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Some Properties of Laplacians on Fractals

Robert S. Strichartz[1]

Department of Mathematics, Cornell University, Ithaca, New York 14853
E-mail: str@math.cornell.edu

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Kigami has defined an analog of the Laplacian on a class of self-similar fractals, including the familiar Sierpinski gasket. We study properties of this operator. We show that there is a maximal principle for solutions of certain nonlinear equations of the form $\Delta u(x) = F(x, u(x))$. We discuss the extension of the Laplacian to non-compact fractal blow-ups, and show that it is essentially self-adjoint, and we prove an analog of Liouville's theorem in some cases. We also give an explicit algorithm for solving the Dirichlet problem for certain domains in the Sierpinski gasket and give a characterization of all harmonic functions on those domains.

1. INTRODUCTION

Analysis on fractals has been made possible by the definition of operators that play the role of the Laplacian. Originally produced as a by-product of the construction of the analog of Brownian motion ([BP, G, Ku1, Ku2, L1]), these Laplacians have been given by direct limit-of-difference-quotient definitions in the work of Kigami ([Ki1–6]), for a class of self-similar fractals that includes the Sierpinski gasket. In this paper we will explore some properties of these Laplace operators that are natural analogs of results that are known for the usual Laplacian. Since so much is known about the Laplacian, we can only scratch the surface in attempting the generalization to fractal Laplacians. For related works see [BK, BST, DSV, Fa, FS, KL, La, M, SU, T].

The first topic we discuss, in Section 2, is the maximum principle for solutions of a nonlinear equation $\Delta u(x) = F(x, u(x))$. Assuming that F is continuous and nonnegative for nonnegative values of u, we show that a positive maximum of u can only be attained on a boundary point, unless

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u is constant. This is a general result that holds without essential restrictions on the fractal. It is used to prove uniqueness results, and plays an essential role in some of the other proofs in this paper.

In addition to considering the usual self-similar fractals K, which are compact sets with a largest scale, it is also important to consider noncompact blow-ups K_∞, which have the same structure at all large scales as they have at small scales ([S2]). Suppose that K is defined by the self-similar identity

$$K = \bigcup_{j=1}^{m} S_j K \tag{1.1}$$

where $\{S_j\}$ is an iterated function system of contractive similarities on $\mathbb{R}^n$. Then we may define K_∞ by

$$K_\infty = \bigcup_{N=1}^{\infty} S_{j_1}^{-1} \cdots S_{j_N}^{-1} K \tag{1.2}$$

for some sequence $j_1, j_2, \ldots$ of indices. It is easy to extend the definition of a Laplacian Δ on K to K_∞. For a generic choice of the sequence of indices, K_∞ will have no boundary and plays the role of a noncompact complete Riemannian manifold. An important fact in the classical context is that the Laplacian is essentially self-adjoint (see [C] or [S1] for a proof). In Section 3 we prove the analogous result, under some additional assumptions. In Section 4 we take up the analog of the Liouville theorems. We show that if K is the Sierpinski gasket, nonconstant harmonic functions cannot be bounded, or even nonnegative, on K_∞. In the classical theorem there needs to be an assumption of nonnegative curvature. This result can thus be taken as another small piece of evidence that fractals tend to behave like Riemannian manifolds of positive curvature. At present this is only a heuristic principle, however. Although the results in Sections 3 and 4 are only proved under special assumptions, we conjecture that they are generally valid.

In Section 5 we return to the context of the fractal K, and specifically the Sierpinski gasket, and we study the Dirichlet problem for certain domains in K. The goal here is to find algorithmic formulas for solving $\Delta u = 0$ in an open set Ω with the boundary values of u on $\partial\Omega$ given. These are the analogs of the classical Poisson integral formulas for domains in $\mathbb{R}^n$. We are able to get an essentially complete solution when the domain Ω is a triangle obtained by cutting the Sierpinski gasket with a horizontal line at any vertical height below the top vertex. We also show how to generate all harmonic functions on these domains as Poisson integrals involving boundary values that are measures or more generally finitely additive set functions. For some reason, the Dirichlet problem for domains that are the complements of the above triangular domains is much harder. We show how to solve it in only a very special case.

We now briefly review the definitions and properties of the Laplacian that we will use. We assume that K is a self-similar fractal in $\mathbb{R}^n$ given by (1.1), and that it is post-critically finite (p.c.f.), meaning that K is connected, the intersections $S_j K \cap S_k K$ are finite sets, and the pre-images of the intersection points under all iterated mappings (the post-critical set) also form a finite set. (The Sierpinski gasket is the example in $\mathbb{R}^2$ with each S_j being a dilation with factor 1/2 centered at a vertex of a triangle.) We use multiindex notation $J=(j_1, ..., j_m)$ with $|J|=m$ the length of J, and write $S_J=S_{j_1}...S_{j_m}$ for iterations of the mappings. We call $S_J K$ the images of K of order m. It is possible to construct a similar theory in an abstract setting, but all known examples can be realized in $\mathbb{R}^n$, and we will use this embedding to define the fractal blow-ups.

We approximate K by a sequence of graphs $G_0, G_1, ...,$ where G_0 is the complete graph with vertices the post-critical set, and G_m is obtained from G_{m-1} by applying the mappings S_j to G_{m-1} and identifying points that are identical in K. The vertices of G_0 form the boundary of K in this theory (see the discussion at the end of Section 2 on how to delete boundary points).

The Laplacian on K is obtained as a limit of graph Laplacians on G_m. We begin with a symmetric matrix D with row sums zero, and entries that are positive off the diagonal and negative on the diagonal (we could assume merely that the off diagonal entries are nonnegative and D is irreducible, but this does not yield any new examples), that provides a difference operator on G_0. We also choose positive weights $(r_1, ..., r_m)$ and use r_j^{-1} to weight the difference operator when we move to a smaller scale by applying S_j. The matrix and weights must satisfy a consistency condition which says that a harmonic function G_{m-1} has a unique harmonic extension to G_m. See [Ki2] for the details. We will assume that the renormalization constant λ in Definition 4.4 of [Ki2] is one, which can always be achieved by rescaling the weights. When the consistency condition is satisfied, the result is called a *harmonic structure* on K, which allows us to define a sequence of Dirichlet forms $\mathscr{E}_m$ on G_m and a Dirichlet form $\mathscr{E}$ on K in the limit. This gives rise to a space (of dimension $\#G_0$) of harmonic functions on K, each determined uniquely by boundary data, and each restricting to a harmonic function on G_m. The Dirichlet form is the analog of $\int |\nabla f|^2\, dx$ in the classical case. We will consider only *regular* harmonic structures, which are defined by the condition that $r_j < 1$ for all j. This implies that points have positive capacity, and so the domain of the Dirichlet form consists of only continuous functions. In the case of the Sierpinski gasket, each vertex in G_m that is not a boundary point has exactly four neighbors, and the standard harmonic structure has the property that a function on G_m is harmonic if and only if the value at every nonboundary point is the average of the values at the 4 neighboring points.

To define a Laplacian we need to choose a harmonic structure on K and a measure μ. We will always take μ to be a self-similar measure, meaning that

$$\mu = \sum_{j=1}^{m} p_j \mu \circ S_j^{-1} \tag{1.3}$$

for certain probability weights p_j. This uniquely determines μ if we normalize it to be a probability measure. We will call μ the *balanced measure* on K if all p_j are equal. The equation that relates the Laplacian to measure and the Dirichlet form is the analog of

$$\int \nabla f \cdot \nabla g \, dx = -\int f \Delta g \, dx + \text{boundary terms}$$

where the measure plays the role of dx on the right side, but not the left. Thus we will have

$$\mathscr{E}(f, g) = -\int_K f \Delta g \, d\mu \tag{1.4}$$

if f vanishes at the boundary points. The definition of Δg is given as a limit of $\Delta_m g$, where Δ_m is a graph Laplacian on G_m. See [Ki2] for the details in the general case. For the example of the Sierpinski gasket with the standard harmonic structure and the balanced measure, the formula for Δ_m is

$$\Delta_m g(x) = -5^m (g(x) - \tfrac{1}{4}(g(y_1) + g(y_2) + g(y_3) + g(y_4))) \tag{1.5}$$

for x a nonboundary vertex of G_m and y_1, y_2, y_3, y_4 the four neighboring points in G_m. In this paper we always mean this Laplacian when we say "the Laplacian on the Sierpinski gasket." The definition of the domain of the Laplacian requires that both g and Δg be continuous functions on K, and $\Delta_m g \to \Delta g$ uniformly.

We also note that there is a notion of a normal derivative $\partial_n u$ at boundary points, defined in terms of the harmonic structure (the measure is not involved in the definition), and the analog of the Gauss–Green formula holds:

$$\int_K (f \Delta g - g \Delta f) \, d\mu = \sum_{\partial K} (f(x) \, \partial_n g(x) - g(x) \, \partial_n f(x)). \tag{1.6}$$

For the example of the Sierpinski gasket,

$$\partial_n u(x) = \lim_{m\to\infty} (5/3)^m \,(f(x) - 1/2(f(a_m) + f(b_m))) \tag{1.7}$$

where x is a boundary point and a_m and b_m are the two neighboring points of x in G_m. Note that the factors in (1.5) and (1.7) are different. For a harmonic function, the limit is unnecessary since all the terms on the right side of (1.7) are equal. The notion of a normal derivative can be localized to a boundary point of any image $S_J K$. We have the following important criterion for patching together functions on different images $S_J K$. Suppose u is harmonic on $S_J K$ and on $S_{j'} K$, and they intersect at a point x. Then u is harmonic at x if and only if it is continuous at x and the sum of the normal derivatives localized to $S_J K$ and $S_{J'} K$ is zero. It is not necessary that $|J| = |J'|$ for this to hold, so $S_J K$ and $S_{J'} K$ may have quite different sizes.

When considering harmonic functions on general domains Ω in K or in the blow-ups K_∞, we note that the problems are more combinatoric than analytic. For any image $S_J K$ contained in Ω, a harmonic function is determined on $S_J K$ by its values on the boundary points of $S_J K$. In fact, there is a simple algorithm, which we call the *harmonic algorithm*, for doing this. Since every domain is just a finite or countable union of such images $S_J K$, the only question is how to patch together the pieces. For this we will use the normal derivative criterion.

2. THE MAXIMAL PRINCIPLE

Let $F(x, u)$ denote a continuous function from $K \times \mathbb{R}$ to $\mathbb{R}$. We are interested in solutions of the fractal differential equation

$$\Delta u = F(x, u). \tag{2.1}$$

This means that u is in the domain of Δ, and $\Delta u(x)$ and $F(x, u(x))$ are equal as continuous functions on K. The *Dirichlet problem* is the equation (2.1) together with the boundary conditions

$$u(q_j) = a_j \tag{2.2}$$

at the boundary points q_j, $j = 1, ..., N$. We observe that under the uniform Lipschitz condition

$$|F(x, u) - F(x, v)| \leqslant M\, |u - v| \tag{2.3}$$

for all $u, v \in \mathbb{R}$, $x \in K$, we have local existence and uniqueness for the Dirichlet problem, where "local" means either that M is sufficiently small or that we restrict to a small image of K. (Global uniqueness may fail if F is negative, according to a recent preprint of Falconer [Fa].) The proof is essentially the same as for ordinary differential equations. We first reduce to the case of zero boundary conditions by subtracting a harmonic function with the boundary conditions (2.2). Then (2.1) is equivalent to the integral equation

$$u(x)=\int_K G(x, y)\, F(y, u(y))\, d\mu(y), \tag{2.4}$$

where G denotes the Green's function [Ki2]. The assumption that the harmonic structure is regular implies that G is bounded, and we can apply the Picard iteration method.

We can also localize (2.1) to any open set $\Omega \subseteq K$. The equation now is interpreted to mean that the restriction of u to any small copy $S_J K$ of K contained in Ω belongs to the domain of the Laplacian on $S_J K$. In particular, we do not necessarily assume that it has a continuous extension up to the boundary of Ω

We will say that F is *nonnegative* if

$$F(x, u) \geqslant 0 \qquad \text{whenever} \quad u \geqslant 0. \tag{2.5}$$

THEOREM 2.1 (Maximum Principle). *Let u be a solution of* (2.1) *on Ω for nonnegative F. If u is continuous up to the boundary and*

$$\sup_{x \in \Omega} u(x) > 0, \tag{2.6}$$

then

$$\sup_{x \in \Omega} u(x) \leqslant \sup_{y \in \partial\Omega} u(y) \tag{2.7}$$

with equality in (2.7) *only if u is constant on a component of Ω.*

For the proof we will need the following fact, which is essentially Theorem 5.8(2) in [Ki2]:

LEMMA 2.2. *For any nonboundary point y in G_m, let $\psi_{m,y}$ denote the continuous function that is harmonic on the complement of the vertices of G_m, and is equal to the delta function at y on the vertices of G_m. Then*

$$\int \psi_{m,y}(x)\, \Delta u(x)\, d\mu(x) = \Delta_m u(y) \tag{2.8}$$

for any u in the domain of the Laplacian. Also, $\psi_{m,y}$ is a nonnegative function.

Proof of Theorem 2.1. Suppose there were an interior maximum point $\bar{x}$. Then by (2.6) and continuity there exists a neighborhood Ω_1 of $\bar{x}$ on which u is positive. If $\psi_{m,y}$ has support in Ω_1, then the left side of (2.8) is nonnegative, and in fact strictly positive unless $\Delta u \equiv 0$ on the support of $\psi_{m,y}$. Then (2.8) implies $\Delta_m u(y) \geqslant 0$, hence $u(y)$ is bounded above by the maximum of u on the points neighboring y in G_m.

Suppose first that $\bar{x} \in S_J K \subseteq \Omega_1$ and $\bar{x}$ is not a boundary point of $S_J K$, where $|J| = m-1$. Let $\bar{u}$ denote the maximum value of u at the boundary points of $S_J K$. Among all the vertices of G_m interior to $S_J K$, let $\bar{y}$ be the one where u achieves its maximum. Then by the above $u(\bar{y}) \leqslant \bar{u}$, and so $u(y) \leqslant \bar{u}$ for all G_m vertices in $S_J K$. We can repeat the argument on smaller scales to obtain $u(y) \leqslant \bar{u}$ on a dense subset of $S_J K$, hence throughout $S_J K$ by continuity. Moreover, we can only have equality at the first step if $\Delta u = 0$ in $S_J K$ and u attains the same value $\bar{u}$ at all the boundary points, which implies that u is constant on $S_J K$. But once we have strict inequality at the first step, we contradict the possibility that u attains its maximum at an interior point of $S_J K$.

Next we consider the possibility that u attains its maximum at a boundary point of some $S_J K$. By taking m large enough we have $\Delta_m u(\bar{x}) \geqslant 0$, and so u must assume the same value at all neighboring points to $\bar{x}$ in G_m, and must be harmonic in a neighborhood of $\bar{x}$. Again this implies that u is constant in that neighborhood.

So in either case, the only way u can attain its maximum at an interior point $\bar{x}$ of Ω is for u to be constant in a neighborhood of $\bar{x}$. By repeating the argument we can show that u must be constant in the connected component of Ω containing $\bar{x}$. Q.E.D.

COROLLARY 2.3. *The linear equation*

$$\Delta u(x) = a(x)\, u(x) + b(x) \qquad \text{on } K, \tag{2.9}$$

where $a(x)$ and $b(x)$ are continuous and $a(x)$ is non-negative, with given Dirichlet data

$$u|_{\partial K} = f, \tag{2.10}$$

has at most one solution.

Proof. Suppose there were two distinct solutions u and v. Then we may assume without loss of generality that $u - v$ attains positive values. But $u - v$ satisfies $\Delta(u - v) = a(u - v)$ and has zero Dirichlet data, and this contradicts the Maximal Principle. Q.E.D.

The construction of the Laplacian on K given in [Ki2] assumes that the points in the initial graph G_0 will be treated as boundary points. This is natural in some examples, such as the unit interval or the Sierpinski gasket, but not so natural in other examples, where the local geometry of the boundary points is no different than at other nonboundary points. In fact it is always possible to delete points from the boundary, and this is equivalent to imposing Neumann conditions at the boundary points. This gives rise to a different Laplacian with a different domain. Let us denote by Δ_0 the Laplacian on K treated as having no boundary points. Then u is in the domain of Δ_0 if and only if u is in the domain of Δ and the normal derivatives $\partial_n u$ vanish at all boundary points; in that case $\Delta_0 u = \Delta u$. This idea is implicit in [Ki2], and will be explained in detail in the forthcoming book [Ki6].

COROLLARY 2.4. *Let u belong to the domain of Δ_0 on K and satisfy $\Delta_0 u(x) = F(x, u(x))$ on K, where F is nonnegative. Then if u assumes a positive value, u is constant.*

Proof. By the theorem, u attains its maximum at a boundary point. Suppose v_0 is a boundary point where u attains its maximum. We also know, $\partial_n u(v_0) = 0$. Choose m large enough that u is positive on $S_J K$ for $|J| = m$, with $v_0 \in S_J K$. Then by the analog of (2.8) at the boundary point v_0, we have $\Delta_m u(v_0) \leqslant 0$. By the same reasoning as before, u must be constant on all the boundary points of $S_J K$. This implies that u is constant as before. Q.E.D.

3. ESSENTIAL SELF-ADJOINTNESS

We begin with some results about the Laplacian on K that will be needed for the analysis on K_∞. These results appear to be of independent interest and include the equality of weak and strong forms of Δ. One definition of the domain of the Laplacian, $\operatorname{dom} \Delta(K)$, is that g belongs to $\operatorname{dom} \mathscr{E}(K)$ and (1.4) holds for a continuous function Δg and all $f \in \operatorname{dom} \mathscr{E}(K)$ with f vanishing on the boundary. This is a "semi-weak" formulation, which is shown to be equivalent to the strong (pointwise) formulation in [Ki2]. For the full weak formulation we define $\mathscr{D}(K)$ to be the subset of $\operatorname{dom} \Delta(K)$ of functions v such that both v and its normal derivatives $\partial_n v$ vanish on the boundary.

DEFINITION 3.1. We say $u \in \operatorname{dom} \Delta^*(K)$ and $\Delta^* u = f$ for u and f in L^2 if

$$\int_K u\, \Delta v \, d\mu = \int_K f v \, d\mu \qquad \text{for all} \quad v \in \mathscr{D}(K). \tag{3.1}$$

Note that this is essentially (1.6) in view of the vanishing of the boundary terms. Clearly, strong implies weak: if $u \in \operatorname{dom} \Delta(K)$ with $\Delta u = f$ then $u \in \operatorname{dom} \Delta^*(K)$ with $\Delta^* u = f$. We will prove the converse under the assumption that f is continuous. Because we are assuming that the harmonic structure is regular, the assumption $u \in \operatorname{dom} \mathscr{E}(K)$ already implies that u is continuous.

It is also possible to consider the adjoint of Δ defined on the smaller domain $\mathscr{D}_0(K)$, defined to be the function in $\operatorname{dom} \Delta(K)$ that vanishes in a neighborhood of the boundary. In fact, $\mathscr{D}_0(K)$ is dense in $\mathscr{D}(K)$ in an appropriate sense so that the same adjoint arises. We will not need this fact here. A proof will be given in [SU].

THEOREM 3.2 (Weak = Strong). *If $u \in \operatorname{dom} \Delta^*(K)$ with $\Delta^* u = f$, then there exists a harmonic function h such that*

$$u(x) = -\int_K G(x, y)\, f(y)\, d\mu + h(x), \tag{3.2}$$

where $G(x, y)$ denotes the Green's function. In particular, $u \in \operatorname{dom} \mathscr{E}(K)$ and

$$\mathscr{E}(u, v) = -\int_K fv\, d\mu \tag{3.3}$$

for every $v \in \operatorname{dom} \mathscr{E}(K)$ vanishing on the boundary. If in addition f is continuous, then $u \in \operatorname{dom} \Delta(K)$ and $\Delta u = f$.

Proof. Consider any continuous function w that is orthogonal to all harmonic functions,

$$\int_K wh\, d\mu = 0 \qquad \text{for } h \text{ harmonic.} \tag{3.4}$$

We claim

$$v(x) = \int_K G(x, y)\, w(y)\, d\mu(y) \tag{3.5}$$

belongs to $\mathscr{D}(K)$. The basic properties of the Green function imply that $v \in \operatorname{dom} \Delta(K)$ with $\Delta v = -w$ and v vanishes on the boundary, so it remains to check the vanishing of $\partial_n v$. For $x \in \partial K$, $\partial_n v(x)$ is the inner product of w with $\partial_n G(x, \cdot)$. But $\partial_n G(x, \cdot)$ is a harmonic function, in fact the one assuming boundary value 1 at x and 0 otherwise. This follows from Lemma 5.8.2 in [Ki2], and in fact is part of the general folklore relating Poisson kernels and Green's functions. Thus $\partial_n v(x) = 0$ by (3.4).

Now we apply (3.1) to this v, obtaining

$$-\int_K uw \, d\mu = \int_K \int_K f(x)\, G(x, y)\, w(y)\, d\mu(y)\, d\mu(x).$$

Since the Green's function is continuous and symmetric, this means

$$u(x) + \int_K G(x, y)\, f(y)\, d\mu(y)$$

is orthogonal to w. By adding an appropriate harmonic function it becomes orthogonal to all continuous functions, proving (3.2). If f is continuous then (3.2) gives $u \in \operatorname{dom} \Delta(K)$ and $\Delta u = f$, and (3.3) holds. If f is only in L^2, approximate it by continuous functions, then pass to the limit in (3.3).
Q.E.D.

Corollary 3.3. *If* $u \in \operatorname{dom} \Delta^*(K)$ *and* $\Delta^* u = \lambda u$ *for* $\lambda \neq 0$, *then* $u \in \operatorname{dom} \Delta(K)$ *and* $\Delta u = \lambda u$.

Proof. By the theorem, (3.2) holds with $f = \lambda u$. This implies that u is continuous, hence f is continuous, and the result follows by the last statement of the theorem. Q.E.D.

Now let K_∞ denote any fractal blow-up of K (see [S2]). That is,

$$K_\infty = \bigcup_{n=0}^{\infty} S_{j_1}^{-1} S_{j_2}^{-1} \cdots S_{j_n}^{-1} K \tag{3.6}$$

for some sequences, $j_1, j_2, \ldots$ of indices. The union in (3.6) is increasing. The space K_∞ is noncompact, and for a generic choice of the sequence of indices it will have no boundary. Any compact subset of K_∞ will lie in one of the sets in the union (3.6), and so will be similar to a subset of K. This enables us to extend the definition of Δ to functions on K_∞. More precisely, let $x \in S_{j_1}^{-1} \cdots S_{j_n}^{-1} K$ and let u be defined in a neighborhood of x. Then $u \circ (S_{j_1}^{-1} \cdots S_{j_n}^{-1})$ is defined in a neighborhood of $S_{j_n} \cdots S_{j_1} x$ in K, so we may set

$$\Delta u(x) = (r_J p_J)(\Delta(u \circ S_{j_1}^{-1} \cdots S_{j_n}^{-1}))(S_{j_n} \cdots S_{j_1} x), \tag{3.7}$$

where

$$r_J = r_{j_1} \cdots r_{j_n} \qquad \text{and} \qquad p_J = p_{j_1} \cdots p_{j_n}.$$

This definition is easily seen to be independent of the choice of n (provided n is large enough) because of the dilation property of Δ. In a similar way, we can extend the self-similar measure μ to K_∞.

Let $\mathscr{D}_0(K_\infty)$ denote the space of functions of compact support on K_∞, vanishing in a neighborhood of the boundary, that are in the domain of the Laplacian. The statement that Δ is *essentially self-adjoint on* $\mathscr{D}_0(K_\infty)$ means that Δ has a unique extension to an unbounded self-adjoint operator on $L^2(K_\infty, d\mu)$. It is the uniqueness that is in question here; there is always a Friedrichs extension since Δ is a nonpositive operator. If K_∞ has a boundary point then essential self-adjointness will fail, since there can be both Dirichlet and Neumann boundary conditions at the boundary point, leading to distinct self-adjoint extensions. To describe essential self-adjointness when K_∞ has no boundary points we will appeal to the following well-known criterion [RS].

THEOREM 3.4. *Let A be a nonpositive symmetric operator defined on a dense domain in a Hilbert space. Then A is essentially self-adjoint if there are no nonzero solutions to the eigenvalue equation*

$$A^* u = u.$$

Given a harmonic structure on K, there is a natural choice of self-similar measure μ satisfying (1.3) where the weights are chosen so that

$$p_j = r_j^s, \tag{3.8}$$

where s is the unique positive value that makes $\Sigma_j p_j = 1$. In [KL], s is called the similarity dimension of the harmonic structure, and it is shown that the Laplacian associated with μ maximizes the Weyl asymptotic growth rates for the eigenvalues of the Laplacian over all choices of self-similar measure. Roughly speaking, the choice (3.8) means that if we sort the sets $S_J K$ according to either the size of $\mu(S_J K)$ or the strength of the Dirichlet form on $S_J K$, we obtain the same result.

Now the dilation property (3.7) implies that if $\Delta u = \lambda u$, then

$$u \circ (S_{j_1}^{-1} \cdots S_{j_n}^{-1} S_{j'_1} \cdots S_{j'_m}) \tag{3.9}$$

is an eigenfunction with eigenvalue

$$(p_J r_J)^{-1} \, p_{J'} r_{J'} \lambda.$$

Under the assumption (3.8) this becomes

$$(p_{J'}/p_J)^{1+1/s} \, \lambda. \tag{3.10}$$

We will be choosing the indices so that $p_{J'}/p_J$ is bounded above and below. This will mean that if we take an eigenfunction on K_∞ and restrict it to

$$S_{j_1}^{-1} \cdots S_{j_n}^{-1} S_{j'_1} \cdots S_{j'_m} K,$$

we can pull it back to K via (3.9) without substantially distorting either the measure of the set or the eigenvalue.

LEMMA 3.5. *Let u be a solution of $\Delta^* u = u$ on K_∞. If K_∞ has no boundary, then $u \in \operatorname{dom} \Delta(K_\infty)$ and $\Delta u = u$.*

Proof. Suppose $v \in \mathcal{D}(K)$, and denote by $\tilde{v}$ the extension of v to K_∞ obtained by setting $\tilde{v} = 0$ outside K. Because v and $\partial_n v$ vanish at the boundary of K, $\tilde{v} \in \operatorname{dom}(K_\infty)$ and $\Delta \tilde{v}$ is the extension of Δv also equal to 0 outside K. Now if $\Delta^* u = u$ on K_∞ we have

$$\int_{K_\infty} u \, \Delta \tilde{v} \, d\mu = \int_{K_\infty} u \tilde{v} \, d\mu$$

hence

$$\int_K u \, \Delta v \, d\mu = \int_K uv \, d\mu.$$

This means $u|_K \in \operatorname{dom} \Delta^*(K)$ with $\Delta^*(u|_K) = u|_K$. By Corollary 3.3, $u|_K \in \operatorname{dom} \Delta(K)$ with $\Delta u|_K = u|_K$. We can apply the same argument to any of the images of K on the right side of (3.6) that make up K_∞, and the result follows. Q.E.D.

THEOREM 3.6. *Assume Δ is defined in terms of μ satisfying* (3.8). *If K_∞ has no boundary points, then Δ on K_∞ is essentially self-adjoint on $\mathcal{D}$.*

Proof. To apply Theorem 3.4 we consider solutions of $\Delta^* u = u$. By Lemma 3.5, u is in the domain of Δ and $\Delta u = u$. If u is not identically zero, we may assume without loss of generality that it assumes a positive value (if not, multiply by -1). Then by the maximum principle it assumes a value at least $\varepsilon > 0$ on at least one of the boundary points of $S_{j_1}^{-1} \cdots S_{j_n}^{-1} K$ for all large n. Altogether there are an infinite number of distinct points such as this, because if the sets $S_{j_1}^{-1} \cdots S_{j_n}^{-1} K$ had a common boundary point for all large n, this point would be a boundary point of K_∞.

Now each boundary point of $S_{k_1}^{-1} \cdots S_{j_n}^{-1} K$ is a boundary point of $S_{j_1}^{-1} \cdots S_{j_n}^{-1} S_{j'_1} \cdots S_{j'_m} K$ for some sequence $j'_1, \ldots, j'_m$ and each m. Choose m to be the smallest value that makes $p_{J'}/p_J \leqslant 1$. Then we also have $p_{J'}/p_J \geqslant p_{\min}$, where $p_{\min}$ is the minimum value of $\{p_j\}$. By passing to a subsequence if necessary, we can arrange for all these sets to be disjoint. Again this is a compactness argument based on the fact that K_∞ has no boundary, and each time we apply a blow-up we increase the measure by a fixed amount. So we have an infinite collection of disjoint sets whose measures are bounded above and below, and when we pull back u to K via (3.9), we obtain an eigenfunction with eigenvalue bounded above and below. Now on K we have the estimate

$$\max_{\partial K} |u| \leqslant c_1(c_2 |\lambda| + 1) \|u\|_2 \qquad \text{if} \quad \Delta u = \lambda u, \tag{3.11}$$

for positive constants c_1 and c_2 [Ki6]. We apply (3.11) to the pullback function (3.9). We have the lower bound ε for the maximum on the boundary, and the L^2 norm is only changed by a constant if we pass from K to $S_{j_1}^{-1} \cdots S_{j_n}^{-1} S_{j'_1} \cdots S_{j'_m} K$, hence

$$\left(\int_{S_{j_1}^{-1} \cdots S_{j_n}^{-1} S_{j'_1} \cdots S_{j'_m} K} |u|^2 \, d\mu \right)^{1/2} \geqslant c\varepsilon$$

for some fixed $c > 0$. Thus we have a uniform lower bound for the L^2 norm on an infinite number of disjoint sets, so u cannot be in L^2 on K_∞. Q.E.D.

Another situation in which we have essential self-adjointness is if we consider the Laplacian Δ_0 on K considered to be a compact space with no boundary. In this case there is no need to assume that μ satisfies (3.8). The statement is that Δ_0 defined on its domain is essentially self-adjoint. The proof is similar, using Corollary 2.4 to show that $\Delta_0 u = u$ has no solutions. Of course, the essential self-adjointness of Δ_0 is the same as the essential self-adjointness of the Neumann Laplacian, which is known from [Ki2].

4. LIOUVILLE THEOREMS

In this section we assume that K_∞ is a fractal blow-up of the standard Sierpinski gasket. It is likely that similar results hold in more general settings, but our arguments are based on specific information about harmonic functions on SG from [Ki1] or [DSV]. Suppose T is any triangle in the graph G_m, and let T_1, T_2, T_3 be the three triangles in G_{m+1} that make up T. For any function on the vertices of such triangles we denote the values by a 3-vector w, going counterclockwise from the top vertex (all triangles have the same orientation). Let u be a harmonic function, and let w, w_1, w_2, w_3 denote the 3-vectors of the values of u on the vertices of triangles T, T_1, T_2, T_3. The result we need is the following.

Lemma 4.1. *We have*

$$w_j = M_j w \tag{4.1}$$

where

$$M_1 = \frac{1}{5}\begin{pmatrix} 5 & 0 & 0 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}, \qquad M_1 = \frac{1}{5}\begin{pmatrix} 2 & 2 & 1 \\ 0 & 5 & 0 \\ 1 & 2 & 2 \end{pmatrix}, \qquad M_3 = \frac{1}{5}\begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 5 \end{pmatrix}. \tag{4.2}$$

We observe that all these matrices preserve the constant 3-vectors, which just corresponds to the fact that a harmonic function constant on the boundary vertices of K is constant. The important observation is that once we factor out by the constant vectors we obtain contractive mappings on the two-dimensional quotient space. For example, if we take $(1, -1, 0)$ and $(0, 1, -1)$ as a basis for the quotient space, then the action of M_1 is represented by the matrix $\frac{1}{5}\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$ with eigenvalues 3/5 and 1/5, and similarly for M_2 and M_3. It follows that the inverse matrices are expansive. This leads easily to our first Liouville theorem.

THEOREM 4.2. *A bounded harmonic function on K_∞ is constant.*

Proof. Let u be a nonconstant harmonic function. Then on some copy of K in K_∞ it is nonconstant, so the 3-vector of boundary values is nonconstant. As we extend u to larger expansions of K we successively multiply by matrices M_j^{-1}, the choice of j depending on the relative positions of each triangle in the next larger one. Since these matrices are expansive in the quotient 2-space, the boundary values grow without bound, so u is not bounded. Q.E.D.

The corresponding fact for nonnegative harmonic functions is more subtle, since it is not true if K_∞ has a boundary point. Indeed, we observe that M_1 has a nonnegative eigenvector $(0, 1, 1)$ with eigenvalue 3/5, so this is also a nonnegative eigenvector for M_1^{-1} with eigenvalue 5/3. Thus if K_∞ is the blow-up with all indices $j_k = 1$, we can take the harmonic function with boundary data $(0, 1, 1)$ on K and extend it to K_∞ so that it has boundary data $(5/3)^n (0, 1, 1)$ on $S_1^{-n}K$. The result is a nonnegative harmonic function on K_∞ that is not constant. The next theorem shows that this cannot happen if K_∞ has no boundary point, which is equivalent to the statement that the sequence of indices $j_1, j_2, \dots$ is not eventually constant.

THEOREM 4.3. *Suppose K_∞ has no boundary points. Then any nonnegative harmonic function is constant.*

Proof. Let u be a nonnegative harmonic function, and let w be the vector of boundary values of u on K. Then $M_{j_n}^{-1} \cdots M_{j_1}^{-1} w$ is the vector of boundary values on $S_{j_1} \cdots S_{j_n} K$, and hence must be nonnegative. We will show that this is impossible unless w is constant. We will study the dynamical system obtained by multiplying by the matrices M_j and renormalizing the length. A convenient normalization is the ℓ^1 condition $x+y+z=3$. With this choice the induced action of multiplication by M_1 is

$$\tilde{M}_1(x, y, z) = \left(\frac{5x}{3x+y+z}, \frac{2x+2y+z}{3x+y+z}, \frac{2x+y+2z}{3x+y+z} \right), \tag{4.2}$$

and similarly for $\tilde{M}_2$ and $\tilde{M}_3$. Note that $\tilde{M}_1$ has two fixed-points, $(1, 1, 1)$ and $(0, 3/2, 3/2)$. The key technical observation is that $\tilde{M}_j$ does not increase distances to the common fixed-point $(1, 1, 1)$, measured in the Euclidean metric on the plane triangle

$$x+y+z=3, \qquad x \geqslant 0, \quad y \geqslant 0, \quad z \geqslant 0. \tag{4.4}$$

Lemma 4.4. *For any vector* (x, y, z) *satisfying* (4.4), *we have*

$$\|\tilde{M}_1(x, y, z) - (1, 1, 1)\| \leqslant \|(x, y, z) - (1, 1, 1)\| \tag{4.5}$$

with equality only if (x, y, z) *is* $(1, 1, 1)$ *or* $(0, 3/2, 3/2)$.

Proof. Introduce coordinates $u = y+z-2$ and $v = y-z$, and similarly u', v' for $\tilde{M}_1(x, y, z)$. Note that $-2 \leqslant u \leqslant 1$ and $(x, y, z) = (1-u, 1+(u+v)/2, 1+(u-v)/2)$ so that

$$\|(x, y, z) - (1, 1, 1)\|^2 = (3/2)\, u^2 + (1/2)\, v^2.$$

Also $(u', v') = (3u/(5-2u), v/(5-2u))$. Thus (4.5) is equivalent to

$$(3/2)(3u/(5-2u))^2 + (1/2)(v/(5-2u))^2 \leqslant (3/2)\, u^2 + (1/2)\, v^2. \tag{4.6}$$

But $1/(5-2u) \leqslant 1/3$ for $-2 \leqslant u \leqslant 1$ with equality only at $u=1$, so (4.6) follows, with equality only at $(u, v) = (1, 0)$ or $(u, v) = (0, 0)$. Q.E.D.

Returning to the proof of Theorem 4.3, we observe that the analog of (4.5) holds for $\tilde{M}_2$ and $\tilde{M}_3$, and that aside from the common fixed point $(1, 1, 1)$, the other point for which equality holds is different for each of the maps $\tilde{M}_1$, $\tilde{M}_2$, $\tilde{M}_3$. This means that if we compose two distinct maps $\tilde{M}_j \tilde{M}_k$. for $j \neq k$, then we have strict inequality, and by compactness we may conclude that there exists $r < 1$ with

$$\|\tilde{M}_j \tilde{M}_k(x, y, z) - (1, 1, 1)\| \leqslant r\, \|(x, y, z) - (1, 1, 1)\| \tag{4.7}$$

if (x, y, z) satisfies (4.4), using (4.6) for the estimate near $(x, y, z) = (1, 1, 1)$.

We may now easily derive a contradiction. If the vectors $M_{j_n}^{-1} \cdots M_{j_1}^{-1} w$ are all nonnegative, then $\tilde{M}_{j_n}^{-1} \cdots \tilde{M}_{j_1}^{-1} w$ all satisfy (4.4), and by applying (4.5) and (4.7) we can make w arbitrarily close to (1, 1, 1) since there will eventually be arbitrarily many consecutive indices that are distinct. Taking the limit, w is constant. Q.E.D.

Another interesting related question concerns the rate of growth of unbounded harmonic functions on K_∞. Furstenberg's theorem [Fu] on products of random matrices gives some information about this in the generic case. To make a more precise statement would require in particular the computation of the top Lyapanov index the family of matrices M_1, M_2, M_3, with equal probability.

5. DIRICHLET PROBLEMS

In this section we give explicit solutions to the Dirichlet problem for harmonic functions on certain domains in the Sierpinski gasket. Since existence and uniqueness is known, the issue here is the explicit formulas, which should be thought of as analogs of explicit Poisson integral formulas in Euclidean domains. But what should we mean by an explicit formula for a function defined on a self-similar fractal? Since points on the fractal are defined as limits of iterations of transformations from the defining i.f.s., or equivalently, as images of infinite sequences from a coding space under a coding map, it would seem reasonable to define an explicit function as one that can be computed by iterating some algorithm based on the coding sequence of the point. We will not attempt a formal definition of these ideas here. The explicit formulas that enter into our solutions of the Dirichlet problem will easily be seen to possess the required iterative descriptions. They will not be easily described in terms of restriction to the Sierpinski gasket of elementary analytic functions on the ambient Euclidean plane; such a description would be implausible in this context. On the other hand, the domains we consider will have a simple description in terms of the standard embedding of the Sierpinski gasket in the plane.

For $0 \leqslant x \leqslant 1$, we let T_x denote an open triangular domain of height x. The boundary of T_x consists of a single vertex v_0, which we take to be the top vertex of G_0, and a horizontal section S_x at vertical distance x to v_0 (we normalize the total vertical distance from the bottom vertices to the top to be 1, so the closure of T_1 is all of K). Thus S_x will be a Cantor set for generic x, and a union of intervals if x is a dyadic rational. We may assume without loss of generality that $\frac{1}{2} < x < 1$, for if not we may first solve the Dirichlet problem for T_{2x}, and then simply dilate the solution to T_x.

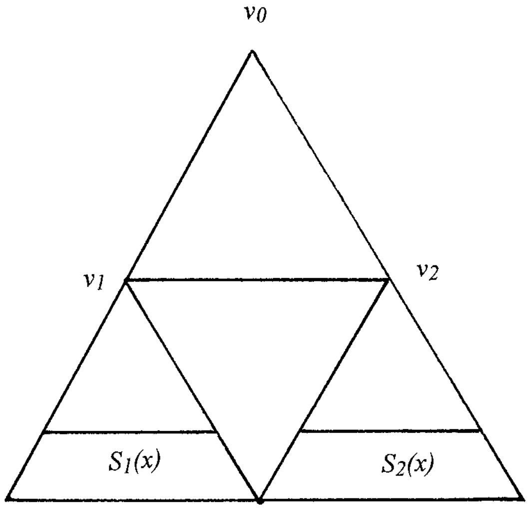

FIGURE 1

The Dirichlet problem for T_x we consider first is

$$\Delta u = 0 \qquad \text{on } T_x \tag{5.1}$$

$$u(v_0) = a_0 \tag{5.2}$$

$$u = f \qquad \text{on } S_x, \tag{5.3}$$

where f is a given continuous function on S_x, and u is assumed to be continuous on the closure of T_x ($T_x \cup S_x \cup \{v_0\}$). Let v_1 and v_2 denote the two other vertices of the triangle in G_1 with top vertex v_0, and let $S_1(x)$ and $S_2(x)$ denote the portions of S_x lying below v_1 and v_2 (see Fig. 1). We claim that it suffices to find a formula for $u(v_1)$ and $u(v_2)$ in terms of the data a_0 and f. For if we can do this, then the value of u inside the (v_0, v_1, v_2) triangle is determined from the values $u(v_0)$, $u(v_1)$, $u(v_2)$, by the harmonic algorithm, and then the problem of finding the values of u in the remaining triangular regions, between v_1 and $S_1(x)$, and v_2 and $S_2(x)$, is essentially the same, after dilation. We also know that the solution must have the form

$$u(v_1) = m_0(x)\, u(v_0) + m_1(x) \int_{S_1(x)} f\, d\mu_1 + m_2(x) \int_{S_2(x)} f\, d\mu_2 \tag{5.4}$$

where μ_1 and μ_2 are the balanced probability measures on $S_1(x)$ and $S_2(x)$, and $m_0(x)$, $m_1(x)$, and $m_2(x)$ are positive numbers satisfying

$$m_0(x)+m_1(x)+m_2(x)=1, \tag{5.5}$$

with a similar formula for $u(v_2)$ with $m_1(x)$ and $m_2(x)$ interchanged. Here we are using symmetry conditions to force μ_1 and μ_2 to be balanced, and to argue that the formula for $u(v_2)$ is symmetric with the formula for $u(v_1)$. Thus our problem is reduced to finding explicitly the functions $m_0(x)$, $m_1(x)$, $m_2(x)$. To do this we introduce the following notation for the unique binary representation of a number in $0<x<1$.

DEFINITION 5.1. For $0<x\leqslant 1$, let $0<n_1<n_2<\cdots$ be the unique increasing sequence of positive integers such that

$$x=\sum_{k=1}^{\infty} 2^{-n_k}. \tag{5.6}$$

Also define

$$Rx=\sum_{k=2}^{\infty} 2^{-n_k}=x-2^{-n_1}. \tag{5.7}$$

THEOREM 5.2. *The function $m_0(x)$ is characterized by the identity*

$$m_0(x)=\frac{1}{1+2(\frac{5}{3})^{n_2-n_1}(1-m_0(Rx))} \tag{5.8}$$

which leads to a variant of a continued fraction representation

$$m_0(x)=\lim_{k\to\infty} m_0^{(k)}(x) \tag{5.9}$$

for

$$m_0^{(k)}=\cfrac{1}{1+2(\frac{5}{3})^{n_2-n_1}\left(1-\cfrac{1}{1+2(\frac{5}{3})^{n_3-n_2}\left(1-\cfrac{1}{\ddots\,\cfrac{}{1+2(\frac{5}{3})^{n_k-n_{k-1}}}}\right)}\right)}. \tag{5.10}$$

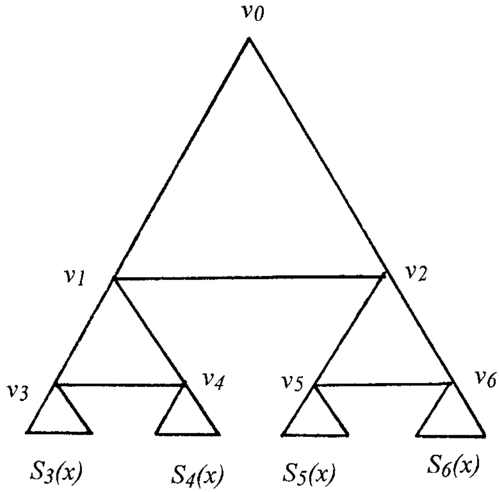

FIGURE 2

We also have

$$m_1(x) = \frac{1 - m_0(x)^2}{2m_0(x)+1}, \qquad m_2(x) = \frac{m_0(x) - m_0(x)^2}{2m_0(x)+1}. \tag{5.11}$$

Proof. We have $n_1 = 1$ by the assumption $x > \frac{1}{2}$. We write $b_1 = \int_{S_1(x)} f \, d\mu_1$ and $b_2 = \int_{S_2(x)} f \, d\mu_2$. We compute the normal derivative of u at v_1 with respect to the upper triangle (v_0, v_1, v_2) to be

$$\begin{aligned}
&(\tfrac{5}{3})^{n_1} \left(u(v_1) - \tfrac{1}{2}u(v_0) - \tfrac{1}{2}u(v_2)\right) \\
&\quad = (\tfrac{5}{3})^{n_1} \left(\tfrac{1}{2}(m_0(x) - 1)\, a_0 + (m_1(x) - \tfrac{1}{2}\, m_2(x))\, b_1 \right. \\
&\qquad \left. + (m_2(x) - \tfrac{1}{2}\, m_1(x))\, b_2\right).
\end{aligned} \tag{5.12}$$

Next we pass to a smaller triangle in the graph G_{n_2} with upper vertex v_1. We label the other vertices v_3 and v_4, and we split the section $S_1(x)$ into $S_3(x) \cup S_4(x)$, with $S_j(x)$ lying below v_j for $j = 3, 4$ (see Fig. 2). Then $\mu_1 = \frac{1}{2}\mu_3 + \frac{1}{2}\mu_4$ where μ_3 and μ_4 are the balanced probability measures on $S_3(x)$ and $S_4(x)$. We write $b_3 = \int_{S_3(x)} f \, d\mu_3$, and $b_4 = \int_{S_4(x)} f \, d\mu_4$, so that $b_1 = \frac{1}{2}(b_3 + b_4)$. The analog of (5.4) is then

$$\begin{aligned}
u(v_3) &= m_0(Rx)\, u(v_1) + m_1(Rx)\, b_3 + m_2(Rx)\, b_4 \\
u(v_4) &= m_0(Rx)\, u(v_1) + m_2(Rx)\, b_3 + m_1(Rx)\, b_4
\end{aligned} \tag{5.13}$$

because Rx is the vertical distance between v_1 and $S_1(x)$. Using (5.13) we compute the normal derivative of u at v_1 with respect to the lower triangle (v_1, v_2, v_3) to be

$$\begin{aligned}
&(\tfrac{5}{3})^{n_2}\,(u(v_1)-\tfrac{1}{2}u(v_3)-\tfrac{1}{2}(v_4))\\
&\quad=(\tfrac{5}{3})^{n_2}\,((m_0(x)\,a_0)+m_1(x)\,b_1+m_2(x)\,b_2)(1-m_0(Rx))\\
&\qquad-(m_1(Rx)+m_2(Rx)\,b_1)\\
&\quad=(\tfrac{5}{3})^{n_2}\,(m_0(x)(1-m_0(Rx)))\,a_0\\
&\qquad+(m_1(x)-1)(1-m_0(Rx))\,b_1+m_1(x)(1-m_0(Rx))\,b_2
\end{aligned}\tag{5.14}$$

since $m_1(Rx)+m_2(Rx)=(1-m_0(Rx))$. Now u will be harmonic at v_1, if and only if the sum of the two normal derivatives (5.12) and (5.14) vanishes. This yields an identity that must be valid for all choices of a_0, b_1, b_2. Thus we may equate to zero the coefficient of a_0, obtaining an equation involving m_0 alone,

$$(\tfrac{5}{3})^{n_2}\,m_0(x)(1-m_0(Rx))+(\tfrac{5}{3})^{n_1}\,\tfrac{1}{2}(m_0(x)-1)=0,$$

which is equivalent to (5.8). By iterating (5.8) we obtain (5.9) and (5.10).

When we equate to zero the coefficients of b_1 and b_2 we obtain

$$\begin{aligned}
(\tfrac{5}{3})^{n_2}\,(m_1(x)-1)(1-m_0(Rx))+(\tfrac{5}{3})^{n_1}\,(m_1(x)-\tfrac{1}{2}\,m_2(x))&=0\\
(\tfrac{5}{3})^{n_2}\,m_1(x)(1-m_0(Rx))+(\tfrac{5}{3})^{n_1}\,(m_2(x)-\tfrac{1}{2}m_1(x))&=0.
\end{aligned}\tag{5.15}$$

We use (5.8) in the form

$$\left(\frac{5}{3}\right)^{n_2-n_1}(1-m_0(Rx))=\frac{1}{2}\left(\frac{1}{m_0(x)}-1\right)\tag{5.16}$$

to simplify (5.15) to

$$\begin{aligned}
\frac{1}{2}(m_1(x)-1)\left(\frac{1}{m_0(x)}-1\right)+m_1(x)-\frac{1}{2}\,m_2(x)&=0\\
\frac{1}{2}\,m_1(x)\left(\frac{1}{m_0(x)}-1\right)+m_2(x)-\frac{1}{2}\,m_1(x)&=0.
\end{aligned}\tag{5.17}$$

But with $m_0(x)$ known, (5.17) is just a pair of linear equations for $m_1(x)$ and $m_2(x)$, whose solution is (5.11). Q.E.D.

Note that the function $m_0(x)$ depends only on the sequence of differences $n_1-n_2, n_2-n_3, \ldots$. When $x=1$ we have $(n_1, n_2, n_3, \ldots)=(1, 2, 3, \ldots)$ so that

$m_0(1) = m_0(R1)$, and so $m_0(1) = 3/10$ since this is the unique fixed point of $\varphi(t) = 1/(1+2(5/3)(1-t))$. Also, $m_0(x)$ is increasing on $\frac{1}{2} < x \leqslant 1$, so 3/10 is its maximum value. It has jump discontinuities at dyadic nationals, but it is continuous from below.

We consider next the problem of describing all harmonic functions on T_x, without any requirement about boundary values. We will do this by extending the recipe (5.4). To begin with, we will assume that u is continuous up to the vertex v_0. This will certainly be true when we pass to smaller triangles in the iteration argument. If ν is any measure (real-valued) on S_x, we consider the algorithm

$$\begin{aligned} u(v_1) &= m_0(x)\, u(v_0) + m_1(x)\, \nu(S_1(x)) + m_2(x)\, \nu(S_2(x)) \\ u(v_2) &= m_0(x)\, u(v_0) + m_2(x)\, \nu(S_1(x)) + m_1(x)\, \nu(S_2(x)) \end{aligned} \tag{5.18}$$

and its iteration to smaller scales. In fact, we may even allow ν to be a finite-valued, finitely additive function on the field of subsets of S_x obtained by repeated division. (When x is a dyadic rational, it is necessary to interpret S_x as a coding space rather than a union of intervals. This will be discussed more thoroughly in the proof of Theorem 5.3.) The arguments in the proof of Theorem 5.2 may be run in reverse to show that the function u obtained is harmonic on T_x. Indeed, the construction makes u harmonic on the interior of each of the triangles that make up T_x, so the only issue is whether u is also harmonic at the junction points where the triangles meet. But this is equivalent to the vanishing of the sum of the normal derivatives (5.12) and (5.14), and so it holds because of the construction of the functions $m_0(x)$, $m_1(x)$, $m_2(x)$. If ν is positive (this implies that it is a finite positive measure) then u is nonnegative. We will show that conversely all nonnegative harmonic functions on T_x arise from this construction.

To obtain all harmonic functions on T_x without the nonnegativity assumption we must allow for functions with a pole at v_0. If we restrict u to the (v_0, v_1, v_2) triangle and dilate we obtain a harmonic function on the interior of K. In [DSV] we showed that this is a six-dimensional space, and since u is continuous at two of the three boundary points, the space is reduced to four dimensions, the spanned by the three-dimensional space of continuous harmonic functions, and a single function $\tilde{u}$ which has a pole at v_0. This function, shown in Fig. 3, extends to all of K, hence to T_x. So if u is harmonic in T_x, there must exist a constant c such that $u - c\tilde{u}$ is continuous at v_0. The claim is that this function arises from our construction. Then the general harmonic function is a sum of $c\tilde{u}$ and one of these. Note that for $c \neq 0$ these functions cannot be nonnegative.

THEOREM 5.3. (a) *Every nonnegative harmonic function on T_x is constructed by* (5.18) *for some choice of $u(v_0) \geqslant 0$ and a finite positive measure*

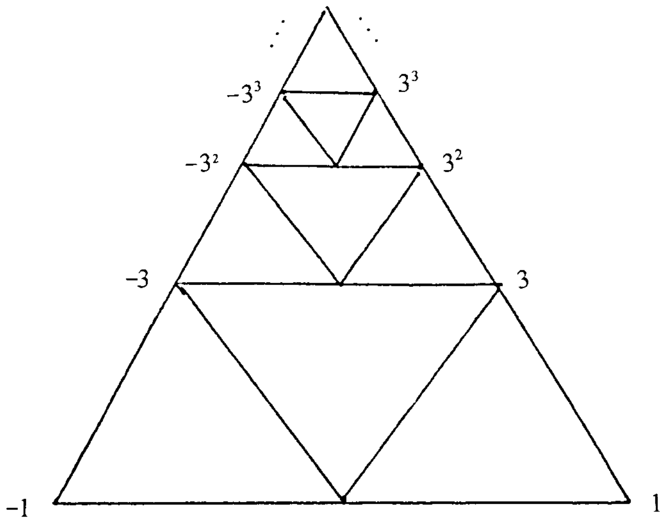

FIGURE 3

v *on* S_x. (b) *Every harmonic function on* T_x *is the sum of* $c\tilde{u}$ *and a function constructed by* (5.18) *for some choice of* $u(v_0)$ *and a finite-valued finitely additive function* v *on the field of subsets of* S_x *generated by repeated division.*

Proof. For simplicity we consider first the case when x is not a dyadic rational. As we have seen, we may assume without loss of generality that u is continuous at v_0. We follow the notation in Figs. 1 and 2. As before, the condition that u be harmonic at v_1 is expressible as the vanishing of the sum of normal derivatives:

$$\left(\tfrac{5}{3}\right)^{n_1} \left(u(v_1) - \tfrac{1}{2}u(v_0) - \tfrac{1}{2}u(v_2)\right) + \left(\tfrac{5}{3}\right)^{n_2} \left(u(v_1) - \tfrac{1}{2}u(v_3) - \tfrac{1}{2}u(v_4)\right) = 0.$$

We have a similar equation holding at v_2. We add these equations and simplify to obtain

$$I_2 = I_1 + \tfrac{1}{2}\left(\tfrac{3}{5}\right)^{n_2 - n_1} A \tag{5.19}$$

with the abbreviations

$$A = u(v_0) - \tfrac{1}{2}u(v_1) - \tfrac{1}{2}u(v_2)$$

$$I_1 = \tfrac{1}{2}(u(v_1) + u(v_2))$$

$$I_2 = \tfrac{1}{2}(u(v_3) + u(v_4) + u(v_5) + u(v_6))$$

where v_5 and v_6 are situated below v_2 in the same way v_3 and v_4 are below v_1.

We now iterate the argument. We consider the sequence of horizontal sections of vertical distance $2^{-n_1} + \cdots + 2^{-n_k}$ from v_0. These sections intersect K in 2^{k-1} intervals with a total of 2^k endpoints (these were v_1, v_2 for $k=1$, and v_3, v_4, v_5, v_6 for $k=2$). Denote these points now by v_{kj}, $j=1, ..., 2^k$. We let I_k denote the average value of $u(v_{kj})$,

$$I_k = 2^{-k}(u(v_{k1}) + \cdots + u(v_{k2^k})). \tag{5.20}$$

The analog of (5.19) is

$$I_k = I_1 + \tfrac{1}{2}A(\tfrac{5}{3})^{n_1}\,((\tfrac{3}{5})^{n_2} + \cdots + (\tfrac{3}{5})^{n_k}). \tag{5.21}$$

It is clear that I_k is uniformly bounded and converges to a finite limit as $k \to \infty$.

Assume u is nonnegative. We now construct a sequence of functions f_k on S_x such that the measures $f_k\, d\mu$, where μ is the balanced measure on S_x, converge weakly to the desired measure ν. We simply take f_k to be piecewise constant on each of the 2^k pieces of S_x lying below each point v_{kj}, and let f_k take on the value $u(v_{kj})$ on the corresponding piece of S_x. Note that I_k is exactly the measure norm of $f_k\, d\mu$, since the balanced measure of each piece is 2^{-k}. The uniform boundedness of I_k and the local version of (5.19) shows that the measures $f_k\, d\mu$ converge weakly to a finite positive measure ν. We claim that the harmonic function constructed by (5.18) from $u(v_0)$ and this measure ν is equal to u. It suffices to do this at the points v_1 and v_2, for then the same argument works for all the points v_{kj}.

Let u_k denote the harmonic function constructed by (5.18) from $u(v_0)$ and the measure $f_k\, d\mu$. By (5.18) and the weak convergence of $f_k\, d\mu$ to ν we know that u_k converges to the constructed harmonic function. Thus we need to show that u_k also converges to the original harmonic function u. Now u is harmonic on the smaller domain T_{x_k}, where

$$x_k = \sum_{j=1}^{k} 2^{-n_j},$$

and so (5.4) holds on T_{x_k}. But in fact u is harmonic below the bottom segment of T_{x_k}, so there is a discrete analog of (5.4) where the integrals are replaced by averages of u over the points v_{kj}. By symmetry we know this must have the form

$$u(v_1) = \tilde{m}_0(x_k)\, u(v_0) + \tilde{m}_1(x_k)\, \tilde{I}_1(u) + \tilde{m}_2(x_k)\, \tilde{I}_2(u), \tag{5.22}$$

where

$$\begin{cases} \tilde{I}_1(u) = 2^{-k+1} \sum_{j=1}^{2^{k-1}} u(v_{kj}) \\ \tilde{I}_2(u) = 2^{-k+1} \sum_{j=2^{k-1}+1}^{2^k} u(v_{kj}), \end{cases} \tag{5.23}$$

and $u(v_2)$ is obtained by interchanging $\tilde{m}_1$ and $\tilde{m}_2$. We use the same reasoning as in the proof of Theorem 5.2 to determine the coefficients $\tilde{m}_0$, $\tilde{m}_1$, $\tilde{m}_2$. This time the process terminates after a finite number of iterations, and we find

$$\tilde{m}_0(x_k) = m_0^{(k)}(x)$$

where $m_0^{(k)}(x)$ is defined by (5.10), and the analog of (5.11) gives $\tilde{m}_1(x_k)$ and $\tilde{m}_2(x_k)$ in terms of $\tilde{m}_0(x_k)$.

Comparing (5.22) with (5.18) for the measure $f_k\, d\mu$, the only difference is that $m_0(x)$, $m_1(x)$, $m_2(x)$ are replaced by $\tilde{m}_0(x_k)$, $\tilde{m}_1(x_k)$, $\tilde{m}_2(x_k)$. But by (5.9) the difference goes to zero as $k \to \infty$, so $u_k(v_1) \to u(v_1)$ as required.

If we drop the assumption that u is nonnegative, the argument is very similar, except we do not obtain a measure as the limit of $f_k\, d\mu$. The only thing that we can assert is that $\lim_{k \to \infty} \int_J f_k\, d\mu$ converges for every interval J that lies below one of the points v_{kj}. We use this to define $v(J)$, and extend it by additivity to the field of sets generated by such intervals. The rest of the proof is the same.

Finally, we discuss the situation when x is a dyadic rational. In that case the section S_x is a finite union of intervals, and the pieces denoted $S_1(x)$ and $S_2(x)$ in (5.18) may overlap at a common point. Since this point may have nonzero measure, we have to allow the mass to be split among the two pieces. When we iterate (5.18) this overlapping will occur infinitely often. The easiest way to remedy this problem is to identify the measure space S_x not with the finite union of intervals but with a Cantor set that contains two distinct points for each overlap. This allows a larger class of set functions v, and the algorithm (5.18), with the correct interpretation of the splitting $S_x = S_1(x) \cup S_2(x)$, defines a harmonic function for such v. The proof of the theorem is then essentially the same, since the sequence of measures $f_k\, d\mu$ will converge when evaluated on each interval J. Q.E.D.

So far we have considered only domains that could be described as lying above a linear section S_x. If we consider domains lying below S_x, the problem becomes much harder. We will only give one example of a domain of this type, where the boundary is a finite set. If $\tilde{T}_x$ denotes the complement of the closure of T_x, the domain Ω_k we consider is essentially the interior of $\tilde{T}_x$ for the value $x = 1 - 2^{-k}$. Thus Ω_k is made up of 2^k adjacent triangles of size 2^{-k} joining the bottom boundary points of K. The boundary of Ω_k consists of these two boundary points of K together with the 2^k top vertices of the triangles. We label the top vertices $b_1, b_2, ..., b_{2^k}$ and the bottom vertices of the triangle $a_0, a_1, ..., a_{2^k}$, so that the boundary of Ω_k is $a_0, a_{2^k}, b_1, b_2, ..., b_{2^k}$. Figure 4 shows Ω_3. Given a harmonic function on Ω_k, we need to find the values of $u(a_j)$ in terms of the data $u(a_0)$, $u(a_{2^k})$, $u(b_1)$, ..., $u(b_{2^k})$, for then u is determined on each of the triangles. In fact it suffices to do this for $j = 2^{k-1}$, for then we may iterate the algorithm to find the other values.

THEOREM 5.4. *Let u be harmonic on Ω_k. Then*

$$u(a_{2^{k-1}}) = \frac{\lambda_1(u(a_0) + u(a_{2^k})) + \sum_{j=1}^{2^{k-1}} \lambda_j(u(b_j) + u(b_{2^k - j + 1}))}{2\lambda_1 + \sum_{j=1}^{2^{k-1}} \lambda_j} \tag{5.24}$$

where $\lambda_1 = 1$, $\lambda_2 = 5$ and

$$\lambda_{j+1} = 4\lambda_j - \lambda_{j-1}. \tag{5.25}$$

Proof. The result is obvious when $k = 1$, for then it is just $u(a_1) = 1/4(u(a_0) + u(a_2) + u(b_1) + u(b_2))$, which is just the statement that u is harmonic at a_1. For $k = 2$, we obtain three equations for u to be harmonic at a_1, a_2, a_3:

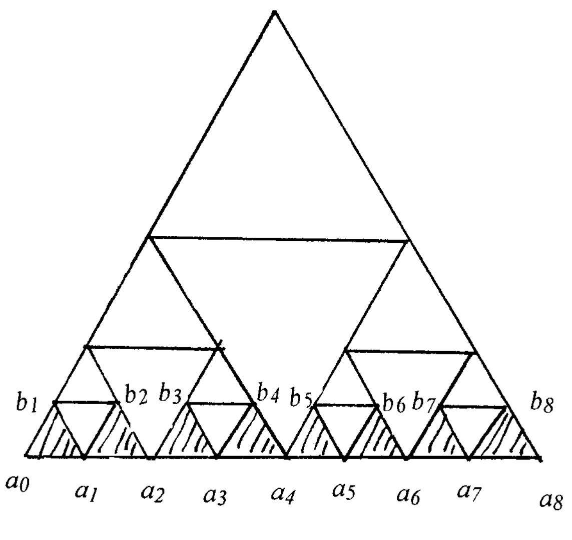

FIGURE 4

$$4u(a_1) = u(a_0) + u(a_2) + u(b_1) + u(b_2)$$
$$4u(a_2) = u(a_1) + u(a_3) + u(b_2) + u(b_3)$$
$$4u(a_3) = u(a_2) + u(a_4) + u(b_3) + u(b_4).$$

We add the first and third,

$$4(u(a_1) + u(a_3)) = u(a_0) + 2u(a_2) + u(a_4) + u(b_1) + u(b_2) + u(b_3) + a(b_4)),$$

and use the second to eliminate $u(a_1) + u(a_3)$, to obtain

$$14u(a_2) = u(a_0) + u(a_4) + u(b_1) + u(b_4) + 5(u(b_2) + u(b_3)).$$

This is (5.24) for $k = 2$ with $\lambda_1 = 1$, $\lambda_2 = 5$.

We now consider the general case. We have $2^k - 1$ equations that express the fact that u is harmonic at a_j, namely

$$4u(a_j) = u(a_{j-1}) + u(a_{j+1}) + u(b_j) + u(b_{j+1}) \tag{5.26}$$

for $j = 1, 2, ..., 2^k - 1$. To simplify the discussion we introduce the abbreviations

$$\begin{aligned} c_j &= u(a_j) + u(a_{2^k - j}) \\ d_j &= u(b_j) + u(b_{2^k + 1 - j}) \end{aligned} \tag{5.27}$$

(note that $c_{2^{k-1}} = 2u(a_{2^{k-1}})$). We add the equations (5.26) for j and $2^k - j$ to obtain

$$4c_j - c_{j-1} - c_{j+1} = d_j + d_{j+1} \tag{5.28}$$

for $j = 1, ..., 2^{k-1} - 1$. We also note that (5.26) for $j = 2^{k-1}$ becomes

$$2c_{2^{k-1}} - c_{2^{k-1}-1} = d_{2^{k-1}}. \tag{5.29}$$

Now let $\Lambda_j = \lambda_j - \lambda_{j-1} + \lambda_{j-2} - \cdots$. Multiply each equation (5.28), (5.29) by the corresponding Λ_j and add to obtain

$$\sum_{j=1}^{2^{k-1}-1} (4c_j - c_{j-1} - c_{j+1})\,\Lambda_j + (2c_{2^{k-1}} - c_{2^{k-1}-1})\,\Lambda_{2^{k-1}} = \sum_{j=1}^{2^{k-1}-1} (d_j + d_{j+1})\,\Lambda_j + d_{2^{k-1}}\Lambda_{2^{k-1}}. \tag{5.30}$$

Now the right side of (5.30) is easily seen to be $\sum_{j=1}^{2^{k-1}} d_j \lambda_j$, because the coefficient of d_j is $\Lambda_j + \Lambda_{j-1}$ for $j = 2, ..., 2^{k-1}$, and the coefficient of d_1 is Λ_1. On the left side, the coefficient of c_j is $4\Lambda_j - \Lambda_{j+1} - \Lambda_{j-1}$ for $j = 2, ..., 2^{k-1} - 1$, and this vanishes by (5.25). The coefficient of c_1 is $4\Lambda_1 - \Lambda_2$

and this vanishes since $\lambda_1 = 1$, $\lambda_2 = 5$. The coefficient of c_0 is $-\Lambda_1 = -1$, and the coefficient of $c_{2^{k-1}}$ is $2\Lambda_{2^{k-1}} - \Lambda_{2^{k-1}-1} = 2\lambda_1 + 2\sum_{j=1}^{2^{k-1}} \lambda_j$. Altogether (5.30) becomes

$$\left(2\lambda_1 + 2\sum_{j=1}^{2^{k-1}} \lambda_j\right) c_{2^{k-1}} - c_0 = \sum_{j=1}^{2^{k-1}} d_j \lambda_j,$$

which is equivalent to (5.24). Q.E.D.

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