Stochastic Processes and their Applications 92 (2001) 61–85

stochastic processes and their applications

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Fluctuation of the transition density for Brownian motion on random recursive Sierpinski gaskets ☆

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Received 18 April 2000; received in revised form 2 August 2000; accepted 3 August 2000

Abstract

We consider a class of random recursive Sierpinski gaskets and examine the short-time asymptotics of the on-diagonal transition density for a natural Brownian motion. In contrast to the case of divergence form operators in $\mathbb{R}^n$ or regular fractals we show that there are unbounded fluctuations in the leading order term. Using the resolvent density we are able to explicitly describe the fluctuations in time at typical points in the fractal and, by considering the supremum and infimum of the on-diagonal transition density over all points in the fractal, we can also describe the fluctuations in space. © 2001 Elsevier Science B.V. All rights reserved.

MSC: primary 60J60; secondary 60J25; 60J65

Keywords: Heat kernel; Laplace operator; Resolvent density; General branching process; Random recursive fractals

1. Introduction

The fundamental work of Aronson (1967), established upper and lower estimates on the heat kernel for an elliptic operator in $\mathbb{R}^n$. There is now a substantial literature on the behaviour of the heat kernel for elliptic operators on manifolds, and that of the transition kernel for random walks on groups or graphs (see for instance Coulhon and Grigoryan, 1998; Davies, 1991). There are two components to the estimate, an on diagonal term, which is usually determined by the volume growth of the space, and the off diagonal term, where there is Gaussian decay.

The study of fractals has shown that the behaviour may be different when the geometry is not smooth. We state here the results for regular fractals F such as the Sierpinski gasket or the Sierpinski carpet. If $p_t(x, y)$ denotes the transition density for the natural Brownian motion on the fractal F (or the heat kernel for the corresponding

☆ Research partly supported by a Royal Society Industry Fellowship at BRIMS, Hewlett-Packard and Grant-in-Aid for Scientific Research (B)(2) 10440029.

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PII: S0304-4149(00)00073-9

Laplace operator on F), then there exist constants $c_{1.1}, c_{1.2}$ such that

$$p_t(x,y) \leqslant c_{1.1} t^{-d_s/2} \exp\left(-c_{1.2} \left(\frac{d(x,y)^{d_w}}{t} \right)^{1/(d_w-1)} \right), \quad \forall x,y \in F, \ 0<t<1. \tag{1.1}$$

The exponent d_s is called the spectral dimension and governs the asymptotics of the spectral counting function, d_w is called the walk dimension and is determined by the time to distance scaling in the fractal, and $d(\cdot,\cdot)$ is an intrinsic shortest path metric (in the case of Sierpinski carpet or gasket it is equivalent to the Euclidean distance). There is a corresponding lower bound of the same form but different constants. Note that if $d_s = n$ and $d_w = 2$ we recover the usual Gaussian bounds of $\mathbb{R}^n$. We will call such upper and lower bounds Aronson-type estimates on the transition density (or heat kernel). For a discussion of these estimates and background results concerning diffusion on fractals see Barlow (1998).

We will be interested in the situation where the geometry of the fractal is generated in a random way, and to determine the effect this has on the on-diagonal transition density. In a previous paper (Hambly, 1997), a natural Brownian motion on a random recursive Sierpinski gasket was constructed and relatively crude estimates obtained on its transition density. The estimates were not tight and indicated that it might not be possible to obtain the uniform Aronson type estimates of (1.1) in this setting.

One situation where fractals with irregular geometry were analysed in detail is the case of scale irregular fractals, discussed in Barlow and Hambly (1997) and Hambly et al. (2000b). These fractals are spatially homogeneous but not scale invariant with the irregularity given by an environment sequence. It is known that there is typically fluctuation in the short-time asymptotics of the heat kernel and, in the Sierpinski gasket case, if the environment is generated by an iid sequence, an explicit description of the fluctuation can be established. Using the relationship between the spectral counting function and the trace of the heat semigroup it can be shown that the spectral counting function also exhibits fluctuation in its asymptotics.

The spectral counting function for random recursive Sierpinski gaskets was the subject of Hambly (2000). It was shown that if $N(\lambda)$ denotes the number of eigenvalues of the Laplacian (Dirichlet or Neumann), then under a certain non-lattice assumption, there exists a non-zero mean one random variable W, and a constant $c_{1.3}$ such that

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{d_s/2}} = c_{1.3} W^{1-d_s/2}, \quad \mathbb{P}\text{-a.s.}$$

This raises the question of whether there really are fluctuations in the short-time asymptotics of the heat kernel. In this paper we will show that there are fluctuations and we identify their functional form (which is determined by the tails of the random variable W). As the spectral counting function can be recovered from the trace of the heat semigroup, this shows that integrating over the fractal leads to the cancellation of these fluctuations.

We will consider the class of random recursive Sierpinski gaskets of Hambly (2000) and state our main result here for a particular example. For random recursive fractals generated from fractals SG(2) and SG(3) (SG(2) is the Sierpinski gasket and SG(3)

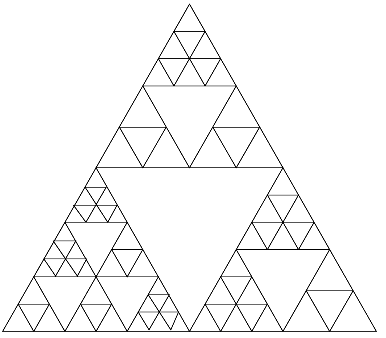

Fig. 1. A graph approximation to a random recursive Sierpinski gasket.

is a triangular fractal with generator consisting of 6 upward pointing triangles and 3 downward ones which are removed, for definitions see Hambly (2000)) we describe explicitly the fluctuations in time and space. Let $(\Omega, \mathscr{F}, \mathbb{P})$ denote the probability space of random recursive fractals $F(\omega)$ built from the two fractals as in Hambly (1997) (a realization is shown in Fig. 1), where we choose type SG(2) with probability p and SG(3) with probability $1-p$. The spectral dimension for the random fractal is given by $d_s/2 = \alpha/(\alpha+1)$, almost surely, where

$$\alpha := \{s\colon\ p3(3/5)^s + (1-p)6(7/15)^s = 1\}.$$

Note that if we define $\alpha_2 = \log 3/\log(5/3)$ and $\alpha_3 = \log 6/\log(15/7)$ the spectral dimension of SG(2) is given by $\alpha_2/(\alpha_2+1) = 2\log 3/\log 5$ and for SG(3) is $\alpha_3/(\alpha_3+1) = 2\log 6/\log(90/7)$. We also need two correction exponents, $\beta' = \alpha/\alpha_2 - 1$, $\beta = 1 - \alpha/\alpha_3$. The Laplace operator is defined with respect to a measure μ induced by a suitable general branching process. This measure is equivalent to the Hausdorff measure in the resistance metric (see Sections 3 and 4 for details).

Theorem 1.1. (1) *There exists a jointly continuous transition density $p_t(x,y)$ for $x, y \in F$ and $t > 0$.*

(2) *There exist constants $c_{1.4}, c_{1.5}$ such that*

$$c_{1.4} \leqslant \limsup_{t\to 0} \frac{p_t(x,x)}{t^{-\alpha/(\alpha+1)}(\log|\log t|)^{\beta/(\alpha+1)}} \leqslant c_{1.5}, \quad \mu\text{-a.e. } x \in F, \quad \mathbb{P}\text{-a.s.}$$

(3) *There exist constants $c_{1.6}, c_{1.7}$ such that*

$$c_{1.6} \leqslant \lim_{t\to 0} \sup_{x\in F} \frac{p_t(x,x)}{t^{-\alpha/(\alpha+1)}(|\log t|)^{\beta/(\alpha+1)}} \leqslant c_{1.7}, \quad \mathbb{P}\text{-a.s.}$$

(4) *There exists a constant $c_{1.8}$ such that*

$$\liminf_{t\to 0} \frac{p_t(x,x)}{t^{-\alpha/(\alpha+1)}(\log|\log t|)^{\beta'/(\alpha+1)}} \leqslant c_{1.8}, \quad \mu\text{-a.e. } x \in F, \quad \mathbb{P}\text{-a.s.}$$

(5) *There exists a constant $c_{1.9}$ such that*

$$\lim_{t\to 0} \inf_{x\in F} \frac{p_t(x,x)}{t^{-\alpha/(\alpha+1)}(|\log t|)^{\beta'/(\alpha+1)}} \leqslant c_{1.9}, \quad \mathbb{P}\text{-a.s.}$$

It is possible to obtain lower bounds in cases (4) and (5) but, though they have the same number of logarithms, we require a further assumption on the class of fractals and the exponent in the logarithmic terms differs.

This result is quite different to that for elliptic operators in divergence form on a bounded domain $D \subset \mathbb{R}^n$, where

$$c_{1.12} \leqslant \lim_{t\to 0} \frac{p_t(x,x)}{t^{n/2}} \leqslant c_{1.13}, \quad \forall x \in D.$$

In the case of regular fractals F such as nested fractals (Lindstrøm, 1990), or Sierpinski carpets we have

$$c_{1.10} \leqslant \lim_{t\to 0} \frac{p_t(x,x)}{t^{d_s/2}} \leqslant c_{1.11}, \quad \forall x \in F.$$

In these settings any fluctuations for the leading order term in the transition density must be bounded.

We note here that extending these fluctuation results to a wider class of random fractals, such as random recursive nested fractals not based on d-dimensional tetrahedra, is a non-trivial problem. The main difficulty lies in establishing the existence of a Brownian motion on such fractals. It can be shown that there is no uniform Harnack inequality in that setting and hence the existence of the process is a serious difficulty.

The outline of the paper is as follows. In Sections 2 and 3 we introduce the random recursive Sierpinski gaskets and give a description of these sets via general branching processes. In Section 4 we introduce the natural Laplace operator on these fractals via its Dirichlet form and a natural measure. We also derive the crucial properties of these quantities. In Section 5 we show fluctuations in the limiting random variable of the general branching process. Section 6 will show the fluctuation in the on-diagonal transition density via a corresponding fluctuation in the Green density. Throughout the paper we will label the ith fixed constant in Section n by $c_{n.i}$, other constants c_i may be used in different proofs but will be fixed within a given proof.

2. Random recursive Sierpinski gaskets

We construct our random recursive fractals from the class of affine nested Sierpinski gaskets and begin by recalling the definitions of such fractals (Fitzsimmons et al., 1994; Lindstrøm, 1990). For $l > 1$, an *l-similitude* is a map $\psi : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\psi(x) = l^{-1}U(x) + x_0, \tag{2.1}$$

where U is a unitary, linear map and $x_0 \in \mathbb{R}^d$. Let $\psi = \{\psi_1, \ldots, \psi_m\}$ be a finite family of maps where ψ_i is an l_i-similitude. For $B \subset \mathbb{R}^d$, define

$$\Psi(B) = \bigcup_{i=1}^{m} \psi_i(B),$$

and let

$$\Psi_n(B) = \Psi \circ \cdots \circ \Psi(B).$$

The map Ψ on the set of compact subsets of $\mathbb{R}^d$ has a unique fixed point F, which is a self-similar set satisfying $F = \Psi(F)$.

As each ψ_i is a contraction, it has a unique fixed point. Let F_0' be the set of fixed points of the mappings ψ_i, $1 \leqslant i \leqslant m$. A point $x \in F_0'$ is called an essential fixed point if there exist $i, j \in \{1, \dots, m\}, i \neq j$ and $y \in F_0'$ such that $\psi_i(x) = \psi_j(y)$. We write F_0 for the set of essential fixed points. Now define

$$\psi_{i_1,\dots,i_n}(B) = \psi_{i_1} \circ \cdots \circ \psi_{i_n}(B), \quad B \subset \mathbb{R}^D.$$

The set $F_{i_1,\dots,i_n} = \psi_{i_1,\dots,i_n}(F_0)$ is called an n-cell and the set $E_{i_1,\dots,i_n} = \psi_{i_1,\dots,i_n}(F)$ an n-complex. The lattice of fixed points F_n is defined by

$$F_n = \Psi_n(F_0) \tag{2.2}$$

and the set F can be recovered from the essential fixed points by setting

$$F = \mathrm{cl}\left(\bigcup_{n=0}^{\infty} F_n\right).$$

We can now define an affine nested fractal as follows.

Definition 2.1. The set F is an affine nested fractal if $\{\psi_1, \dots, \psi_m\}$ satisfy:

(A1) (*Connectivity*) For any 1-cells C and C', there is a sequence $\{C_i\colon i = 0, \dots, n\}$ of 1-cells such that $C_0 = C$, $C_n = C'$ and $C_{i-1} \cap C_i \neq \emptyset$, $i = 1, \dots, n$.

(A2) (*Symmetry*) If $x, y \in F_0$, then reflection in the hyperplane $H_{xy} = \{z\colon |z - x| = |z - y|\}$ maps F_n to itself.

(A3) (*Nesting*) If $\{i_1, \dots, i_n\}, \{j_1, \dots, j_n\}$ are distinct sequences, then

$$\psi_{i_1,\dots,i_n}(F) \cap \psi_{j_1,\dots,j_n}(F) = \psi_{i_1,\dots,i_n}(F_0) \cap \psi_{j_1,\dots,j_n}(F_0).$$

(A4) (*Open set condition*) There is a non-empty, bounded, open set V such that the $\psi_i(V)$ are disjoint and $\bigcup_{i=1}^{m} \psi_i(V) \subset V$.

Note that the difference between nested and affine nested fractals is that affine nested fractals can have similitudes with different scale factors. We define a size class for an affine nested fractal to consist of those sets that can be mapped to each other by composition of the reflection maps in (A2). An affine nested fractal contains k size classes and, as each set in a size class must have the same length scale factor, there are k length scale factors (not necessarily different).

We fix a dimension $d > 1$ and define the family of affine nested random recursive Sierpinski gaskets based on tetrahedra in $\mathbb{R}^d$. Let $F_0 = \{z_0, \dots, z_d\}$ be the vertices of the unit equilateral tetrahedron in $\mathbb{R}^d$. Let A be a finite set and for each $a \in A$, let B_a be a bounded subset of $\mathbb{R}_+^{k_a}$ for some $k_a \in \mathbb{N}$. For each $a \in A, b \in B_a$, let

$$\psi^{a,b} = \{\psi_i^{a,b};\ i = 1, \dots, m_a\}$$

be a family of similitudes on $\mathbb{R}^d$ containing the $d+1$ essential fixed points given by F_0. The similitudes can be divided into k_a size classes and for $j \in \{1, \dots, k_a\}$ we write $m_a(j)$ or sometimes $m(a, j)$, for the number of similitudes in class j and write $l_{a,b}(j)$ or $l(a, b, j)$ for the length scale factors of the similitudes. We only allow a finite number of possible configurations of size classes but, for each possible configuration, the set of length scale factors for the similitudes lies in the possibly uncountable subset B_a

(which must be compatible with the geometry). As above there is a unique compact subset $K_{a,b}$ of $\mathbb{R}^d$ which satisfies

$$K_{a,b} = \bigcup_{i=1}^{m_a} \psi_i^{a,b}(K_{a,b}).$$

Under the open set condition (A4), this set will have Hausdorff dimension

$$d_f(K_{a,b}) = \left\{ \alpha\colon \sum_{j=1}^{k_a} m_a(j) l_{a,b}(j)^{-\alpha} = 1 \right\}.$$

We will now set up our class of random recursive Sierpinski gaskets, which is the same as that of Hambly (2000). Let $I_n = \bigcup_{k=0}^n \mathbb{N}^k$ and let $I = \bigcup_k I_k$ be the space of arbitrary length sequences. We will write $\boldsymbol{i}, \boldsymbol{j}$ for concatenation of sequences. For a point $\boldsymbol{i} \in I \backslash I_n$ denote by $\boldsymbol{i}|n$ the sequence of length n such that $\boldsymbol{i} = \boldsymbol{i}|n, \boldsymbol{k}$ for a sequence $\boldsymbol{k}$. We write $\boldsymbol{j} \leqslant \boldsymbol{i}$, if $\boldsymbol{i} = \boldsymbol{j}, \boldsymbol{k}$ for some $\boldsymbol{k}$, which provides a natural ordering on branches. Also denote by $|\boldsymbol{i}|$ the length of the sequence $\boldsymbol{i}$.

The infinite random tree, T, is a subset of the space I, defined as the sample path of a Galton–Watson process. Let the root be $T_0 = I_0 = \emptyset$, the empty sequence. Let $U_{\boldsymbol{i}}$, $\boldsymbol{i} \in I$ be independent and identically distributed A-valued random variables, indicating the type of affine nested fractal to be used, with probability distribution

$$P(U_{\boldsymbol{i}} = a) = p_a, \quad a \in A, \ \forall \boldsymbol{i} \in I.$$

Then $\boldsymbol{i} \in T$ if $\boldsymbol{i}|n \in T_n \subset I_n$ for each $1 \leqslant n \leqslant |\boldsymbol{i}|$, where $\boldsymbol{i}|n \in T_n$ if

1. $\boldsymbol{i}|n-1 \in T_{n-1}$,
2. there is a j: $1 \leqslant j \leqslant m(U_{\boldsymbol{i}|n-1})$ such that $\boldsymbol{i}|n-1, j = \boldsymbol{i}|n$.

Let $s(\boldsymbol{i})$ be the projection map which allocates to each address $\boldsymbol{i}$ the size class of the similitude $\boldsymbol{i}||\boldsymbol{i}|$. We need another random variable $V(a,\boldsymbol{i}) \in \mathbb{R}_+^{k_a}$, chosen according to Φ_a, which specifies the length scale factor. Thus the length scale factor for the ith similitude is the $s(i)$th coordinate of V, $l(U_{\boldsymbol{i}}, V(U_{\boldsymbol{i}}, \boldsymbol{i}), i) = V_{s(i)}(U_{\boldsymbol{i}}, \boldsymbol{i})$ and this is a label for each node in the tree. Let $(\Omega, \mathscr{B}, \mathbb{P})$ be a probability space. We will now denote a random tree T as a sample point $\omega \in \Omega$. The σ-algebras are defined as

$$\mathscr{B}_n = \sigma(U_{\boldsymbol{i}}, V(U_{\boldsymbol{i}}, \boldsymbol{i}); \boldsymbol{i} \in T_{n-1}(\omega)), \quad \mathscr{B} = \bigcup_{n=1}^{\infty} \mathscr{B}_n$$

and the probability measure, $\mathbb{P}$, is determined by both a Galton–Watson process, in which an individual has m_a offspring with probability p_a for $a \in A$, and a labelling process, in which each individual has a label according to Φ_U.

Let $E = E_\emptyset$ be the unit equilateral tetrahedron. Then set $E_{\boldsymbol{i}}$, $\boldsymbol{i} \in T_n$, geometrically similar to E, to be

$$E_{\boldsymbol{i}} = \psi_{\boldsymbol{i}}(E) = \psi_{\boldsymbol{i}|1}^{U_\emptyset, V_{s(\boldsymbol{i}|1)}(U_\emptyset, \emptyset)}(\cdots(\psi_{\boldsymbol{i}|n}^{U_{\boldsymbol{i}|n-1}, V_{s(\boldsymbol{i}|n)}(U_{\boldsymbol{i}|n-1}, \boldsymbol{i}|n-1)}(E))).$$

We regard $\boldsymbol{i}$ as the address of the set $E_{\boldsymbol{i}}$ and will use this notation for any sequence $\boldsymbol{i}$. A random gasket can then be defined by

$$F^\omega = \bigcap_{n=0}^{\infty} \bigcup_{\boldsymbol{i} \in T_n(\omega)} E_{\boldsymbol{i}}.$$

The Hausdorff dimension of the set F^{ω} can be found by applying the results of Falconer (1986), Mauldin and Williams (1986) and Graf (1987) and is given by

$$d_f(F^{\omega}) = \inf\left\{\alpha\colon \mathbb{E}\left(\sum_{i=1}^{m(U_\emptyset)} l(U_\emptyset, V(U_\emptyset,\emptyset), i)^{-\alpha}\right) = 1\right\} \quad \text{for a.e. } \omega \in \Omega. \qquad (2.3)$$

We conclude this section with some more notation. Firstly, we note that there is a natural projection map $\pi : T \to F$ given by $\pi(\boldsymbol{i}) = \bigcap_{j=1}^{|\boldsymbol{i}|} E_{\boldsymbol{i}|j}$. We will write $E_n(x) = E_{\boldsymbol{i}|n}$, for $\pi(\boldsymbol{i}) = x$ and $\boldsymbol{i} \in T_\infty$. We also denote a neighbourhood for a point x by

$$D_n(x) = E_n(x) \cup \bigcup_{E_n(y)\cap E_n(x)\neq\emptyset} E_n(y).$$

When on the address space we write $N_n(\boldsymbol{i}) = \{\boldsymbol{j}|n\colon \pi(\boldsymbol{j}|n) \in D_n(x), \pi(\boldsymbol{i}) = x\}$.

It will be convenient for us to approximate the fractal with a sequence of graphs and we will write F_n for the nth graph approximation, where

$$F_n = \bigcup_{\boldsymbol{i}|n\in T_n} \psi_{\boldsymbol{i}}(F_0).$$

In the next section we will construct a general branching process with ancestry described by T and such that the resistance of each edge in the graph F_n is of resistance approximately e^{-n}.

3. General branching processes

We introduce briefly C–M–J branching processes as it is the behaviour of the normalized limit of their growth rate which will provide the fluctuation of the transition density of the Brownian motion on the random fractal.

Let ξ be a point process which describes the birth events, L the life-length and ϕ, a function on $[0,\infty)$, called a random characteristic of the process. We make no assumptions about the joint distributions of these quantities. We write $\xi(t)$ for the ξ-measure of $[0,t]$ and $v(t) = \mathbb{E}\xi(t)$ for the mean reproduction measure. The basic probability space is now

$$(\Omega, \mathscr{B}, \mathbb{P}) = \prod_{\boldsymbol{i}\in I} (\Omega_{\boldsymbol{i}}, \mathscr{B}_{\boldsymbol{i}}, \mathbb{P}_{\boldsymbol{i}}),$$

where the spaces $(\Omega_{\boldsymbol{i}}, \mathscr{B}_{\boldsymbol{i}}, \mathbb{P}_{\boldsymbol{i}})$ are identical and contain independent copies of (ξ, L, ϕ). We now denote a random tree by $\omega \in \Omega$ and we will write $\theta_{\boldsymbol{i}}(\omega)$ for the subtree of ω rooted at individual $\boldsymbol{i}$. The attributes of the individual $\boldsymbol{i}$ are denoted by $(\xi_{\boldsymbol{i}}, L_{\boldsymbol{i}}, \phi_{\boldsymbol{i}})$ and the birth time of the individual is denoted by $\sigma_{\boldsymbol{i}}$.

Let $\{\sigma_{(n)}\}$ be the sequence of ordered birth times and write $(\xi_{(n)}, L_{(n)}, \phi_{(n)})$ when we refer to this time-ordered sequence. Note $\{\sigma_{(n)}\}$ is not a strictly increasing sequence. Let $\sigma_{(1)} = \sigma_\emptyset = 0$. We consider the process

$$Z^{\phi}(t) = \sum_{n:\sigma_{(n)}\leqslant t} \phi_{(n)}(t - \sigma_{(n)}).$$

That is the individuals in the population are counted according to the random characteristic ϕ.

We will assume that $\nu(0)=0$ and there exists a Malthusian parameter $\alpha>0$, such that

$$\int_0^\infty \mathrm{e}^{-\alpha t}\nu(\mathrm{d}t)=1 \quad \text{and} \quad \int_0^\infty t\mathrm{e}^{-\alpha t}\nu(\mathrm{d}t)<\infty.$$

Let $\xi_\alpha(t)=\int_0^t \mathrm{e}^{-\alpha s}\xi(\mathrm{d}s)$, and define the probability measure $\nu_\alpha(\mathrm{d}t)=\mathbb{E}(\xi_\alpha(\mathrm{d}t))$. We also assume that each individual has at least two offspring so there is no possibility of extinction and the process will be strictly supercritical. We will write

$$\nu_\alpha^\phi(t)=\mathbb{E}(\mathrm{e}^{-\alpha t}Z^\phi(t))$$

for the discounted mean of the process with random characteristic ϕ.

We define the σ-algebra determined by the first n individuals and their characteristics as

$$\mathscr{A}_n=\sigma((\xi_{(k)},L_{(k)},\phi_{(k)})\colon 1\leqslant k\leqslant n).$$

The birth times $\sigma_{(k)}$ are $\mathscr{A}_{k-1}$ measurable. Now define

$$R_n=\sum_{l=n+1}^\infty \mathrm{e}^{-\alpha\sigma_{(l)}}I_{\{l \text{ is a child of the first } n \text{ individuals}\}}.$$

Then, in our setting, $\{R_n\}_{n=1}^\infty$ is a non-negative martingale with respect to $\mathscr{A}_n$ and hence $\lim_{n\to\infty}R_n=W>0$ exists. We also state a Theorem concerning the limiting behaviour of $Z^\phi(t)$ which is a version of Nerman (1981) Theorem 5.4.

Theorem 3.1. *Let $D[0,\infty)$ denote the set of $\mathbb{R}_+$-valued cadlag paths and let ϕ be a $D[0,\infty)$-valued characteristic. We assume that*
(1) *There exists a non-increasing, bounded positive integrable function g, such that*

$$\mathbb{E}\sup_{t\geq 0}\left(\frac{\xi_\alpha(\infty)-\xi_\alpha(t)}{g(t)}\right)<\infty.$$

(2) *There exists a non-increasing, bounded positive integrable function h, such that*

$$\mathbb{E}\sup_{t>0}\left(\frac{\mathrm{e}^{-\alpha t}\phi(t)}{h(t)}\right)<\infty.$$

Then, if the reproduction process is non-lattice,

$$\lim_{t\to\infty}\mathrm{e}^{-\alpha t}Z^\phi(t)=W\nu_\alpha^\phi(\infty) \quad a.s. \tag{3.1}$$

If the lifelength distribution is lattice, then there exists a periodic function G_α^ϕ, such that for large t,

$$Z^\phi(t)=W\mathrm{e}^{\alpha t}(G_\alpha^\phi(t)+\mathrm{o}(1)) \quad a.s. \tag{3.2}$$

We define a specific general branching process to describe the fractal. Let the reproduction and lifelength be given by

$$(\xi(ds),L)=\left(\sum_{i=1}^{k_a}m_a(i)\delta_{\log x_i}(ds),\max_i \log x_i\right) \text{ with probability } p_a\Phi_a(\mathrm{d}x_1,\ldots,\mathrm{d}x_k),$$

then, if we let ϕ denote the characteristic

$$\phi_{\boldsymbol{i}}(t) = \xi_{\boldsymbol{i}}(\infty) - \xi_{\boldsymbol{i}}(t), \tag{3.3}$$

which counts the individuals born after time t to mothers born at or before time t, then the process $Z^{\phi}(t)$ is the number of sets in a e^{-t}-cover for the fractal.

4. Laplacians on random recursive Sierpinski gaskets

We define a natural Laplace operator on each possible random fractal $\omega \in \Omega$ and give some properties. For more discussion see Hambly (2000). Note that for affine nested fractals based upon the Sierpinski gasket there is no difficulty in solving the fixed point problem of Lindstrøm (1990). Recall that there are k_a size classes of set in the affine nested fractal (some of these could be the same size) and recall that $s(\boldsymbol{i}) \in \{1,\dots,k_a\}$ denotes the size class of the set with address $\boldsymbol{i}$. We can allocate a fixed resistance $r_a(j)$, $j = 1,\dots,k_a$ to all cells in a given class in the fractal K_a. Let F_0 denote the complete graph on the essential fixed points and define

$$\mathscr{E}_0(f,g) = \frac{1}{2} \sum_{x,y \in F_0} (f(x) - f(y))(g(x) - g(y))$$

for $f,g \in C(F_0)$. If we let

$$\tilde{\mathscr{E}}_1^{(a)}(f,f) = \sum_{i=1}^{m_a} r_a(s(i))^{-1} \mathscr{E}_0(f \circ \psi_i, f \circ \psi_i) = \sum_{j=1}^{k_a} \sum_{i=1}^{m(a,j)} r_a(j)^{-1} \mathscr{E}_0(f \circ \psi_i, f \circ \psi_i)$$

for $f \in C(F_1^a)$, then there is a constant λ_a such that

$$\mathscr{E}_0(f,f) = \lambda_a \inf\{\tilde{\mathscr{E}}_1^{(a)}(g,g)\colon g = f|_{F_0}\}.$$

This allows us to define the Dirichlet form for each fractal in our family A, for details see Barlow (1998). We will let $\rho_a(j) = \rho(a,j) = \lambda_a / r_a(j)$ denote the conductance of a cell of class j in the fractal.

We can define a Dirichlet form $(\mathscr{E},\mathscr{F})$ on an appropriate $L^2(F,\mu)$ for the random fractal for each $\omega \in \Omega$. As usual we build this up from a sequence of approximating Dirichlet forms on the lattice approximations to the fractal. We define the resistance of a cell with address $\boldsymbol{i}$, by

$$R(\boldsymbol{i})^{-1} = \prod_{i=1}^{|\boldsymbol{i}|} \rho(U_{\boldsymbol{i}|i-1}, s(\boldsymbol{i}|i)).$$

We can then write

$$\mathscr{E}_n^{\omega}(f,g) = \sum_{\boldsymbol{i} \in \omega_n} R(\boldsymbol{i})^{-1} \mathscr{E}_0(f \circ \psi_{\boldsymbol{i}}, f \circ \psi_{\boldsymbol{i}}).$$

By the construction of the conductances the sequence of Dirichlet forms is monotone increasing as, for $f : F \to \mathbb{R}$, we have the property that

$$\mathscr{E}_n^{\omega}(f|_{F_n}, f|_{F_n}) = \inf\{\mathscr{E}_{n+1}^{\omega}(g,g)\colon g \in C(F_{n+1}), g = f|_{F_n}\}.$$

Once we have such a sequence of Dirichlet forms we can clearly define the limiting Dirichlet form as the limit of the sequence.

To define the associated Laplace operator, we need a measure. As in Hambly (1997, 2000) we choose a measure, equivalent to the Hausdorff measure of the fractal in the resistance metric, as the limit of the invariant measures for the Markov chains on the sequence of lattice approximations in which each edge has roughly the same resistance. We modify the general branching process description of the fractal to describe this new approximation to the fractal and to obtain the measure. Let the vector of conductances $\rho_a = \{\rho_a(i), 1 \leqslant i \leqslant k_a\}$ be chosen according to the random variable $V(a, \boldsymbol{i})$ with probability measure Φ_a. As in Hambly (2000) we restrict the support of the measure to ensure that the resistance and conductance can be controlled uniformly.

Assumption 4.1. For each $a \in A$, the support B_a of the measure Φ_a, for the distribution of conductances on the cells in the fractal K_a, has each coordinate bounded away from 0 and ∞ in $\mathbb{R}_+^{k_a}$.

Note that the resistance of a component of the fractal does not have to depend on its length scale. Let

$$(\xi(ds), L) = \left(\sum_{i=1}^{k_a} m_a(i) \delta_{\log x_i}(\mathrm{d}s), \max_i \, \log x_i \right) \text{with probability } \; p_a \Phi_a(\mathrm{d}x_1, \ldots, \mathrm{d}x_{k_a}),$$

so that the offspring of an individual are born at times given by $\log \rho_a(i)$. Let ϕ denote the characteristic, defined in (3.3), which counts the number of individuals in the population born after time t to mothers born before or at time t, and denote the corresponding general branching process by $z_t^{\phi} = Z^{\phi}(t)$.

Let

$$\Lambda_n = \{ \boldsymbol{i} \in z_n^{\phi} \},$$

identify an individual with their line of descent, and then define

$$\tilde{F}_n = \bigcup_{\boldsymbol{i} \in \Lambda_n} \psi_{\boldsymbol{i}}(F_0).$$

The graph based on $\tilde{F}_n$ has approximately the same resistance for the edge of each tetrahedron, in that there exists a constant $c_{4.1} > 0$ such that $c_{4.1}\mathrm{e}^{-n} \leqslant R(\boldsymbol{i}) \leqslant \mathrm{e}^{-n}$. We will refer to the sets $E_{\boldsymbol{i}}$ for $\boldsymbol{i} \in \Lambda_n$ as n-cells.

We now define the measure μ^{ω} as a limit of a sequence of measures μ_n^{ω}. We specify the measure μ_n^{ω} on each m-complex $E_{\boldsymbol{i}}$ as

$$\mu_n^{\omega}(E_{\boldsymbol{i}}) = \frac{\sum_{\boldsymbol{j} \in \Lambda_{n-m}} R(\boldsymbol{i}, \boldsymbol{j})^{-1}}{\sum_{\boldsymbol{j} \in \Lambda_n} R(\boldsymbol{j})^{-1}}. \tag{4.1}$$

As the fractal F^{ω} is compact, by tightness there is a subsequence of the measures μ_n^{ω} which converges weakly to a limit measure μ^{ω} on the fractal F^{ω}. We can then define the Dirichlet form $(\mathscr{E}^{\omega}, \mathscr{F}^{\omega})$ on $L^2(F^{\omega}, \mu^{\omega})$ for each $\omega \in \Omega$. Note that this measure could be defined as the projection onto the fractal of a natural measure on the boundary of the tree T.

We define the Dirichlet form $(\mathscr{E}^{\omega},\mathscr{F}^{\omega})$ on the space $L^2(F^{\omega},\mu^{\omega})$ as

$$\mathscr{F}^{\omega}=\left\{f:\ \sup_n \mathscr{E}^{\omega}_n(f,f)<\infty\right\}$$

and

$$\mathscr{E}^{\omega}(f,f)=\lim_{n\to\infty}\mathscr{E}^{\omega}_n(f,f),\quad \forall f\in\mathscr{F}^{\omega}.$$

The effective resistance between two points in the random fractal F is defined by

$$r^{\omega}(x,y)=(\inf\{\mathscr{E}^{\omega}(f,f):\ f(x)=0,\ f(y)=1,\ f\in\mathscr{F}^{\omega}\})^{-1}.$$

As in Hambly (1997) we have the following estimate on the effective resistance.

Lemma 4.2. *There exist constants $c_{4.2},c_{4.3}$ such that for each edge $(x,y)\in\tilde{F}^{\omega}_n$,*

$$c_{4.2}\mathrm{e}^{-n}\leqslant r^{\omega}(x,y)\leqslant c_{4.3}\mathrm{e}^{-n},\quad \forall\omega\in\Omega.$$

From this result it is not difficult to see that the measure μ^{ω} is equivalent to the α-dimensional Hausdorff measure in the effective resistance metric.

We note that using our conductivity coordinates, and the definition of effective resistance, we can prove the following estimate on the continuity of functions in the domain $\mathscr{F}^{\omega}$.

Lemma 4.3. *There exists a constant $c_{4.4}$ such that*

$$\sup_{x,y\in E_i}|f(x)-f(y)|\leqslant c_{4.4}R(\boldsymbol{i})\mathscr{E}^{\omega}(f,f),\quad \forall f\in\mathscr{F}^{\omega},\ \forall\boldsymbol{i}\in\Lambda_m,\ \forall\omega\in\Omega.$$

By construction we have $c_{4.1}\mathrm{e}^{-m}\leqslant R(\boldsymbol{i})\leqslant\mathrm{e}^{-m}$ for $\boldsymbol{i}\in\Lambda_m$ and this shows that the domain $\mathscr{F}^{\omega}\subset C(F^{\omega})$. The first part of the following theorem follows from Lemma 4.3 and the second from the proof of Hambly (2000) Lemma 4.6.

Theorem 4.4. *The bilinear form $(\mathscr{E}^{\omega},\mathscr{F}^{\omega})$ is a local regular Dirichlet form on $L^2(F^{\omega},\mu^{\omega})$ and has the property that there exist constants $c_{4.5},c_{4.6}$ such that for all $\omega\in\Omega$, we have*

$$\sup_{x,y\in F^{\omega}}|f(x)-f(y)|\leqslant c_{4.5}\mathscr{E}^{\omega}(f,f)\quad\text{for all } f\in\mathscr{F}^{\omega}, \tag{4.2}$$

$$||f||_{\infty}\leqslant c_{4.6}\left(\mathscr{E}^{\omega}(f,f)+||f||_2^2\right)\quad\text{for all } f\in\mathscr{F}^{\omega}. \tag{4.3}$$

We can also observe a scaling property of the Dirichlet form. We write $\rho_{(1)}(j)$ for the conductance of the sets of size class j in the first division of the fractal. In the corresponding branching process the first individual has $m(U_{\emptyset},j)$ offspring at times $\log\rho_{\emptyset}(j)$.

Lemma 4.5. *We can write for all $f,g\in\mathscr{F}^{\omega}$,*

$$\mathscr{E}^{\omega}(f,g)=\sum_{i=1}^{\xi_{(1)}(\infty)}\rho_{(1)}(s(i))\mathscr{E}^{\theta_i(\omega)}(f\circ\psi_i,g\circ\psi_i).$$

5. Fluctuations in the branching process limit

We now work on $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ where the general branching process converges. By Theorem 3.1 we have that for all $\omega \in \Omega'$,

$$e^{-\alpha t} z_t^{\phi}(\omega) \to v_\alpha^{\phi}(\infty) W(\omega),$$

where α satisfies the equation

$$\mathbb{E}\left(\sum_{i=1}^{\xi_{(1)}(\infty)} \rho_{(1)}(s(i))^{-\alpha} \right) = \sum_{a \in A} \sum_{j=1}^{k_a} m_a(j) \rho_j^{-\alpha} p_a = 1. \tag{5.1}$$

The branching process with characteristic ϕ can be written, for a fixed m, by taking t large enough (using our Assumption 4.1), as

$$z_t^{\phi} = \sum_{\boldsymbol{i} \in \Lambda_m} z_{t-\sigma_i}^{\phi}(\boldsymbol{i}),$$

where $z^{\phi}(\boldsymbol{i})$ are iid copies of z^{ϕ}. Substituting the convergence result into the above, and using the definition of Λ_m we see that

$$W = \sum_{\boldsymbol{i} \in \Lambda_m} R(\boldsymbol{i})^{\alpha} W_{\boldsymbol{i}},$$

where

$$W_{\boldsymbol{i}} = W(\theta_{\boldsymbol{i}}(\omega)) = \lim_{s \to \infty} e^{-\alpha s} z_s^{\phi}(\boldsymbol{i}) / v_\alpha^{\phi}(\infty).$$

Hence, for an m-complex $E_{\boldsymbol{i}}$ in conductivity coordinates, we have

$$\mu^{\omega}(E_{\boldsymbol{i}}) = \frac{R(\boldsymbol{i})^{\alpha} W(\theta_{\boldsymbol{i}}(\omega))}{W(\omega)}. \tag{5.2}$$

By taking the characteristic $\phi_{\boldsymbol{i}}(t) = R(\boldsymbol{i})^{-1}$ and using Theorem 3.1 we can see that this is the behaviour of the limit of the sequence of measures defined by (4.1). Note that we can decompose W and hence the measure using any section of the tree ω, in particular, by looking at the offspring of the first born individual,

$$W = \sum_{i=1}^{\xi_{(1)}(\infty)} \rho_{(1)}^{-\alpha}(s(i)) W_i, \tag{5.3}$$

$$\int_{F^{\omega}} f(x) \mu^{\omega}(\mathrm{d}x) = \sum_{i=1}^{\xi_{(1)}(\infty)} \mu^{\omega}(\psi_i(F^{\theta_i(\omega)})) \int_{F^{\theta_i(\omega)}} f(\psi_i(x)) \mu^{\theta_i(\omega)}(\mathrm{d}x),$$

$$f \in C(F^{\omega}). \tag{5.4}$$

We now note some bounds on the random variable W that are essential for establishing the results of Section 6. Let $\alpha_a = \{x\colon \sum_{i=1}^{m_a} (\rho_i^{(a)})^{-x} = 1\}$ and set $\bar{\alpha} = \max_{a \in A} \alpha_a$ and $\underline{\alpha} = \min_{a \in A} \alpha_a$. As the set A is finite we have $0 < \underline{\alpha} \leqslant \bar{\alpha} < \infty$. Define

$$\beta' = \frac{\alpha}{\underline{\alpha}} - 1 \quad \text{and} \quad \beta = 1 - \frac{\alpha}{\bar{\alpha}}.$$

Note that $0 < \beta' < \infty$ and $0 < \beta < 1$.

Theorem 5.1. *There exist constants* $c_{5.i} > 0,\ i = 1,\dots,8$ *such that*

$$c_{5.1}\exp(-c_{5.2}\delta^{-1/\beta'}) \leqslant \mathbb{P}(W < \delta) \leqslant c_{5.3}\exp(-c_{5.4}\delta^{-1/\beta'}), \quad \forall \delta > 0 \tag{5.5}$$

and

$$c_{5.5}\exp(-c_{5.6}\delta^{1/\beta}) \leqslant \mathbb{P}(W > \delta) \leqslant c_{5.7}\exp(-c_{5.8}\delta^{1/\beta}), \quad \forall \delta > 0. \tag{5.6}$$

Proof. The upper bounds on both tails were given for a subclass of these fractals in Hambly (1997). The arguments used there are easily extended to the affine nested fractals discussed here.

For the lower bounds we can bound the right tail using (Liu, 1996) where exactly this problem is analysed using characteristic functions and the above lower bound obtained.

For the lower bound on the left tail we begin by estimating the Laplace transform for W, $\Phi(u) = \mathbb{E}(\exp(-uW))$. Using the worst case behaviour of the possible offspring distribution, in the same way as in the proof of Hambly (1997) Lemma 3.6 we have the existence of constants c_1, c_2 such that

$$\Phi(u) \geq c_1 \exp(-c_2 u^{\underline{\alpha}/\alpha}).$$

Now observe that

$$\begin{aligned}\Phi(u) &= \mathbb{E}(\exp(-uW)I_{\{W \geq x\}}) + \mathbb{E}(\exp(-uW)I_{\{W < x\}})\\ &\leqslant e^{-ux} + \mathbb{P}(W < x)\end{aligned}$$

and hence

$$\mathbb{P}(W < x) \geq c_1 \exp(-c_2 u^{\underline{\alpha}/\alpha}) - e^{-ux} = (c_1\exp(-c_2 u^{\underline{\alpha}/\alpha} + ux) - 1)e^{-ux} \quad \forall u > 0.$$

Choosing $u = c_3 x^{\underline{\alpha}/(\underline{\alpha}-\alpha)}$ and making suitable adjustments to the constants we have the result. □

Using these estimates we will prove bounds on the fluctuation in W; more precise estimates on the tails of W and finer results for this fluctuation (and that of the measure) can be found in Hambly and Jones (2000).

Firstly, we require two preliminary lemmas. Let $T_{k,k-1}(\boldsymbol{i}) = T_{n_k - n_{k-1}}$ be the tree with the root at $\boldsymbol{i}|n_{k-1}$ for any subsequence $\{n_k\}$ and write $\mathbb{P}_{T_k}$ for the probability measure $\mathbb{P}$ conditioned on the tree $T_{k,k-1}(\boldsymbol{i})$ and $\mathbb{E}_{T_k}$ the corresponding expectation.

Lemma 5.2. *There exist constants* $c_{5.9}, c_{5.10}$ *and* $M \in \mathbf{N}$ *such that if* $x_k = c_{5.9}(\log k)^\beta$ *and* $n_k = Mk$ *for each* $k \in \mathbf{N}$, *then*

$$\mathbb{P}_{T_k}(W_{\boldsymbol{i}|n_k} > x_k, W_{\boldsymbol{i}|n_{k-1}} < x_{k-1}) \geq c_{5.10}k^{-1}.$$

Proof. This is proved using our tail estimates on W. Define $\boldsymbol{i}(k, k-1) = \boldsymbol{j} : \boldsymbol{i}|n_k = \boldsymbol{i}|n_{k-1}, \boldsymbol{j}$, then

$$\begin{aligned}&\mathbb{P}_{T_k}(W_{\boldsymbol{i}|n_k} > x_k, W_{\boldsymbol{i}|n_{k-1}} \leqslant x_{k-1})\\ &\quad = \mathbb{P}_{T_k}\left(W_{\boldsymbol{i}|n_k} > x_k, \sum_{\boldsymbol{j} \in T_{k,k-1}(\boldsymbol{i})} R(\boldsymbol{j})^\alpha W_{\boldsymbol{i}|n_{k-1},\boldsymbol{j}} \leqslant x_{k-1}\right)\end{aligned}$$

$$= \int_{x_k}^{\infty} \mathbb{P}_{T_k}\left(\sum_{\boldsymbol{j} \neq \boldsymbol{i}(k,k-1)} R(\boldsymbol{j})^{\alpha} W_{\boldsymbol{i}|n_{k-1},\boldsymbol{j}} + R(\boldsymbol{i}(k,k-1))^{\alpha} y \leqslant x_{k-1} \right) \mathbb{P}(W_{\boldsymbol{i}|n_k} \in \mathrm{d}y)$$

$$\geq \mathbb{P}(W_{\boldsymbol{i}|n_k} \in [x_k, c_1 R(\boldsymbol{i}(k,k-1))^{-\alpha} x_{k-1}])$$

$$\times \mathbb{P}_{T_k}\left(\sum_{\boldsymbol{j} \neq \boldsymbol{i}(k,k-1)} R(\boldsymbol{j})^{\alpha} W_{\boldsymbol{i}|n_{k-1},\boldsymbol{j}} \leqslant (1-c_1) x_{k-1} \right)$$

for some constant $c_1 < 1$. Observing that the second term in the product will be bounded below by a constant c_2, setting $c_3 = (c_1 R(\boldsymbol{i}(k,k-1))^{-\alpha})^{1/\beta}$ and applying the tail estimates in (5.6), gives

$$\begin{aligned} &\mathbb{P}_{T_k}(W_{\boldsymbol{i}|n_k} > x_k, W_{\boldsymbol{i}|n_{k-1}} \leqslant x_{k-1}) \\ &\quad \geqslant c_2 (c_{5.5} \exp(-c_{5.6} x_k^{1/\beta}) - c_{5.7} \exp(-c_{5.8} c_3 x_{k-1}^{1/\beta})) \\ &\quad = c_4 k^{-c_5} (1 - c_6 \exp(-(c_{5.8} c_3 \log(k-1) - c_{5.6} \log k))). \end{aligned}$$

We can choose $c_{5.9}$ in order that $c_5 = 1$, and also M large so that c_3 is sufficiently large to make $c_6 \exp(-(c_{5.8} c_3 \log(k-1) - c_{5.6} \log k)) \leqslant \frac{1}{2}$ for all k, and hence we have the result. □

Lemma 5.3. *There exist constants $c_{5.11}, c_{5.12}$ and $M \in \boldsymbol{N}$ such that if $y_k = c_{5.11} (\log k)^{-\beta'}$ and $n_k = Mk$ for each $k \in \boldsymbol{N}$, then*

$$\mathbb{E}_{T_k}(W_{\boldsymbol{i}|n_k} I(\{W_{\boldsymbol{i}|n_k} < y_k, W_{\boldsymbol{i}|n_{k-1}} > y_{k-1}\})). \geqslant c_{5.12} y_k k^{-1}.$$

Proof. We begin by estimating the conditional distribution of W, for $0 < x < y_k$,

$$\begin{aligned} &\mathbb{P}_{T_k}(W_{\boldsymbol{i}|n_k} \in \mathrm{d}x | W_{\boldsymbol{i}|n_k} < y_k, W_{\boldsymbol{i}|n_{k-1}} > y_{k-1}) \\ &\quad = \frac{\mathbb{P}_{T_k}(W_{\boldsymbol{i}|n_k} \in \mathrm{d}x, W_{\boldsymbol{i}|n_k} < y_k, W_{\boldsymbol{i}|n_{k-1}} > y_{k-1})}{\mathbb{P}_{T_k}(W_{\boldsymbol{i}|n_k} < y_k, W_{\boldsymbol{i}|n_{k-1}} > y_{k-1})} \\ &\quad = \frac{\mathbb{P}_{T_k}(W_{\boldsymbol{i}|n_k} \in \mathrm{d}x, \sum_{\boldsymbol{j} \in T_{k,k-1}(\boldsymbol{i})} R(\boldsymbol{j})^{\alpha} W_{\boldsymbol{i}|n_{k-1},\boldsymbol{j}} > y_{k-1})}{\mathbb{P}_{T_k}(W_{\boldsymbol{i}|n_k} < y_k, \sum_{\boldsymbol{j} \in T_{k,k-1}(\boldsymbol{i})} R(\boldsymbol{j})^{\alpha} W_{\boldsymbol{i}|n_{k-1},\boldsymbol{j}} > y_{k-1})} \\ &\quad \geqslant \frac{\mathbb{P}(W_{\boldsymbol{i}|n_k} \in \mathrm{d}x)}{\mathbb{P}(W_{\boldsymbol{i}|n_k} < y_k)} \frac{\mathbb{P}_{T_k}(\sum_{\boldsymbol{j} \neq \boldsymbol{i}(k,k-1)} R(\boldsymbol{j})^{\alpha} W_{\boldsymbol{i}|n_{k-1},\boldsymbol{j}} > y_{k-1})}{\mathbb{P}_{T_k}(\sum_{\boldsymbol{j} \in T_{k,k-1}(\boldsymbol{i})} R(\boldsymbol{j})^{\alpha} W_{\boldsymbol{i}|n_{k-1},\boldsymbol{j}} > y_{k-1} | W_{\boldsymbol{i}|n_k} < x_k)}. \end{aligned}$$

As $\mathbb{P}_{T_k}(\sum_{\boldsymbol{j} \neq \boldsymbol{i}(k,k-1)} R(\boldsymbol{j})^{\alpha} W_{\boldsymbol{i}|n_{k-1},\boldsymbol{j}} > y_{k-1}) \to 1$ as $k \to \infty$ there is a constant c_1 such that for $0 < x < y_k$, for all $k > 0$,

$$\mathbb{P}_{T_k}(W_{\boldsymbol{i}|n_k} \in \mathrm{d}x | W_{\boldsymbol{i}|n_k} < y_k, W_{\boldsymbol{i}|n_{k-1}} > y_{k-1}) \geqslant c_1 \mathbb{P}(W_{\boldsymbol{i}|n_k} \in \mathrm{d}x | W_{\boldsymbol{i}|n_k} < y_k)$$

and thus

$$\mathbb{E}_{T_k}(W_{\boldsymbol{i}|n_k} | W_{\boldsymbol{i}|n_k} < y_k, W_{\boldsymbol{i}|n_{k-1}} > y_{k-1}) \geqslant c_1 \mathbb{E}(W_{\boldsymbol{i}|n_k} | W_{\boldsymbol{i}|n_k} < y_k). \tag{5.7}$$

Now

$$\mathbb{E}(W_{\boldsymbol{i}|n_k}|W_{\boldsymbol{i}|n_k} < y_k) = \int_0^{y_k} \mathbb{P}(W_{\boldsymbol{i}|n_k} > x|W_{\boldsymbol{i}|n_k} < y_k)\,\mathrm{d}x,$$
$$\geqslant c_2 y_k \mathbb{P}(W_{\boldsymbol{i}|n_k} > c_2 y_k|W_{\boldsymbol{i}|n_k} < y_k).$$

Finally, using the left tail estimates on W in (5.5), we see that if $c_2 < (c_{5.4}/c_{5.2})^{\beta'}$, then there exists a constant c_3 such that

$$\mathbb{E}(W_{\boldsymbol{i}|n_k}|W_{\boldsymbol{i}|n_k} < y_k) \geqslant c_3 y_k. \tag{5.8}$$

To complete the proof we follow the same arguments as in Lemma 5.2 and hence choose the constants $c_{5.11}$ and M to obtain the estimate

$$\mathbb{P}_{T_k}(W_{\boldsymbol{i}|n_k} < y_k, W_{\boldsymbol{i}|n_{k-1}} > y_{k-1}) \geqslant c_4 k^{-1}.$$

Putting this, (5.7) and (5.8) together gives the result. □

Theorem 5.4. *There exist constants* $c_{5.i} > 0$, $i = 13, \ldots, 16$ *such that* $\mathbb{P}$*-a.s.*

$$c_{5.13} \leqslant \liminf_{n\to\infty} \frac{W_{\boldsymbol{i}|n}}{(\log n)^{-\beta'}} \leqslant c_{5.14}, \quad \mu\text{-a.e. } \boldsymbol{i} \in T$$

and

$$c_{5.15} \leqslant \limsup_{n\to\infty} \frac{W_{\boldsymbol{i}|n}}{(\log n)^{\beta}} \leqslant c_{5.16}, \quad \mu\text{-a.e. } \boldsymbol{i} \in T.$$

Proof. We begin with the lim sup case. Recall that we have assumed that we are working in Ω' in which the $W_{\boldsymbol{i}}$ exist and are non-zero.

For the upper bound we use the first Borel–Cantelli lemma and need to show that almost surely under $\mathbb{P}$ there is a constant c_1 such that

$$\sum_n \mu\left(\frac{W_{\boldsymbol{i}|n}}{(\log n)^{\beta}} > c_1\right) < \infty.$$

By definition of the measure it is enough to show that

$$\mathbb{E}\sum_n \sum_{\boldsymbol{i}\in T_n} R(\boldsymbol{i}|n)^{\alpha} W_{\boldsymbol{i}|n} I_{\{W_{\boldsymbol{i}|n} > c_1(\log n)^{\beta}\}} < \infty.$$

By construction we have an upper bound on $R(\boldsymbol{i}|n) \leqslant c_2 \mathrm{e}^{-\alpha n}$. Now, conditioning on the tree T_n and using the independence of T_n and $W_{\boldsymbol{i}|n}$, we have

$$\mathbb{E}\sum_n \sum_{\boldsymbol{i}\in T_n} R(\boldsymbol{i}|n)^{\alpha} W_{\boldsymbol{i}|n} I_{\{W_{\boldsymbol{i}|n} > c_1(\log n)^{\beta}\}} \leqslant \sum_n \mathrm{e}^{-n\alpha}\mathbb{E}(z_n^{\phi})\mathbb{E}(W I_{\{W > c_1(\log n)^{\beta}\}}).$$

As $\mathrm{e}^{-n\alpha}\mathbb{E}(z_n^{\phi}) \leqslant c_3$, we just require an estimate on $\mathbb{E}(W I_{\{W > c_1(\log n)^{\beta}\}})$. Using an integration by parts and the estimate given in Theorem 5.1, we have

$$\mathbb{E}(W I_{\{W > c_1(\log n)^{\beta}\}}) \leqslant c_4 (\log n)^{1-1/\beta} n^{-c_5}.$$

Thus by a suitable adjustment of the constant c_5 we have the result.

For the lower bound, it is enough to prove that almost surely under $\mathbb{P}$,

$$\mu(W_{\boldsymbol{i}|n_k} \geqslant c_{5.15}(\log k)^{\beta} \text{ i.o.}) = 1,$$

where $\{n_k\}$ is the subsequence which appeared in Lemma 5.2. Using the proof of the second Borel–Cantelli lemma, if F_n is a sequence of events, then

$$\mu(\limsup F_n) = 1, \quad \text{if } \sum_n \mu(F_n|F_{n-1}^c, \dots, F_1^c) = \infty,$$

where $A_1, \dots, A_n \equiv A_1 \cap \dots \cap A_n$. Let the events $F_k = \{\boldsymbol{i} \in T_{n_k}\colon W_{\boldsymbol{i}|n_k} \geqslant c_1(\log k)^\beta)\}$. We can write

$$\mu(F_k|F_{k-1}^c, \dots, F_1^c) = \left(1 + \frac{\mu(F_{k-1}^c, \dots, F_1^c|F_k^c)\mu(F_k^c)}{\mu(F_{k-1}^c, \dots, F_1^c|F_k)\mu(F_k)}\right)^{-1}. \tag{5.9}$$

As the sequence $W_{\boldsymbol{i}|n}$ has a Markov structure, in that

$$W_{\boldsymbol{i}|n_{k-1}} = \sum_{\boldsymbol{j} \in T_{k,k-1}(\boldsymbol{i})} R(\boldsymbol{j})^\alpha W_{\boldsymbol{i}|n_{k-1}, \boldsymbol{j}},$$

we can estimate the term

$$\frac{\mu(F_{k-1}^c, \dots, F_1^c|F_k^c)\mu(F_k^c)}{\mu(F_{k-1}^c, \dots, F_1^c|F_k)\mu(F_k)}. \tag{5.10}$$

This gives, for $1 \leqslant i \leqslant k-2$,

$$\mu(F_i^c|F_{i+1}^c, \dots, F_k^c) = \mu(F_i^c|F_{i+1}^c, \dots, F_k) = \mu(F_i^c|F_{i+1}^c).$$

Substituting this into (5.10) and cancelling leaves

$$\frac{\mu(F_{k-1}^c, \dots, F_1^c|F_k^c)\mu(F_k^c)}{\mu(F_{k-1}^c, \dots, F_1^c|F_k)\mu(F_k)} = \frac{\mu(F_k^c, F_{k-1}^c)}{\mu(F_k, F_{k-1}^c)}.$$

Note that the top term is bounded above by 1. For the bottom term we will write $x_k = c_1(\log k)^\beta$, and observe that

$$\mu(F_k, F_{k-1}^c) = \sum_{\boldsymbol{i} \in T_{n_k}} \mathrm{e}^{-\alpha\sigma_{\boldsymbol{i}|n_k}} W_{\boldsymbol{i}|n_k} I(\{W_{\boldsymbol{i}|n_k} > x_k, W_{\boldsymbol{i}|n_{k-1}} \leqslant x_{k-1}\}). \tag{5.11}$$

We will prove that there exists a constant c_6 such that

$$\mu(F_k, F_{k-1}^c) \geqslant c_6 k^{-1}, \quad \forall k \in \mathbb{N}, \quad \mathbb{P}\text{-a.s.} \tag{5.12}$$

With this bound we have that

$$\sum_k \mu(F_k|F_{k-1}^c, \dots, F_1^c) = \infty, \quad \mathbb{P}\text{-a.s.},$$

which gives the result.

Thus all we need to establish is (5.12). Rewriting (5.11), we have

$$\mu(F_k, F_{k-1}^c) \geqslant x_k |T_{n_{k-1}}| \mathrm{e}^{-\alpha n_k} B_k,$$

where $|T_n|$ denotes the size of the tree T_n and

$$B_k = \frac{1}{|T_{n_{k-1}}|} \sum_{\boldsymbol{i} \in T_{n_k}} I(\{W_{\boldsymbol{i}|n_k} > x_k, W_{\boldsymbol{i}|n_{k-1}} \leqslant x_{k-1}\}).$$

By the convergence of the general branching process we have $|T_{n_{k-1}}| \mathrm{e}^{-\alpha n_{k-1}} \to c_7$ as $k \to \infty$. We will use the independence of the $W_{\boldsymbol{i}|n}$ for fixed n and a straightforward large deviation approach to estimate the behaviour of B_k. Let

$$X_{\boldsymbol{i}|n_{k-1}} = \mathrm{e}^{-\alpha(n_k - n_{k-1})} \sum_{j \in T_{k,k-1}(\boldsymbol{i})} I_{\{W_{\boldsymbol{i}|n_{k-1},j} > x_k, W_{\boldsymbol{i}|n_{k-1}} \leqslant x_{k-1}\}}$$

and also let $\tilde{X}_{\boldsymbol{i}|n_{k-1}} = X_{\boldsymbol{i}|n_{k-1}} - \mathbb{E}(X_{\boldsymbol{i}|n_{k-1}})$ and $\tilde{B}_k = \sum_{\boldsymbol{i}\in T_{n_{k-1}}} \tilde{X}_{\boldsymbol{i}|n_{k-1}}$. With these definitions we have

$$B_k = \frac{\tilde{B}_k}{|T_{n_{k-1}}|} + \mathbb{E}(X_{\boldsymbol{i}|n_{k-1}}). \tag{5.13}$$

Now, using Markov's inequality and the independence,

$$\begin{aligned}\mathbb{P}(\tilde{B}_k < -x) &= \mathbb{P}(\mathrm{e}^{-\theta\tilde{B}_k} > \mathrm{e}^{\theta x}),\\ &\leqslant \mathrm{e}^{-\theta x}\mathbb{E}(\mathrm{e}^{-\theta\tilde{B}_k})\\ &= \mathrm{e}^{-\theta x}\phi_k(\theta)^{|T_{n_{k-1}}|},\end{aligned}$$

where $\phi_k(\theta) = E\exp(-\theta\tilde{X}_{\boldsymbol{i}|n_{k-1}})$ (which exists as $\tilde{X}_{\boldsymbol{i}|n_{k-1}}$ is bounded below). We now recall the elementary inequalities that $1-x \leqslant \mathrm{e}^{-x}$ for $x\in\mathbb{R}$ and $\mathrm{e}^{-x} \leqslant 1 - x + \frac{1}{2}x^2$ for $x\in\mathbb{R}_+$. Applying the upper bound we have

$$\begin{aligned}\phi_k(\theta) &= \mathbb{E}\mathrm{e}^{-\theta(X_{\boldsymbol{i}|n_{k-1}} - \mathbb{E}(X_{\boldsymbol{i}|n_{k-1}}))}\\ &\leqslant E\left(1 - \theta(X_{\boldsymbol{i}|n_{k-1}} - \mathbb{E}(X_{\boldsymbol{i}|n_{k-1}})) + \tfrac{1}{2}\theta^2(X_{\boldsymbol{i}|n_{k-1}} - \mathbb{E}(X_{\boldsymbol{i}|n_{k-1}}))^2\right)\\ &\leqslant 1 + \tfrac{1}{2}\theta^2 \mathrm{var}\,(X_{\boldsymbol{i}|n_{k-1}})\end{aligned}$$

and then the lower bound,

$$\mathbb{P}(\tilde{B}_k < -x) \leqslant \exp(-\theta x + \tfrac{1}{2}\theta^2\,\mathrm{var}(X_{\boldsymbol{i}|n_{k-1}})|T_{n_{k-1}}|).$$

Optimizing over θ we have

$$\mathbb{P}(\tilde{B}_k < -x) \leqslant \exp\left(-\frac{1}{2}\frac{x^2}{\mathrm{var}(X_{\boldsymbol{i}|n_{k-1}})|T_{n_{k-1}}|}\right).$$

Hence, choosing $x = c_8\mathbb{E}X_{\boldsymbol{i}|n_{k-1}}|T_{n_{k-1}}|$ for some constant $c_8 < 1$, and using the relationship in (5.13), we have a constant $c_9 = \frac{1}{2}(1-c_8)^2 > 0$, such that

$$\mathbb{P}(B_k < (1-c_8)\mathbb{E}X_{\boldsymbol{i}|n_{k-1}}) \leqslant \exp\left(-c_9|T_{n_{k-1}}|\frac{(\mathbb{E}X_{\boldsymbol{i}|n_{k-1}})^2}{\mathrm{var}(X_{\boldsymbol{i}|n_{k-1}})}\right). \tag{5.14}$$

We estimate $\mathbb{E}X_{\boldsymbol{i}|n_{k-1}}$ by conditioning on the tree to get

$$\mathbb{E}X_{\boldsymbol{i}|n_{k-1}} \geqslant E(T_{k,k-1}(\boldsymbol{i}))\mathrm{e}^{-\alpha(n_k - n_{k-1})}\mathbb{P}_{T_k}(W_{\boldsymbol{i}|n_k} > x_k, W_{\boldsymbol{i}|n_{k-1}} < x_{k-1}).$$

Thus, using Lemma 5.2, there is a subsequence such that

$$\mathbb{E}X_{\boldsymbol{i}|n_{k-1}} \geqslant c_{10}k^{-1}.$$

As $\mathrm{var}\,(X_{\boldsymbol{i}|n_{k-1}})$ is bounded above by a constant we see that there is exponential convergence in (5.14) and hence $B_k \geqslant c_{11}k^{-1}$ almost surely. Thus we have established (5.12) which concludes the proof for the lim sup result.

We now turn to the lim inf case, which is similar but requires some modification. For the lower bound on the lim inf we use the same argument as for the upper bound in the lim sup case. For the upper bound we can argue as in the lim sup case and hence we need to establish (5.12) for the appropriate choice of events $F_k = \{W_{\boldsymbol{i}|n_k} < y_k, W_{\boldsymbol{i}|n_{k-1}} > y_{k-1}\}$ where $y_k = c_{12}(\log k)^{-\beta'}$.

However, if we write (5.11) in this case, we only have

$$\mu(F_k, F_{k-1}^c) \geqslant |T_{n_{k-1}}| e^{-\alpha n_k} B'_k,$$

where

$$B'_k = \frac{1}{|T_{n_{k-1}}|} \sum_{\boldsymbol{i} \in T_{n_k}} W_{\boldsymbol{i}|n_k} I(\{W_{\boldsymbol{i}|n_k} > y_k, W_{\boldsymbol{i}|n_{k-1}} \leqslant y_{k-1}\}).$$

To apply the large deviation argument we write

$$X'_{\boldsymbol{i}|n_{k-1}} = e^{-\alpha(n_k - n_{k-1})} \sum_{\boldsymbol{j} \in T_{k,k-1}(\boldsymbol{i})} W_{\boldsymbol{i}|n_{k-1},\boldsymbol{j}} I_{\{W_{\boldsymbol{i}|n_{k-1},\boldsymbol{j}} > y_k, W_{\boldsymbol{i}|n_{k-1}} \leqslant y_{k-1}\}}$$

and $\tilde{X}'_{\boldsymbol{i}|n_{k-1}} = X'_{\boldsymbol{i}|n_{k-1}} - \mathbb{E}(X'_{\boldsymbol{i}|n_{k-1}})$ and $\tilde{B}'_k = \sum_{\boldsymbol{i} \in T_{n_{k-1}}} \tilde{X}'_{\boldsymbol{i}|n_{k-1}}$. Now, as before we can show using elementary inequalities and optimizing over θ, that

$$\mathbb{P}(\tilde{B}'_k < -x) \leqslant \exp\left(-\frac{1}{2} \frac{x^2}{\mathrm{var}(X'_{\boldsymbol{i}|n_{k-1}})|T_{n_{k-1}}|} \right). \tag{5.15}$$

Hence, choosing $x = c_{13} \mathbb{E} X'_{\boldsymbol{i}|n_{k-1}} |T_{n_{k-1}}|$ for some constant c_{13}, we have a constant $c_{14} > 0$, such that

$$\mathbb{P}(B'_k < (1 - c_{13}) \mathbb{E} X'_{\boldsymbol{i}|n_{k-1}}) \leqslant \exp\left(-c_{14} \frac{(\mathbb{E} X'_{\boldsymbol{i}|n_{k-1}})^2}{\mathrm{var}(X'_{\boldsymbol{i}|n_{k-1}})|T_{n_{k-1}}|} \right).$$

We condition on the tree and on this occasion use Lemma 5.3 to show that there is a constant c_{15} and a subsequence such that

$$\mathbb{E} X'_{\boldsymbol{i}|n_{k-1}} \geqslant c_{15} y_k k^{-1}.$$

Thus

$$\sum_k \mathbb{P}(B'_k < (1 - c_{13}) y_k k^{-1}) < \infty$$

and $B'_k \geqslant (1 - c_{13}) y_k k^{-1}$ eventually almost surely. As y_k decreases as a logarithm, we can adjust the constants to ensure that $\sum_k \mu(F_k, F_{k-1}^c)$ diverges and hence we have the lim inf result. □

For the spatial fluctuation we can prove the following result in a similar but much more straightforward manner, see Liu (1999) and Hambly and Jones (2000).

Theorem 5.5. *There exist constants* $c_{5.i} > 0$, $i = 17, \ldots, 20$ *such that* $\mathbb{P}$*-a.s.*

$$c_{5.17} \leqslant \lim_{n \to \infty} \inf_{\boldsymbol{i}|n \in T_n} \frac{W_{\boldsymbol{i}|n}}{n^{-\beta'}} \leqslant c_{5.18}$$

and

$$c_{5.19} \leqslant \lim_{n\to\infty} \sup_{\boldsymbol{i}|n\in T_n} \frac{W_{\boldsymbol{i}|n}}{n^{\beta}} \leqslant c_{5.20}.$$

6. Fluctuation in the transition density

In this section we will omit reference to the underlying probability space unless required. There is a transition semigroup associated with the Dirichlet form $(\mathscr{E}, \mathscr{F})$ and it will have a transition density $p_t(x, y)$ with respect to the measure μ. As the Dirichlet form is local and regular there is also an associated Feller diffusion process $(\{X_t\}_{t\geqslant 0}, P^x,\ x \in F)$. In Hambly (1997) bounds were found on $p_t(x, y)$ which were not tight and indicated that there might be fluctuation in the heat kernel in space. Here we will show that this fluctuation occurs in both time and space.

We will take our first step toward uncovering the temporal fluctuation in the heat kernel by proving that there is some fluctuation in the on-diagonal Green density. The results for the heat kernel will then follow from Tauberian theorems.

Let $g_A(x, y)$ denote the Green density for the process killed upon leaving the set A, and $g_\lambda(x, y)$ denote the λ-Green density. Let $T_{D_n(x)} = \inf\{t \geqslant 0\colon X_t \notin D_n(x)\}$ be the exit time for the set $D_n(x)$. Observe that from Hambly (1997) we have the following estimate on the λ-Green density in terms of the killed Green density,

$$g_\lambda(x,x) \geqslant g_{D_n(x)}(x,x) P^x(T_{D_n(x)} < \zeta_\lambda),$$

where ζ_λ is an independent exponentially distributed random variable with mean $1/\lambda$. As it is possible to estimate the Green density for the process killed on exiting the set $D_n(x)$ we have a constant $c_{6.1}$ such that

$$g_\lambda(x,x) \geqslant c_{6.1}\mathrm{e}^{-n} \quad \text{if } P^x(T_{D_n} > \zeta_\lambda) \leqslant \tfrac{1}{2}. \tag{6.1}$$

By an elementary modification in the proof of Lemma 7.9 of Hambly (1997) we have the following.

Lemma 6.1. *There exists a constant* $c_{6.2} > 0$ *such that*

$$P^x(T_{D_n} > \zeta_\lambda) \leqslant \tfrac{1}{2} \quad \textit{if } \lambda E^x T_{D_n(x)} \leqslant c_{6.2}. \tag{6.2}$$

The following lemma gives control on the exit time from a neighbourhood.

Lemma 6.2. *There exist constants* $c_{6.3}, c_{6.4} > 0$ *such that*

$$c_{6.3}\mathrm{e}^{-(\alpha+1)n} W_{\boldsymbol{i}|n} \leqslant E^x T_{D_n(x)} \leqslant c_{6.4}\mathrm{e}^{-(\alpha+1)n} \left(\sum_{\boldsymbol{j}\in N_n(\boldsymbol{i})} W_{\boldsymbol{j}|n} \right), \quad \forall x \in F.$$

Proof. From Hambly (1997) Section 6 we can control the supremum of the exit times from a cell. Observe that there exists a fixed constant c_1 such that

$$\sup_{y,z\in D_n(x)} g_{D_n(x)}(y,z) \leqslant \sup_{z\in D_n(x)} g_{D_n(x)}(z,z) \leqslant c_1 \mathrm{e}^{-n}.$$

Thus, for the upper bound, we have

$$\sup_{y\in D_n(x)} E^y T_{D_n(x)} = \sup_{y\in D_n(x)} \int_{D_n(x)} g_{D_n(x)}(y,z)\mu(\mathrm{d}z)$$

$$\leqslant \int_{D_n(x)} \sup_{y\in D_n(x)} g_{D_n(x)}(y,y)\mu(\mathrm{d}z)$$

$$\leqslant c_1 \mathrm{e}^{-n}\mu(D_n(x)).$$

By using the results of Section 6 of Hambly (1997) it can be shown that there exists a constant c_2 such that

$$g_{D_n(x)}(x,y) \geqslant c_2 \mathrm{e}^{-n}, \quad \forall x,y \in E_n(x).$$

Using this there is a lower bound on the exit time,

$$E^x T_{D_n(x)} \geqslant c_3 \mathrm{e}^{-n}\mu(E_n(x)), \quad \forall x \in F,$$

and using (5.2) gives the result. □

We prove our fluctuation result in two parts, the upper and the lower bounds.

Proposition 6.3. *There exist constants $c_{6.5}, c_{6.6} > 0$ such that $\mathbb{P}$-a.s.*

$$c_{6.5} \leqslant \liminf_{\lambda\to\infty} \frac{g_\lambda(x,x)}{\lambda^{-1/(\alpha+1)}(\log\log\lambda)^{-\beta'/(\alpha+1)}}, \quad \mu\text{-a.e. } x \in F \tag{6.3}$$

and

$$c_{6.6} \leqslant \limsup_{\lambda\to\infty} \frac{g_\lambda(x,x)}{\lambda^{-1/(\alpha+1)}(\log\log\lambda)^{\beta/(\alpha+1)}}, \quad \mu\text{-a.e. } x \in F, \tag{6.4}$$

Proof. We first prove (6.4). Using (6.2) in (6.1), we see that

$$g_\lambda(x,x) \geqslant c_{6.1}\mathrm{e}^{-n} \quad \text{if } \lambda E^x T_{D_n(x)} \leqslant c_{6.2}. \tag{6.5}$$

Now, from applying Theorem 5.4 in Lemma 6.2, for μ-a.e. $x \in F$ we can take a subsequence m_k such that

$$E^x(T_{D_{m_k}}) \geqslant c_1(\log m_k)^\beta \exp(-(\alpha+1)m_k).$$

Further, if we define the sequence $\{\lambda_k\}$ by

$$\lambda_k \leqslant c_1^{-1}(\log m_k)^{-\beta}\exp((\alpha+1)m_k),$$

we have

$$\mathrm{e}^{-m_k} \geqslant \lambda_k^{-1/(\alpha+1)}(\log\log\lambda_k)^{\beta/(\alpha+1)}.$$

Replacing this in (6.5) we have (6.4), the lower bound on the lim sup.

For the lower bound on the lim inf we take the full sequence $W_{\boldsymbol{i}|n}$ so that

$$E^x(T_{D_n}) \geqslant c_2(\log n)^{-\beta'}\exp(-(\alpha+1)n) \quad \text{for } \mu\text{-a.e. } x \in F.$$

Then, if

$$\lambda_n \leqslant c_2^{-1}(\log n)^{\beta'}\exp((\alpha+1)n),$$

we have

$$e^{-n} \geqslant \lambda_n^{-1/(\alpha+1)}(\log\log\lambda_n)^{-\beta'/(\alpha+1)}.$$

Replacing this in (6.5) we have (6.3). □

We can also tackle the upper bound using the scaling argument of Hambly et al. (2000a). Let

$$\mathscr{F}_0^{\omega} = \{f \in \mathscr{F}^{\omega}: f|_{F_0} = 0\}, \quad \mathscr{F}_i^{\theta_i(\omega)} = \{f \in \mathscr{F}^{\omega}: f \circ F_i \in \mathscr{F}^{\omega}\},$$

$$\hat{\mathscr{F}}^{\omega} = \{f \in \boldsymbol{L}^2(F,\mu): \exists f_i \in \mathscr{F}_i^{\theta_i(\omega)}, f|_{\psi_i(F)\setminus F_1} = f_i\},$$

$$\hat{\mathscr{F}}_0^{\omega} = \{f \in \mathscr{F}^{\omega}: f|_{F_1} = 0\}.$$

Set $\hat{\mathscr{E}}^{\omega}(f,g) = \sum_i \rho_\emptyset \mathscr{E}^{\theta_i(\omega)}(f_i,g_i)$ for $f,g \in \hat{\mathscr{F}}^{\omega}$. Then, $(\hat{\mathscr{E}}^{\omega}, \hat{\mathscr{F}}^{\omega})$ is a regular local Dirichlet form on $\boldsymbol{L}^2(F,\mu)$. Note that

$$\hat{\mathscr{F}}_0^{\omega} \subset \mathscr{F}_0^{\omega} \subset \mathscr{F}^{\omega} \subset \hat{\mathscr{F}}^{\omega}. \tag{6.6}$$

Let $g_\lambda^{0,\omega}, g_\lambda^{\omega}$ be the λ-order reproducing kernels for $(\mathscr{E}^{\omega}, \mathscr{F}_0^{\omega}), (\mathscr{E}^{\omega}, \mathscr{F}^{\omega})$, respectively. Also, let $\hat{g}_\lambda^{0,\omega}, \hat{g}_\lambda^{\omega}$ be the λ-order reproducing kernels for $(\mathscr{E}^{\omega}, \hat{\mathscr{F}}_0^{\omega}), (\hat{\mathscr{E}}^{\omega}, \hat{\mathscr{F}}^{\omega})$, respectively. Note that by the same proof as that of Lemma 4.3, the former kernels are continuous on $F \times F$ and the latter are continuous on $\bigcup_{i=1}^{N_\emptyset} \psi_i(F) \times \psi_i(F)$.

Lemma 6.4. *For each $x \in F\setminus F_1$,*

$$\hat{g}_\lambda^{0,\omega}(x,x) \leqslant g_\lambda^{0,\omega}(x,x) \leqslant g_\lambda^{\omega}(x,x) \leqslant \hat{g}_\lambda^{\omega}(x,x).$$

Proof. Noting that

$$\frac{1}{g_\lambda^{\omega}(x,x)} = \inf_{u \in \mathscr{L}_x} \mathscr{E}_\lambda^{\omega}(u,u), \tag{6.7}$$

where $\mathscr{L}_x = \{u \in \mathscr{F}^{\omega}: u(x) \geqslant 1\}$ and $\mathscr{E}_\lambda^{\omega}(u,u) = \mathscr{E}^{\omega}(u,u) + \lambda||u||_2^2$ (similar formulae also hold for $\hat{g}_\lambda^{0,\omega}, g_\lambda^{0,\omega}, \hat{g}_\lambda^{\omega}$), we obtain the result using (6.6) (see Hambly et al., 2000a). □

Lemma 6.5. *For $x \in F\setminus F_1$, we have*

$$g_\lambda^{0,\omega}(x,x) = \rho_\emptyset \hat{g}^{0,\omega}_{\rho_\emptyset \mu(E_i)^{-1}\lambda}(\psi_i(x), \psi_i(x)),$$

$$g_\lambda^{\omega}(x,x) = \rho_\emptyset \hat{g}^{\omega}_{\rho_\emptyset \mu(E_i)^{-1}\lambda}(\psi_i(x), \psi_i(x)).$$

Proof. This can be proved by an application of the decomposition of the Dirichlet form in Lemma 4.5, the decomposition of the L^2 norm of functions in domain (5.4), with the definition of the λ-Green density in (6.7). □

Iterating Lemma 6.4 using Lemma 6.5 and setting $\tau_{\boldsymbol{i}|n} = \rho_{\boldsymbol{i}|n}\mu(\psi_{\boldsymbol{i}|n}(F))^{-1}$, we have for $\bar{x}_n \in F\setminus F_0$,

$$\begin{aligned} g_\lambda^{0,\omega}(\bar{x}_n,\bar{x}_n) &\leqslant \rho_{\boldsymbol{i}|n} g^{0,\omega}_{\tau_{\boldsymbol{i}|n}\lambda}(\psi_{\boldsymbol{i}|n}(\bar{x}_n), \psi_{\boldsymbol{i}|n}(\bar{x}_n)) \\ &\leqslant \rho_{\boldsymbol{i}|n} g^{\omega}_{\tau_{\boldsymbol{i}|n}\lambda}(\psi_{\boldsymbol{i}|n}(\bar{x}_n), \psi_{\boldsymbol{i}|n}(\bar{x}_n)) \leqslant g_\lambda^{\omega}(\bar{x}_n,\bar{x}_n). \end{aligned} \tag{6.8}$$

Now for $x \in F \backslash F_\infty$ and $\lambda > 1$ we can choose n such that $\tau_{\boldsymbol{i}|n-1} \leqslant \lambda < \tau_{\boldsymbol{i}|n}$. By our choice of λ, and the fact that g_λ is decreasing in λ, we have that

$$\rho_{\boldsymbol{i}|n-1} g^{\omega}_{\tau_{\boldsymbol{i}|n}}(x,x) \leqslant \rho_{\boldsymbol{i}|n} g^{\omega}_{\lambda}(x,x) \leqslant \rho_{\boldsymbol{i}|n} g^{\omega}_{\tau_{\boldsymbol{i}|n-1}}(x,x).$$

If we write $x = \psi_{\boldsymbol{i}|n}(\bar{x}_n)$ and apply (6.8) with $\lambda = 1$ to both sides of the above inequality, there are constants $c_{6.7}, c_{6.8}$ such that

$$g_1^{0,\omega}(\bar{x}_n, \bar{x}_n) \leqslant \rho_{\boldsymbol{i}|n} g^{\omega}_{\lambda}(x,x) \leqslant c_{6.7} g^{\omega}_1(\psi_{i_n}(\bar{x}_n), \psi_{i_n}(\bar{x}_n)) \leqslant c_{6.8}, \tag{6.9}$$

where $c_{6.8} = c_{6.7} \max_{x \in F^\omega} g^{\omega}_1(x,x)$. We can check that this constant is deterministic following the proof of Fitzsimmons et al. (1994) Theorem 4.1. As, applying (4.3), we see

$$\sup_{x \in F^\omega} \sqrt{g^{\omega}_1(x,x)} \leqslant \sup_{x \in F^\omega} \frac{\sup_{y \in F^\omega} g^{\omega}_1(x,y)}{\sqrt{g^{\omega}_1(x,x)}} \leqslant \sup_{f \in \mathscr{F}^\omega} \frac{\sup_{y \in F^\omega} |f(y)|}{\sqrt{\mathscr{E}^\omega(f,f) + ||f||_2^2}} \leqslant c_{4.6}.$$

We summarize in the following lemma.

Lemma 6.6. *If $\tau_{\boldsymbol{i}|n-1} = \mathrm{e}^{(\alpha+1)n} W^{-1}_{\boldsymbol{i}|n} \leqslant \lambda$, then there exists a constant $c_{6.9}$ such that $g^{\omega}_{\lambda}(x,x) \leqslant c_{6.9} \rho^{-1}_{\boldsymbol{i}|n}$.*

Proposition 6.7. *There are constants $c_{6.10}, c_{6.11} > 0$ such that $\mathbb{P}$-a.s.*

$$\liminf_{\lambda \to \infty} \frac{g_\lambda(x,x)}{\lambda^{-1/(\alpha+1)} (\log\log\lambda)^{-\beta'/(\alpha+1)}} \leqslant c_{6.10}, \quad \mu\text{-a.e. } x \in F$$

and

$$\limsup_{\lambda \to \infty} \frac{g_\lambda(x,x)}{\lambda^{-1/(\alpha+1)} (\log\log\lambda)^{\beta/(\alpha+1)}} \leqslant c_{6.11}, \quad \mu\text{-a.e. } x \in F.$$

Proof. Observe that due to the estimates we have on $\rho_{\boldsymbol{i}|n}$ and μ, there exist constants such that

$$c_1 W^{-1}_{\boldsymbol{i}|n} \mathrm{e}^{(\alpha+1)n} \leqslant \tau_{\boldsymbol{i}|n} \leqslant c_2 W^{-1}_{\boldsymbol{i}|n} \mathrm{e}^{(\alpha+1)n}, \quad \forall \boldsymbol{i},\ \forall n.$$

By Theorem 5.4, for μ-a.e. $x \in F$ we can choose a subsequence $\{n_k\}$ such that

$$W^{-1}_{\boldsymbol{i}|n_k} \geqslant c_3 (\log n_k)^{\beta'}.$$

Thus, if

$$\lambda_k \geqslant c_3^{-1} (\log n_k)^{\beta'} \exp(-(\alpha+1) n_k),$$

we have for μ-a.e. $x \in F$,

$$g_{\lambda_k}(x,x) \leqslant \rho^{-1}_{\boldsymbol{i}|n_k} \leqslant \mathrm{e}^{-n_k} \leqslant \lambda_k^{-1/(\alpha+1)} (\log\log\lambda_k)^{-\beta'/(\alpha+1)}.$$

Replacing this in the above we have the bound on the lim inf. If we just use the worst-case bound for $W^{-1}_{\boldsymbol{i}|n}$ in Theorem 5.4, as in the demonstration of (6.3), we have the lim sup upper bound. $\square$

We combine the above bounds in order to state a theorem for the fluctuation in the Green density.

Theorem 6.8. *There exist constants* $c_{6.i} > 0,\ i = 5, 6, 10, 11$ *such that* $\mathbb{P}$*-a.s.*

$$c_{6.5} \leqslant \liminf_{\lambda\to\infty} \frac{g_\lambda(x,x)}{\lambda^{-1/(\alpha+1)}(\log\log\lambda)^{-\beta'/(\alpha+1)}} \leqslant c_{6.10}, \quad \mu\text{-a.e. } x \in F \tag{6.10}$$

and

$$c_{6.6} \leqslant \limsup_{\lambda\to\infty} \frac{g_\lambda(x,x)}{\lambda^{-1/(\alpha+1)}(\log\log\lambda)^{\beta/(\alpha+1)}} \leqslant c_{6.11}, \quad \mu\text{-a.e. } x \in F. \tag{6.11}$$

We now need Tauberian-type arguments to obtain the limit result for the transition density. The fluctuation prevents us from using Karamata's Tauberian theorem and we thus take a bare hands approach.

Theorem 6.9. *There exist constants* $c_{6.12}, c_{6.13} > 0$ *such that* $\mathbb{P}$*-a.s.*

$$c_{6.12} \leqslant \limsup_{t\to 0} \frac{p_t(x,x)}{t^{-\alpha/(\alpha+1)}(\log|\log t|)^{\beta/(\alpha+1)}} \leqslant c_{6.13}, \quad \mu\text{-a.e. } x \in F.$$

Proof. The upper bound is easy. As $p_t(x,x)$ is non-increasing w.r.t. t, we have

$$g_\lambda(x,x) \geqslant p_t(x,x)\int_0^t \mathrm{e}^{-\lambda s}\,\mathrm{d}s = p_t(x,x)(1-\mathrm{e}^{-\lambda t})/\lambda.$$

Taking $t = 1/\lambda$, the result easily follows using (6.11). For the lower bound, as $p_t(x,x)$ is non-increasing w.r.t. t, using the upper bound just obtained, we have for small $t > 0$,

$$\begin{aligned} g_\lambda(x,x) &\leqslant \int_0^t p_s(x,x)\mathrm{e}^{-\lambda s}\,\mathrm{d}s + p_t(x,x)\mathrm{e}^{-\lambda t}/\lambda \\ &\leqslant c_1\int_0^t s^{-\gamma} l(s)\mathrm{e}^{-\lambda s}\,\mathrm{d}s + p_t(x,x)\mathrm{e}^{-\lambda t}/\lambda \end{aligned}$$

for some $c_1 > 0$ where we set $\gamma = \alpha/(\alpha+1) \in (0,1)$ and $l(s) = (\log|\log s|)^{\beta/(\alpha+1)}$. Note that $l(s) = l(1/s)$ for $s > 0$ and l is a slowly varying function, i.e. $\lim_{x\to\infty} l(cx)/l(x) = 1$ for all $c > 0$. Now,

$$\begin{aligned} \int_0^t s^{-\gamma} l(s)\mathrm{e}^{-\lambda s}\,\mathrm{d}s &= \lambda^{-(1-\gamma)}\int_0^{\lambda t} s^{-\gamma} l(s/\lambda)\mathrm{e}^{-s}\,\mathrm{d}s \\ &\leqslant \lambda^{-(1-\gamma)} l(\lambda)\int_0^{\lambda t} s^{-\gamma-\varepsilon}\mathrm{e}^{-s}\,\mathrm{d}s \equiv \lambda^{-(1-\gamma)} l(\lambda) h(\lambda t) \end{aligned}$$

for all $\varepsilon > 0$ where we use the fact $l(s/\lambda) \leqslant l(s)l(\lambda) \leqslant t^{-\varepsilon} l(\lambda)$ for s small and λ large. As $h(\infty) = \Gamma(1-\gamma-\varepsilon)$, $h(\lambda t) \to 0$ as $\lambda t \to 0$. Take $c_2 > 0$ small enough so that $c_1 h(c_2) < c_{6.6}/2$ and take t so that $\lambda t = c_2$. By the above, we then have

$$\frac{p_t(x,x)}{\lambda^\gamma l(\lambda)} = \frac{p_t(x,x)}{(c_2/t)^\gamma l(c_2/t)} \geqslant \mathrm{e}^{c_2}\left\{\frac{g_\lambda(x,x)}{\lambda^{-(1-\gamma)} l(\lambda)} - c_1 h(c_2)\right\}.$$

By taking the lim sup as $\lambda \to \infty$, we obtain the desired lower bound. □

For the lim inf behaviour we do not yet have a completely sharp result. The following is obtained in the same way as the upper bound of the above theorem using (6.10).

Theorem 6.10. *There exists* $c_{6.14} > 0$ *such that* $\mathbb{P}$*-a.s.*

$$\liminf_{t\to 0} \frac{p_t(x,x)}{t^{-\alpha/(\alpha+1)}(\log|\log t|)^{-\beta'/(\alpha+1)}} \leqslant c_{6.14}, \quad \mu\text{-a.e. } x \in F.$$

We remark that it is possible to get the following lower bound: if $\beta' < 1$, then there exist constants $c_{6.15} > 0, \beta''$ such that

$$c_{6.15} \leqslant \liminf_{t\to 0} \frac{p_t(x,x)}{t^{-\alpha/(\alpha+1)}(\log|\log t|)^{-\beta''/(\alpha+1)}}.$$

This requires the use of a sharper estimate on the tail of the hitting time distribution than that found in Hambly (1997) Lemma 7.7 and we do not give the argument here. Note that the exponent for the iterated logarithm correction term are different in the two bounds. We anticipate that it is the upper bound which is tight.

We now discuss the spatial fluctuation. From the results of Hambly (1997) there are upper and lower bounds on the lim sup and lim inf, respectively. In order to establish the corresponding lower bounds we follow the same approach as above. We have an expression for the local Green density in terms of the sequence of random variables $W_{\mathbf{i}|n}$. If we choose a subsequence which approaches the worst-case behaviour of this sequence it will demonstrate the required worst-case behaviour of the Green density.

Theorem 6.11. *There exist constants* $c_{6.16},\ldots,c_{6.19} > 0$ *such that* $\mathbb{P}$*-a.s.*

$$c_{6.16} \leqslant \lim_{\lambda\to\infty} \inf_{x\in F} \frac{g_\lambda(x,x)}{\lambda^{-1/(\alpha+1)}(\log\lambda)^{-\beta'/(\alpha+1)}} \leqslant c_{6.17} \tag{6.12}$$

and

$$c_{6.18} \leqslant \lim_{\lambda\to\infty} \sup_{x\in F} \frac{g_\lambda(x,x)}{\lambda^{-1/(\alpha+1)}(\log\lambda)^{\beta/(\alpha+1)}} \leqslant c_{6.19}. \tag{6.13}$$

As before we can apply Tauberian theorems to obtain the spatial fluctuation in the heat kernel.

Theorem 6.12. *There exist constants* $c_{6.20}, c_{6.21} > 0$ *such that* $\mathbb{P}$*-a.s.*

$$c_{6.20} \leqslant \lim_{t\to 0} \sup_{x\in F} \frac{p_t(x,x)}{t^{-\alpha/(\alpha+1)}(|\log t|)^{\beta/(\alpha+1)}} \leqslant c_{6.21}.$$

For the lim inf result we can use (Hambly, 1997) Lemma 8.3 to get control on the lower bound.

Theorem 6.13. *There exists a constant* $c_{6.22} > 0$ *such that* $\mathbb{P}$*-a.s.*

$$\lim_{t\to 0} \inf_{x\in F} \frac{p_t(x,x)}{t^{-\alpha/(\alpha+1)}(|\log t|)^{-\beta'/(\alpha+1)}} \leqslant c_{6.22}$$

and constants $c_{6.23}, \beta''' > 0$ *such that*

$$c_{6.23} \leqslant \lim_{t\to 0} \inf_{x\in F} \frac{p_t(x,x)}{t^{-\alpha/(\alpha+1)}(|\log t|)^{-\beta'''/(\alpha+1)}}.$$

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