THE EXACT HAUSDORFF MEASURE OF RANDOM SETS CONNECTED WITH ISOTROPIC GAUSSIAN RANDOM FIELDS

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Abstract.

Let $\mathcal{X}$ be the compact symmetric space of rank one, i.e., the sphere S^n, $n \geq 1$, the real projective space $\mathbb{R}P^n$, $n \geq 2$, the complex projective space $\mathbb{C}P^{2n}$, $n \geq 2$, the quaternion projective space $\mathbb{H}P^{4n}$, $n \geq 2$, or the projective plane over Cayley numbers $\mathbb{C}aP^{16}$. Let $(\Omega, \mathfrak{F}, \mathrm{P})$ be a probability space, let $\boldsymbol{\xi}(x) = \boldsymbol{\xi}(x, \omega) \colon \mathcal{X} \times \Omega \mapsto \mathbb{R}^d$ be the separable and measurable random field on $\mathcal{X}$ with independent and identically distributed Gaussian isotropic components.

Three kinds of random sets are connected with such a field:

(1) the image set $\boldsymbol{\xi}(\mathcal{X}) \subset \mathbb{R}^d$;
(2) the level set $\boldsymbol{\xi}^{-1}(\boldsymbol{x}) \subset \mathcal{X}$, $\boldsymbol{x} \in \mathbb{R}^d$;
(3) the graph $\operatorname{Gr} \boldsymbol{\xi} = \{\, (x, \boldsymbol{\xi}(x)) \colon x \in \mathcal{X} \,\} \subset \mathcal{X} \times \mathbb{R}^d$.

For every kind of a random set under some restrictions concerning the correlation function of the random field $\boldsymbol{\xi}(x)$, we calculate the function φ, for which the Hausdorff measure of the corresponding random set is positive and finite P-almost surely.

1. Introduction

Let X be the metric space. Let $\mathfrak{F}$ be a family of subsets of the space X and let $\varphi \colon \mathfrak{F} \mapsto [0, +\infty]$. Let $\cup\mathfrak{F}$ denotes the union of all sets of the family $\mathfrak{F}$. For any $t \in (0, +\infty]$ we define an auxiliary measure μ_t as

$$\mu_t(A) = \inf_{\substack{\operatorname{card} \mathfrak{G} \leq \aleph_0 \\ \mathfrak{G} \subset \{\, S : \operatorname{diam} S \leq t \,\} \\ A \subset \cup \mathfrak{G}}} \sum_{S \in \mathfrak{G}} \varphi(S),$$

where card denotes the cardinal number of the set, and diam denotes the diameter of the set. If $0 < t < s \leq +\infty$, then $\mu_t \geq \mu_s$. That's why the limit measure

$$\mu_\varphi(A) = \lim_{t \downarrow 0} \mu_t(A)$$

exists. In what follows only the case when $\mathfrak{F}$ is the family of all closed balls in X and the function φ depends only on the diameter of the ball will be considered. We restrict attention to the class Φ of functions $\varphi \colon (0, \delta) \mapsto [0, +\infty)$ which are right continuous, monotone increasing with $\lim_{t \downarrow 0} \varphi(t) = 0$ and smooth in the sense that there is a finite constant K with

$$\frac{\varphi(2t)}{\varphi(t)} \leq K, \qquad 0 < t < \frac{\delta}{2}.$$

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We will say that the function φ is the *exact Hausdorff measure* of the set S, if $0 < \mu_\varphi(S) < +\infty$.

Let $\dim S$ be the topological dimension of the set S.

Definition 1 [1]**.** A non-countable bounded set S of the metric space (X, ρ) is called a *fractal*, if the function $t^{\dim S}$ is not an exact Hausdorff measure of the set S.

We are interested in fractal properties of random sets connected with isotropic Gaussian random fields on compact rank one symmetric spaces. A full list of such spaces $\mathcal{X}$ is [2]: the sphere S^n, $n \geq 1$, the real projective space $\mathbb{R}P^n$, $n \geq 2$, the complex projective space $\mathbb{C}P^{2n}$, $n \geq 2$, the quaternion projective space $\mathbb{H}P^{4n}$, $n \geq 2$, and the projective plane over Cayley numbers $\mathbb{C}aP^{16}$. Here n denotes the dimension of the space over the corresponding algebra. In all the subsequent we will denote by N the *topological* dimension of the corresponding space. Let $\xi(x)$, $x \in \mathcal{X}$ be the second order mean square continuous real valued zero mean random field with the correlation function

$$B(x, y) = E\xi(x)\xi(y), \qquad x, y \in \mathcal{X}.$$

Let $\mathcal{G}$ be the group of isometries of the space $\mathcal{X}$. The next two definitions are equivalent:

Definition 2$'$. A random field $\xi(x)$ is called *isotropic*, if for any $g \in \mathcal{G}$

$$B(gx, gy) = B(x, y).$$

Definition 2$''$. A random field $\xi(x)$ is called *isotropic*, if its correlation function $B(x, y)$ depends only on the distance between the points x and y.

In the classical case of $\mathcal{X} = \mathbb{R}^N$ [3] such a field is called *homogeneous and isotropic*, because the corresponding group of isometries is the semi-direct product of the group of shifts by the group of rotations. The term *homogeneous* corresponds to shifts and the term *isotropic* corresponds to rotations. In our case the group of isometries is a simple Lie group. That's why we use only one term "isotropic random field". Such fields were firstly defined in [4].

Let $(\Omega, \mathfrak{F}, \mathrm{P})$ be a probability space, let $\boldsymbol{\xi}(x) = \boldsymbol{\xi}(x, \omega) \colon \mathcal{X} \times \Omega \mapsto \mathbb{R}^d$ be the separable and measurable random field on $\mathcal{X}$ with independent and identically distributed Gaussian isotropic components. Three kinds of random sets are connected with such a field:

(1) the image set $\boldsymbol{\xi}(\mathcal{X}) \subset \mathbb{R}^d$;
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(3) the graph $\mathrm{Gr}\,\boldsymbol{\xi} = \{\, (x, \boldsymbol{\xi}(x)) \colon x \in \mathcal{X} \,\} \subset \mathcal{X} \times \mathbb{R}^d$.

An excellent survey of the developments in the area of calculating the Hausdorff measure for such sets before 1987 is published in [5]. Later M.Talagrand proposed new method in [6]. His ideas were recently developed by Y.Xiao ([7–9]). We follow their ideas in our proofs.

2. Formulation of theorems

In order to formulate our results we need more definitions. Let μ denotes the probability $\mathcal{G}$-invariant measure on the space $\mathcal{X}$. The natural unitary representation

$$(U(g)f)(x) = f(g^{-1}x), \qquad g \in \mathcal{G}$$

in the space $L^2(\mathcal{G},\mu)$ is the direct sum of the irreducible unitary representations U_n, $n\geq 0$. Let $h(\mathcal{X},n)$ denotes the dimension of the representation U_n. According to [10] we have:

$$
\begin{aligned}
h(S^N,n)&=\frac{(n+2N-1)(n+N-2)!}{(n-1)!N!},\\
h(\mathbb{R}P^N,n)&=\frac{(2n+2N-1)(2n+N-2)!}{(2n-1)!N!},\\
h(\mathbb{C}P^N,n)&=\frac{(2n+N/2)[(n+N/2-1)!]^2}{[n!]^2(N/2)!(N/2-1)!},\\
h(\mathbb{H}P^N,n)&=\frac{(2n+N/2+1)(n+N/2+4)!(n+N/2-1)!}{(n+1)!n!(N/2+1)!(N/2-1)!},\\
h(\mathbb{C}aP^{16},n)&=\frac{(2n+7)(n+6)!}{7!}.
\end{aligned}
$$

Let ρ be the $\mathcal{G}$-invariant metrics on $\mathcal{X}$ satisfying the condition $\operatorname{diam}\mathcal{X}=\pi$. Let o denotes some fixed point in $\mathcal{X}$. Let $\theta_{xy}=\rho(x,y)$, $\theta_x=\rho(x,o)$. Let $P_n^{(\alpha,\beta)}(\cos\theta)$ denotes Jacobi polynomial. According to [4] the correlation function of the isotropic random field on the space $\mathcal{X}$ has the form

$$B(x,y)=\sum_{n=0}^{\infty}b_nh(\mathcal{X},n)\hat{P}_n^{(\alpha,\beta)}(\cos\theta_{xy}),$$

where

$$\hat{P}_n^{(\alpha,\beta)}(\cos\theta)=\frac{P_n^{(\alpha,\beta)}(\cos\theta)}{P_n^{(\alpha,\beta)}(1)}$$

and $\{b_n,n\geq 0\}$ is the sequence of non-negative real numbers satisfying the condition

$$\sum_{n=0}^{\infty}b_nh(\mathcal{X},n)<\infty.$$

The coefficients α and β depend on $\mathcal{X}$. In any case $\alpha(\mathcal{X})=(N-2)/2$. The coefficient β can be calculated as: $\beta(S^N)=(N-2)/2$, $\beta(\mathbb{R}P^N)=-1/2$, $\beta(\mathbb{C}P^N)=0$, $\beta(\mathbb{H}P^N)=1$, $\beta(\mathbb{C}aP^{16})=3$.

Let $\sigma^2(\theta)=\operatorname{Var}(\xi(x)-\xi(o))$ be the incremental variance of the random field $\xi(x)$. Here x is an arbitrary point in $\mathcal{X}$ satisfying the condition $\theta_x=\theta$. Let ψ denotes the inverse function of σ. Let

$$\mathcal{B}(o,\varepsilon)=\{\,x\in\mathcal{X}\colon\theta_x<\varepsilon\,\}$$

be the open ball with center at o and radius ε. Let $[\mathcal{X}\setminus\mathcal{B}(o,\varepsilon)]^n_{\neq}$ denotes the subset of points $(x_1,\dots,x_n)$ in the nth Cartesian power of the set $\mathcal{X}\setminus\mathcal{B}(o,\varepsilon)$ satisfying the condition $x_j\neq x_k$ for $j\neq k$. Denote

$$\hat{\sigma}^2_n(\varepsilon)=\inf_{(x_1,\dots,x_n)\in[\mathcal{X}\setminus\mathcal{B}(o,\varepsilon)]^n_{\neq}}\operatorname{Var}(\xi(o)|\xi(x_1),\dots,\xi(x_n)).$$

This quantity is essentially the infimum of the mean square error of the optimal linear prediction using n observations at points outside of sphere $\mathcal{B}(o,\varepsilon)$.

Definition 3 [11]**.** A random field $\xi(x)$ is called *locally nondeterministic*, if for any $n \geq 1$

$$\liminf_{\varepsilon\downarrow 0} \frac{\hat{\sigma}_n^2(\varepsilon)}{\sigma^2(\varepsilon)} > 0.$$

We will call a d-dimensional random field $\boldsymbol{\xi}(x)$ with independent components locally nondeterministic, if any of its component is locally nondeterministic.

Let $f(t)$ be a measurable positive function defined on some interval $(0, \delta)$, let $\gamma \in \mathbb{R}$ be some fixed number.

Definition 4 [12]**.** A function $f(t)$ is called *regularly varying* of index γ, if for any $\lambda > 0$ it satisfies the condition

$$\lim_{t\downarrow 0} \frac{f(\lambda t)}{f(t)} = \lambda^\gamma.$$

Now we are ready to formulate our results.

Theorem 1. *Let $\boldsymbol{\xi}(x)\colon \mathcal{X} \mapsto \mathbb{R}^d$ be the separable and measurable locally nondeterministic random field on $\mathcal{X}$ with independent and identically distributed Gaussian isotropic components satisfying the next conditions:*

(1) *the function $\sigma(\theta)$ is regularly varying of index $\gamma > 0$;*
(2) *the function $\theta\sigma(\theta^{-1})$ is monotone non-increasing for all sufficiently large θ;*
(3) *$N < \gamma d$;*
(4) *$\sum_{n=1}^\infty b_n h(\mathcal{X}, n)\sigma^{-2}(n^{-1}) < \infty$;*

Then we have

$$\mathrm{P}\{0 < \mu_{\varphi_1}(\boldsymbol{\xi}(\mathcal{X})) < \infty\} = 1,$$

where $\varphi_1(\theta) = \psi^N(\theta) \log\log(\theta^{-1})$.

Theorem 2. *Let $\boldsymbol{\xi}(x)\colon \mathcal{X} \mapsto \mathbb{R}^d$ be the separable and measurable locally nondeterministic random field on $\mathcal{X}$ with independent and identically distributed Gaussian isotropic components satisfying the next conditions:*

(1) *the function $\sigma(\theta)$ is regularly varying of index $\gamma > 0$;*
(2) *$N > \gamma d$.*

Then for any $\boldsymbol{x} \in \boldsymbol{\xi}(\mathcal{X})$ we have

$$\mathrm{P}\{0 < \mu_{\varphi_2}(\boldsymbol{\xi}^{-1}(\boldsymbol{x})) < \infty\} = 1,$$

where $\varphi_2(\theta) = \theta^N(\sigma(\theta(\log\log\theta^{-1})^{-1/N}))^{-d}$.

Theorem 3. *Let $\boldsymbol{\xi}(x)\colon \mathcal{X} \mapsto \mathbb{R}^d$ be the separable and measurable locally nondeterministic random field on $\mathcal{X}$ with independent and identically distributed Gaussian isotropic components satisfying the next conditions:*

(1) *the function $\sigma(\theta)$ is regularly varying of index $\gamma > 0$;*
(2) *$N > \gamma d$.*

Then we have

$$\mathrm{P}\{0 < \mu_{\varphi_3}(\mathrm{Gr}\,\boldsymbol{\xi}) < \infty\} = 1,$$

where $\varphi_3(\theta) = \theta^{N+d}(\sigma(\theta(\log\log\theta^{-1})^{-1/N}))^{-d}$.

Theorem 4. *Let $\boldsymbol{\xi}(x)\colon \mathcal{X} \mapsto \mathbb{R}^d$ be the separable and measurable locally nondeterministic random field on $\mathcal{X}$ with independent and identically distributed Gaussian isotropic components satisfying the next conditions:*

(1) *the function $\sigma(\theta)$ is regularly varying of index $\gamma > 0$;*
(2) *the function $\theta\sigma(\theta^{-1})$ is monotone non-increasing for all sufficiently large θ;*
(3) *$N < \gamma d$;*
(4) *$\sum_{n=1}^{\infty} b_n h(\mathcal{X}, n)\sigma^{-2}(n^{-1}) < \infty$;*

Then we have

$$\mathrm{P}\{0 < \mu_{\varphi_1}(\mathrm{Gr}\,\boldsymbol{\xi}) < \infty\} = 1.$$

These theorems were proved in [13–15].

3. Sketch of proofs

We will use the letter K to denote an unspecified positive constant which may be different in different appearances. Proofs of Theorems 1–4 can be naturally divided into two parts. Let Ξ denotes any of the mentioned above random sets. In the first part of the proof one should prove that

$$\mathrm{P}\{0 < \mu_{\varphi_j}(\Xi)\} = 1.$$

In order to prove this equality one can use the density theorem of Rogers and Taylor [16], see also [17], theorem 2.10.17(2). For any finite Borel measure ν on the metric space (X, ρ) and any function $\varphi \in \Phi$ consider the upper spherical density at the point $x \in X$ defined as

$$\overline{D}_{\varphi}(\nu, x) = \limsup_{\varepsilon \downarrow 0} \frac{\nu(\mathcal{B}(x, \varepsilon))}{\varphi(2\varepsilon)}.$$

The density theorem suggests that there exist constants c_1 and c_2, depending only on φ and X, such that for any Borel set $E \subset X$ and any finite Borel measure ν one has

$$c_1 \inf_{x \in E} \overline{D}_{\varphi}(\nu, x)\nu(E) \le \mu_{\varphi}(E) \le c_2 \sup_{x \in E} \overline{D}_{\varphi}(\nu, x)\nu(E).$$

This theorem allows us to prove positiveness of measure by defining a suitable Borel measure concentrated on Ξ for which the upper spherical density is bounded below at all points of some subset $\Xi_1 \subset \Xi$ of positive ν-measure. For example, in the case of Theorem 1 we define the measure ν as

$$\nu(E) = \mu(\boldsymbol{\xi}^{-1}(E)).$$

This measure is concentrated on the set $\Xi = \boldsymbol{\xi}(\mathcal{X})$. Now it is enough to prove that there exists such a constant $b > 0$ that for any fixed point $x \in \mathcal{X}$ the next equality holds with probability 1:

$$\limsup_{\varepsilon \downarrow 0} \frac{\nu(\mathcal{B}(\boldsymbol{\xi}(x), \varepsilon))}{\varphi_1(\varepsilon)} \le b^{-1}. \tag{1}$$

Indeed, let $\Xi_1(\omega)$ be the random subset in Ξ defined as

$$\Xi_1(\omega) = \{\, \boldsymbol{\xi}(x)\colon (1) \text{ is true for } x \,\}.$$

According to Fubini's theorem we have $\nu(\Xi_1(\omega)) = 1$ P-almost surely. By density theorem $\mu_{\varphi_1}(\Xi_1(\omega)) \geq c_1 b$ and we are done. We omit technical details of proof of (1).

In the second part of the proof one should prove that

$$\mathrm{P}\{\mu_{\varphi_j}(\Xi) < \infty\} = 1.$$

At a first glance it is enough to prove an inequality opposite to (1), namely

$$\limsup_{\varepsilon\downarrow 0} \frac{\nu(\mathcal{B}(\boldsymbol{\xi}(x), \varepsilon))}{\varphi_1(\varepsilon)} \geq c^{-1}.$$

We have no ideas how to prove this inequality. But even in the case of having such a proof this inequality does not give a technique for finding a finite upper bound for $\mu_{\varphi_1}(\Xi)$. It is not enough to show that $\overline{D}_{\varphi_1}(\nu, x) \leq K$ ν-almost everywhere on Ξ since the ν-exceptional set may well contribute to $\mu_{\varphi_1}(\Xi)$. A different method is needed to attack the set $\{\, x \in \Xi \colon \overline{D}_{\varphi_1}(\nu, x) > K \,\}$. The corresponding calculations are very complicated. See, for example, the cases of Brownian motion in $\mathbb{R}^d$, $d \geq 3$ in [18], Brownian motion in $\mathbb{R}^2$ in [19] or multiparameter Brownian motion in [20].

A key estimate which simplified calculations was proved by M.Talagrand in [6] for the case of multiparameter fractional Brownian motion. Its main ingredient was proved earlier in [21]. Namely, let $\zeta(t)$ be the Gaussian random function defined on an arbitrary set S. Let ρ denotes the Dudley distance $\rho(s,t) = \sqrt{\mathrm{E}(\zeta(s) - \zeta(t))^2}$. Let $N_\rho(S, \varepsilon)$ denotes the smallest number of balls of radius ε needed to cover the set S. Let D be the diameter of S. In most cases the quantity $N_\rho(S, \varepsilon)$ can not be exactly calculated. But suppose, that one can prove an estimate

$$N_\rho(S, \varepsilon) \leq \Psi(\varepsilon), \qquad \varepsilon > 0,$$

and there exists a constant $C > 0$ such that for any $\varepsilon > 0$ the following inequality holds true:

$$C^{-1}\Psi(\varepsilon) \leq \Psi(\varepsilon/2) \leq C\Psi(\varepsilon).$$

Then there exists such a constant $K = K(C) > 0$ that

$$\mathrm{P}\{\sup_{s,t\in S} |\zeta(s) - \zeta(t)| \leq u\} \geq \exp(-K\Psi(u)).$$

Using the last estimate and Talagrand's technique, one can prove that there exist such constants $\delta > 0$ and $K > 0$, that for any $0 < \varepsilon_0 < \delta$ the inequality

$$\mathrm{P}\left\{\exists\varepsilon \in [\varepsilon_0^2, \varepsilon_0] \colon \sup_{\theta_{xy}\leq\varepsilon} \|\boldsymbol{\xi}(x) - \boldsymbol{\xi}(y)\| \leq K\sigma(r(\log\log\varepsilon^{-1})^{-1/N})\right\} \geq 1 - \exp(-(\log\varepsilon_0^{-1})^{1/2}). \tag{2}$$

holds true. Once more we omit the details.

Now we can prove the second part of Theorem 1. For such natural k that $2^{-k} < \delta$, consider the random set $\mathcal{R}_k(\omega) \subset \mathcal{X}$:

$$\mathcal{R}_k(\omega) = \left\{ x \in \mathcal{M} \colon \exists\varepsilon \in [2^{-2k}, 2^{-k}] \colon \sup_{\theta_{xy}\leq\varepsilon} \|\boldsymbol{\xi}(x) - \boldsymbol{\xi}(y)\| \leq K\sigma(\varepsilon(\log\log\varepsilon^{-1})^{-1/N}) \right\}.$$

By (2) we have

$$P\{x \in \mathcal{R}_k(\omega)\} \geq 1 - \exp(-k/2).$$

Denote by Ω_0 the event

$$\Omega_0 = \{\, \omega \in \Omega \colon \mu(\mathcal{R}_k(\omega)) \geq 1 - \exp(-k/4) \text{ infinitely often} \,\}.$$

It follows from Fubini's theorem that $P(\Omega_0) = 1$.

Now we build some special construction. For any natural number k consider a covering of the space $\mathcal{X}$ by various closed balls of radius 2^{-k}. According to Theorem 2.8.14 from [17] there exists a natural number M and M families of disjoint balls which cover the space $\mathcal{X}$. Let

$$V(\varepsilon) = \mu(\mathcal{B}(x, \varepsilon)).$$

Due to the invariance of the measure μ this quantity does not depend on $x \in \mathcal{X}$. According to [2] we have

$$V(\varepsilon) = \int_0^\varepsilon A(\theta)\, d\theta,$$

where $A(\theta)$ denotes the surface measure of the sphere of radius θ induced by the standard Riemann metric on the space $\mathcal{X}$. According to Lemma 4.10 from [2]

$$A(\theta) = K \sin^{N-1}(\theta/2) \cos^q(\theta/2),$$

and $q \geq 0$ for any space $\mathcal{X}$. Using inequalities $\sin\theta \geq 2\theta/\pi$ for $0 \leq \theta \leq \pi/2$ and $\cos\theta \geq \cos(1/2)$ for $0 \leq \theta \leq 2^{-k}$, we obtain the inequality

$$V(\theta) \geq K\theta^N.$$

That's why any family consists of no more then $K^{-1}2^{Nk}$ balls, and the constructed covering contains no more than $K \cdot 2^{Nk}$ balls. We will call these balls the *dyadic balls* of the kth order and denote them by $\mathcal{C}_{kj}$, $1 \leq j \leq K \cdot 2^{Nk}$.

Let the event Ω_1 consists of elementary events $\omega \in \Omega$, for which there exists a number $n_1 = n_1(\omega)$ large enough such that for all $n \geq n_1(\omega)$ and any dyadic ball $\mathcal{C}_n$ of order n in $\mathcal{X}$, we have

$$\sup_{x,y \in \mathcal{C}_n} \|\boldsymbol{\xi}(x) - \boldsymbol{\xi}(y)\| \leq K\sigma(2^{-n})\sqrt{n}.$$

According to [22], p. 174 the function $\sigma(t)\sqrt{\log t^{-1}}$ is a uniform modulus of the random field $\boldsymbol{\xi}(x)$. It means that there exists such a constant c that the inequality

$$\lim_{\varepsilon \downarrow 0} \sup_{\theta_{xy} \leq \varepsilon} \frac{\|\boldsymbol{\xi}(x) - \boldsymbol{\xi}(y)\|}{\sigma(\varepsilon)\sqrt{2c\log\varepsilon^{-1}}} \leq 1$$

hold true almost surely. Consequently we obtain $P(\Omega_1) = 1$.

Now fix an $\omega \in \Omega_0 \cap \Omega_1$, we show that $\mu_{\varphi_1}(\boldsymbol{\xi}(\mathcal{X}, \omega)) < \infty$. Consider $k \geq 1$ such that

$$\mu(\mathcal{R}_k(\omega)) \geq 1 - \exp(-\sqrt{k/4}).$$

Let $z \in \mathcal{R}_k(\omega)$ be some fixed point. Fix some number ε which proves that z belongs to the set $\mathcal{R}_k(\omega)$. Let n be the smallest natural number for which any dyadic ball $\mathcal{C}_{nj}$ of order n containing z lies inside the ball $\mathcal{B}(z,\varepsilon)$. Let $\mathcal{C}_n(z)$ denotes the union of all these balls. By construction we can a number $k_0 = k_0(\mathcal{X})$ with $k \leq n \leq 2k + k_0$ and

$$\sup_{x,y\in\mathcal{C}_n(z)} \|\boldsymbol{\xi}(x) - \boldsymbol{\xi}(y)\| \leq K\sigma(2^{-n}(\log\log 2^n)^{-1/N}). \tag{3}$$

Let $\mathcal{V}_n$ denotes the union of all dyadic balls of order n for which (3) holds. Thus we have

$$\mathcal{R}_k \subseteq \mathcal{V} = \cup_{n=k}^{2k+k_0} \mathcal{V}_k.$$

Clearly $\boldsymbol{\xi}(\mathcal{C}_n)$ can be covered by a ball of radius

$$\rho_n = K\sigma(2^{-n}(\log\log 2^n)^{-1/N}).$$

Since $\varphi_1(2\rho_n) \leq K \cdot 2^{-Nn} \leq K\mu(\mathcal{C}_n)$, we have

$$\sum_n \sum_{\mathcal{C}_n\in\mathcal{V}_n} \varphi_1(2\rho_n) \leq \sum_n \sum_{\mathcal{C}_n\in\mathcal{V}_n} K\mu(\mathcal{C}_n) = K\mu(\mathcal{V}) < \infty. \tag{4}$$

On the other hand, $\mathcal{X} \setminus \mathcal{V}$ is contained in a union of dyadic balls of order $q = 2k + k_0$, none of which meets $\mathcal{R}_k$. For k large enough there can be at most

$$K \cdot 2^{Nq}\mu(\mathcal{X} \setminus \mathcal{V}) \leq K \cdot 2^{Nq} \exp(-\sqrt{k}/4)$$

of such balls. For each of these balls, $\boldsymbol{\xi}(\mathcal{C}_q)$ is contained in a ball of radius $\rho = K\sigma(2^{-q})\sqrt{q}$. Thus for any $\delta > 0$ and for k large enough

$$\sum \varphi_1(2\rho) \leq K \cdot 2^{Nq} \exp(-\sqrt{k}/4) 2^{-Nq} q^{N/(2\gamma)+\delta} \leq 1. \tag{5}$$

Since k can be arbitrarily large, the upper bound follows from (4) and (5).

4. Some problems

We note two interesting problems. The first one is connected with the so called critical case $N = \gamma d$. The answer is known only for planar Brownian motion [19] and the answer is

$$\varphi(t) = t^2 \log t^{-1} \log\log\log t^{-1}.$$

In order to formulate the second problem consider the spectral representation of the random field $\xi(x)$. Let H_n be the space of the representation U_n. It consists of the real-valued eigenfunctions of the self-adjoint Laplace–Beltrami operator Δ on X [2]. Let ψ_l, $1 \leq l \leq h(\mathcal{X}, n)$ be some fixed orthonormal basis in the space H_n. According to Peter–Weyl theorem [2] the functions

$$S_n^l(x) = \sqrt{h(\mathcal{X}, n)}\psi_l(x)$$

form the orthonormal basis of the space $L^2(\mathcal{X}, \mu)$. Using Kahrunen's theorem, we obtain

$$\xi(x) = \sum_{n=0}^{\infty} \sqrt{b_n} \sum_{l=1}^{h(\mathcal{X},n)} S_n^l(x)\xi_n^l,$$

where ξ_n^l is the sequence of independent standard Gaussian random variables. In the case $\mathcal{X} = S^1$ this is the ordinary Fourier random series. That's why our investigation can be considered as studying of local properties of sample functions of generalized Gaussian Fourier series. It would be interesting to prove analogous results for sub-Gaussian series [23].

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