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# The multifractal nature of Lévy processes

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**Abstract.** We show that the sample paths of most Lévy processes are multifractal functions and we determine their spectrum of singularities.

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A Lévy process $X_t$ $(t \geq 0)$ valued in $\mathbb{R}^d$ is, by definition, a stochastic process with stationary independent increments: $X_{t+s} - X_t$ is independent of the $(X_v)_{0 \leq v \leq t}$ and has the same law as $X_s$. Brownian motion and Poisson processes are examples of Lévy processes that can be qualified as *monofractal*; for instance the Hölder exponent of the Brownian motion is everywhere $1/2$ (the variations of its regularity are only of a logarithmic order of magnitude). These two examples are not typical: we will see that the other Lévy processes are multifractal provided that their Lévy measure is neither too small nor too large near zero. Furthermore their spectrum of singularities depends precisely on the growth of the Lévy measure near the origin. Before stating our main result, we need to recall some basic definitions and properties of Lévy processes and multifractal functions.

The characteristic function of a Lévy process $X_t$ (valued in $\mathbb{R}^d$) satisfies $\mathbb{E}(e^{i\langle\lambda|X_t\rangle}) = e^{-t\psi(\lambda)}$ where

$$\psi(\lambda) = i\langle a|\lambda\rangle + \frac{1}{2}Q(\lambda) + \int_{\mathbb{R}^d} \left(1 - e^{i\langle\lambda|x\rangle} + i\langle\lambda|x\rangle 1_{|x|<1}\right)\pi(dx) \quad ; \quad (1)$$

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$Q$ is a positive quadratic form and $\pi(dx)$ is the Lévy measure of $X_t$, i.e. a positive Radon measure defined on $\mathbb{R}^d - \{0\}$ satisfying

$$\int (1 \wedge |x|^2)\pi(dx) < \infty \ . \tag{2}$$

The Lévy measure is usually not integrable in the neighbourhood of the origin; this is in particular the case for stable Lévy processes of index $\beta$ which satisfy (in polar coordinates) $\pi(dr, d\theta) = r^{-\beta-1} dr \nu(d\theta)$ where $\nu$ is a finite measure on the unit sphere. When $\pi(\mathbb{R}^d) = +\infty$, the growth of the Lévy measure near the origin can be estimated using the upper index

$$\beta = \inf \left\{ \gamma \geq 0 : \int_{|x| \leq 1} |x|^\gamma \pi(dx) < \infty \right\} \ .$$

This index was introduced by R. Blumenthal and R. Getoor in [3]. W. Pruitt in [14] showed that the Hölder exponent of Lévy processes (without Brownian component) at $t = 0$ is $1/\beta$. Condition (2) implies that $0 \leq \beta \leq 2$, and when $X_t$ is a stable process, this definition coincides with the definition of the stability index.

Let us recall the basic definitions concerning multifractal functions. The starting point is the definition of *pointwise* regularity $C^l(t_0)$. Let $t_0 \in \mathbb{R}$ and let $l$ be a positive real number. A function $f(t)$ is $C^l(t_0)$ if there exists a constant $C > 0$ and a polynomial $P_{t_0}$ of degree at most $[l]$ such that in a neighbourhood of $t_0$,

$$|f(t) - P_{t_0}(t)| \leq C|t - t_0|^l \ .$$

Note that this definition is local and involves no uniform regularity; furthermore, $f$ can be $C^l(t_0)$ for a large $l$ without being continuously differentiable at $t_0$: Indeed continuous differentiability at $t_0$ implies differentiabilty in a neighbourhhod of $t_0$ which is not implied by this definition. The Hölder exponent of $f$ at $t_0$ is

$$h_f(t_0) = \sup\{l : f \in C^l(t_0)\}$$

(we emphasize that this definition is not sensitive to logarithmic corrections in the modulus of continuity so that, for instance, with probability 1 the Hölder exponent of a sample path of the Brownian motion is *everywhere* 1/2).

The *multifractal analysis* is concerned in the study of the (usually fractal) sets $S_h$ where a function $f$ has a given Hölder exponent $h$ and in particular in the determination of the Hausdorff dimension $d(h)$ of $S_h$. (Recall that $dim(\emptyset) = -\infty$, so that, for instance, $d(h) = 0$ implies that there exists at least one point of Hölder exponent $h$.) The function $d(h)$ is called the *spectrum of singularities* of $f$. The notion of 'multifractal functions' was first introduced by physicists in the context of fully developed turbulence, see [8]. Since then several mathematical functions were shown to be multifractal, *i.e.* were shown to have a spectrum of singularities supported on an interval of non empty interior, see for instance [9] and references therein.

We can determine immediately the spectrum of singularities of Lévy processes in four cases:

- $X_t$ is deterministic; then $X_t = Ct$ and $d(h) = -\infty \ \forall h$.
- $X_t$ is a compound Poisson process with drift; then $X_t$ is piecewise linear, with a finite number of jumps on any bounded interval, so that $d(0) = 0$ and $d(h) = -\infty$ else.
- $X_t$ is a Brownian motion; then $d(1/2) = 1$ and $d(h) = -\infty$ else (see [4], [6] and [13]).
- $X_t$ is the superposition of a Brownian motion and a compound Poisson process with drift; one easily checks that $d(0) = 0$, $d(1/2) = 1$ and $d(h) = -\infty$ else.

Let

$$\begin{aligned} d_\beta(h) &= \beta h \quad \text{if} \quad h \in [0, 1/\beta] \\ &= -\infty \text{ else;} \end{aligned}$$

$$\begin{aligned} \overline{d_\beta}(h) &= \beta h \quad \text{if} \quad h \in [0, 1/2] \\ &= 1 \qquad \text{if} \quad h = 1/2 \\ &= -\infty \text{ else.} \end{aligned}$$

Let

$$C_j = \int_{2^{-j-1} \le |x| \le 2^{-j}} \pi(dx) \ \ ;$$

the exponent $\beta$ can also be defined using the $C_j$'s by

$$\beta = \sup\left(0, \limsup_{j \to \infty} \frac{\log C_j}{j \log 2}\right) \ \ .$$

Our purpose in this paper is to prove the following theorem.

**Theorem 1.** *Let $X_t$ be a Lévy process of Lévy measure $\pi(dx)$ satisfying $\beta > 0$ and*

$$\sum 2^{-j} \sqrt{C_j \log(1 + C_j)} < \infty \ \ . \tag{3}$$

- *If $X_t$ has no Brownian component ($Q \equiv 0$), the spectrum of singularities of almost every sample path of $X_t$ is $d_\beta(h)$.*
- *If $X_t$ has a Brownian component ($Q \not\equiv 0$), the spectrum of singularities of almost every sample path of $X_t$ is $\overline{d_\beta}(h)$.*

*If $\beta = 0$ but $\pi(\mathbb{R}^d) = +\infty$, for each $h$, with probability $1$, $d(h) = 0$.*

*Remarks.*

1. If Condition (3) fails, there exists a subsequence $j_n$ such that

$$2^{-j_n}\sqrt{C_{j_n}\log(1+C_{j_n})} \geq 1/j_n^2 \quad .$$

Since $\pi$ is a Lévy measure, $C_{j_n} \leq 2^{2j_n}$ for $n$ large enough, so that

$$2^{-j_n}\sqrt{C_{j_n}2j_n} \geq 1/j_n^2 \quad ,$$

hence $C_{j_n} \geq 2^{2j_n}/2j_n^3$, so that $\beta = 2$; thus all Lévy processes of upper index $\beta < 2$ satisfy the assumptions of Theorem 1, or fall into one of the cases we already considered. In fact Condition (3) is slightly stronger than stating that $\pi$ is a Lévy measure (which, near 0, is equivalent to the requirement $\sum C_j 2^{-2j} < \infty$). In particular all stable Lévy processes are covered by this theorem.
2. In [13], S. Orey and S.J. Taylor proved that if $X_t$ is a *stable symetric* Lévy process, the Hausdorff dimension of the set of points where the Hölder exponent of $X_t$ is *at most* $h$ is $\beta h$. Note however that their method cannot give regularity results at these points.
3. When $\beta > 0$, the assertion expressed in the theorem is stronger than stating that, for each $h$, $d(h)$ has almost surely a given value, which would not be sufficient to determine the spectrum of singularities of almost every sample path.
4. The almost everywhere Hölder exponent of Lévy processes without Brownian component is $1/\beta$, see [14], which of course agrees with the theorem (case where $h = 1/\beta$).
5. Many results have been proved concerning the fractal nature of the *range* of Lévy processes, see for instance [15], or [12] for references concerning 'Lévy flights', or [1] for results concerning the range of subordinators.

At the end of the paper we will also answer a question of Jean Bertoin concerning the existence of moduli of continuity for Lévy processes outside the jump points.

## 1. Preliminaries

Since Lévy processes have independent increments, we can restrict our study to the interval [0, 1]; indeed, if Theorem 1 is proved for $t \in [0, 1]$, the spectrum on any other interval $[k, k+1]$ will be the same, and it will thus also be the spectrum on $\mathbb{R}^+$.

Any Lévy process can be decomposed as a sum of three independent processes:

- A Brownian motion with drift of covariance matrix $Q$.
- A compound Poisson process, of Lévy measure $1_{|x|>1}\pi(dx)$
- A Lévy process of Lévy measure $1_{|x|\le 1}\pi(dx)$.

We can clearly forget the second term, since it is piecewise linear, and won't affect the spectra given by Theorem 1. We will momentarily forget the Brownian component, and we will see at the very end of this paper how adding this component affects the spectrum. Thus, we now focus on the study of the last term, that we also denote by $X_t$.

Up to a linear term (which does not affect the regularity), these Lévy processes can be constructed as a superposition of independent compensated Poisson processes $X_t^j$ which have jumps the size of which belongs to

$$\Gamma_j = \{x:\ \ 2^{-j-1} < |x| \le 2^{-j}\}\ \ .$$

Let $Y_t^j$ be the compound Poisson process of Lévy measure

$$\pi_j(dx) = 1_{\Gamma_j}(x)\pi(dx)$$

and let $X_t^j$ be the compound compensated Poisson process

$$X_t^j = Y_t^j - t\int_{\mathbb{R}^d} x\pi_j(dx)\ \ ;$$

the $X_t^j$ are independent processes and $X_t = \sum_{j=1}^{\infty} X_t^j$.

Denote by $N_j$ the number of jumps of $Y_t^j$ (hence of $X_t^j$) in [0, 1]. It is a Poisson variable of intensity $C_j$ (and thus of expectation $C_j$):

$$\mathbb{P}(N_j = N) = e^{-C_j}{C_j}^N/N!$$

We will use repeatedly the following lemma.

**Lemma 1.** *There exists $C'$ such that, if $N$ is a Poisson variable of intensity $C \ge C'$,*

$$\mathbb{P}\left(|N-C| \ge \sqrt{C}(\log C)^2\right) \le e^{-(\log C)^3}\ \ , \tag{4}$$

$$\mathbb{P}\left(|N-C| \geq 4\sqrt{C\log C}\right) \leq 1/C^7 \quad , \tag{5}$$

*and there exists* $D > 0$ *such that*

$$\mathbb{P}\left(|N-C| \geq C/2\right) \leq e^{-DC\log C} \quad . \tag{6}$$

This lemma is a direct consequence of Stirling's formula, however we sketch its proof for the sake of completeness; (6) is derived by summing the probabilities for $|N-C| \geq C/2$ which is straightforward to bound because these probabilities decay geometrically.

Suppose now that $|N-C| < C/2$; by Stirling's formula,

$$\mathbb{P}(N=n) = \frac{e^{n-C+n(\log C-\log n)}}{\sqrt{n}}(1+o(1)) \quad ,$$

thus, if $|N-C| \leq \sqrt{C}(\log C)^2$,

$$\mathbb{P}(N-C=a) = \frac{e^{-a^2/2C}}{\sqrt{C}}(1+o(1)) \quad . \tag{7}$$

Since $\mathbb{P}(N=n)$ decays with $|n-C|$, $\mathbb{P}(\sqrt{C}(\log C)^2 \leq |N-C| \leq |C|/2)$ is bounded by $2\frac{C}{\sqrt{C}}e^{-(\log C)^4/2}$, hence (4) holds.

Similarly, the sum of the probabilities for $4\sqrt{C\log C} \leq |N-C| \leq \sqrt{C}(\log C)^2$ is bounded by

$$3\sqrt{C}(\log C)^2\frac{e^{-(4\sqrt{C\log C})^2/2C}}{\sqrt{C}} \leq \frac{1}{C^7}$$

hence (5) holds.

It follows immediately from Lemma 1 that, for every given $j$,

$$\mathbb{P}(N_j \geq 2C_j + j) \leq e^{-5j} \tag{8}$$

(consider separately the cases $2C_j \geq j$ and $2C_j < j$); and similarly,

$$\mathbb{P}\left(|N_j - C_j| \geq 4\left(\sqrt{C_j\log C_j} + j\right)\right) \leq 1/j^7 \quad . \tag{9}$$

We now define the random fractal sets on which the Hölder singularities of $X_t$ will be situated. Let $F_j$ be the set of the jumps of $X_t^j$, and let $\delta > 0$; denote by $A_\delta^j$ the union of the intervals of length $2\cdot 2^{-\delta j}$ centered at the points of $F_j$ and by $E_\delta$ the random set

$$E_\delta = \limsup_{j\to\infty} A_\delta^j \quad .$$

**Lemma 2.** *Almost surely*, $\forall \delta < \beta$, *every point of* $[0,1]$ *belongs to* $E_\delta$.

This is a consequence of a result of Shepp concerning random coverings of the circle, see [16]. We will actually rather use the following equivalent formulation given by Lemma 3 (see [2], where Bertoin uses this lemma in order to determine which Lévy processes with unbounded variation have exceptional points of differentiability).

Denote by $\lambda$ the Lebesgue measure on $\mathbb{R}$ and let $\mu$ be an arbitrary measure on $(0,1)$. We consider a Poisson point process $\mathscr{P}$ with intensity $\lambda \otimes \mu$. Corresponding to each point $(x,y)$ in $\mathscr{P}$ we associate the interval $(x-y, x+y)$ of the real line, and we consider the set of points covered by these intervals

$$V = \bigcup_{(x,y)\in\mathscr{P}} (x-y, x+y) \ .$$

**Lemma 3.** *If the integral*

$$\int_0^1 \exp\left\{2\int_t^1 \mu((y,1))\,dy\right\} dt$$

*diverges*, $V = \mathbb{R}$ *almost surely.*

Let us now prove Lemma 2. The process of the jumps of a Lévy process $Y_t$ of Lévy measure $\mu$ is a Poisson point process with intensity $\lambda\otimes\mu$. We now consider the Poisson point process of intensity $\lambda \otimes \pi_\delta^J$, where $\pi_\delta^J$ denotes the image of $\pi 1_{|x|<2^{-J}}$ by the mapping $y \to |y|^\delta$. The corresponding set $V$ is contained in $\bigcup_{j\geq J} A_\delta^j$. Thus, in order to prove Lemma 2, it is sufficient to prove the divergence of the integral

$$\int_0^1 \exp\left\{2\int_t^1 \pi_\delta^J((y,1))\,dy\right\} dt \ . \tag{10}$$

Note that

$$\int_t^1 \pi_\delta^J((y,1))\,dy = \int_{t^{1/\delta}}^1 \left(\int_{u<|x|<2^{-J}} \pi(dx)\right) \delta u^{\delta-1}\,du \ .$$

Let us now prove that (10) is divergent when $\delta < \beta$. Let $\omega(u) = \int_{u<|x|<2^{-J}} \pi(dx)$; $\omega$ is decreasing and if $u \in [2^{-j-2}, 2^{-j-1}]$, $\omega(u) \geq C_j$. Denote by $j(t)$ the largest integer $j$ such that $t^{1/\delta} \leq 2^{-j(t)-2}$;

$$\begin{aligned}\int_{t^{1/\delta}}^1 \omega(u)\delta u^{\delta-1}\,du &\geq \int_{2^{-j(t)-2}}^{2^{-j(t)-1}} \omega(u)\delta u^{\delta-1}\,du \\ &\geq C_j\delta\left(2^{-j(t)-2}\right)^{\delta-1} 2^{-j(t)-2} \\ &= C_j\delta 2^{-\delta(j(t)+2)} \ .\end{aligned}$$

Thus the function $2\int_t^1 \pi_\delta^J((y,1))dy$ is larger than $\frac{C_j\delta}{2}2^{-\delta j}$ on the interval $[(2^{-j-3})^\delta, (2^{-j-2})^\delta]$. Let $r$ be such that $\delta < r < \beta$. If $j$ is such that $C_j \geq 2^{rj}$,

$$\int_{(2^{-j-3})^\delta}^{(2^{-j-2})^\delta} \exp\left\{2\int_t^1 \pi_\delta^J((y,1))\,dy\right\} \geq 2^{-(j+3)\delta} \exp\left\{\frac{\delta}{2}2^{(r-\delta)j}\right\} \quad .$$

Since there exists an infinite number of such $j$s, the integral (10) is divergent; hence Lemma 2 holds for a fixed value of $\delta$ picked smaller than $\beta$, hence for a sequence $\delta_n \to \beta$. The result follows for any $\delta$ smaller than $\beta$ because the $E_\delta$ are decreasing.

The following lemma of [9] yields an upper bound for the Hölder exponent of $X_t$.

**Lemma 4.** *Let $f$ be a function discontinuous on a dense set of points, $t \in \mathbb{R}$ and let $r_n$ be a sequence of points of discontinuity of $f$ converging to $t$ such that, at each point $r_n$, $f$ has a right limit and a left limit; denote by $\Delta(f)(r_n)$ the jump of $f$ at $r_n$. Then*

$$h_f(t) \leq \liminf \frac{\log|\Delta(f)(r_n)|}{\log|r_n - t|} \quad .$$

Since Lévy processes are right-continuous with left limits, this lemma can be applied to $X_t$ and yields the following bound for the Hölder exponent of $X_t$:

$$\text{If} \quad t \in E_\delta \quad \text{then} \quad h_X(t) \leq 1/\delta \quad . \tag{11}$$

Note that (11) together with Lemma 2 implies that almost surely

$$\forall t \in \mathbb{R}^+ \quad h(t) \leq 1/\beta \quad . \tag{12}$$

Denote by $R_\delta$ ($\delta > 0$) the set of $t \in [0,1]$ such that the Hölder exponent $h(t)$ of $X_t$ satisfies $h(t) = 1/\delta$. The following proposition (which is a direct consequence of (11) and of Proposition 2 below) compares the $R_\delta$ with the $E_\delta$.

**Proposition 1.** *Let $S$ be the countable set of all jumps of $X_t$; if $0 < \delta < \infty$,*

$$R_\delta = \left(\bigcap_{a<\delta} E_a\right) - \left(\bigcup_{b>\delta} E_b\right) - S \quad . \tag{13}$$

*If $\delta = \infty$,*

$$R_\infty = \left(\bigcap_{a>0} E_a\right) \bigcup S \quad . \tag{14}$$

Note that, since the $E_\delta$ are decreasing (in $\delta$), the $a$ and $b$ in (13) and (14) can be chosen to belong to a fixed countable set.

Let us first obtain an upper bound for the dimension of $R_\delta$. Using (8), with probability at least $1-2e^{-5j}$, $\forall\delta > \beta$, $A_\delta^j$ is a union of at most $2C_j + j$ intervals of length $2 \cdot 2^{-\delta j}$; using these intervals for $j \geq J$ as a covering, we obtain that, with probability 1, $\forall\delta > \beta$, the Hausdorff dimension of $E_\delta$ is bounded by $\beta/\delta$. This implies that with probability 1,

$$\forall\delta > \beta \quad dim_H(R_\delta) \leq \beta/\delta \ .$$

In order to obtain a lower bound for the dimension of $R_\delta$ when $\beta > 0$, we will show Section 3 that a certain $\beta/\delta$-dimensional measure $\mu_\delta$ supported by $E_\delta$ satisfies

$$0 < \mu_\delta(E_\delta) < +\infty$$

and

$$\forall\delta' > \delta, \quad \mu_\delta(E_{\delta'}) = 0 \ ;$$

this implies that $\mu_\delta(R_\delta) > 0$, hence that $dim_H(R_\delta) \geq \beta/\delta$. The case $\beta = 0$ will be treated separately at the end of Section 3. Thus the proof of the first part of the theorem is reduced to proving Proposition 1, which will be done in Section 2, and to obtaining a Hausdorff $\beta/\delta$-dimensional function for the $E_\delta$, which is done in Section 3.

## 2. A lower bound of pointwise regularity

Our purpose in this section is to show that the apparently crude upper bound of regularity given by Lemma 4 is actually optimal for Lévy processes.

**Proposition 2.** *Suppose that (3) holds, and let $\delta > \beta$ be a fixed number. For almost every sample path of $X_t$, if $t_0$ is not a jump point of $X_t$,*

$$t_0 \notin E_\delta \implies h_X(t_0) \geq 1/\delta \ . \tag{15}$$

Note that Proposition 1 immediately follows from Proposition 2. We will prove the regularity of $X_t$ by estimating the increments of the $X_t^j$ on intervals of length between $2^{-m}$ and $2^{-m+1} (= l)$. We will first prove uniform (i.e. independent of $t$) bounds on such increments. Two cases have to be considered depending on whether many or few points of jump fall in such an interval. The first case will be considered in Lemma 5, and the second case in Lemma 8.

The constant $C_1'$ which appears in the following lemma is a universal constant which will be defined in Lemma 6.

**Lemma 5.** *There exists $J_0 \geq 0$ such that $\forall j \geq J_0$, the following event holds with probability at least $1 - 2/j^7$:*

*$\forall m$ satisfying*

$$C_j \geq \frac{(32)^2}{C_1'^2} 2^m j \sqrt{m} \quad , \tag{16}$$

*with probability larger than $1 - e^{-2j\sqrt{m}}$, $\forall s, t \in [0, 1]$ such that $2^{-m} \leq |s - t| \leq 2^{-m+1}$,*

$$|X_t^j - X_s^j| \leq 16(d+2)2^{-j} m \left( \sqrt{C_j l j} + |t - s| \sqrt{C_j \log C_j} \right) \quad . \tag{17}$$

Note that, if $m$ and $C_j$ satisfy (16), there exists $D > 0$ such that

$$C_j \geq Dj, \tag{18}$$

and Lemma 1 implies that, for $j$ large enough,

$$\mathbb{P}(|N_j - C_j| \geq C_j/2) \leq e^{-j} \quad . \tag{19}$$

Note also that (16) implies that, if $j$ is large enough, $m \leq 3j$.

*Proof of Lemma 5.* The process $X_t^j$ can be written as the sum of two (dependent) compound compensated Poisson processes

$$X_t^j = Q_t^j + R_t^j$$

where $Q_t^j$ and $R_t^j$ have their jumps at the same time as $X_t^j$, but $Q_t^j$ has jumps of constant size

$$A_j = \frac{1}{C_j} \int x \pi_j(dx)$$

while the expectation of the jumps of $R_t^j$ vanishes. (Note that $|A_j| \leq 2^{-j}$.)

Let us first estimate the increments of $Q_t^j$:

$$Q_t^j = A_j (P_t^j - C_j t) \tag{20}$$

where $P_t^j$ is a Poisson process (with jumps of size 1). Since $C_j \geq Dj$, Lemma 1 can be applied.

We condition the Poisson process $P_t^j$ by the event

$$\left\{ P_t^j \quad \text{has} \quad \text{exactly} \quad N \quad \text{jumps} \quad \text{on} \quad [0, 1] \right\} \quad ,$$

and we pick $N$ in the interval $[C_j - \sqrt{C_j \log C_j}, C_j + \sqrt{C_j \log C_j}]$, which holds with probability $1 - C'/j^7$ by (5) and (18). The $N$ times of jump are now $N$ independent uniformly distributed random variables on [0, 1], and thus the process

$$\alpha_t^{j,N} = \sqrt{N}\left(\frac{P_t^{j,N}}{N} - t\right) \tag{21}$$

is an empirical process on [0, 1] (the letter $N$ in the notation $P_t^{j,N}$ is a reminder of the conditioning). The increments of the empirical process can be estimated using the following result which is a particular case of Lemma 2.4 of Stute [17]. $\square$

**Lemma 6.** *There exist two positive constants $C_1'$ and $C_2'$ such that, if $0 < l < 1/8$, $Nl \geq 1$ and $8 \leq A \leq C_1'\sqrt{Nl}$,*

$$\mathbb{P}\left(\sup_{|t-s|\leq l} |\alpha_t^{j,N} - \alpha_s^{j,N}| > A\sqrt{l}\right) \leq \frac{C_2'}{l} e^{-A^2/64} \ . \tag{22}$$

Using the definition of $\alpha_t^{j,N}$,

$$\begin{aligned} |Q_t^{j,N} - Q_s^{j,N}| &\leq |A_j||P_t^{j,N} - P_s^{j,N} - C_j(t-s)| \\ &\leq |A_j|(\sqrt{N}|\alpha_t^{j,N} - \alpha_s^{j,N}| + |t-s||C_j - N|) \ . \end{aligned}$$

We apply Lemma 6 with $A = 16 j^{1/2} m^{1/4}$ and $l = 2^{-m+1}$ in (22). Since $N \geq C_j/2$, Condition $A \leq C_1'\sqrt{Nl}$ holds for $j$ large enough because

$$C_1'\sqrt{Nl} \geq C_1'\sqrt{\frac{C_j}{2} l} \geq 16 j^{1/2} m^{1/4}$$

(using (16)), and $Nl \geq 1$ also holds because of (16); so that, with a probability larger than $1 - e^{-4j\sqrt{m}}$, $\forall t, s$ such that $|t-s| \leq l$,

$$|Q_t^{j,N} - Q_s^{j,N}| \leq 32 \cdot m^{1/4} 2^{-j} \left(\sqrt{C_j l j} + |t-s|\sqrt{C_j \log C_j}\right) \ . \tag{23}$$

We now estimate the increments of $R_t^{j,N}$. Recall that $R_t^j$ is a compound Poisson process; denote by $Z_n$ the size of its jumps. The $Z_n$ are independent centered variables and $|Z_n| \leq 2^{-j}$. In order to bound the increments of $R_t^{j,N}$, we have to bound partial sums of the $Z_n$. We will use the following lemma (see [11] Lemma 1.5, Chap. 1).

**Lemma 7.** *Let the $u_i$ be independent centered real random variables satisfying $|u_i| \leq 1$. For all $n \geq 1$ and all $\lambda > 0$*

$$\mathbb{P}(|u_1 + \cdots + u_n| \geq \lambda\sqrt{n}) \leq 2e^{-\lambda^2/2} \ .$$

Thus if the $u_i$ are independent centered random variables in $\mathbb{R}^d$ satisfying $|u_i| \leq 1$, for all $n \geq 1$ and all $\lambda > 0$,

$$\mathbb{P}(|u_1 + \cdots + u_n| \geq \lambda\sqrt{n}) \leq 2de^{-\lambda^2/2d^2} \ . \tag{24}$$

We have estimated above the increments of $Q_t^{j,N}$; an estimate for the increments of $P_t^{j,N}$, hence for the number of jumps of $R_t^{j,N}$, immediately follows: Uniformly on all dyadic intervals of length $l$, with a probability larger than $1 - e^{-4j\sqrt{m}}$, $\forall t, s$ such that $|t - s| \leq l$,

$$|P_t^{j,N} - P_s^{j,N}| \leq 32m^{1/4}\left(\sqrt{C_j l j} + |t-s|\sqrt{C_j \log C_j} + |t-s|C_j\right)$$

which is bounded by $3C_j l$ because of (16).

Let now $I$ be any dyadic interval of length $2^{-m+1}$; let us estimate the increments of $R_t^{j,N}$ between the beginning of $I$ and another point of $I$. We must estimate the maximum of $|Z_p + \cdots + Z_q|$, where $t_p$ is the first jump in $I$ and $t_q$ is another jump in $I$, so that $q - p \leq 3C_j l$. We use (24) with $\lambda = 4dm\sqrt{j}$, which yields

$$\mathbb{P}\left(|Z_p + \cdots + Z_q| \geq 8dm\sqrt{C_j l j}\right) \leq 2de^{-8jm^2} \ .$$

We now add the unfavorable probabilities corresponding to all possible values of $q$, and to all possible locations of the dyadic interval $[k2^{-m+1}, (k+1)2^{-m+1}]$ in $[0, 1]$, which yields

$$\begin{aligned}&\mathbb{P}\left(\sup_{|t-s|\leq l} |R_t^{j,N} - R_s^{j,N}| \geq 16dm2^{-j}\sqrt{C_j l j}\right)\\ &\leq 2^{m+1}3C_j l \cdot 2de^{-8jm^2} \leq e^{-2jm^2}\end{aligned}$$

(because $C_j \leq 2^{2j}$). Thus, with a probability larger than $1 - e^{-2jm^2}$,

$$\sup_{|t-s|\leq l} |R_t^{j,N} - R_s^{j,N}| \leq 16dm2^{-j}\sqrt{C_j l j} \ . \tag{25}$$

Note that the bounds (23) and (25) are independent of $N$. Since the only asumption we made on $N$ is $|C_j - N| \leq \sqrt{C_j \log C_j}$ which holds with probability at least $1 - C'/j^7$, finally, with probability at least $1 - C'/j^7$: $\forall m$, with probability $1 - e^{-2j\sqrt{m}}$,

$$\sup_{|t-s|\leq l} |Q_t^j - Q_s^j| + \sup_{|t-s|\leq l} |R_t^j - R_s^j|$$
$$\leq 16(d+2)m \cdot 2^{-j} \left(\sqrt{C_j l j} + |t-s|\sqrt{C_j \log C_j}\right) \quad ;$$

hence Lemma 5 holds.

We now consider the case of few jumps in each interval of length $l$.

**Lemma 8.** *There exists $J_0$ such that $\forall j \geq J_0$ and $\forall m \geq 0$ satisfying*

$$C_j \leq \frac{(32)^2}{{C_1'}^2} 2^m j\sqrt{m} \quad , \tag{26}$$

*the probability that $Y_t^j$ has on any of the $2^m$ dyadic intervals of length $2^{-m+1}$ more than $mj^2$ jumps is bounded by $e^{-mj^2}$, and therefore, if this event does not happen, $\forall s, t$ such that $2^{-m} \leq |s-t| \leq 2^{-m+1}$,*

$$|Y_t^j - Y_s^j| \leq 2mj^2 2^{-j} \quad . \tag{27}$$

*Proof of Lemma 8.* The number of jumps of $Y_t^j$ on an interval of length $2^{-m+1}$ is a Poisson variable of parameter $\lambda_j = C_j 2^{-m+1} \leq Dj\sqrt{m}$. We split the interval [0, 1] into $2^{m-1}$ dyadic intervals of length $2^{-m+1}$. The probability that $Y_t^j$ has on any of these $2^{m-1}$ intervals more than $mj^2$ jumps is bounded by

$$2^{m-1} \sum_{k=mj^2}^{\infty} \frac{\lambda_j^k}{k!} \leq 2^{m-1} \sum_{k=mj^2}^{\infty} \frac{(Dj\sqrt{m})^k}{k!} \leq e^{-mj^2} \quad .$$

Thus, if $t$ and $s$ belong to the same dyadic interval of length $2^{-m}$,

$$\mathbb{P}(|Y_t^j - Y_s^j| \leq mj^2 2^{-j}) \geq 1 - e^{-mj^2} \quad .$$

If $t$ and $s$ belong to two adjacent intervals, since there are at most $j^2$ jumps on each interval, with probability at least $1 - e^{-mj^2}$, $|Y_t^j - Y_s^j| \leq 2mj^2 2^{-j}$.

□

*Proof of Proposition 2.* Let $t_0$ be such that $t_0 \notin E_\delta$ and $t_0 \notin S$. Since $t_0 \notin E_\delta$, there exists $J_0$ such that $\forall j \geq J_0$, $t_0$ belongs to no set $A_\delta^j$. Since $t_0 \notin S$, $\sum_{j \leq J_0} X_t^j$ is linear in a neighbourhood of $t_0$, and, in order to estimate the regularity of $X_t = \sum X_t^j$ at $t_0$, we only have to consider the values of $j$ larger than $J_0$.

From Lemma 5 and Lemma 8, we deduce that (17) and (27) hold $\forall j \geq J$ and $\forall m \geq M$ with probability at least

$$1 - \sum_{j \geq J} \frac{C'}{j^7} \left( \sum_{m \geq M} 2e^{-2j\sqrt{m}} \right) \quad ;$$

since this series is convergent and since the event we consider in Proposition 2 do not depend on the first values of $j$ and $m$, we can suppose in the following that the uniform estimates (17) and (27) hold with probability 1.

Let $\gamma$ be such that $\beta < \gamma < 1$ if $\beta < 1$, $\beta < \gamma < 2$ if $1 \leq \beta < 2$, and $\gamma = 2$ if $\beta = 2$. Let $m \geq 1$ and $t$ be such that $2^{-m} \leq |t - t_0| < 2^{-m+1}$, and let $j_1 = [\frac{m}{\delta}]$.

**First case:** $\beta \geq 1$.

If $j \leq j_1$, $X_t^j$ has no jump between $t_0$ and $t$, so that

$$|X_t^j - X_{t_0}^j| = |(t - t_0) \int x\pi_j(dx)| \leq |t - t_0| 2^{-j} C_j$$

and the sum on the corresponding $j$s is bounded by

$$C|t - t_0| 2^{(\gamma - 1)j_1} \leq C|t - t_0|^{1 - \frac{\gamma}{\delta} + \frac{1}{\delta}} \tag{28}$$

(we sum geometrically decreasing series if $\beta \neq 2$, and the result holds also for $\beta = 2$ because of (3)).

If $j \geq j_1$ and (26) holds,

$$|X_t^j - X_{t_0}^j| \leq |Y_t^j - Y_{t_0}^j| + |t - t_0| 2^{-j} C_j$$

$$\leq 2mj^2 2^{-j} + C|t - t_0| 2^{-j} 2^m j \sqrt{m}$$

(using (26) and (27)); and the sum for $j \geq j_1$ is bounded by

$$Cmj_1^2 2^{-j_1} \leq C|t - t_0|^{\frac{1}{\delta}} |\log(|t - t_0|)|^3 \quad . \tag{29}$$

If $j \geq j_1$ and (16) holds, we use (17): If $\gamma < 2$, the sum of the $|X_t^j - X_{t_0}^j|$ taken on the corresponding $j$s is bounded by

$$\begin{aligned} &C2^{-j_1} |\log(|t - t_0|)| 2^{(\gamma/2)j_1} \sqrt{j_1} \sqrt{|t - t_0|} \\ &\qquad + C2^{-j_1} |\log(|t - t_0|)| |t - t_0| 2^{(\gamma/2)j_1} \sqrt{j_1} \\ &\leq C|t - t_0|^{\frac{1}{\delta} - \frac{\gamma}{2\delta} + \frac{1}{2}} |\log(|t - t_0|)|^2 \\ &\qquad + C|t - t_0|^{\frac{1}{\delta} + 1 - \frac{\gamma}{2}} |\log(|t - t_0|)|^{3/2} \quad ; \end{aligned} \tag{30}$$

if $\gamma = 2$, using (3), the sum is bounded by $\sqrt{|t-t_0|}$. Since $\delta \geq \beta$ and since $\gamma$ can be chosen arbitrarily close to $\beta$, Proposition 2 follows in this case follows from these estimates.

**Second case:** $\beta < 1$.
In this case, we rather estimate the increments of $Y_t = \sum Y_t^j$ (i.e. we do not compensate the compound Poisson processes).

We separate the subcases as above: If $j \leq j_1$, each $Y_t^j$ is constant between $t$ and $t_0$, so that $|Y_t^j - Y_{t_0}^j| = 0$.

If $j \geq j_1$,

$$|Y_t^j - Y_{t_0}^j| \leq |X_t^j - X_{t_0}^j| + |t-t_0| C_j 2^{-j} \quad .$$

The sum of the $|X_t^j - X_{t_0}^j|$ is estimated as in the first case; and, since $\gamma < 1$,

$$\sum_{j \geq j_1} |t-t_0| C_j 2^{-j} \leq |t-t_0| 2^{(\gamma-1)j_1} \leq C|t-t_0|^{1+\frac{1}{\delta}-\frac{\gamma}{\delta}}$$

hence Proposition 2 in that case. □

## 3. The Hausdorff measure of $R_\delta$

We suppose now that $\beta > 0$; the case $\beta = 0$ (where the spectrum of singularities vanishes everywhere) will be treated separately at the end of this section.

Deriving a lower bound for the Hausdorff dimension of $R_\delta$ will be the consequence of Theorem 2 proved in [10] concerning a rather general type of fractal sets. Let us first recall the statement of this result in its full generality.

Let $\lambda_n$ be a sequence of points in [0, 1] and $\epsilon_n > 0$. We define

$$G_a = \limsup_{N\to\infty} \bigcup_{n \geq N} [\lambda_n - \epsilon_n^a,\ \lambda_n + \epsilon_n^a] \ ;$$

($G_a$ is the set of points that belong to an infinite number of intervals $[\lambda_n - \epsilon_n^a,\ \lambda_n + \epsilon_n^a]$). The function which associates to every $a$ the Hausdorff dimension of $G_a$ (denoted in the following by $dim_H(G_a)$) is decreasing. We may know that for an $a$ small enough, almost every point of [0, 1] belongs to $G_a$. This sole information yields a lower bound on $dim_H(G_b)$ for $b > a$.

We start we a few classical notions and results related with Hausdorff dimensions. Let $h(x)$ be a continuous increasing function defined on $[0, +\infty)$ such that

$$\left.\begin{array}{l} h(0)=0 \\ x/h(x) \text{ is increasing} \\ \lim_{x\to 0} x/h(x)=0; \end{array}\right\} \tag{31}$$

define, if $E$ is a subset of $\mathbb{R}$,

$$mes_h(E)=\lim_{\epsilon\to 0}\ \inf\left\{\sum_{n=0}^{\infty} h(|I_n|),\quad (I_n)_{n\in\mathbb{N}}\in\mathscr{P}_\epsilon\right\} \tag{32}$$

where $\mathscr{P}_\epsilon$ is the set of all coverings de $E$ by intervals $I_n$ of length at most $\epsilon$.

**Theorem 2.** *Let $h_d$ be the dimension function $h_d(x)=(\log x)^2|x|^d$, and let $\mathscr{H}^d$ be the Hausdorff measure constructed with the help of this dimension function. If almost every $x$ belongs to $G_a$,*

$$\forall b>a \quad \mathscr{H}^{a/b}(G_b)>0\ .$$

(*In particular, the Hausdorff dimension of $G_b$ is larger than $a/b$.*)

Suppose that $\beta>0$. There exists $j_n\to\infty$ such that $C_{j_n}$ is increasing and

$$\frac{\log C_{j_n}}{\log 2^{j_n}}\to\beta \tag{33}$$

By the Borel-Cantelli lemma, almost evry point of [0, 1] belongs to

$$\limsup_{n\to\infty}\bigcup_{t\in F_{j_n}}\left[t-\frac{1}{C_{j_n}},t+\frac{1}{C_{j_n}}\right]\ .$$

We can thus apply Theorem 2 to almost every sample path, choosing for sequence $\lambda_k$ the union of the $F_{j_n}$, and $\epsilon_k=1/C_{j_n}$ if $\lambda_k\in F_{j_n}$; we also choose $a=1$. Thus for any $b$ larger than 1, the set $H_b$ of points covered by an infinite number of the intervals

$$\left[t-\left(\frac{1}{C_{j_n}}\right)^b,t+\left(\frac{1}{C_{j_n}}\right)^b\right],\quad t\in F_{j_n}$$

satisfies $\mathscr{H}^{1/b}(H_b)>0$. By (33), $H_b\subset\bigcap_{a<\beta b}E_a$; furthermore, since $\mathscr{H}^{1/b}(E_a)=0\ \forall a>\beta b$, it follows that $\mathscr{H}^{1/b}(R_{\beta b})>0$ so that $dim_H(R_\delta)\geq\beta/\delta$.

We consider now the case where $\beta=0$. In that case we have to prove that $D(h)=0\ \forall h$. Since we already know that $D(h)\leq 0$, we only have to prove that for each $h$ there exists at least one point where the Hölder exponent is $h$ (so that $D(h)\neq-\infty$). This point will be obtained as an intersection of compact intervals.

**Lemma 9.** *If $\beta = 0$ but $\pi(\mathbb{R}^d) = +\infty$, $\forall \delta > 0$, with probability 1, the set $R_\delta$ is not empty.*

*Proof of Lemma 9.* Let $D_j$ be a non-decreasing sequence such that

$$\sum_{i=1}^{j}(C_i + i) = o(D_j) \quad \text{when} \quad j \to +\infty \tag{34}$$

and

$$\forall \epsilon > 0 \;\; \exists C \;\; D_j \leq C \exp(\epsilon j) \tag{35}$$

(such a sequence $D_j$ exists because $\beta = 0$). Let $\delta > 0$. We will construct an increasing sequence of integers $j_n$ and a decreasing sequence of intervals

$$I^\delta_{j_n} = [S + 2^{-\delta j_n}, S + D_{j_n} 2^{-\delta j_n}]$$

or

$$I^\delta_{j_n} = [S - D_{j_n} 2^{-\delta j_n}, S - 2^{-\delta j_n}]$$

such that

$$\text{if} \quad n \geq 2, \quad S \quad \text{is one of the jump points of} \quad \sum_{j=j_{n-1}}^{j_n} X^j_t \ . \tag{36}$$

Let $j_0$ be a (large enough) integer that will be fixed later. Let $j_1$ be an integer larger than $j_0$ and which satisfies simultaneously the following conditions:

- $I^\delta_{j_1} \subset [0, 1]$
- The sets $F_j$ of jumps of $X_j$ satisfy

$$\forall j \in \{1, \ldots, j_0\} \quad \forall S \in F_j \quad [S - 2^{-\delta j_0}, S + 2^{-\delta j_0}] \cap I^\delta_{j_1} = \emptyset \ . \tag{37}$$

  and

$$\forall j \in \{j_0 + 1, \ldots, j_1\} \quad \forall S \in F_j \quad [S - 2^{-\delta j}, S + 2^{-\delta j}] \cap I^\delta_{j_1} = \emptyset \ .$$

Once the a.s. existence of such an interval $I^\delta_{j_1}$ will have been proved, the construction will be continued as follows: We now look for an integer $j_2 > j_1$ such that $X_{j_2}$ has a jump at a point $S_{j_2}$ and satisfies the conditions:

- $I^\delta_{j_2} \subset I^\delta_{j_1}$
- The set $F_j$ of jumps of $X_j$ satisfies

$$\forall j \in \{j_1 + 1, \ldots, j_2\} \quad \forall S \in F_j \quad [S - 2^{-\delta j}, S + 2^{-\delta j}] \cap I^\delta_{j_1} = \emptyset \ .$$

Let us suppose for the moment that a whole sequence of imbedded intervals $I^{\delta}_{j_n}$ is thus constructed and let us prove that the Hölder exponent of $X_t$ is $1/\delta$ at the point $t_0$ which is the intersection of the $I^{\delta}_{j_n}$. First note that, if $S_{j_n}$ is the jump of $\sum_{j' \leq j_n} X_t^{j'}$ used in the definition of $I^{\delta}_{j_n}$, $dist(t_0, S_{j_n}) \leq D_{j_n} 2^{-\delta j_n}$, so that, using (36), (35) and Lemma 4, the Hölder exponent of $X_t$ at $t_0$ is at most $1/\delta$. The proof of the regularity of $X_t$ at $t_0$ follows the proof of Proposition 2, so we just sketch it.

As in the case $\beta < 1$ above, we do not compensate the jumps of $X_t^j$; $\sum_{j \leq j_0} X_t^j$ is $C^\infty$ at $t_0$ by (37), so we estimate the increments of $X_t^j$ only for $j \geq j_0$. Let $t \neq t_0$ and let $J$ be defined by $2^{-\delta J} \leq |t - t_0| \leq 2^{-\delta(J+1)}$. If $j_0 < j < J$, $|X_t^j - X_{t_0}^j| = 0$. If $j \geq \sup(J, j_0)$, using (8), with probability at least $1 - 2e^{-5j}$, $X_t^j$ jumps at most $2C_j|t - t_0| + j$ times on $[t_0, t]$; so that, with probability at least $1 - 2 \cdot e^{-5j_0}$,

$$\sum_{j \geq J} |X_t^j - X_{t_0}^j| \leq C \sum_{j \geq J} 2.2^{-j}(C_j|t-t_0| + j) \leq C|t-t_0|^{1/\delta} |\log(|t-t_0|)| \ .$$

The result holds almost surely because $j_0$ can be chosen large enough.

We now come back to the construction of $j_1$. Let us consider an integer $j' \geq j_0$ such that $X_t^{j'}$ has a jump on $[0, 1/2]$. By (8), with probability at least $1 - 2e^{-5j'}$, $\sum_{j \leq j'} X_t^j$ has at most $\sum_{j \leq j'}(2C_j + j)$ jumps on $[0, 1]$. If $j < j_0$, we exclude around each of jump of $X_t^j$ an interval of length $2^{-\delta j_0}$, and if $j \geq j_0$, we exclude an interval of length $2^{-\delta j}$. There remains in $[0, 1]$ at least one interval of length at least

$$\frac{1 - 2\sum_1^{j_0-1}(2C_j + j)2^{-\delta j_0} - 2\sum_{j_0}^{j'}(2C_j + j)2^{-\delta j}}{\sum_1^{j'} 2C_j + j} \geq \frac{1}{2\left(\sum_1^{j'} 2C_j + j\right)}$$

because of (34) and (35). For each $j'$ we have thus obtained a random interval $L_{j'}$ of length at least

$$l_{j'} = \frac{1}{2\sum_{j_0}^{j'} 2C_j + j} \ .$$

Let us show that we can insert an interval $I^{\delta}_{j'}$ of length $D_{j'}2^{-\delta j'}$ inside $L_{j'}$; this is possible if $D_{j'}2^{-\delta j'} \leq l_{j'}$, i.e. if

$$D_{j'} \leq \frac{2^{\delta(j'-j_0)}}{2\sum_{j_0}^{j'} 2C_j + j}$$

which will hold for $j'$ large enough because the sequences $D_j$ and $C_j$ do not increase at an exponential rate; we pick for $j_1$ the first $j'$ such that the above properties hold.

Let us now construct $j_2$. If $j_2$ is large enough, using (8), with probability at least $1 - 2e^{-5j_2}$, the number of jumps of the $(X_t^j)_{j\in[j_1,j_2]}$ which belong to $I_{j_1}^\delta$ is bounded by $\sum_{j_1}^{j_2} 2C_j + j$. Let us exclude around each of these jumps an interval of length $2^{-\delta j}$. There remains in $I_{j_1}$ at least one interval of length at least

$$\frac{(D_{j_1} - 1)2^{-\delta j_1} - 2\sum_{j_1}^{j_2}(2C_j + j)2^{-\delta j}}{\sum_{j_1}^{j_2} 2C_j + j} \geq \frac{D_{j_1}2^{-\delta j_1}}{2\left(\sum_{j_1}^{j_2} 2C_j + j\right)}$$

because of (34). For each $j_2$ we have thus obtained a random interval $L_{j_2}$ of length at least

$$l_{j_2} = \frac{D_{j_1}2^{-\delta j_1}}{2\left(\sum_{j_1}^{j_2} 2C_j + j\right)} \quad .$$

Since $\sum_{j\geq j_1} X_t^j$ has a dense set of jumps, if $j_2$ is large enough, at least one new jump has appeared in $I_{j_1}^\delta$ so that (36) will hold. Thus, we can insert an interval $I_{j_2}^\delta$ inside $L_{j_2}$ if $D_{j_2}2^{-\delta j_2} \leq l_{j_2}$, i.e. if

$$\frac{D_{j_2}}{D_{j_1}} \leq \frac{2^{\delta(j_2 - j_1)}}{2\sum_{j_1}^{j_2} 2C_j + j}$$

which will hold for $j_2$ large enough; the $j_n$ and the corresponding $I_{j_n}^\delta$ are constructed as $j_2$; hence Lemma 9 holds, and the first part of Theorem 1 is proved.

The second part of Theorem 1 (case with a Brownian component) is a direct consequence of the following remark: the Hölder exponent of the sum of two functions is the infimum of the exponents, except perhaps when the two exponents coincide, in which case the exponent of the sum may be larger. In the case of the sum of a Brownian motion $B_t$ and a Lévy process $X_t$ without Brownian component, $X_t + B_t$ has the same jumps as $X_t$, and Lemma 4 thus gives the same upper bound for the Hölder exponents of $X_t$ and $X_t + B_t$; thus

$$\forall t, \quad h_{X+B}(t) = \inf(h_X(t), h_B(t)) \quad ,$$

and the second part of Theorem 1 follows. $\square$

*Remark*. The methods we introduced in this paper allow to answer a problem posed by Jean Bertoin. If $X_t$ is a composed Poisson process, and $t_0$ is not a time of jump, $X_t$ is $C^\infty$ at $t_0$. The problem is to determine if a similar property holds for Lévy processes satisfying $\pi(\mathbb{R}^d) = +\infty$. Does there exist in this case a modulus of continuity which holds apart from all points of jump? We will prove that the answer is negative.

A function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is a modulus of continuity at $t_0$ if $\omega$ is a continuous strictly increasing function satisfying $\omega(0) = 0$ and, for $|t - t_0|$ small enough,

$$|X_t - X_{t_0}| \leq \omega(|t - t_0|) \ .$$

**Proposition 3.** *Let $X_t$ be a Lévy process satisfying $\pi(\mathbb{R}^d) = +\infty$. For any continuous strictly increasing function satisfying $\omega(0) = 0$, there exists $t_0$ which is not a jump point of $X_t$, and is such that $\omega$ is not a modulus of continuity at $t_0$.*

We can suppose that $X_t$ has no Brownian part (since the Brownian motion is, say, $C^{1/3}$ at every point, subtracting this part won't change the muduli of continuity larger than $t^{1/3}$. Let $\omega$ be a modulus of continuity and let $\omega'$ be a continuous strictly increasing function satisfying $\omega'(0) = 0$ and $\omega = o(\omega')$ near 0. Let $\eta$ be the invert of $\omega'$. The function $\eta$ is also continuous strictly increasing and satisfies $\eta(0) = 0$. We consider as usual the decomposition $X_t = \sum X_t^j$ where the $X_t^j$ are independent compensated compound Poisson processes with jumps of size satisfying

$$2^{-j+1} \leq |x| < 2^{-j+2} \ .$$

Let $j_1$ be the first positive $j$ such that $(X_t^j)_{j\geq 0}$ has a jump at (at least) one point $t_1 \in [0, 1]$. Note that $j_1 < +\infty$ a.s.; let

$$I_1 = \left[ t_1 - \frac{\eta(2^{-j_1})}{2}, t_1 + \frac{\eta(2^{-j_1})}{2} \right] \ .$$

Let now $j_2$ be the first $j$ larger than $j_1$ such that $(X_t^j)$ has a jump at (at least) one point $t_1 \in I_1$; $j_2 < +\infty$ a.s. We continue this procedure, thus constructing a decreasing sequence of imbedded closed intervals. Let now $t_0 = \cap I_n$; $X_t^{j_n}$ (hence $X_t$) has a jump of amplitude larger than $2 \cdot 2^{-j_n}$, at a distance at most $\eta(2^{-j_n})/2$ from $t_0$. Thus we can pick $u_n$ arbitrarily close to $t_n$ such that $|t_0 - u_n| \leq \eta(2^{-j_n})$ and $|X_{t_0} - X_{u_n}| \geq 2^{-j_n}$ (pick $t_n$ immediately on the left or on the right hand side of the jump). Thus $|X_{t_0} - X_{t_n}| \geq \omega'(|t_0 - t_n|)$, and $|X_{t_0} - X_{t_n}|$ cannot be bounded by $\omega(|t_0 - t_n|)$, since $\omega = o(\omega')$.

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