point may be represented $F_w v_k$ in more than one way and so there are different normal derivatives associated to each such representation. When this happens we call such a point a *junction point*. For each junction point x we denote by $J_m(x)$ the set of all pairs (w, k) where w is a word of length m and $x = F_w v_k$.

We are now ready to define the space of splines $S(\mathscr{H}_j, V_m)$ based on $\mathscr{H}_j$ of level m. These will be functions that belong to $\mathscr{H}_j$ when restricted to $F_w K$ for all words of length m, and satisfy appropriate matching conditions at junction points.

Definition 4·3. We say $f \in S(\mathscr{H}_j, V_m)$ if $f \circ F_w \in \mathscr{H}_j$ for all words with $|w| = m$ and for all junction points x in V_m the following matching conditions hold:

$$(r_w \mu_w)^{-\ell} \Delta^\ell (f \circ F_w)(v_k) \tag{4·5}$$

is the same for all $(w, k) \in J_m(x)$, for each $\ell \leqslant j/2$ and

$$\sum_{(w,k) \in J_m(x)} r_w^{-1} (r_w \mu_w)^{-\ell} \partial_n \Delta^\ell (f \circ F_w)(v_k) = 0 \tag{4·6}$$

for each $\ell < j/2$.

Note that (4·5) just says that a unique value for $\Delta^\ell f(x)$ exists at a junction point and (4·6) says that the sum of all the normal derivatives of $\Delta^\ell f$ at x is zero. These conditions suffice to obtain a certain order of 'smoothness' for f, except at points in V_m that are not junction or boundary points.

Theorem 4·4. *Let $f \in S(\mathscr{H}_j, V_m)$ and suppose that for any $x \in V_m$ that is not a* *junction or boundary point we have*

$$\partial_n \Delta^\ell f(x) = 0 \quad \textit{for all} \quad \ell < j/2. \tag{4·7}$$

Then for j odd we have $f \in \mathrm{dom}_{L,\infty}(\Delta^{(j+1)/2})$, while for j even $f \in \mathrm{dom}_C(\Delta^{j/2})$.

Proof. For each w with $|w| = m$, the restriction of f to F_wK is in $\mathrm{dom}_C(\Delta^\ell|_{F_wK})$ for every ℓ. We can then create a function f_ℓ on K by piecing together $\Delta^\ell f$ on each F_wK. By (4·5) this will be a continuous function for $\ell \leqslant j/2$, but it will only be L^∞ when $\ell = (j+1)/2$ for j odd. It remains to show $\Delta^\ell f = f_\ell$, which by Definition 4·1 means

$$\mathscr{E}(f_\ell, v) = -\int f_{\ell+1} v d\mu \tag{4·8}$$

for all $v \in \mathrm{dom}\,\mathscr{E}$ vanishing on V_0, for $\ell < j/2$, where $f_0 = f$. Now

$$\int f_{\ell+1} v d\mu = \sum_{|w|=m} \mu_w \int (f_{\ell+1} \circ F_w)(v \circ F_w) d\mu \tag{4·9}$$

by (2·2) and

$$\mathscr{E}(f_\ell, v) = \sum_{|w|=m} r_w^{-1} \mathscr{E}(f_\ell \circ F_w, v \circ F_w) \tag{4·10}$$

by (2·1). Since $\Delta f_\ell = f_{\ell+1}$ on F_wK, we have

$$\left.\begin{aligned} \mathscr{E}(f_\ell \circ F_w, v \circ F_w) &= -\int (\Delta(f_\ell \circ F_w)) v \circ F_w d\mu \\ &\quad + \sum_{k=1}^{N_0} v \circ F_w(x_k) \partial_n (f_\ell \circ F_w)(x_k) \\ &= -\mu_w r_w \int (f_{\ell+1} \circ F_w)(v \circ F_w) d\mu \\ &\quad + (\mu_w r_w)^{-\ell} \sum_{k=1}^{N_0} v(F_w x_k) \partial_n \Delta^\ell (f \circ F_w)(v_k). \end{aligned}\right\} \tag{4·11}$$

We multiply (4·11) by r_w^{-1} and sum over w. Taking into account (4·9) and (4·10), we find

$$\mathscr{E}(f_\ell, v) = -\int f_{\ell+1}\, v d\mu + \sum_{|w|=m} \sum_{k=1}^{N_0} r_w^{-1} (\mu_w r_w)^{-\ell} v(F_w x_k) \partial_n \Delta^\ell (f \circ F_w)(v_k).$$

Thus it suffices to show that the sum vanishes. If $F_w x_k \in V_0$ then v vanishes. If $F_w x_k$ is neither a boundary or a junction point then $\partial_n \Delta^\ell (f \circ F_w)(v_k) = 0$ by (4·7). Thus there remain only terms involving junction points. We can rearrange the sum to vary over $(w, k) \in J_m(x)$ for each junction point x. The value of $v(F_w x_k)$ is the same, $v(x)$, for each $(w, k) \in J_m(x)$, so we can factor this out and we are left with the sum (4·6) which vanishes.

THEOREM 4·5. *Under the hypotheses of Theorem 2·5, the space $S(\mathscr{H}_j, V_m)$ for j odd has dimension*

$$(1 + [j/2])(\#V_m) + [(j+1)/2](N^m N_0 - \#J_m), \tag{4·12}$$

where J_m denotes the set of junction points in V_m and each element of $S(\mathscr{H}_j, V_m)$ is uniquely determined by specifying

$$\Delta^\ell f(x) \quad \textit{for} \quad x \in V_m \quad \textit{and} \quad \ell \leqslant j/2 \tag{4·13}$$

and

$$\partial_n \Delta^\ell (f \circ F_w)(v_k) \quad \textit{for} \quad |w| = m, \quad v_k \in V_0 \quad \textit{and} \quad \ell < j/2, \tag{4·14}$$

subject to the conditions (4·6) *for all $x \in J_m$ and $\ell < j/2$. The same result holds for j even in the cases when the construction in Section* 3 *can be carried out.*

Proof. From the construction in Section 3 and Definition 4·3 it is clear that $f \in S(\mathscr{H}_j, V_m)$ is uniquely specified by the data (4·13) and (4·14) subject to the conditions (4·6), since (4·5) is exactly the condition that $\Delta^\ell f(x)$ depends only on x and not its particular representation $F_w v_k$. So it remains to verify the dimension formula (4·12). The data (4·13) involves $1+[j/2]$ choices of ℓ and $\#V_m$ choices of x and so contributes $(1 + [j/2])(\#V_m)$ to the dimension. The data (4·14) involves $[(j + 1)/2]$ choices of ℓ, N^m choices of w and N_0 choices of k, while the number of conditions of the form (4·6) is $[(j + 1)/2](\#J_m)$.

It is clear how to construct a basis for $S(\mathscr{H}_j, V_m)$ by localizing the basis for $\mathscr{H}_j$ constructed in Section 3, but there are many ways to incorporate the matching conditions (4·6). We will not give a description in the general setting to avoid a notational thicket of questionable value. In the next section we give an explicit construction for the case of the Sierpinski gasket.

Now we establish the basic approximation properties of the spline spaces in the energy norm $\mathscr{E}(u, u)^{\frac{1}{2}}$. We know that functions in dom$(\mathscr{E})$ are continuous and we have the basic estimate

$$|u(x) - u(y)| \leqslant c\mathscr{E}(u, u)^{\frac{1}{2}} \tag{4·15}$$

for any x, y. In particular

$$\|u\|_\infty \leqslant c\mathscr{E}(u, u)^{\frac{1}{2}} \quad \text{if } u|_{\partial K} = 0. \tag{4·16}$$

In what follows we will also use the weaker estimate

$$\|u\|_2 \leqslant c\mathscr{E}(u, u)^{\frac{1}{2}} \quad \text{if } u|_{\partial K} = 0. \tag{4·17}$$

The significance of (4·16) is that any estimate for the energy norm implies the same estimate in the uniform norm. We first establish global estimates and then scale them down to get spline approximation estimates.

Lemma 4·6. *If $u \in \mathrm{dom}_{L^2}\Delta$ and $u|_{\partial K} = 0$, then*

$$\mathscr{E}(u, u)^{\frac{1}{2}} \leqslant c\|\Delta u\|_2, \tag{4·18}$$

with the same constant in (4·18) *as* (4·17).

Proof. By the definition of $\mathrm{dom}_{L^2}\Delta$ we have $u \in \mathrm{dom}\,(\mathscr{E})$, so $\mathscr{E}(u, u)$ is finite. Now $\mathscr{E}(u, u) = -\int u\Delta u d\mu$ by (4·1), since $u|_{\partial K} = 0$. Thus $\mathscr{E}(u, u) \leqslant \|u\|_2\|\Delta u\|_2$ by Cauchy–Schwartz and substituting (4·17) yields

$$\mathscr{E}(u, u) \leqslant c\mathscr{E}(u, u)^{\frac{1}{2}}\|\Delta u\|_2$$

and (4·18) follows.

THEOREM 4·7. *Suppose* $u \in \mathrm{dom}_{L^2}\Delta^{2^n}$ *and* $\Delta^\ell u\big|_{\partial K} = 0$ *and* $\partial_n \Delta^\ell u\big|_{\partial K} = 0$ *for all* $\ell \leqslant 2^{n-1} - 1$. *Then*

$$\mathscr{E}(u,u)^{\frac{1}{2}} \leqslant c_n \|\Delta^{2^n} u\|_2. \tag{4·19}$$

Proof. Consider first the case $n = 1$. Since $u|_{\partial k} = 0$ and $\partial_n u|_{\partial K} = 0$ we have

$$\int (\Delta u)^2 d\mu = \int u \Delta^2 u d\mu$$

by the Gauss–Green formula. Thus

$$\|\Delta u\|_2^2 \leqslant \|u\|_2 \|\Delta^2 u\|_2$$

by Cauchy–Schwartz. We substitute (4·17) and (4·18) to obtain

$$c^{-2}\mathscr{E}(u,u) \leqslant \|\Delta u\|_2^2 \leqslant c\mathscr{E}(u,u)^{\frac{1}{2}} \|\Delta^2 u\|_2$$

and so (4·19) follows with $c_1 = c^3$.

We prove the general case by induction, so suppose (4·18) holds for $n-1$. Then

$$\int (\Delta^{2^{n-1}} u)^2 d\mu = \int u \Delta^{2^n} u d\mu$$

follows by applying the Gauss–Green formula 2^{n-1} times. There are no boundary terms because of the vanishing of $\Delta^\ell u$ and $\partial_n \Delta^\ell u$ on the boundary for $\ell \leqslant 2^{n-1} - 1$. We obtain

$$\|\Delta^{2^{n-1}} u\|_2^2 \leqslant \|u\|_2 \|\Delta^{2^n} u\|_2$$

by Cauchy–Schwartz and then substitute (4·17) and the induction hypothesis to obtain

$$c_{n-1}^{-2}\mathscr{E}(u,u) \leqslant \|\Delta^{2^{n-1}} u\|_2^2 \leqslant c\mathscr{E}(u,u)^{\frac{1}{2}} \|\Delta^{2^n} u\|_2$$

which yields (4·19) with $c_n = c c_{n-1}^2$.

Remark. It is reasonable to conjecture that analogous results hold for integers not necessarily powers of 2. Thus if $\Delta^\ell u\big|_{\partial K} = 0$ and $\partial_n \Delta^\ell u|_{\partial K} = 0$ for all $\ell \leqslant k$ we should have

$$\mathscr{E}(u,u)^{\frac{1}{2}} \leqslant c_k' \|\Delta^{2k} u\|_2.$$

Similarly, if $\Delta^\ell u|_{\partial K} = 0$ for all $\ell \leqslant k+1$ and $\partial_n \Delta^\ell u|_{\partial K} = 0$ for all $\ell \leqslant k$, then we should have

$$\mathscr{E}(u,u)^{\frac{1}{2}} \leqslant c_k'' \|\Delta^{2k+1} u\|_2.$$

THEOREM 4·8. *For* $j = 2^n - 1$ *there exists a constant* C_j *such that for any* $u \in \mathrm{dom}_{L^2}(\Delta^{j+1})$ *and any* m *there exists* $u_m \in S(\mathscr{H}_j, V_m)$ *with*

$$\mathscr{E}(u - u_m, u - u_m)^{\frac{1}{2}} \leqslant C_j \|\Delta^{j+1} u\|_2 \rho^{(j+\frac{1}{2})m}, \tag{4·20}$$

where $\rho = \max\{r_i \mu_i : 1 \leqslant i \leqslant N\}$. *In fact,* u_m *may be taken to be the spline that interpolates* u *on* V_m, *meaning* $\Delta^\ell u_m(x) = \Delta^\ell u(x)$ *and* $\partial_n \Delta^\ell u_m(x) = \partial_n \Delta^\ell u(x)$ *for all* $\ell \leqslant (j-1)/2$ *and all* $x \in V_m$ (*the equality for normal derivatives refers to all the different normal derivatives at* x).

Proof. For each word w of length m, we consider $(u - u_m) \circ F_w$. When u_m is the interpolating spline, this function satisfies the hypothesis of Theorem 4·7. (Note that for $n \geqslant 1$ we have j odd, so the existence of the interpolating splines follows from Theorem 4·5, while for $j = 0$ it is easy.) So (4·19) yields

$$\begin{aligned}\mathscr{E}((u-u_m)\circ F_w,(u-u_m)\circ F_w) &\leqslant c_n^2\|\Delta^{2^n}((u-u_m)\circ F_w)\|_2^2\\ &= c_n^2\|\Delta^{2^n}(u\circ F_w)\|_2^2\end{aligned}$$

since $\Delta^{2^n}(u_m\circ F_w) = 0$. Using the self-similarity identity (2·1) for $\mathscr{E}$ and (2·6) for Δ we obtain

$$\begin{aligned}\mathscr{E}(u-u_m,u-u_m) &= \sum_{|w|=m} r_w^{-1}\mathscr{E}((u-u_m)\circ F_w,(u-u_m)\circ F_w)\\ &\leqslant c_n^2\sum_{|w|=m} r_w^{-1}\|\Delta^{2^n}(u\circ F_w)\|_2^2\\ &= c_n^2\sum_{|w|=m} r_w^{-1}(r_w\mu_w)^{2^{n+1}}\|(\Delta^{2^n}u)\circ F_w\|_2^2.\end{aligned}$$

But $r_w^{-1}(r_w\mu_w)^{2^{n+1}} = (r_w\mu_w)^{2^{n+1}-1}\mu_w \leqslant \rho^{(2^{n+1}-1)m}\mu_w$. Thus

$$\begin{aligned}\mathscr{E}(u-u_m,u-u_m) &\leqslant c_n^2\rho^{(2^{n+1}-1)m}\sum_{|w|=m}\mu_w\|(\Delta^{2^n}u)\circ F_w\|_2^2\\ &= c_n^2\rho^{(2^{n+1}-1)m}\|\Delta^{2^n}u\|_2^2\end{aligned}$$

by (2·2) and this yields (4·20).

Corollary 4·9. *Assume the hypotheses of Theorem* 4·8 *and* $u \in \mathrm{dom}_{L^\infty}(\Delta^{j+1})$. *Then*

$$\|u-u_m\|_\infty \leqslant C_j\|\Delta^{j+1}u\|_\infty\rho^{(j+1)m}. \qquad (4·21)$$

Proof. Every point x belongs to some F_wK. Then

$$|u(x)-u_m(x)| \leqslant c\mathscr{E}((u-u_m)\circ F_w,(u-u_m)\circ F_w)^{\frac{1}{2}}$$

by (4·16), since $(u-u_m)\circ F_w$ vanishes on ∂K. But the proof of Theorem 4·8 gives

$$\mathscr{E}((u-u_m)\circ F_w,(u-u_m)\circ F_w) \leqslant c_n^2(r_w\mu_w)^{2^{n+1}}\|(\Delta^{2^n}u)\circ F_w\|_2^2$$

and now we use $\|(\Delta^{2^n}u)\circ F_w\|_2 \leqslant \|\Delta^{2^n}u\|_\infty$ to obtain

$$|u(x)-u_m(x)| \leqslant cc_n(r_w\mu_w)^{2^n}\|\Delta^{2^n}\|_\infty \qquad (4·22)$$

for $x \in F_wK$. This yields (4·21) and in fact gives a more precise estimate when not all the values of $r_i\mu_i$ are the same.

5. *The Sierpinski gasket*

We now describe explicitly the algorithms of the previous sections for the case of the standard Laplacian on SG. Because of the high degree of symmetry, there is much simplification. We have $\mu_i = \frac{1}{3}$ and $r_i = \frac{3}{5}$ for all i,

$$\left.\begin{aligned} &A_1 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ 0 & 1 & 0 \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{pmatrix}, \quad A_3 = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ 0 & 0 & 1 \end{pmatrix}, \\ &G_{pq} = \begin{cases} \frac{9}{50} & \text{if } p = q \\ \frac{3}{50} & \text{if } p \neq q. \end{cases} \end{aligned}\right\} \tag{5·1}$$

Because of the symmetry we only have to determine 4 types of quantities:

$$\left.\begin{aligned} a_\ell &= I(\ell k, 0k) \\ b_\ell &= I(\ell k, 0n) \quad \text{for } k \neq n \\ p_\ell &= 5^\ell f_{\ell k}(F_i v_k) = 5^\ell f_{\ell k}(F_k v_i), \quad i \neq k \\ q_\ell &= 5^\ell f_{\ell k}(F_i v_n)\ i, k, n \quad \text{distinct.} \end{aligned}\right\} \tag{5·2}$$

The initial values are

$$a_0 = \tfrac{7}{45}, \quad b_0 = \tfrac{4}{45}, \quad p_0 = \tfrac{2}{5}, \quad q_0 = \tfrac{1}{5}. \tag{5·3}$$

The recursion relations (2·10) for a_ℓ and b_ℓ are

$$\left.\begin{aligned} 5^j a_j &= \tfrac{43}{75} a_j + \tfrac{56}{75} b_j + \sum_{\ell=0}^{j-1} \tfrac{2}{15}(4p_{j-\ell} + q_{j-\ell})(a_\ell + 2b_\ell) \\ 5^j b_j &= \tfrac{16}{75} a_j + \tfrac{47}{45} b_j + \sum_{\ell=0}^{j-1} \tfrac{2}{15}(3p_{j-\ell} + 2q_{j-\ell})(a_\ell + 2b_\ell). \end{aligned}\right\} \tag{5·4}$$

These can be simplified if we express them in terms of $a_j + 2b_j$ and $-4a_j + 7b_j$:

$$\left.\begin{aligned} a_j + 2b_j &= \frac{2}{3(5^j - 1)} \sum_{\ell=0}^{j-1} (2p_{j-\ell} + q_{j-\ell})(a_\ell + 2b_\ell) \\ -4a_j + 7b_j &= \frac{10}{3(5^{j+1} - 1)} \sum_{\ell=0}^{j-1} (p_{j-\ell} + 2q_{j-\ell})(a_\ell + 2b_\ell). \end{aligned}\right\} \tag{5·5}$$

Note that $a_j + 2b_j$ is just equal to the integral of f_{jk}, since $f_{00} + f_{01} + f_{02} = 1$. The recursion relations (2·18) for p_ℓ and q_ℓ are

$$\left.\begin{aligned} p_j &= -\tfrac{2}{5} b_{j-1} - \tfrac{1}{5} \sum_{\ell=0}^{j-1} (4a_{j-1-\ell} + 3b_{j-1-\ell}) p_\ell + (a_{j-1-\ell} + 2b_{j-1-\ell}) q_\ell \\ q_j &= -\tfrac{1}{5} b_{j-1} - \tfrac{1}{5} \sum_{\ell=0}^{j-1} (2a_{j-1-\ell} + 4b_{j-1-\ell}) p_\ell + (3a_{j-1-\ell} + b_{j-1-\ell}) q_\ell. \end{aligned}\right\} \tag{5·6}$$

Table 5·1. *The values of p_j, q_j, a_j, b_j for $0 \leqslant j \leqslant 6$*

j	p_j	q_j	a_j	b_j
0	0.4	0.2	0.1555...	0.0888...
1	−0.12	−0.09333...	$-6.7489711 \times 10^{-3}$	$-5.8847736 \times 10^{-3}$
2	0.034222...	0.031555...	3.7547384×10^{-4}	3.6095992×10^{-4}
3	−0.01	$-9.7530864 \times 10^{-3}$	$-2.1918331 \times 10^{-5}$	$-2.1662729 \times 10^{-5}$
4	2.9564688×10^{-3}	2.9341351×10^{-3}	1.2963031×10^{-6}	1.2917494×10^{-6}
5	$-8.7752204 \times 10^{-4}$	$-8.7551683 \times 10^{-4}$	$-7.6961127 \times 10^{-8}$	$-7.6879745 \times 10^{-8}$
6	2.6077998×10^{-4}	2.6060037×10^{-4}	4.5744047×10^{-9}	4.5729491×10^{-9}

These can be simplified by expressing them in terms of $2p_j + q_j$ and $p_j - q_j$:

$$\left.\begin{aligned} 2p_j + q_j &= -b_{j-1} - \sum_{\ell=0}^{j-1}(2a_{j-1-\ell} + 2b_{j-1-\ell})(2p_\ell + q_\ell) \\ p_j - q_j &= -\tfrac{1}{5}b_{j-1} - \tfrac{1}{5}\sum_{\ell=0}^{j-1}(2a_{j-1-\ell} - b_{j-1-\ell})(p_\ell - q_\ell). \end{aligned}\right\} \tag{5·7}$$

Table 5·1 gives the values of these constants for small values of j, obtained using Maple.

We may compute the values of any function in $\mathscr{H}_j$ efficiently by using (2·7) and (2·8) and appropriately scaled versions. For $u \in \mathscr{H}_j$ we have the following local algorithm for computing $\Delta^k u(x)$ for $k \leqslant j$ and all $x \in V_m$ in terms of the values on V_{m-I}. It is local in the sense that the values at x depend only on the values at the vertices of the level $m-1$ triangle containing x. Specifically, if we call these vertices v_0, v_1, v_2 and x is the midpoint between v_0 and v_1, then

$$\Delta^k u(x) = \sum_{\ell=k}^{j} \frac{1}{5^{m(\ell-k)}}(p_{\ell-k}(\Delta^\ell u(v_0) + \Delta^\ell u(v_1)) + q_{\ell-k}\Delta^\ell u(v_2)). \tag{5·8}$$

Next we give the explicit coefficients for the bases for $\mathscr{H}_j$ described in Section 3. Because of the symmetry, all the matrices $b_{k\ell}$, etc. have the simple form that all diagonal entries have one value and all off-diagonal entries have another value. We report these values by giving b_{00} for the diagonal and b_{01} for the off diagonal. When $j = 1$, (3·5) holds with

$$b_{00} = -30, \quad b_{01} = 15, \quad d_{00} = 11, \quad d_{01} = -4. \tag{5·9}$$

When $j = 2$, (3·19) holds with

$$\left.\begin{aligned} &a_{00} = \tfrac{16200}{7}, \quad a_{01} = -\tfrac{8100}{7}, \quad b_{00} = \tfrac{402}{7}, \quad b_{01} = -\tfrac{138}{7}, \\ &c_{00} = -\tfrac{5526}{7}, \quad c_{01} = \tfrac{2574}{7}. \end{aligned}\right\} \tag{5·10}$$

In this case the 3×3 matrix $I(1\ell, 0m)$ has a_1 on the diagonal and b_1 off the diagonal

Table 5·2. *Inner products and energies for the better basis for $j = 1$*

$\int f^1_{00}\, f^1_{00}\, d\mu = \frac{190}{837}$	$\int f^1_{00}\, g^1_{00}\, d\mu = -\frac{47}{1395}$	$\int g^1_{00}\, g^1_{00}\, d\mu = \frac{206}{37665}$
$\mathscr{E}(f^1_{00},\, f^1_{00}) = \frac{19}{6}$	$\mathscr{E}(f^1_{00},\, g^1_{00}) = -\frac{7}{18}$	$\mathscr{E}(g^1_{00},\, g^1_{00}) = \frac{5}{27}$
$\int f^1_{00}\, f^1_{01} d\mu = \frac{89}{1674}$	$\int f^1_{00}\, g^1_{01} d\mu = -\frac{61}{5580}$	$\int g^1_{00}\, g^1_{01} d\mu = \frac{83}{37665}$
$\mathscr{E}(f^1_{00},\, f^1_{01}) = -\frac{19}{12}$	$\mathscr{E}(f^1_{00},\, g^1_{01}) = \frac{7}{36}$	$\mathscr{E}(g^1_{00},\, g^1_{01}) = -\frac{1}{108}$

and is invertible. When $j = 3$, (3·12) holds with

$$\left.\begin{aligned}
a_{00} &= -\tfrac{3515400}{449}, & a_{01} &= \tfrac{1757700}{449}\\
b_{00} &= -\tfrac{271188000}{449}, & b_{01} &= \tfrac{135594000}{449}\\
c_{00} &= -\tfrac{177570}{449}, & c_{01} &= \tfrac{28170}{449}\\
d_{00} &= -\tfrac{10269180}{449}, & d_{01} &= \tfrac{4043520}{449}\\
a'_{00} &= \tfrac{1293030}{449}, & a'_{01} &= -\tfrac{464670}{449}\\
b'_{00} &= \tfrac{92578140}{449}, & b'_{01} &= -\tfrac{43015860}{449}\\
c'_{00} &= \tfrac{26864}{449}, & c'_{01} &= -\tfrac{2656}{449}\\
d'_{00} &= \tfrac{1293030}{449}, & d'_{01} &= -\tfrac{464670}{449}.
\end{aligned}\right\} \tag{5·11}$$

We do not have an explanation for the equalities $a'_{00} = d'_{00}$ and $a'_{01} = d'_{01}$. The large size of some of these coeffcients may seem alarming, but it should be kept in mind that they are multiplying functions whose values are relatively small. We have also computed the coefficients for $j = 4$, but we will not give the results here. When j is even, the systems of equations involve the matrix $I((j/2)\ell, 0m)$, which has entries $a_{(j/2)}$ on the diagonal and $b_{(j/2)}$ off the diagonal. Since the determinant of a 3×3 matrix with x on the diagonal and y off the diagonal is $x^3-3xy^2+2y^3 = (x-y)^2(x+2y)$, the matrix will be invertible unless $x = y$ or $x = -2y$. It is apparent from the values given in Table 5·1 that a_ℓ and b_ℓ are close, but presumably never equal, so the better basis for $\mathscr{H}_j$ exists for all even j. However, for large j, the computation becomes unstable because the determinant is so close to 0.

It will be useful to have the values of the inner products $\int f^{(j)}_{\ell n}\, f^{(j)}_{\ell' n'}\, d\mu$, $\int f^{(j)}_{\ell n}\, g^{(j)}_{\ell' n'} d\mu$, $\int g^{(j)}_{\ell n} g^{(j)}_{\ell' n'}\, d\mu$ and the energies $\mathscr{E}(f^{(j)}_{\ell n}, f^{(j)}_{\ell' n'})$, $\mathscr{E}(f^{(j)}_{\ell n}, g^{(j)}_{\ell n})$, $\mathscr{E}(g^{(j)}_{\ell n}, g^{(j)}_{\ell' n'})$. Clearly the values depend on whether or not $n = n'$, but not on the specific values of n, n'. The results for $j = 1$ are given in Table 5·2.

Next we describe a specific basis for the spline spaces $S(\mathscr{H}_j, V_m)$. For each vertex $y \in V_m$ we will have functions $\varphi^{(j)}_{\ell y}$ for $\ell \leqslant j/2$ and $\psi^{(j)}_{\ell y}$ for $\ell < j/2$ such that the values (4·13) and (4·14) vanish at all other points in V_m and at y exactly one of the values is 1 and the others vanish. For $\varphi^{(j)}_{\ell y}$ we have (4·14) vanish and $\Delta^k \varphi^{(j)}_{\ell y}(y) = \delta_{k\ell}$ in (4·13). For $\psi^{(j)}_{\ell y}$ we have to resolve the ambiguity in the normal derivative at y. If y is not one of the three boundary points, then $y = F_w v_n = F_{w'} v_{n'}$ for two distinct choices of words with $|w| = |w'| = m$. We make the convention that w comes before w' in lexicographic order (or $w < w'$ if we interpret them as base 3 integers). Then for any function u, the normal derivative $\partial_n u(y)$ is defined with respect to $F_w K$:

$$\partial_n u(y) = (\tfrac{5}{3})^m \partial_n (u \circ F_w)(v_n). \tag{5·12}$$

With this convention, the spline $\psi_{\ell y}^{(j)}$ will be determined by the conditions that (4·13) always vanishes and (4·14) vanishes at all other points of V_n, while $\partial_n \Delta^k \psi_{\ell y}^{(j)}(y) = \delta_{k\ell}$.

The explicit expressions for $\varphi_{\ell y}^{(j)}$ and $\psi_{\ell y}^{(j)}$ when $y = F_w V_n = F_{w'} V_{n'}$ are

$$\varphi_{\ell y}^{(j)} = \begin{cases} 5^{-m\ell} f_{\ell n}^{(j)} \circ F_w^{-1} & \text{on } F_w K \\ 5^{-m\ell} f_{\ell n'}^{(j)} \circ F_{w'}^{-1} & \text{on } F_{w'} K \\ 0 & \text{otherwise,} \end{cases} \tag{5·13}$$

$$\psi_{\ell y}^{(j)} = \begin{cases} 3^m 5^{-m(\ell+1)} g_{\ell n}^{(j)} \circ F_w^{-1} & \text{on } F_w K \\ -3^m 5^{-m(\ell+1)} g_{\ell n'}^{(j)} \circ F_{w'}^{-1} & \text{on } F_{w'} K \\ 0 & \text{otherwise.} \end{cases} \tag{5·14}$$

If $y = v_n$ is a boundary point, the expressions are slightly different:

$$\varphi_{\ell y}^{(j)} = \begin{cases} 5^{-m\ell} f_{\ell n}^{(j)} \circ F_n^{-m} & \text{on } F_n^m K \\ 0 & \text{otherwise,} \end{cases} \tag{5·15}$$

$$\psi_{\ell y}^{(j)} = \begin{cases} 3^m 5^{-m(\ell+1)} g_{\ell n}^{(j)} \circ F_n^{-m} & \text{on } F_n^m K \\ 0 & \text{otherwise.} \end{cases} \tag{5·16}$$

An arbitrary function $u \in S(\mathscr{H}_j, V_m)$ can then be written

$$u = \sum_{y \in V_m} \left(\sum_{\ell \leqslant j/2} \Delta^\ell u(y) \varphi_{\ell y}^{(j)} + \sum_{\ell < j/2} \partial_n \Delta^\ell u(y) \psi_{\ell y}^{(j)} \right). \tag{5·17}$$

It is straightforward to compute inner products and energies involving basis elements by using definitions (5·13)–(5·16), scaling properties and the inner products and energies given in Table 5·2. It is clear that we get sparse matrices because basis elements with disjoint supports will have zero inner product and energy.

As an application, we now give schemes for numerical integration analogous to the trapezoidal rule and Simpson's rule. The trapezoidal rule will provide exact values for $S(\mathscr{H}_0, V_m)$ splines, while Simpson's rule will exactly integrate $S(\mathscr{H}_1, V_{m-1})$ splines.

The trapezoidal rule is the same as the obvious choice, based on the idea that the average of the 3 values at the boundary provides the best estimate for the integral based on boundary values alone and this choice is simply scaled down to each cell of the decomposition $K = \bigcup_{|w|=m} F_w K$. This leads to the approximation for $\int f d\mu$ given by

$$I_0^m(f) = 3^{-m-1} \left(\sum_{x \in V_m \setminus V_0} 2f(x) + \sum_{x \in V_0} f(x) \right). \tag{5·18}$$

It is easy to see that this is exact for $S(\mathscr{H}_0, V_m)$ splines. It follows from Corollary

4·9 that if $f \in \mathrm{dom}_{L^\infty}(\Delta)$ then the error bound is

$$\left| I_0^m(f) - \int f d\mu \right| \leqslant c_0 5^{-m} ||\Delta f||_\infty. \tag{5·19}$$

To obtain Simpson's method we first need to find the exact integral of functions in $\mathscr{H}_1$ in terms of the values on V_1 (note that dim $\mathscr{H}_1 = \#V_1 = 6$ and it is easy to see that a function in $\mathscr{H}_1$ is uniquely determined by prescribing arbitrary values at points of V_1). By symmetry the expression must be

$$d_1 \sum_{x \in V_1 \setminus V_0} f(x) + d_2 \sum_{x \in V_0} f(x) \tag{5·20}$$

and to integrate constants we must have $3d_1 + 3d_2 = 1$. Now the function $f = f_{10} + f_{11} + f_{12}$ takes values 0 on V_0 and $2p_1 + q_1$ on $V_1 \setminus V_0$, while its integral is $3(a_1 + 2b_1)$. Thus for (5·20) to be exact we must have $3d_1(2p_1 + q_1) = 3(a_1 + 2b_1)$, so $d_1 = \frac{5}{18}$ and $d_2 = \frac{1}{18}$. We then scale this down to cells $F_w K$ with $|w| = m - 1$ and sum. Each vertex in $V_m \setminus V_{m-1}$ will appear once with weight $\frac{5}{6} \times 3^m$. Each vertex in $V_{m-1} \setminus V_0$ will appear twice, each time with weight $\frac{1}{6} \times 3^m$. Boundary vertices appear only once with weight $\frac{1}{6} \times 3^m$. Thus we set

$$I_1^m(f) = \frac{1}{6 \times 3^m} \left(5 \sum_{x \in V_m \setminus V_{m-1}} f(x) + 2 \sum_{x \in V_{m-1} \setminus V_0} f(x) + \sum_{x \in V_0} f(x) \right). \tag{5·21}$$

Theorem 5·1 (Simpson's rule). *If* $f \in \mathrm{dom}_{L^\infty}(\Delta^2)$, *then*

$$\left| I_1^m(f) - \int f d\mu \right| \leqslant c_1 5^{-2m} ||\Delta^2 f||_\infty. \tag{5·22}$$

Proof. We have already seen that $I_1^1(f)$ is exact for $f \in \mathscr{H}_1$. To show (5·22) we break the integral up into the sum over all cells $F_w K$ with $|w| = m - 1$. Fix such a cell and compare $f \circ F_w$ with the function $g_w \in \mathscr{H}_1$ that assumes the same values on the 6 points in V_1. We have $I_1^1(f \circ F_w) = I_1^1(g_w) = \int g_w d\mu$, so

$$\left| I_1^1(f \circ F_w) - \int f \circ F_w d\mu \right| = \left| \int (g_w - f \circ F_w) d\mu \right| \leqslant ||g_w - f \circ F_w||_\infty, \tag{5·23}$$

where $g_w - f \circ F_w$ vanishes on V_1. Since a function u in $\mathrm{dom}_{L^\infty} = (\Delta^2)$ vanishing on V_1 with $\Delta^2 u = 0$ must be identically zero, it follows by standard functional analysis principles that there must be an estimate of the form

$$||u||_\infty \leqslant c_1 ||\Delta^2 u||_\infty \quad if \quad u|_{V_1} = 0. \tag{5·24}$$

Combining (5·23) and (5·24) yields

$$\left| I_1^1(f \circ F_w) - \int f \circ F_w d\mu \right| \leqslant c_1 ||\Delta^2 (f \circ F_w)||_\infty. \tag{5·25}$$

Since we have

$$I_1^m(f) - \int f d\mu = \sum_{|w|=m-1} 3^{-m+1} \left(I_1^1(f \circ F_w) - \int (f \circ F_w) d\mu \right)$$

and there are 3^{m-1} terms in the sum,

$$\left| I_1^m(f) - \int f d\mu \right| \leqslant c_1 \sup_{|w|=m-1} \|\Delta^2(f \circ F_w)\|_\infty \leqslant c_1 5^{-2(m-1)} \|\Delta^2 f\|_\infty. \quad \square$$

The proof shows that Simpson's method gives the exact integral for functions in $S(\mathscr{H}_1, V_{m-1})$, and more generally for piecewise $\mathscr{H}_1$ functions that are only continuous at the V_{m-1} nodes. It would be useful to have the optimal constants in (5·19) and (5·22). We can obtain a plausible guess by assuming the maximum error occurs when $\Delta f = 1$ in (5·19) and $\Delta^2 f = 1$ in (5·22). Thus, for $f = f_{10} + f_{11} + f_{12}$ we have $I_0^0(f) = 0$ and $\int f d\mu = 3(a_1 + 2b_1) = \frac{1}{6}$ so the constant c_0 in (5·19) is at least $\frac{1}{6}$ and we conjecture this in the correct bound. Similarly, the function

$$f = f_{20} + f_{21} + f_{22} - \left(\frac{2p_2 + q_2}{2p_1 + q_1}\right)(f_{10} + f_{11} + f_{12})$$

vanishes on V_1 and $\Delta^2 f = 1$, so $I_1^1(f) = 0$ while

$$\int f d\mu = 3\left((a_2 + 2b_2) - \left(\frac{2p_2 + q_2}{2p_1 + q_1}\right)(a_1 + 2b_1)\right).$$

6. *The finite element method*

We consider a simple class of fractal differential equations,

$$-\Delta u + qu = f \tag{6·1}$$

for q and f in $C(K)$, with boundary conditions $u\big|_{V_0} = 0$. Under the assumption $q \geqslant 0$ it is easy to see that there exists a unique solution in $\mathrm{dom}_C(\Delta)$, using the theory of self-adjoint operators. There is in fact quite a difference between the two cases q constant and q nonconstant. For q constant we have a kind of hypoellipticity, in that $u \in \mathrm{dom}_C(\Delta^n)$ for any n as long as $f \in \mathrm{dom}_C(\Delta^{n-1})$. But for q nonconstant, it follows from [**BST**] that qu is never in $\mathrm{dom}_C(\Delta)$, so we will not even have $u \in \mathrm{dom}_C(\Delta^2)$ if $f \in \mathrm{dom}_C(\Delta)$.

To use the finite element method [**BS**] we incorporate the boundary conditions in the spline space. So we define $S_o(\mathscr{H}_j, V_m)$ to be the subspace of $S(\mathscr{H}_j, V_m)$ consisting of functions vanishing on V_0. The spline approximation $P_m^j u$ to the solution to (6·1) is defined to be the function in $S_0(\mathscr{H}_j, V_m)$ satisfying

$$\mathscr{E}(P_m^j u, v) + \int q v P_m^j u d\mu = \int f v d\mu \tag{6·2}$$

for all $v \in S_0(\mathscr{H}_j, V_m)$. It is easy to see that $P_m^j u$ is the orthogonal projection of u onto $S_0(\mathscr{H}_j, V_m)$ with respect to the inner product

$$\langle u, v\rangle = \mathscr{E}(u, v) + \int q u v d\mu. \tag{6·3}$$

The associated norm is equivalent to the energy norm, since $\int q u^2 d\mu \leqslant c||u||_2^2 \leqslant c\mathscr{E}(u, u)$ on the space of functions vanishing on the boundary.

We can obtain an easy estimate for the rate of convergency of $P_m^j u$ to u. Since P_m^j is an orthogonal projection we have

$$\langle u - P_m^j u, u - P_m^j u\rangle \leqslant \langle u - u_m, u - u_m\rangle$$

for any $u_m \in S_0(\mathscr{H}_j, V_m)$, in particular the interpolant in Theorem 4·8. Thus

$$\left.\begin{aligned} \|u - P_m^j u\|_\infty &\leqslant c\mathscr{E}(u - P_m^j u, u - P_m^j u)^{\frac{1}{2}} \\ &\leqslant c\mathscr{E}(u - u_m, u - u_m)^{\frac{1}{2}} \\ &\leqslant c\|\Delta^{j+1}u\|_{2\rho^{(j+\frac{1}{2})m}} \end{aligned}\right\} \tag{6·4}$$

when $j+1$ is a power of 2, if $u \in \operatorname{dom}_{L^2}(\Delta^{j+1})$. It seems plausible that the rate of convergence could be improved to $O(\rho^{(j+1)m})$ with the assumption that $u \in \operatorname{dom}_C(\Delta^{j+1})$. In any case, we are not predicted to obtain faster convergence by increasing j above 0 except when q is constant.

Another easy observation is that for the case $q = 0$ we obtain the exact solution at points of V_m for any j. In fact we know the solution is given by

$$u(x) = \int G(x, y) f(y) d\mu(y) \tag{6·5}$$

and for $x \in V_m$ the function $G(x, \cdot)$ belongs to $S_0(\mathscr{H}_0, V_m)$ hence $S_0(\mathscr{H}_j, V_m)$ for any j. In particular, choosing $v = G(x, \cdot)$ in (6·2) yields

$$\mathscr{E}(P_n^j u, G(x, \cdot)) = \int G(x, y) f(y) d\mu(y) = u(x)$$

by (6·5). On the other hand

$$\mathscr{E}(P_n^j u, G(x, \cdot)) = -\int G(x, y) \Delta P_m^j u(y) d\mu(y) = P_m^j u(x)$$

because $P_m^j u$ vanishes on the boundary. Of course in this case we can also use (6·5) to approximate the solution. Some of the computational aspects of this approach are discussed in [**KSS**].

If we write $P_m^j u$ in terms of a basis for $S_0(\mathscr{H}_j, V_m)$ then (6·2) becomes a sparse system of linear equations for the coefficients. The computation of the energy term on the left-hand side can be done theoretically and the same is true for the second term on the left-hand side if q is constant. It is necessary to use numerical integration for the right-hand side and for the second term on the left if q is not constant.

A full implementation of this method and tests of accuracy have been carried out by the first author in collaboration with M. Gibbons and A. Raj [**GRS**], and results are available at http://mathlab.cit.cornell.edu/~gibbons.

The finite element method may be adapted to handle a wider class of fractal differential equations, including equations involving powers of the Laplacian, space–time equations such as the heat and wave equations where the time variable is a standard real variable, and some nonlinear equations. There are no really new ideas here, so we will not discuss the details.

7. *Spline cut-offs*

An important technical tool in the study of differential equations in Euclidean space, or on manifolds, is that a function that vanishes to finite order at a point (or on a submanifold) may be approximated, in a suitable sense, by functions vanishing in a neighbourhood of the point (or submanifold). The simplest way to accomplish this is to multiply the function by an appropriate family of cut-off functions. This approach is not available for fractals where the operation of multiplication by a nonconstant function is badly behaved [**BST**]. But we can still obtain analogous results by a more complicated procedure that cuts off the function in small corners and substitutes certain spline cut-off functions, at least for the standard Laplacian on SG. As an application, we improve the 'weak = strong' result for solutions of $\Delta u = f$ from [**S1**].

We begin by proving the result in a simple context, involving just $\mathrm{dom}\,\mathscr{E}$ and using just $\mathscr{H}_0$ splines. A multiplication by cut-off functions argument could be used here, but our purpose is to prepare the way for the context of $\mathrm{dom}\,\Delta$ where this is not possible.

Theorem 7·1. *Let $f \in \mathrm{dom}\,\mathscr{E}$ and suppose f vanishes on the boundary of K. Then there exists a sequence of functions $\{f_m\}$ with each $f_m \in \mathrm{dom}\,\mathscr{E}$ vanishing in a neighbourhood of the boundary, and*

$$f_m \to f \text{ uniformly} \tag{7·1}$$

and

$$\mathscr{E}(f_m - f, f_m - f) \to 0 \quad as \quad m \to \infty. \tag{7·2}$$

Proof. We first prove the result under the simplifying assumption that every boundary point is the fixed point of one of the mappings F_k (we may arrange that $F_i v_i = v_i$ for $1 \leqslant i \leqslant N_0$). Let $\Omega_m = K \backslash \bigcup_{i=1}^{N_0} F_i^m K$. We will choose f_m so that $f_m = f$ on Ω_m and f_m has support in Ω_{m+1}, hence vanishes in a neighbourhood of the boundary. On each of the sets $F_i^m K$ we define f_m to be the spline locally in $S(\mathscr{H}_0, V_{m+1})$ with $f_m = f$ on the points in $V_{m+1} \cap F_i^m K \cap \overline{\Omega}_m$ and $f_m = 0$ on all the other points of $V_{m+1} \cap F_i^m K$. In particular, f_m vanishes on the boundary of $F_i^{m+1}K$ and since it is harmonic there it vanishes on all of $F_i^{m+1}K$, as claimed. Also, f_m is continuous, so it is easy to see that $f_m \in \mathrm{dom}\,\mathscr{E}$.

It remains to show (7·2), since this implies (7·1) by (4·16). Now we use a basic fact from the theory of Dirichlet forms ([**BH**], [**FOT**]) that $\mathscr{E}(f, f)$ can be written as the integral over K of a measure ν_f. For simple sets A (such as $A = F_w K$, or finite unions of such sets), it is a simple matter to define $\nu_f(A)$ to be the limit (1·7) with the sum in (1·5) restricted to all words w such that $F_w K \subseteq A$. The nature of these measures is discussed in [**Ku**] and [**BST**]. The only observation we need is that they have no atoms. Thus

$$\lim_{m\to\infty} \nu_f(F_i^m K) = 0. \tag{7·3}$$

Since $f_m - f$ vanishes on Ω_m so does $\nu_{f_m - f}$, hence

$$\mathscr{E}(f_m - f, f_m - f) = \sum_{i=1}^{N_0} \nu_{f_m - f}(F_i^m K). \tag{7·4}$$

Now we claim that $\nu_{f_m}(F_i^m K)$ is bounded by a constant multiple of $\nu_f(F_i^m K)$, with a constant that is independent of m. This is straightforward for any fixed m. Since the limit in (1·7) is increasing, a multiple of $\nu_f(F_i^m K)$ gives an upper bound for all the values $f(F_i^m K)^2$ that enter into the expression $\mathscr{E}_m(f, f)$. These in turn control the values of $\nu_{f_m}(F_i^m K)$. The fact that the constants in the final estimate $\nu_{f_m}(F_i^m K) \leqslant c_m \nu_f(F_i^m K)$ are independent of m follows by a scaling argument. Now (7·2) is a consequence of (7·3), (7·4) and the estimate $\nu_{f_m - f} \leqslant 2(\nu_{f_m} + \nu_f)$.

Without the simplifying assumption, the p.c.f. condition only shows that for each boundary point v_i there exist finite words w and w' such that $v_i = F_{w'}z$ and $F_w z = z$. In place of the sets $F_i^m K$ above we need to consider instead $F_{w'}F_w^m K$. The argument is essentially the same, only the notation gets more complicated. We omit the details.

COROLLARY 7·2. *Definition* 4·1 *remains unchanged if we replace the condition that v vanish on the boundary by the condition that v vanish in a neighbourhood of the boundary.*

Proof. We have to show that if (4·1) holds for all v vanishing in a neighbourhood of the boundary, then it also holds when v just vanishes at the boundary (and similarly for (4·2)). So, given v that vanishes at the boundary, we construct the sequence v_m vanishing in a neighbourhood of the boundary by the theorem. Now (4·1) (or (4·2)) holds for v_m by hypothesis, and we pass to the limit to obtain the same equation for v, using (7·1) for the right-hand side and (7·2) for the left-hand side.

LEMMA 7·3. *For any* $u \in \mathscr{H}_1$ *we have*

$$\|\Delta u|_{F_w K}\|_\infty \leqslant c((r_w\mu_w)^{-1} \max_{\partial F_w K} |u| + \mu_w^{-1} \max_{\partial F_w K} |\partial_n u|). \tag{7·5}$$

Proof. Since Δu is harmonic, it suffices to bound its values on $\partial F_w K = F_w \partial K$. Now for w equal to the empty word, the estimate (7·5) is an immediate consequence of the existence of the basis for $\mathscr{H}_1$ constructed in Section 3. The general case then follows from the scaling identity (2·5) for the Laplacian and the analogous scaling identity (with r_i in place of $r_i\mu_i$) for normal derivatives.

To use the estimate (7·5) effectively requires that we have tight control over the rate of decay of the function and its normal derivative near the boundary, as a consequence of the vanishing of the function and its normal derivative at the boundary. This is difficult to obtain in general, but works out quite well on SG.

LEMMA 7·4. *Let* Δ *be the standard Laplacian on* SG *and suppose* $f \in \mathrm{dom}_C(\Delta)$ *vanishes together with its normal derivatives as the boundary. Then*

$$f\big|_{\partial F_i^m K} = O(m(\tfrac{1}{5})^m) \tag{7·6}$$

and

$$\partial_n f\big|_{\partial F_i^m K} = O(m(\tfrac{1}{3})^m) \tag{7·7}$$

and $\mu_i = \frac{1}{3}$, $r_i\mu_i = \frac{1}{5}$.

Proof. The estimate (7·6) is proved in [**BST**] (for harmonic functions it was observed in [**DSV**] without the m factor and in [**S2**] it is shown to hold without the m factor if we assume Δf satisfies a Hölder condition). To prove (7·7) we use the Gauss–Green formula (1·10) localized to $F_i^m K$ with the functions $u = f$ and $v = h$ where h is the harmonic function taking the values $1, -1, 0$ on the boundary points of $F_i^m K$ (the value 1 at the point v_i). This gives

$$-\int_{F_i^m K} h\Delta f d\mu = \sum_{x \in V_0} f(F_i^m x)\partial_n h(F_i^m x) - h(F_i^m x)\partial_n f(F_i^m x). \tag{7·8}$$

By assumption $\partial_n f(v_i) = 0$, so the only term of the form $-h(F_i^m x)\partial_n f(F_i^m x)$ that occurs is the single value $\partial_n f(F_i^m x)$ at the vertex where h assumes the value -1. The integral on the left side is $O((\frac{1}{3})^m)$ since h and f are uniformly bounded and the measure is $(\frac{1}{3})^m$, and the terms of the form $f(F_i^m x)\partial_n h(F_i^m x)$ are $O(m(\frac{1}{3})^m)$ since $\partial_n h(F_i^m x) = O((\frac{5}{3})^m)$ and we have the estimate (7·6) for $f(F_i^m x)$.

Theorem 7·5. *For the standard Laplacian on* SG, *suppose* $f \in \mathrm{dom}_C(\Delta)$ *vanishes together with its normal derivatives on the boundary. Then there exists a sequence of functions* $\{f_m\}$ *with each* $f_m \in \mathrm{dom}\,\mathscr{E}$ *vanishing in a neighbourhood of the boundary, with* (7·1), (7·2) *and*

$$\Delta f_m \to \Delta f \quad in \quad L^2(d\mu) \quad as \quad m \to \infty \tag{7·9}$$

(*also in* $L^p(d\mu)$ *for any* $p < \infty$).

Remark. We cannot expect uniform convergence in (7·9) because we may not have $\Delta f = 0$ at the boundary.

Proof. As in the proof of Theorem 7·1 we choose f_m so that $f_m = f$ on $\mathbf{\Omega}_m$ with support in $\mathbf{\Omega}_{m+1}$. On each of the sets $F_i^m K$ we take f_m to be the spline locally in $S(\mathscr{H}_1, V_{m+1})$ so that $f_m = f$ and $\partial_n f_m = \partial_n f$ at the two boundary points of $F_i^m K$ not equal to v_i, and $f_m = 0$ and $\partial_n f_m = 0$ at the other 4 vertices in $V_{m+1} \cap F_i^m K$. Because we have matched the values of the functions and the normal derivatives, the functions f_m will be in $\mathrm{dom}_C(\Delta)$. We will show

$$\int_{F_i^m K} |\Delta f_m|^p d\mu \to 0 \quad \text{as} \quad m \to \infty \quad \text{for any} \quad p < \infty. \tag{7·10}$$

This easily implies (7·9) as before and the proof of (7·1) and (7·2) as before.

To prove (7·10) we use the estimates (7·6) and (7·7) from Lemma 7·4. Then we apply Lemma 7·3 to the function f_m on each of the sets $F_i^m F_j K$ for $j = 1, 2, 3$. The estimate (7·5) then yields

$$\|\Delta f_m|_{F_i^m K}\|_\infty \leqslant c(5^{m+1} \max_{\partial F_i^m K} |f| + 3^{m+1} \max_{\partial F_i^m K} |\partial_n f|) \leqslant cm$$

which suffices to prove (7·10) since $\mu(F_i^m K) = (\frac{1}{3})^m$.

Corollary 7·6 (weak = strong). *Let* Δ *be the standard Laplacian on* SG. *Suppose* $u \in L^2(d\mu)$ *and* $f \in L^2(d\mu)$ (*respectively,* f *is continuous*) *and*

$$\int_K u\Delta v d\mu = \int_K f v d\mu \tag{7·11}$$

for all $v \in \mathrm{dom}_C(\Delta)$ *vanishing on a neighbourhood of the boundary. Then* $u \in \mathrm{dom}_{L^2}(\Delta)$ (*respectively,* $u \in \mathrm{dom}_C(\Delta)$) *and* $\Delta u = f$.

Proof. In [**S1**] the same result is shown under the stronger hypothesis that (7·11) holds for all $v \in \mathrm{dom}_C(\Delta)$ such that v and $\partial_n v$ vanish at the boundary. The argument given there, which does not require that $K = \mathrm{SG}$, is that this implies

$$u(x) = -\int_K G(x, y) f(y) d\mu(y) + h(x) \tag{7·12}$$

for some harmonic function h. So we need to show that if (7·11) holds for the smaller class of functions v, it holds for the larger class. For any $v \in \mathrm{dom}_C(\Delta)$ such that v and $\partial_n v$ vanish at the boundary, we apply the theorem to obtain the approximating sequence $\{v_m\}$ and (7·11) holds for v_m in place of v. We then pass to the limit to obtain (7·11) for v, using (7·9) for the left-hand side and (7·1) for the right-hand side.

In the case of a more general fractal we will not have as strong a result as Lemma 7·4. For this discussion we again adopt the simplifying assumptions from the proof of Theorem 7·1. For harmonic functions, the estimate analogous to (7·6) will be

$$f|_{\partial F_i^m K} = O(m^{k_i}\lambda_i^m), \quad 1 \leqslant i \leqslant N_0, \tag{7·13}$$

for some $\lambda_i < r_i$ that can be explicitly computed from the matrix A_i in (2·16). If A_i is diagonalizable then λ_i is the third largest eigenvalue (in absolute value), the first eigenvalue being 1 and the second being r_i (if the third eigenvalue has multiplicity 1 then we can take $k_i = 0$). The Perron–Frobenius theorem implies $\lambda_i < r_i$, but it does not imply that $\lambda_i = r_i\mu_i$, and in fact this does not hold in two examples, the hexagasket and the level 3 Sierpinski gasket, that are worked out in detail in [**S2**].

It seems plausible, although we do not have a proof, that the estimate (7·13) can be transferred to functions satisfying the hypotheses of Lemma 7·4, allowing an increase in the value of k_i. The proof of this for SG in [**BST**] uses many specific facts, so it is not immediately apparent how to extend it. If this conjecture holds, the same argument as in the proof of Lemma 7·4 can be used to show

$$\partial_n f|_{\partial F_i^m K} = O(m^{k_i}\max(\lambda_i/r_i, \mu_i)^m) \tag{7·14}$$

(we expect that $\max(\lambda_i/r_i, \mu_i) = \lambda_i/r_i$, but there is no harm in the other case since $O(m^{k_i}\mu_i^m)$ is the exact analogue of (7·7)).

Modulo our conjecture, we have the analogue of Lemma 7·4 with (7·6) and (7·7) replaced by (7·13) and (7·14). For the analogue of Theorem 7·5 to hold (meaning L^2 convergence in (7·9)) we need the condition

$$\lambda_i < r_i\mu_i^{\frac{1}{2}} \quad \text{for} \quad 1 \leqslant i \leqslant N_0. \tag{7·15}$$

For L^p convergence we need

$$\lambda_i < r_i\mu_i^{1/p'} \quad \text{for} \quad 1 \leqslant i \leqslant N_0, \tag{7·16}$$

where p' denotes the dual index. The proof is essentially the same. We note that (7·16) will always hold if p is chosen close enough to 1. We do not know if (7·15) always holds, but it does hold for the two examples mentioned above. Our conjecture thus leads to the general validity of Corollary 7·6 under the additional hypothesis (7·15).

We conclude with an application of the original 'weak = strong' theorem, showing that $\mathscr{H}_1$ functions may be characterized by a minimization condition analogous to the minimum energy condition for harmonic functions.

THEOREM 7·7. *Let $u \in \mathrm{dom}_C(\Delta)$. Then u minimizes*

$$\int |\Delta v|^2 d\mu$$

over all functions $v \in \mathrm{dom}_C(\Delta)$ with $v = u$ and $\partial_n v = \partial_n u$ on ∂K, if and only if $\Delta^2 u = 0$.

Proof. Let w vary over the functions in $\mathrm{dom}_C(\Delta)$ with $w|_{\partial K} = 0$ and $\partial_n w|_{\partial K} = 0$. Then $v = u + tw$ is an allowable choice for any real t (and conversely). Since

$$\int |\Delta(u + tw)|^2 d\mu = \int |\Delta u|^2 d\mu + 2t\int \Delta u \Delta w d\mu + t^2\int |\Delta w|^2 d\mu$$

the minimization is equivalent to

$$\int \Delta u \Delta w d\mu = 0$$

for all such functions w. By 'weak = strong', this is equivalent to $\Delta^2 u = 0$.

By a similar argument, if $\Delta^{2k} u = 0$ then u minimizes

$$\int |\Delta^k v|^2 d\mu$$

subject to the conditions $v \in \mathrm{dom}_C(\Delta^k)$ and $\Delta^j v = \Delta^j u$, $\partial_n \Delta^j v = \partial_n \Delta^j u$ on ∂K for all $j < k$. It seems plausible that the converse statement is also true.

Acknowledgements. We are grateful to Jun Kigami, Al Schatz and Alexander Teplyaev for advice and encouragement.

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