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What Is Not in the Domain of the Laplacian on Sierpinski Gasket Type Fractals

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We consider the analog of the Laplacian on the Sierpinski gasket and related fractals, constructed by Kigami. A function f is said to belong to the domain of Δ if f is continuous and Δf is defined as a continuous function. We show that if f is a nonconstant function in the domain of Δ, then f^2 is not in the domain of Δ. We give two proofs of this fact. The first is based on the analog of the pointwise identity $\Delta f^2 - 2f\,\Delta f = |\nabla f|^2$, where we show that $|\nabla f|^2$ does not exist as a continuous function. In fact the correct interpretation of Δf^2 is as a singular measure, a result due to Kusuoka; we give a new proof of this fact. The second is based on a dichotomy for the local behavior of a function in the domain of Δ, at a junction point x_0 of the fractal: in the typical case (nonvanishing of the normal derivative) we have upper and lower bounds for $|f(x) - f(x_0)|$ in terms of $d(x, x_0)^\beta$ for a certain value β, and in the nontypical case (vanishing normal derivative) we have an upper bound with an exponent greater than 2. This method allows us to show that general nonlinear functions do not operate on the domain of Δ. © 1999 Academic Press

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1. INTRODUCTION

There exists a well developed theory of Laplacians on a class of fractals including the familiar Sierpinski gasket. This theory may be obtained indirectly through the construction of probabilistic processes analogous to Brownian motion [BP, G, Ku1, Ku2, Li], or directly by taking renormalized limits of graph Laplacians, as in the work of Kigami [K1, K2]. See [BK, DSV, FS, Ki3, Ki4, Ki5, KL, La, S1, S2, SU, T] for a sampling of works on this subject.

To define a Laplacian Δ on a fractal F, we need a Dirichlet form $\mathscr{E}(f, f)$, which is the analog of $\int |\nabla f|^2\, dx$, and a measure μ on F. The Dirichlet form determines the harmonic functions, which are minimizers of $\mathscr{E}(f, f)$ subject to boundary conditions. The Laplacian is determined by the analog of

$$\int \nabla f \cdot \nabla g\, dx = -\int g\, \Delta f\, dx + \text{boundary terms}, \tag{1.1}$$

with $\mathscr{E}(f, g)$ playing the role of the left hand side, and $d\mu$ substituting for dx on the right side. It is possible to interpret $\mathscr{E}(f, g)$ as the total mass of a signed measure $\nu_{f,g}$ defined by

$$\int h\, d\nu_{f,g} = \mathscr{E}(fh, g) + \mathscr{E}(f, gh) - \mathscr{E}(h, fg) \tag{1.2}$$

for h in the domain of $\mathscr{E}$ (see (3.2.15) of [FOT]), but the energy measures $\nu_{f,g}$ may be unrelated to the measure μ used to define the Laplacian. In fact, Kusuoka [Ku2] proves they are singular for many fractals. We will give a new proof of this fact that is considerably shorter, and that works for a larger class of examples. There is no immediate interpretation of the energy measure $\nu_{f,g}$ as an inner product of gradients. A theory of gradients is described in [S2], but it is not clear yet if it can be related to energy measures.

The domain of the Laplacian is defined to be the set of continuous functions f for which Δf is defined as a continuous function. This domain is well behaved in that it is dense in the continuous functions in the uniform norm, and forms a core for defining $-\Delta$ as a self-adjoint positive definite operator on $L^2(d\mu)$ with a discrete spectrum. In this paper we wish to point out that the domain is rather peculiar, however, in that it fails to have properties one might expect it to have by analogy with the usual theory of Laplacians. We will show that the domain is not closed under multiplication; in fact, if f is any nontrivial function in the domain, then f^2 is not in the domain. We will also show that if we take a standard embedding of

F into a Euclidean space, then the restrictions to F of noncontant C^∞ functions are not in the domain.

One way to understand our results is to begin with the identity

$$\Delta f^2 - 2f\,\Delta f = |\nabla f|^2, \tag{1.3}$$

which holds pointwise for the usual Laplacian. There is an analogous result holding for a graph Laplacian. In our case we show that the right side blows up in the limit. Since $f\,\Delta f$ exists, this shows Δf^2 cannot exist. In fact, the identity (1.3) shows that the nonexistence of Δf^2 is essentially equivalent to Kusuoka's singularity result for the energy measure $\nu_{f,f}$ (we are grateful to the referee for pointing this out). Our proof shows in more detail the divergence of Δf^2 at specific points.

Another approach is to study the behavior of functions in the domain of Δ in the neighborhood of a junction point on F (the junction points are the points in the graph approximations to F). We show that there is a dichotomy: either

$$c_1 d(x, x_0)^\beta \leqslant |f(x) - f(x_0)| \leqslant c_2 d(x, x_0)^\beta \tag{1.4}$$

for a certain $\beta < 1$, or

$$|f(x) - f(x_0)| \leqslant c d(x, x_0)^\gamma \log d(x, x_0) \tag{1.5}$$

for a certain $\gamma > 2$, with the first case holding if and only if the normal derivative of f at x_0 is nonzero. (This result was proved for harmonic functions on the Sierpinski gasket in [DSV].) It is then simple to see that when the first case holds for f at x_0, neither case can hold for f^2 at x_0. The argument is then completed by observing that the normal derivative can vanish at every junction point only for a constant function. The same reasoning leads to the conclusion that essentially any nonlinear function, not just the square, will fail to act on the domain of Δ.

What are we to make of these negative results? One point of view is that they indicate certain natural limitations of the theory. For example, one might be tempted to develop a distribution theory on fractals with the role of the space of test functions played by the domain of all powers of Δ. Such a theory would not allow multiplication of a distribution by a test function.

Another point of view is that we need to broaden the definition of Δ. We will show that it is possible to define a Laplacian mapping functions to measures in such a way that Δf^2 is well defined. The drawback of this approach is that the domain and range of this Laplacian are not the same, so natural objects like Δ^2 would not be defined. Still another idea is that we need to pick the initial measure μ more carefully. In [Ki2] a rather broad class of measures is allowed in the definition of Δ (in fact the notation Δ_μ is used there to indicate the independence of the Laplacian on the measure). In most detailed studies, however, the measure is assumed to be self-similar, and sometimes it is even required to be normalized Hausdorff measure (a specific self-similar measure). The rationale for this restriction is that the most natural measures are those that reflect the self-similar property of the fractal. However, as we will show, there is a measure ν such that all the energy measures $\nu_{f,g}$ are absolutely continuous with respect to ν. This allows the definition of a carré du champs operator [BH] $\Gamma(f, g)$ via $d\nu_{f,g} = \Gamma(f, g)\, d\nu$. Thus if we use ν in the definition of Δ, then all the problems disappear, and Δf^2 is well defined. Of course, one must be wary of changing the problem in order to overcome difficulties. In this case there are sufficient doubts that we really know what constitutes "the natural measure" to use on fractals, that it would certainly be interesting to explore the properties of the Laplacian defined with this measure. Although ν is not self-similar in the strict sense, it does satisfy identities of a self-similar nature (involving some negative coefficients and overlaps) that could be used to facilitate computations.

We will present our results in detail for the case of the symmetric Laplacian on the planar Sierpinski gasket. In this case it is very easy to give all definitions explicitly. The same arguments can be extended to many other examples of post critically finite (p.c.f.) self-similar fractals. We do this in Section 5 for our proof of the singularity of energy measures.

In Section 2 we recall the facts about the Laplacian on the Sierpinski gasket, mostly from [Ki1]. In Section 3 we follow the first approach outlined above to show that Δf^2 is undefined as a function, and discuss how to define it as a measure. In Section 4 we prove the dichotomy in the local behavior near a junction point, giving a second proof that Δf^2 is undefined, and also showing the restrictions of smooth functions are not in the domain of Δ.

2. THE LAPLACIAN ON THE SIERPINSKI GASKET

The Sierpinski gasket SG is the attractor of the iterated function system (i.f.s.) in the plane

$$S_j x = \tfrac{1}{2}(x - p_j) + p_j, \qquad j = 1, 2, 3,$$

where p_1, p_2, p_3 are vertices of a triangle T. We regard it as the limit of graphs G_n, where G_0 is just the triangle T, and

$$G_{n+1} = \bigcup_{j=1}^{3} S_j G_n$$

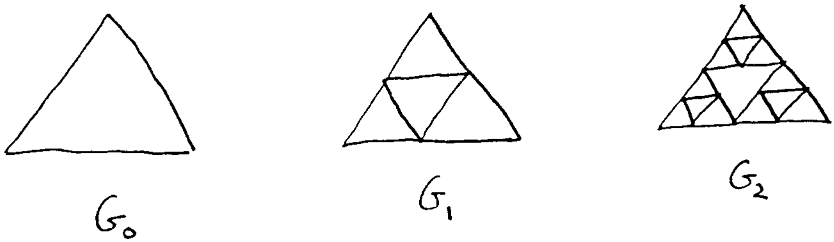

FIG. 2.1. The graphs G_0, G_1, G_2.

with the identification of the three junction points where the images $S_j G_n$ meet (see Fig. 2.1). The three vertices of T will be regarded as boundary points of each graph G_n and SG. Note that every nonboundary vertex of G_n has exactly 4 neighboring vertices, so

$$-\Delta_n f(x) = f(x) - \tfrac{1}{4} \sum_{y \sim x} f(y) \tag{2.1}$$

defines a symmetric graph Laplacian on G_n, and

$$\mathscr{E}_n(f, f) = \tfrac{1}{4} \sum_{x \sim y} (f(x) - f(y))^2 \tag{2.2}$$

the associated energy form. The Dirichlet form on SG is defined to be

$$\mathscr{E}(f, f) = \lim_{n \to \infty} (\tfrac{5}{3})^n \mathscr{E}_n(f, f). \tag{2.3}$$

The choice of the renormalization factor $(\frac{5}{3})^n$ is dictated by the fact

$$(\tfrac{5}{3})^n \mathscr{E}_n(f, f) \geqslant (\tfrac{5}{3})^{n-1} \mathscr{E}_{n-1}(f, f), \tag{2.4}$$

with equality holding if and only if $\Delta_n f(x) = 0$ at each vertex in G_n that is not in G_{n-1}. Thus the limit in (2.3) always exists as an extended real number.

A function on G_n is called *harmonic* if $\Delta_n f(x) = 0$ at every nonboundary vertex x of G_n; equivalently, f minimizes $\mathscr{E}_n(f, f)$ over all functions with the same boundary values. A function that is harmonic on G_{n-1} has a unique extension to a harmonic function on G_n, given by the following *harmonic algorithm*,

$$f(v_{12}) = \tfrac{2}{5} f(v_1) + \tfrac{2}{5} f(v_2) + \tfrac{1}{5} f(v_3) \tag{2.5}$$

if v_1, v_2, v_3 are the vertices of any small triangle in G_{n-1}, and v_{12} is the vertex in G_n between v_1 and v_2 (see Fig. 2.2). A continuous function f on SG is called *harmonic* if its restriction to every G_n is harmonic. The space

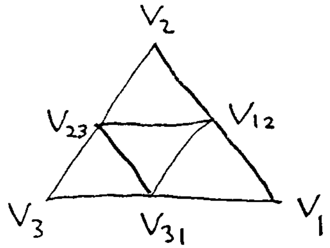

FIG. 2.2. Labeling of vertices in G_n on a small triangle in G_{n-1}.

of harmonic functions is 3-dimensional, and the values of f at the dense set of all junction points is determined by the boundary values $f(p_j)$ by successive applications of the harmonic algorithm.

We choose for the measure μ on SG the symmetric Bernoulli measure, which is the unique probability measure satisfying the self-similar identity

$$\mu = \tfrac{1}{3}\mu \circ S_1^{-1} + \tfrac{1}{3}\mu \circ S_2^{-1} + \tfrac{1}{3}\mu \circ S_3^{-1}. \tag{2.6}$$

This is simply the measure that assigns the weight $(\frac{1}{3})^n$ to each of the 3^n small triangles in G_n (regarded as subsets of SG). With this choice of measure, the Laplacian on SG is just

$$\Delta f(x) = \lim_{n \to \infty} (3/2)\, 5^n \Delta_n f(x). \tag{2.7}$$

This is interpreted in the following sense. Let f and g be continuous functions on SG. We say f belongs to the domain of Δ and $\Delta f = g$ provided $\lim_{n \to \infty} 5^n \Delta_n f(x) = g(x)$ for every nonboundary junction point x (of course $\Delta_n f(x)$ is only defined for n large enough that x is a vertex of G_n). The renormalization constant 5^n is explained as $3^n \cdot (\frac{5}{3})^n$, with 3^n coming from the reciprocal of the measure and $(\frac{5}{3})^n$ being the renormalization factor from the Dirichlet form. The definition is consistent with the definition of harmonic function, in that the harmonic functions are the solutions of $\Delta f = 0$.

We also need the notion of normal derivative at the boundary points. Each boundary point has exactly 2 neighboring vertices in each graph G_n, so we define

$$(\partial_\nu)_n f(p) = \tfrac{1}{2} f(p) - \tfrac{1}{4} \sum_{y \sim p} f(y) \tag{2.8}$$

for the normal derivative in G_n, and

$$\partial_\nu f(p) = \lim_{n \to \infty} (\tfrac{5}{3})^n (\partial_\nu)_n (p) \tag{2.9}$$

for the normal derivative on SG, if the limit exists. On G_n we have the Gauss–Green formula

$$\mathscr{E}_n(f, g) = -\sum_x g(x)\, \Delta_n f(x) + \sum_p g(p)(\partial_v)_n f(p) \tag{2.10}$$

(the x-sum is over nonboundary points, and the p-sum over the 3 boundary points). Multiplying by $(5/3)^n$ and taking the limit we obtain

$$\mathscr{E}(f, g) = -\int g\, \Delta f\, d\mu + \sum_p g(p)\, \partial_v f(p), \tag{2.11}$$

the Gauss–Green formula on SG. This makes sense provided f and g are in the domain of the Dirichlet form and f is in the domain of the Laplacian, and this argument proves that the normal derivatives exist for functions in the domain of the Laplacian. For f and g in the domain of Δ we can also obtain the symmetric variant

$$\int (g\, \Delta f - f\, \Delta g)\, d\mu = \sum_p (g(p)\, \partial_v f(p) - f(p)\, \partial_v g(p)) \tag{2.12}$$

by subtraction.

Now let $T_n = S_{j_1} \cdots S_{j_n} T$ be any small triangle in G_n. For each vertex p of T_n we can define the outward normal derivative by

$$\partial_v f(p) = \lim_{k \to \infty} \left(\tfrac{5}{3}\right)^k \left(\tfrac{1}{2} f(p) - \tfrac{1}{4} \sum_{y \sim p} f(y) \right),$$

where the sum is over the 2 neighboring vertices of G_k that are in T_n. Note that if we take the other triangle that has p as a vertex, the normal derivative will change by a minus sign; and the normal derivative only depends on which side of p the triangle lies on. We then have the existence of normal derivatives at all junction points for functions in the domain of Δ, and the local Gauss–Green formula on T_n

$$\int_{T_n} (g\, \Delta f - f\, \Delta g)\, d\mu = \sum_{\partial T_n} (g(p)\, \partial_v f(p) - f(p)\, \partial_v g(p)). \tag{2.13}$$

3. NO SQUARES

Theorem 3.1. *Let f be in the domain of Δ on SG, and let x be any junction point where $\partial_v f(x) \neq 0$. Then $\Delta f^2(x)$ is undefined, and in fact the limit in* (2.7) *is* $+\infty$.

Proof. On G_n a simple computation yields

$$\Delta_n f^2(x) - 2f(x)\,\Delta_n f(x) = \tfrac{1}{4} \sum_{y \sim x} (f(x) - f(y))^2. \tag{3.1}$$

We multiply by 5^n and try to take the limit. Since $5^n f(x)\,\Delta_n f(x) \to f(x)\,\Delta f(x)$ it suffices to show $5^n \sum_{y \sim x} (f(x) - f(y))^2 \to +\infty$. Now the assumption that $\partial_\nu f(x) \neq 0$ implies that there exists a sequence of neighboring vertices y_n in G_n (for large enough n) such that $|f(x) - f(y_n)| \geqslant c(3/5)^n$, because otherwise $\partial_\nu f(x) = 0$ by (2.9). Thus $5^n \sum_{y \sim x} (f(x) - f(y))^2 \geqslant c((3/5))^2\, 5)^n$ which diverges because $(3/5)^2 \cdot 5 = 9/5 > 1$. ∎

Lemma 3.2. *Let f be a nonconstant function in the domain of Δ. Then there exists a junction point where $\partial_\nu f(x) \neq 0$.*

Proof. Apply the local Gauss–Green formula (9.13) with $g \equiv 1$, to obtain

$$\int_{T_n} \Delta f\, d\mu = \sum_{\partial T_n} \partial_\nu f(p). \tag{3.2}$$

If we had $\partial_\nu f(x) = 0$ at every junction point, this would imply that the integral of Δf vanishes on every triangle T_n. Since Δf is continuous, this can only happen if f is harmonic. But it is easy to check that non-constant harmonic functions have non-zero normal derivative at least at one vertex of every small triangle. ∎

Corollary 3.3. *If f is a nonconstant function in the domain of Δ, then f^2 is not in the domain of Δ.*

Now we indicate how Δf^2 can be defined as a measure. First we observe that there is a positive energy measure ν_f obtained from the Dirichlet form. If A is any polygonal set bounded by edges from one of the graphs G_k, then we let

$$\nu_f(A) = \lim_{n \to \infty} \left(\tfrac{5}{3}\right)^n \tfrac{1}{4} \sum_{\substack{x \sim y \\ x,\, y \in A \cap G_n}} (f(x) - f(y))^2. \tag{3.3}$$

The existence of the limit follows from the same argument that gives the limit in (2.3). It is clear that ν_f is finitely additive, and extends to a finite Borel measure by the Carathéodory extension theorem. It is easy to see that ν_f is non-atomic. In fact $\nu_f = \nu_{f,\,f}$ defined by (1.3).

Now if we multiply (3.1) by $(5/3)^n$ and sum over all x in a polygonal set A, we can pass to the limit to obtain

$$\lim_{n \to \infty} 3^{-n} \sum_{x \in A \cap G_n} 5^n \Delta_n f^2(x) = 2 \int_A f\, \Delta f\, d\mu + \nu_f(A). \tag{3.4}$$

This suggests that we have

$$\Delta f^2 = 2f\,\Delta f\,d\mu + \nu_f \tag{3.5}$$

for f in the domain of Δ, with the following definition for a statement $\Delta F = \rho$ where F is a continuous function and ρ a finite Borel measure.

DEFINITION. We say a continuous function F is in the measure domain of Δ and $\Delta F = \rho$ if there exists a finite Borel measure ρ such that

$$\lim_{n\to\infty} 3^{-n} \sum_{x \in A \cap G_n} 5^n \Delta_n F(x) = \rho(A) \tag{3.6}$$

for all polygonal sets A.

This definition is consistent with the function definition: if F is in the domain of Δ with $\Delta F = g$ then F is in the measure domain with $\Delta F = g\,d\mu$. With this definition, (3.4) implies (3.5).

We show next that ν_f is singular with respect to μ. Because of the net structure of the triangles in SG, the analog of the Lebesgue differentiation of the integral theorem holds for triangular sets. Thus, to show that ν_f is singular with respect to μ, it suffices to show that for μ-a.e. x,

$$3^n \nu_f(T_n) \to 0 \tag{3.7}$$

for T_n a sequence of triangles with $\mu(T_n) = 3^{-n}$ converging to x. For simplicity assume f is harmonic. Then we have simply

$$\nu_f(T_n) = (\tfrac{5}{3})^n \tfrac{1}{4}((f(a_n) - f(b_n))^2 + f(b_n) - f(c_n))^2 + (f(c_n) - f(a_n))^2), \tag{3.8}$$

where a_n, b_n, c_n are the vertices of T_n. The values $f(a_n)$, $f(b_n)$, $f(c_n)$ are derived from the values of f at the boundary points by applying a product of matrices determined by the harmonic algorithm (2.5), depending on the mappings that send T to T_n. Since constants do not contribute to the energy (3.8), it is convenient to factor out by the constants to obtain a 2-dimensional Hilbert space with energy norm. Taking $n = 0$ for simplicity, we have an orthonormal basis of the two harmonic functions h_1 and h_2 with boundary values $(h_1(a), h_1(b), h_1(c)) = (0, \sqrt{2}, \sqrt{2})$ and $(h_2(a), h_2(b), h_2(c)) = (0, \sqrt{2/3}, -\sqrt{2/3})$. With respect to this basis, the matrices have the form

$$M_1 = \begin{pmatrix} 3/5 & 0 \\ 0 & 1/5 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 3/10 & \sqrt{3}/10 \\ \sqrt{3}/10 & 1/2 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 3/10 & -\sqrt{3}/10 \\ -\sqrt{3}/10 & 1/2 \end{pmatrix}.$$

We can then invoke the theory of products of random matrices, and Furstenberg's theorem [Fu]: there exists an exponent $\alpha > \sqrt{3}/5$ such that

$$\log \|M_{j_n} \cdots M_{j_1}\| \sim n \log \alpha \tag{3.9}$$

as $n \to \infty$ for a.e. choice of matrices. But this is exactly the same as μ-a.e. x in (3.7). To obtain the estimate (3.7) from (3.9) we need $\alpha < 1/\sqrt{5}$. This inequality is proved in the next theorem.

The next theorem follows from a more general result proved by S. Kusuoka in [Ku2]. Our proof seems to be shorter and more analytic in nature. Moreover, in Section 5 we show that our method can be applied to general finitely ramified fractals with fewer assumptions than are made in [Ku2]. In the proof of Theorem 5.1 we avoid using Furstenberg's theorem [Fu] although do use this theorem in the proof of Theorem 3.4 in order to shorten the exposition.

In what follows the domain of the Dirichlet form $\mathscr{E}$ is denoted by $\mathscr{F}$.

THEOREM 3.4. *For any $f \in \mathscr{F}$ the measure ν_f is singular with respect to μ. Moreover, there exists a measure ν (singular to μ), such that all the energy measures are absolutely continuous with respect to ν.*

Proof. For μ-a.e. point x we can define a unique sequence of matrices $A_n(x) = M_{j_n}$ as above. Then Furstenberg's theorem implies that

$$\lim_{n \to \infty} \frac{1}{n} \log \|A_n(x) \cdots A_1(x)\, v_0\| = \log \alpha$$

for μ-a.e. x. Here v_0 denotes the components of the harmonic function in the h_1, h_2 basis (mod constants), and $\|\cdot\|$ is now just the Euclidean norm on $\mathbb{R}^2$. Since $M_1^2 + M_2^2 + M_3^2 = \frac{3}{5} I$, it follows that

$$\int_T A_n^*(x)\, A_n(x)\, d\mu(x) = \tfrac{1}{5}\, I.$$

Hence, by Jensen's inequality, for any nonzero vector v we have

$$\int_T \log \|A_n(x)\, v\|\, d\mu(x) < \tfrac{1}{2} \log \int_T \langle v, A_n^*(x)\, A_n(x)\, v \rangle\, d\mu(x) = \tfrac{1}{2} \log(\tfrac{1}{5}\, \|v\|).$$

Thus

$$\beta = \max_{\{v : \|v\| = 1\}} \int_T \log \|A_n(x)\, v\| \, d\mu(x) < \tfrac{1}{2} \log \tfrac{1}{5}. \tag{3.10}$$

Denote $v_n(x) = A_n(x) \cdots A_1(x)\, v_0$. The matrices $A_n(x)$ are statistically independent with respect to μ, and so $A_n(x)$ is statistically independent of $v_{n-1}(x)$. Hence

$$\begin{aligned}\int_T \log \|v_n(x)\| \, d\mu(x) &= \int_T \log \left\| A_n(x) \frac{v_{n-1}(x)}{\|v_{n-1}(x)\|} \right\| d\mu(x) \\ &\quad + \int_T \log \|v_{n-1}(x)\| \, d\mu(x) \\ &\leqslant \beta + \int_T \log \|v_{n-1}(x)\| \, d\mu(x).\end{aligned}$$

By induction this implies $\log \alpha \leqslant \beta$ and so $\alpha < 1/\sqrt{5}$. Therefore ν_h is singular with respect to μ for any harmonic function h.

Suppose now that $f \in \mathscr{E}$. Then f can be approximated by a sequence of functions $\{f_m\}$ that are continuous and piecewise harmonic on the triangles T_m [Ki1, Ki2]. The approximation is in energy norm, $\mathscr{E}(f - f_m, f - f_n) \to 0$ as $m \to \infty$, and also uniformly. Let $\nu = \nu_{h_1} + \nu_{h_2}$. Note that for any harmonic function h the measure ν_h has a bounded density with respect to ν since $\nu_{c_1 h_1 + c_2 h_2} \leqslant 2(c_1^2 \nu_{h_1} + c_2^2 \nu_{h_2})$. The same is true for the functions f_m. We claim that the measures ν_{f_m} form a Cauchy sequence in the space of measures. This will complete the proof that $\nu_f \ll \nu$ because $L^1(\nu)$ is already complete in the measure norm.

To see this we use the general estimate

$$|\nu_g(A) - \nu_{g'}(A)|^2 \leqslant \mathscr{E}(g + g', g + g')\, \mathscr{E}(g - g', g - g') \tag{3.11}$$

for any $g, g' \in \mathscr{F}$ and any polygonal subset A of SG. Taking g and g' to be f_m and f_k shows that $|\nu_{f_m}(A) - \nu_{f_k}(A)| \to 0$ uniformly in A as $m, k \to \infty$. This implies that $\{\nu_{f_m}\}$ is a Cauchy sequence (in particular $\{d\nu_{f_m}/d\nu\}$ is a Cauchy sequence in $L^1(\nu)$).

We prove (3.11) first in the case $A = SG$, when $\nu_g(SG) = \mathscr{E}(g, g)$ and $\nu_{g'}(SG) = \mathscr{E}(g', g')$, so (3.11) is just

$$\begin{aligned}&\mathscr{E}(g, g)^2 + \mathscr{E}(g', g')^2 - 2\mathscr{E}(g, g)\, \mathscr{E}(g', g') \\ &\quad \leqslant (\mathscr{E}(g, g) + 2\mathscr{E}(g, g') + \mathscr{E}(g', g'))(\mathscr{E}(g, g) - 2\mathscr{E}(g, g') + \mathscr{E}(g', g')).\end{aligned} \tag{3.12}$$

Multiplying out the right side of (3.12) and cancelling like terms reduces to

$$0 \leqslant 4\mathscr{E}(g, g)\,\mathscr{E}(g', g') - 4\mathscr{E}(g, g')^2$$

which is just the Cauchy–Schwartz inequality. The modification of the argument for general A is simple. We just restrict all energies to A, to obtain $|\nu_g(A) - \nu_{g'}(A)|^2 \leqslant \nu_{g+g'}(A)\,\nu_{g-g'}(A)$. Since $\nu_{g+g'}$ and $\nu_{g+g'}$ are positive measures, (3.11) follows.

It is clear by polarization that the energy measures $\nu_{f,g}$ are also absolutely continuous with respect to ν. Q.E.D.

The measure ν is independent of the choice of orthonormal basis (h_1, h_2), and so it may be regarded as a natural measure associated to the Dirichlet form.

It is easy to see that the map $f \mapsto (d\nu_f/d\nu)$ is a continuous quadratic map from the domain of $\mathscr{E}$ to $L^1(\nu)$.

THEOREM 3.5. *For any $f \in \mathscr{F}$ the measure ν_f has no atoms.*

Proof. In view of Theorem 3.4 it suffices to prove this when f is harmonic. In fact we will show

$$\nu_f(T_n) \leqslant (3/5)^n\,\mathscr{E}(f, f) \tag{3.13}$$

for any triangle of level n ($T_n = S_{j_1} \cdots S_{j_n} T$). A simple computation shows that for any harmonic function f,

$$\nu_f(S_j T) \leqslant (3/5)\,\nu_f(T), \tag{3.14}$$

and in fact the constant 3/5 is attained when $f(v_k) = \delta_{jk}$. We then obtain (3.13) by iterating (3.14), and it is clear that (3.13) implies ν_f has no atoms. Q.E.D.

4. LOCAL CUSP DICHOTOMY

Let f belong to the domain of Δ on SG, and let x be any nonboundary junction point. Let T_n and T'_n denote the 2 small triangles in G_n that have x as a vertex, and let a_n, b_n and c_n, d_n denote the neighboring vertices to x in T_n and T'_n. We know

$$-\Delta f(x) = \lim_{n\to\infty} \tfrac{3}{2}5^n\left(f(x) - \tfrac{1}{4}(f(a_n) + f(b_n) + f(c_n) + f(d_n))\right). \tag{4.1}$$

But what is the rate of convergence? To answer this question we first use the Gauss–Green formula to obtain an integral expression for the difference. Let h_n denote the piecewise harmonic function supported on the union $T_n \cup T'_n$ which takes the value 1 at x and 0 at a_n, b_n, c_n, d_n.

LEMMA 4.1. *We have*

$$\tfrac{3}{2}\, 5^n \left(f(x) - \tfrac{1}{4}(f(a_n) + f(b_n) + f(c_n) + f(d_n))\right) + \Delta f(x)$$
$$= (3/2)\, 3^n \int_{T_n \cup T'_n} h_n(y)(\Delta f(x) - \Delta f(y))\, d\mu(y). \tag{4.2}$$

Proof. Apply (2.13) to T_n and T'_n and sum to obtain

$$\int_{T_n \cup T'_n} h_n\, \Delta f\, d\mu = \sum_{\partial T_n} h_n\, \partial_\nu f - f\, \partial_\nu h_n + \sum_{\partial T'_n} h_n\, \partial_\nu f - f\, \partial_\nu h_n.$$

Now the terms involving $\partial_\nu f$ cancel, because h_n is 0 except at x where the values of $\partial_\nu f$ differ by a minus sign. On the other hand we have $\partial_\nu h_n(x) = \frac{1}{2}(\frac{5}{3})^n$ and $\partial_\nu h_n(y) = -\frac{1}{4}(\frac{5}{3})^n$ for $y = a_n, b_n, c_n, d_n$ (for harmonic functions $\partial_\nu = (\frac{5}{3})^n (\partial_\nu)_n$ exactly). Thus we have

$$\int_{T_n \cup T'_n} h_n\, \Delta f\, d\mu = (\tfrac{5}{3})^n \left(f(x) - \tfrac{1}{4}\,(f(a_n) + f(b_n) - f(c_n) + f(d_n))\right)$$

and we obtain (4.2) by combining this with the fact that $3^n \int_{T_n \cup T'_n} h_n\, d\mu = 2/3$. ▮

It follows that the convergence in (4.1) is uniform, with the rate depending on the modulus of continuity of Δf. If Δf is Lipschitz, then the error is $O(2^{-n})$.

For the next result we consider any small triangle in G_{n-1} and label the vertices as in (2.5). We have the following extension of the harmonic algorithm:

THEOREM 4.2. *Let f be in the domain of Δ. Then*

$$f(v_{12}) = \frac{2}{5} f(v_1) + \frac{2}{5} f(v_2) + \frac{1}{5} f(v_3)$$
$$+ \frac{2}{3} \frac{1}{5^n} \left(\frac{6}{5} \Delta f(v_1) + \frac{2}{5} \Delta f(v_2) + \frac{2}{5} \Delta f(v_3) \right) + R_n, \tag{4.3}$$

where the remainder R_n satisfies

$$R_n = o(5^{-n}) \tag{4.4}$$

uniformly depending only on the modulus of continuity of Δf. Moreover, if Δf is Lipschitz then

$$R_n = O(10^{-n}). \tag{4.5}$$

Proof. Let $A_n = f(v_{12}) + f(v_{23}) + f(v_{31})$, $B_n = f(v_1) + f(v_2) + f(v_3)$ and $C_n = \Delta f(v_{12}) + \Delta f(v_{23}) + \Delta f(v_{31})$. Apply (4.2) to each to the points v_{12}, v_{21} and v_{31} to obtain

$$f(v_{12}) - \tfrac{1}{4}(f(v_1) + f(v_2) + f(v_{31}) + f(v_{23})) = \tfrac{2}{3} 5^{-n} \Delta f(v_{12}) + o(5^{-n}) \tag{4.6}$$

and so forth, and add to obtain

$$\tfrac{1}{2} A_n - \tfrac{1}{2} B_n = \tfrac{2}{3} 5^{-n} C_n + o(5^{-n}). \tag{4.7}$$

Now the left side of (4.6) is just

$$\tfrac{5}{4} f(v_{12}) - \tfrac{1}{4}(f(v_1) + f(v_2) + A_n),$$

and we can substitute (4.7) to eliminate A_n, so

$$f(v_{12}) = \tfrac{1}{5}(f(v_1) + f(v_2) + B_n + \tfrac{4}{3} \cdot 5^{-n} C_n$$
$$+ o(5^{-n})) + 5^{-n}(4/5)\, \Delta f(v_{12}) + o(5^{-n})$$

which is (4.3). Q.E.D.

THEOREM 4.3. *Let f be in the domain of Δ and let x be any junction point.*

(a) *If $\partial_v f(x) \neq 0$ then there exist positive constants c_1, c_2 such that*

$$c_1(3/5)^n \leqslant |f(x) - f(a_n)| \leqslant c_2(3/5)^n \tag{4.8}$$

(and the same for b_n, c_n, d_n).

(b) *If $\partial_v f(x) = 0$ then*

$$|f(x) - f(a_n)| \leqslant c_2 n 5^{-n} \tag{4.9}$$

(and the same for b_n, c_n, d_n).

Proof. In either case we have

$$f(a_n) - f(b_n) = \tfrac{1}{5}(f(a_{n-1}) - f(b_{n-1})) + O(5^{-n})$$

by subtracting (4.3) and its analog. From this we obtain easily

$$|f(a_n) - f(b_n)| \leqslant cn5^{-n} \tag{4.10}$$

(we can eliminate the factor n from (4.10) and (4.9) if we assume that Δf is Lipschitz continuous).

By applying (4.3) twice and adding we obtain

$$f(x)-\tfrac{1}{2}(f(a_n)+f(b_n))=\tfrac{3}{5}(f(x)-\tfrac{1}{2}(f(a_{n-1})+f(b_{n-1})))+O(5^{-n}).$$

If we write $v_n=(\tfrac{5}{3})^n\,(f(x)-\tfrac{1}{2}(f(a_n)+f(b_n)))$ this is just

$$v_n=v_{n-1}+O(3^{-n}), \tag{4.11}$$

and since $O(3^{-n})$ is a convergent geometric series it follows that v_n is a Cauchy sequence, and the limit is a multiple of the normal derivative. In the case that the normal derivative is nonzero, we obtain $c_1 \leqslant v_n \leqslant c_2$ which yields (4.8) when combined with (4.10). On the other hand, if $v_n \to 0$ then (4.11) implies $v_n=O(3^{-n})$, which yields (4.9) when combined with (4.10). ∎

Since $d(x, a_n)=2^{-n}$, we can express (4.8) as

$$c_1 d(x, y)^{\beta} \leqslant |f(x)-f(y)| \leqslant c_2 d(x, y)^{\beta} \tag{4.12}$$

for $\beta=\log(5/3)/\log 2 \approx .7369655$ and y equal to one of the points a_n, b_n, c_n, d_n. By using similar arguments it is easy to extend (4.12) to all points y. Similarly (4.9) becomes

$$|f(x)-f(y)| \leqslant c d(x, y)^{\gamma} \log d(x, y) \tag{4.13}$$

for $\gamma=\log 5/\log 2 \approx 2.3219281$. This dichotomy was established in [DSV] for harmonic functions (Theorem 4.4), without the logarithm term in (4.13).

It is easy to give another proof of Corollary 3.3 using this dichotomy, although we do not obtain Theorem 3.1 since we need to assume that a function belongs to the domain of the Laplacian in order to obtain the dichotomy at a single point. On the other hand, the dichotomy shows how difficult it is for a function to belong to the domain of the Laplacian, and allows us to deduce more general negative results.

THEOREM 4.4. *Let $\Phi: \mathbb{R} \to \mathbb{R}$ be any C^2 function such that Φ'' only has isolated zeroes. If f is any nonconstant function on SG in the domain of Δ, then $\Phi(f)$ is not in the domain of Δ.*

Proof. By a simple extension of Lemma 3.2, we can find a junction point x_0 where $\partial_\nu f(x_0) \neq 0$ and also $f(x_0)$ is not a zero of Φ. Consider the function $g(x)=\Phi(f(x))-\Phi'(f(x_0))\,f(x)$. If $\Phi(f)$ were in the domain of Δ,

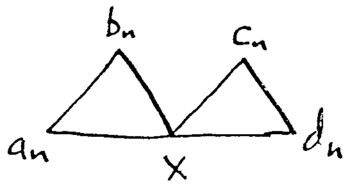

FIGURE. 4.1.

then g would be also. Therefore Theorem 4.3 would apply to g at x_0. But by Taylor's theorem

$$\begin{aligned} g(x)-g(x_0) &= \Phi(f(x))-\Phi(f(x_0))-\Phi'(f(x_0))(f(x)-f(x_0)) \\ &= \tfrac{1}{2}\Phi''(z)(f(x)-f(x_0))^2 \end{aligned}$$

for z between $f(x_0)$ and $f(x)$. Since f is continuous, by taking x close enough to x_0 we can make $\Phi''(z)$ close to $\Phi''(f(x_0))$ which is not zero. Since f satisfies (4.8) at x_0, we obtain from (4.14) $c_1(3/5)^{2n} \leqslant |g(x_0)-g(a_n)| \leqslant c_2(3/5)^{2n}$ for large enough n, so g satisfies neither (4.8) nor (4.9). Q.E.D.

THEOREM 4.5. *Let f be any C^1 on $\mathbb{R}^2$ with nonconstant restriction to SG. Then f is not in the domain of Δ.*

Proof. Suppose f were in the domain of Δ. By Lemma 3.2 there exists a junction point where $\partial_v f(x) \neq 0$. Then we are in part a) of Theorem 4.3, and (4.8) is inconsistent with f being C^1. ∎

We can also observe directly that $\Delta f(x)$ is undefined at a junction point x if f is differentiable at x and the directional derivative in the direction perpendicular to the line segment containing x is non-zero. For example, if x lies on a horizontal line segment as in Fig. 4.1, then

$$f(x)-\frac{1}{4}(f(a_n)+f(b_n)+f(c_n)+f(d_n)) = -\frac{\sqrt{3}}{4}\frac{\partial f}{\partial x_2}(x)\,2^{-n}+o(2^{-n}).$$

So if $(\partial f/\partial x_2)(x) \neq 0$, $\Delta f(x)$ is undefined.

5. SINGULARITY FOR SELF-SIMILAR FRACTALS

Let $(K, S, \{F_s\}_{s\in S})$ be a post critically finite self-similar structure and (D, r) be a harmonic structure as defined in [Ki2]. Here K is a compact metric space, $S=\{1, 2, ..., N\}$, $F_s\colon K\to K$ are continuous injections and $r=(r_1, ..., r_N)$ is a collection of positive numbers. The reader may find all the definitions in [Ki2]. This harmonic structure defines a Dirichlet form $\mathscr{E}$ which satisfies a self-similarity relation

$$\mathscr{E}(f, f) = \lambda \sum_{i=1}^{N} \frac{1}{r_i} \mathscr{E}(f \circ F_i, f \circ F_i), \tag{5.1}$$

where λ is a constant associated with (D, r).

The p.c.f. self-similar set K has a finite boundary $V_0 \subset K$, and the boundary of $K_{\omega_1 \cdots \omega_n} = F_{\omega_1} \cdots F_{\omega_n}(K)$ is $F_{\omega_1} \cdots F_{\omega_n}(V_0)$. The important feature of a p.c.f. structure is that the intersection of the sets $K_{\omega_1 \cdots \omega_n}$ and $K_{\omega'_1 \cdots \omega'_n}$ is contained in the boundary of these sets unless $\omega_i = \omega'_i$, $i = 1, ..., n$.

There are matrices $M_1, ..., M_N$ such that the boundary values of harmonic function h on the boundary of $K_{\omega_1 \cdots \omega_n}$ are equal to $M_{\omega_n} \cdots M_{\omega_1} v_0$ where v_0 is the vector of the boundary values of h. For all $x \in K$, except a countable subset, there corresponds a unique sequence $\{\omega_m\}_{m \geqslant 1}$ such that $\{x\} = \bigcap_{m \geqslant 1} K_{\omega_1 \cdots \omega_m}$. Then we denote $A_m(x) = M_{\omega_m}$.

Let μ be a Bernoulli measure on K such that $\mu(K_{\omega_1 \cdots \omega_m}) = \mu_{\omega_1} \cdots \mu_{\omega_m}$ where $\mu_i = \mu(K_i)$. Then matrices $A_m(x)$ are statistically independent with respect to μ with $\text{Prob}\{A_m(x) = M_i\} = \mu_i$.

For any f from the domain $\mathscr{F}$ of $\mathscr{E}$ we can define the measure ν_f in the same way as it was done for the Sierpinski gasket. Then there is a matrix $Q = (-D)^{1/2}$ such that for any harmonic function h

$$\nu_h(K_{\omega_1 \cdots \omega_m}) = \frac{\lambda^m}{r_{\omega_1} \cdots r_{\omega_m}} \|QM_{\omega_m} \cdots M_{\omega_1} v_0\|^2, \tag{5.2}$$

where v_0 is the vector of the boundary values of h (see Lemma 6.13.1 in [Ki2]).

For the next theorem we assume that

$$\mu_i = \frac{1}{r_i}. \tag{5.3}$$

The same assumption is made in [Ku2]. Note that in Section 3 we have constants $r_1 = r_2 = r_3 = 1$ and $\mu_1 = \mu_2 = \mu_3 = \frac{1}{3}$, the same as (5.3) up to a constant factor.

THEOREM 5.1. *Suppose that for any nonconstant harmonic function with boundary values v_0 there exists m such that function $x \mapsto \|QA_m(x) \cdots A_1(x) v_0\|$ is not constant. Then the measure ν_f is singular with respect to μ for any $f \in \mathscr{F}$.*

Proof. By (5.1) we have that

$$\|Qv_0\|^2 = \lambda \sum_{i=1}^{N} \frac{1}{r_i} \|QM_i v_0\|^2 = \lambda \int_K \|QA_1(x)\, v_0\|^2 \, d\mu(x)$$

$$= \lambda^m \int_K \|QA_m(x) \cdots A_1(x)\, v_0\|^2 \, d\mu(x) \tag{5.4}$$

for any m. This relation is the same as Lemma 6.13.1 in [Ki2]. The assumption of the theorem implies, similar to (3.10), that for some m

$$\sup_{\{v_0 \,:\, \|Qv_0\| = 1\}} \int_K \log \|Qv_m(x)\| \, d\mu(x) = \beta < -\frac{m}{2} \log \lambda, \tag{5.5}$$

where $v_m(x) = A_m(x) \cdots A_1(x)\, v_0$.

In this proof for the sake of simplicity we assume that for any nonconstant harmonic function $\|Qv_m(x)\| \neq 0$ for all m and x. Otherwise one can change the expression under the integral in (5.5) to $\log(\|Qv_m(x)\| + \delta)$. If $\delta > 0$ is small then the inequality (5.5) still holds though with a larger β.

Then, by induction,

$$\int_K \log \|Qv_{mn}(x)\| \, d\mu(x)$$

$$= \int_K \log \left\| QA_{mn}(x) \cdots A_{m(n-1)}(x) \frac{v_{m(n-1)}(x)}{\|Qv_{m(n-1)}(x)\|} \right\| d\mu(x)$$

$$+ \int_K \log \|Qv_{m(n-1)}(x)\| \, d\mu(x)$$

$$\leqslant \beta + \int_K \log \|Qv_{m(n-1)}(x)\| \, d\mu(x) \leqslant n\beta$$

if $\|Qv_0\| = 1$. Moreover, one can see that for any sequence $\omega_1, ..., \omega_k$ we have

$$\int_{K_{\omega_1 \cdots \omega_k}} \log \|Qv_{mn+k}(x)\| \, d\mu(x) \leqslant \mu_{\omega_1} \cdots \mu_{\omega_k} (n\beta + \log \|Qv_k(x)\|).$$

This implies that (at least for a subsequence)

$$\limsup_{n \to \infty} \frac{1}{n} \log \|Qv_n(x)\| \leqslant \frac{1}{m} \beta < -\frac{1}{2} \log \lambda \tag{5.6}$$

for μ-a.e. x.

Inequality (5.6) follows from the fact that the sequence $\{\log \|Qv_{mn}(x)\| - \beta n\}_{n=1}^{\infty}$ is a supermartingale on the probability space (K, μ). To prove it in more elementary terms, define $f_k(x) = \log \|Qv_{mk}(x)\|$, $g_{k+1} = (\mu_{\omega_1} \cdots \mu_{\omega_k})^{-1} \times \int_{K_{\omega_1 \cdots \omega_k}} f_{k+1}(x)\, d\mu(x)$ for $x \in K_{\omega_1 \cdots \omega_k}$ and $h_k(x) = f_k(x) - g_k(x)$. It is easy to see that $\{h_n\}_{n=1}^{\infty}$ is a bounded orthogonal sequence in L^2_μ and so $\|(1/n) \sum_{k=1}^{n} h_n\|_{L^2_\mu} \to 0$ as $n \to \infty$. At the same time $g_{n+1}(x) \leqslant \beta + f_n(x)$, that is $f_{n+1}(x) \leqslant \beta + f_n(x) + h_{n+1}(x)$. Then the L^2-convergence implies that (at least for a subsequence) inequality (5.6) holds for μ-a.e. x.

Thus by (5.2–6) for μ-a.e. sequence $\omega_1, \omega_2, ...$ we have

$$\lim_{n \to \infty} \frac{\nu_h(K_{\omega_1 \cdots \omega_n})}{\mu_{\omega_1} \cdots \mu_{\omega_n}} = 0$$

for any harmonic function h.

To define the measure ν, let $\{h_1, ..., h_p\}$ be an orthonormal basis of the nonconstant harmonic functions in $\|Q \cdot\|$-norm. Then $\nu = \nu_{h_1} + \cdots + \nu_{h_p}$. However, if not all matrices $M_1, ..., M_N$ are invertible, ν-measure of some open sets may not be positive. In this case it is enough to replace ν by the measure $\tilde{\nu} = \sum_{n=1}^{\infty} (1/(2N)^2) \sum_{\omega_1, ..., \omega_n} \nu \circ F_{\omega_n}^{-1} \cdots F_{\omega_1}^{-1}$.

The rest of the proof goes in the same way as in Theorem 3.4. Q.E.D.

Remark. One can see that the assumption of this theorem is not satisfied if and only if there exists a linear subspace $\mathscr{L}$ such that $\mathscr{L}$ is invariant for each M_i and $QM_i|_{\mathscr{L}} = QM_j|_{\mathscr{L}} \neq 0$ for every $i, j \in S$. It is easy to see that for any harmonic function h with boundary values $v_0 \in \mathscr{L}$ the measures ν_h and μ are equivalent (actually $\nu_h = \mathscr{E}(h, h)\, \mu$). The usual harmonic structure on an interval provides such an example. We conjecture that an interval is essentially the only situation when ν_h is not singular with respect to μ.

The singularity of the measures ν_f was proved in [Ku2, Theorem 2.14] under the assumption that the matrices $\{M_1, ..., M_N\}$ are invertible and strongly irreducible, and an additional assumption on a certain index (see Definition 1.10 and assumptions (A-1)–(A-4) in [Ku2]).

THEOREM 5.2. *Under the hypotheses of Theorem* 5.1, *the measure* ν_f *has no atoms, for any* $f \in \mathscr{F}$.

Proof. We claim that there is a constant $\rho < 1$ and a positive integer n such that for any harmonic function f,

$$\nu_f(K_{w_1 \cdots w_n}) \leqslant \rho \nu_f(K) \tag{5.7}$$

for any choice of $(w_1, ..., w_n)$. Once we have (5.7), the proof is the same as Theorem 3.5, using (5.7) in place of (3.14). By a compactness argument, if

(5.7) does not hold then there exists a nonconstant harmonic function and $(w_1, ..., w_n)$ such that

$$\nu_f(K_{w_1 \cdots w_n}) = \nu_f(K). \tag{5.8}$$

We will show that (5.8) is impossible, once n is chosen large enough so that each set $K_{w_1 \cdots w_n}$ contains at most one boundary point. Since f is nonconstant it attains both a maximum and minimum value, and by the weak maximum principle these values are attained at boundary points. It follows that the restriction of f to at least two distinct sets of the form $K_{w_1 \cdots w_n}$ must be nonconstant, hence $\nu_f(K_{w_1 \cdots w_n}) \neq 0$ for these two choices, making (5.8) impossible. ∎

REFERENCES

[BK] M. T. Barlow and J. Kigami, Localized eigenfunctions of the Laplacian on p.c.f. self-similar sets, *J. London Math. Soc. (2)* **56** (1997), 320–332.

[BH] N. Bouleau and F. Hirsch, "Dirichlet Forms and Analysis on Wiener Space," de Gruyter Stud. Math., Vol. 14, de Gruyter, Berlin, 1991.

[BP] M. T. Barlow and E. A. Perkins, Brownian motion on the Sierpinski gasket, *Probab. Theory Related Fields* **79** (1988), 543–623.

[DSV] K. Dalrymple, R. S. Strichartz, and J. P. Vinson, Fractal differential equations on the Sierpinski gasket, *J. Fourier Anal. Appl.* **5** (1999), 203–284.

[FOT] M. Fukushima, Y. Oshima, and M. Takada, "Dirichlet Forms and Symmetric Markov Processes," de Gruyter Stud. Math., Vol. 19, de Gruyter, Berlin, 1994.

[FS] M. Fukushima and T. Shima, On a spectral analysis for the Sierpinski gasket, *Potential Anal.* **1** (1992), 1–35.

[Fu] H. Furstenberg, Noncommuting random products, *Trans. Amer. Math. Soc.* **108** (1963), 377–428.

[G] S. Goldstein, Random walks and diffusions on fractals, *in* "Percolation Theory and Ergodic Theory of Infinite Particle Systems" (H. Kesten, Ed.), IMA Math Appl., Vol. 8, pp. 121–129, Springer-Verlag, New York, 1987.

[Ki1] J. Kigami, A harmonic calculus on the Sierpinski spaces, *Japan J. Appl. Math.* **6** (1989), 259–290.

[Ki2] J. Kigami, Harmonic calculus on p.c.f. self-similar sets, *Trans. Amer. Math. Soc.* **335** (1993), 721–755.

[Ki3] J. Kigami, Laplacians on self-similar sets and their spectral distributions, *in* "Fractal Geometry and Stochastics (Finsterbergen, 1994)," Progr. Probab., Vol. 37, pp. 221–238, Birkhäuser, Basel, 1995.

[Ki4] J. Kigami, Effective resistances for harmonic structures on p.c.f. self-similar sets, *Math. Proc. Cambridge Philos. Soc.* **115** (1994), 291–303.

[Ki5] J. Kigami, Distributions of localized eigenvalues of Laplacian on p.c.f. self-similar sets, preprint.

[KL] J. Kigami and M. L. Lapidus, Weyl's problem for the spectral distribution of Laplacians on p.c.f. self-similar fractals, *Comm. Math. Phys.* **158** (1993), 93–125.

[Ku1] S. Kusuoka, A diffusion process on a fractal, *in* "Probabilistic Methods in Mathematical Physics (Katata/Kyoto, 1985)," pp. 251–274, Academic Press, Boston, 1987.

[Ku2] S. Kusuoka, Dirichlet forms on fractals and products of random matrices, *Publ. Res. Inst. Math. Sci.* **25** (1989), 659–680.

[La] M. L. Lapidus, Analysis on fractals, Laplacians on self-similar sets, noncommutative geometry and spectral dimensions, *Topol. Methods Nonlinear Anal.* **4** (1994), 137–195.

[Li] T. Lindstrom, Brownian motion on nested fractals, *Mem. Amer. Math. Soc.* **420** (1989).

[S1] R. Strichartz, Some properties of Laplacians on fractals, *J. Funct. Anal.* **164** (1999), 181–208.

[S2] R. Strichartz, Taylor approximations on Sierpinski gasket type fractals, preprint.

[SU] R. Strichartz and M. Usher, Splines on fractals, *Math. Proc. Camb. Phil. Soc.*, to appear.

[T] A. Teplyaev, Spectral analysis on infinite Sierpinski gaskets, *J. Funct. Anal.* **159** (1998), 537–567.