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Spectral Analysis on Infinite Sierpiński Gaskets

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We study the spectral properties of the Laplacian on infinite Sierpiński gaskets. We prove that the Laplacian with the Neumann boundary condition has pure point spectrum. Moreover, the set of eigenfunctions with compact support is complete. The same is true if the infinite Sierpiński gasket has no boundary, but is false for the Laplacian with the Dirichlet boundary condition. In all these cases we describe the spectrum of the Laplacian and all the eigenfunctions with compact support. To obtain these results, first we prove certain new formulas for eigenprojectors of the Laplacian on finite Sierpiński pre-gaskets. Then we prove that the spectrum of the discrete Laplacian on a Sierpiński lattice is pure point, and the eigenfunctions are localized.

Key Words: fractals; fractal graphs; Laplacian; pure point spectrum; localization.

1. INTRODUCTION

The study of the Laplacian on fractals was originated in physics literature, where so-called spectral decimation method was developed [Al, Ra, RT]. The Laplacian on the Sierpiński gasket was first constructed as the generator of a diffusion process by S. Kusuoka and S. Goldstein in [Ku1, Go]. Later an analytic approach was developed by J. Kigami, who constructed the Laplacian using the theory of Dirichlet forms [Ki1]. The eigenvalue distribution and eigenfunctions for the Laplacian on the Sierpiński gasket were studied in detail by M. Fukushima and T. Shima in [Sh1, Sh2, FS]. Later the theory of the Laplacian was developed for nested fractals and p.c.f. self-similar sets (finitely ramified fractals) in [Ba, BP, DK, Fu, Ki2, Ki3, KL, Ku2, Ku3, La, Li, Ma]. Many new features of the differential calculus on the Sierpiński gasket were illuminated by R. Strichartz (with co-authors) in [DSV, St1, St2, St3, St4]. Different

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questions related to localized eigenfunctions on fractals were recently studied by M. T. Barlow and J. Kigami in [BK, Ki4]. In our paper localized eigenfunctions are those whose support is compact (finite in the case of a lattice). M. T. Barlow and J. Kigami consider compact fractals and use a different, although related to ours, definition of localized eigenfunctions. The first proof that the localized eigenfunctions are complete (in the case of certain fractal graphs) was given in [MT1]. The completness means that any square integrable (summable in the case of a lattice) function can be approximated by linear combinations of localized eigenfunctions.

In this work we study the spectral properties of the Laplacian on an infinite Sierpiński gasket. There are uncountably many distinct infinite Sierpiński gaskets without boundary, but essentially only one with a non-empty boundary. In the latter case the boundary consists of one point (see Lemma 2.3). We prove that if we impose the Neumann boundary condition at the boundary, or if the boundary is empty, then the Laplacian has pure point spectrum. Moreover, the set of eigenfunctions with compact support is complete. This statement is false for the Laplacian with the Dirichlet boundary condition.

(The spectrum of a selfadjoint operator is called pure point if the eigenfunctions of this operator form a complete set. In this case all the spectral measures are discrete. In other words, the operator does not have singularly continuous or absolutely continuous spectrum. However a pure point spectrum may be continuous if the eigenvalues are not isolated. For the Laplacian on an infinite Sierpiński gasket or lattice all the eigenvalues are of infinite multiplicity and so there is no discrete spectrum, even though the spectrum is pure point).

An infinite Sierpiński gasket is a particular example of fractal blowups described by R. Strichartz in [St2]. The infinite Sierpiński gasket with boundary is an example of expanded nested fractals, which were considered by M. Fukushima in [Fu], and his results hold for any infinite Sierpiński gasket. M. T. Barlow, E. Perkins [BP] obtained extremely accurate estimates for the transition density for the Brownian motion on an infinite fractal which is slightly different from infinite Sierpiński gaskets defined here. Our results are also applicable to the Laplacian on this infinite Sierpiński gasket (see Remark 6.10).

Only two-dimensional Sierpiński gaskets are considered in this paper, although similar results can be proved for an infinite Sierpiński gasket in any dimension, and also for other infinite fractals.

This paper is organized as follows. The basic definitions and facts about Sierpiński lattices and the Laplacian on these lattices are presented in Section 2. In Section 3 we prove formulae, which describe the spectrum and the eigenprojectors of the Laplacian on successive Sierpiński pre-gaskets. These formulae give a refinement of the spectral decimation method and are used later in this paper. As a byproduct we describe all the eigenfunctions and eigenvalues of the Laplacian on the Sierpiński pre-gaskets, which are already well known from the works of T. Shima and M. Fukushima [Sh1, Sh2, FS].

In Section 4 we apply formulae from Section 3, together with some more geometric results, in order to prove that the spectrum of the Laplacian on any Sierpiński lattice is pure point. We also prove that the set of eigenfunctions with finite support is complete. This is an interesting result by itself, and in addition it plays an important role in proving that the spectrum of the Laplacian on the infinite Sierpiński gaskets is pure point. The latter fact as well as the completeness of the set of eigenfunctions with compact support is proved in Section 5. Then we describe the spectrum and all the eigenfunctions.

Up to this point we considered the discrete Laplacian without boundary conditions. It is known from [Ki1, Ki2] that it leads to the Laplacian on the Sierpiński gasket with the Neumann boundary conditions. Similarly, in the case of a nonempty boundary we obtain a Laplacian on an infinite Sierpiński gasket with the Neumann boundary condition.

In Section 6 we describe the spectrum of the Dirichlet Laplacian on a Sierpiński lattice and an infinite Sierpiński gasket. We prove that the spectrum (as a closed set) is the same as the one of the Neumann Laplacian. Moreover, any eigenvalue of the Neumann Laplacian is an eigenvalue of the Dirichlet Laplacian and has infinitely many corresponding localized eigenfunctions. However the set of eigenfunctions with compact (finite in the case of a lattice) support is not complete. The question about the nature of the spectrum of the Dirichlet Laplacian on the infinite Sierpiński gasket or lattice with boundary is still open, although we have some information about the spectral measures, such as the absence of an absolutely continuous component and a nonlinear self-similarity property. There exists a number of very interesting works of B. Simon *et al.* on the relation between pure point and singular continuous spectra of selfadjoint operators (see, for instance, [DRMS, Si1, Si2, Si3, SW]). It is often possible to prove that the spectrum is pure point or singular continuous "generically" in one sense or another, but it might be hard to give a definitive answer for a fixed perturbation, such as one in our situation.

2. LAPLACIAN ON SIERPIŃSKI LATTICE

We fix three contractions $\Psi_1, \Psi_2, \Psi_3 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\Psi_1(x) = \frac{1}{2}x$, $\Psi_2(x) = \frac{1}{2}x + (\frac{1}{2}, 0)$, $\Psi_3(x) = \frac{1}{2}x + (\frac{1}{4}, \sqrt{3}/4)$, $x \in \mathbb{R}^2$. The Sierpiński gasket is defined as a unique compact set S such that $S = \Psi_1(S) \cup \Psi_2(S) \cup \Psi_3(S)$ (Fig. 1).

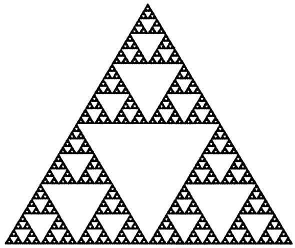

FIG. 1. Sierpiński gasket.

DEFINITION 2.1. For each $n \geqslant 0$ we define V_n inductively by $V_n = \Psi_1(V_{n-1}) \cup \Psi_2(V_{n-1}) \cup \Psi_3(V_{n-1})$ where $V_0 = \{(0, 0), (1, 0), (1/2, \sqrt{3}/2)\}$ is the set of fixed points of Ψ_1, Ψ_2 and Ψ_3.

For a fixed sequence $\mathscr{K} = \{k_n\}_{n \geqslant 1}$, $k_n \in \{1, 2, 3\}$, let $V_{-n} = \Psi^{-1}_{\mathscr{K}, n}(V_n)$ where $\Psi_{\mathscr{K}, n} = \Psi_{k_n}, ..., \Psi_{k_1}$. Then the *Sierpiński lattice* is $V = \bigcup_{m \geqslant 1} V_{-m}$ (see Example 2.6 and Fig. 2).

Notation 2.2. We say that $x \in V$ and $y \in V$ are neighbors and write $x \sim y$ if $x, y \in \Psi^{-1}_{\mathscr{K}, n} \Psi_{l_n}, ..., \Psi_{l_1}(V_0)$ for a sequence $l_1, ..., l_n \in \{1, 2, 3\}$. We denote by $\mathtt{deg}(x)$ the number of neighbors of x. Also we denote by $\mathtt{deg}_n(x)$ the number of neighbors of x which lie in V_{-n}. The boundaries ∂V and ∂V_{-n} are defined as the set of points that have degree two in ∂V and ∂V_{-n} respectively. If $x \sim y$ then $|x - y| = 1$.

Clearly, $V_{-n} \subset V_{-n-1} \subset V$. It is easy to see that ∂V_{-n} always consists of three points, which are the corners of the largest triangle with vertices in V_{-n}, and ∂V consists of at most one point. The degree of any point of V or V_{-n}, which does not belong to the boundary, is four.

The structure of the lattice V depends on the sequence $\mathscr{K}$, although we omit this dependence in our notation. The lemma below shows that there exist uncountably many nonisometric lattices V. However, only one lattice, up to an isometry, has a nonempty boundary.

LEMMA 2.3. (i) *$\partial V \neq \varnothing$ if and only if there is n_0 such that $k_n = k_{n_0}$ for $n \geqslant n_0$.*

(ii) *Let V' be a lattice corresponding to a sequence of indices $\mathscr{K}' = \{k'_n\}_{n \geqslant 1}$, $k'_n \in \{1, 2, 3\}$. Then V is isometric to V' if and only if there exist a permutation $\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$ and n_0 such that $k_n = k'_{\sigma(n)}$ for $n \geqslant n_0$. Word "isometric" means that there exists a distance preserving map from V to V'.*

Proof. First, we prove (ii).

Suppose there exist a permutation $\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$ and n_0 such that $k_n = k'_{\sigma(n)}$ for $n \geqslant n_0$. Then there exists a unique isometry $\bar{\sigma}$ of $\mathbb{R}^2$ which coincides with σ on V_0. It is easy to see that $\Psi \stackrel{\text{def}}{=} \Psi^{-1}_{\mathcal{K}', n} \bar{\sigma} \Psi_{\mathcal{K}, n}$ is an isometry of $\mathbb{R}^2$ which does not depend on $n \geqslant n_0$ because $\Psi^{-1}_{\sigma(i)} \bar{\sigma} = \bar{\sigma} \Psi_i$ for any $i \in \{1, 2, 3\}$. It is evident that $\Psi(V) = V'$.

Suppose now that there exists a distance preserving map $\Psi : V \to V'$. Then there exists an isometry of $\mathbb{R}^2$ which coincides with Ψ on V_0. It is easy to see that then this isometry coincides with Ψ on the whole V. We will denote this extension of Ψ to $\mathbb{R}^2$ by the same symbol Ψ. One can see that there is n_0 such that $\Psi(V_{-n_0}) = (V'_{-n_0})$. Hence $\bar{\sigma} \stackrel{\text{def}}{=} \Psi_{\mathcal{K}', n} \Psi \Psi^{-1}_{\mathcal{K}, n}$ is an isometry of $\mathbb{R}^2$ which maps V_0 to V_0. Let x_m be the fixed point of Ψ_i for $m \in \{1, 2, 3\}$. Then we can define a permutation of $\{1, 2, 3\}$ such that $\sigma(i) = j$ if $\bar{\sigma}(x_i) = x_j$. It is straightforward to check that $k_n = k'_{\sigma(n)}$ for $n \geqslant n_0$.

The statement (i) follows from (ii) and Example 2.6. ∎

Notation 2.4. ℓ^2 denotes a Hilbert space of complex valued functions on V with the scalar product

$$\langle f, g \rangle = \sum_{x \in V} \mathtt{deg}(x) \, f(x) \, \overline{g(x)}.$$

The norm in ℓ^2 is denoted by $\|\cdot\|$. The natural (probabilistic) *Laplacian* on V is defined by

$$\Delta f(x) = \frac{1}{\mathtt{deg}(x)} \sum_{\{y \in V : x \sim y\}} f(y) - f(x).$$

It is a bounded nonpositive selfadjoint operator in ℓ^2. By similar formulae, with $\mathtt{deg}(x)$ replaced by $\mathtt{deg}_n(x)$, we define finite dimensional Hilbert spaces ℓ^2_n of complex valued functions on V_{-n} and Laplacians Δ_n acting on these spaces.

Notation 2.5. Lap The map $\Psi^{-1}_{\mathcal{K}, m} \Psi_{\mathcal{K}, n}$ is a "canonical" injection of V_{-n} into V_{-m} for any $m > n$ (it differs from the inclusion map of V_{-n} into V_{-m}). This injection induces an isometry $J_{n, m} : \ell^2_n \to \ell^2_m$ such that

$$J_{n, m} f(x) = \begin{cases} f(y) & \text{if there exists } y \in V_{-n} \text{ such that } x = \Psi^{-1}_{\mathcal{K}, m} \Psi_{\mathcal{K}, n}(y) \\ 0 & \text{otherwise} \end{cases}.$$

EXAMPLE 2.6. Suppose $k_j = 1$ for each $j \geqslant 1$. Then we have $V_{-k} = 2^k V_k$. In this case $J_{n, m} f(x) = f(2^{n-m} x)$, if $x \in 2^{m-n} V_{-n}$ and zero otherwise. That is $J^*_{n, m} f(x) = f(2^{m-n} x)$.

Also we have an isometry $J : \ell^2 \to \ell^2$ such that $Jf(x) = f(x/2)$, if $x \in 2V$ and zero otherwise. That is $J^* f(x) = f(2x)$.

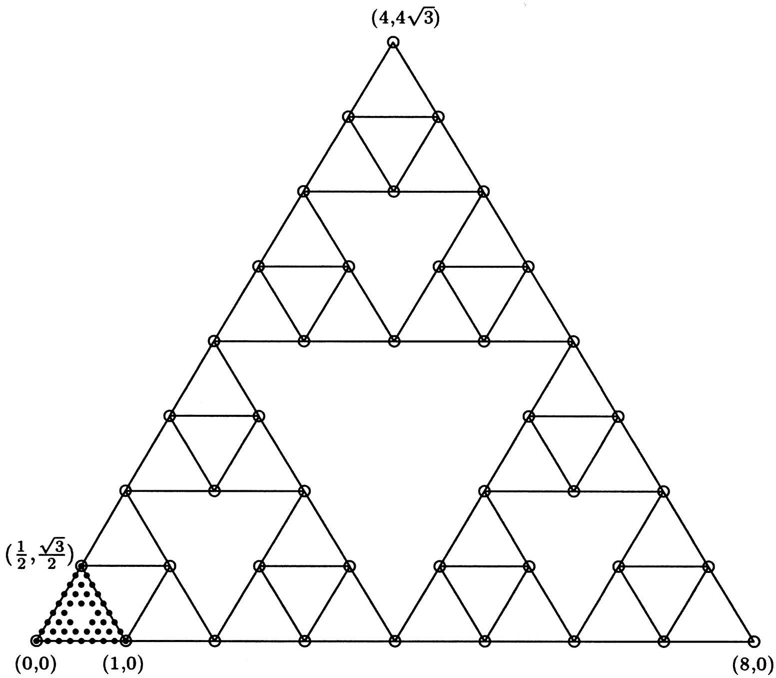

FIG. 2. Sets V_{-3} and V_3.

Figure 2 shows the set V_{-3} corresponding to the sequence $k_j = 1$ for each $j \geqslant 1$. Each point of V_{-3} is denoted by a small circle. We connect the neighbors in order to make the structure of this set more clear. Also we denote each point of V_3 by a small black dot in order to illustrate how V_{-3} and V_3 are different.

Note that $\partial V_{-3} = \{(0, 0), (4, 4\sqrt{3}), (8, 0)\}$ and $\partial V = \{(0, 0)\}$.

3. SPECTRAL SELF-SIMILARITY

Notation 3.1. Let $\mathscr{H}$ and $\mathscr{H}_0$ be Hilbert spaces, and let J_0 be an isometry from $\mathscr{H}_0$ into $\mathscr{H}$. Suppose that H and H_0 are some bounded linear operators on $\mathscr{H}$ and $\mathscr{H}_0$ respectively and that φ_0 and φ_1 are complex-valued functions defined on a set $\Lambda \subseteq \mathbb{C}$.

DEFINITION 3.2. We call an operator H *spectrally similar* to an operator H_0 with functions φ_0 and φ_1 and an isometry J_0 if

$$J_0^*(H - z)^{-1} J_0 = (\varphi_0(z) H_0 - \varphi_1(z))^{-1} \tag{3.1}$$

on $\mathscr{H}_0$ for any $z \in \Lambda_0$, where Λ_0 is the set of those $z \in \Lambda$ for which the both sides of (3.1) are well defined bounded linear operators. We always assume that Λ_0 is not empty.

Notation 3.3. Let P_0 be the orthogonal projector onto $J_0(\mathscr{H}_0)$, $\mathscr{H}_1$ be the orthogonal complement to $J_0(\mathscr{H}_0)$, P_1 be the orthogonal projector onto $\mathscr{H}_1$, and let J_1 be the operator of inclusion of $\mathscr{H}_1$ into $\mathscr{H}$.

Operators $S : \mathscr{H}_0 \to \mathscr{H}_0$, $X : \mathscr{H}_0 \to \mathscr{H}_1$, $\bar{X} : \mathscr{H}_1 \to \mathscr{H}_0$ and $Q : \mathscr{H}_1 \to \mathscr{H}_1$ are defined by $S = J_0^* H J_0$, $X = J_1^* H J_0$, $\bar{X} = J_0^* H J_1$ and $Q = J_1^* H J_1$. For $i = 1, 2$, the identity operator on $\mathscr{H}_i$ is denoted by I_i. The resolvent set of an operator A is denoted by $\rho(A)$.

LEMMA 3.4. *For $z \in \rho(H) \cap \rho(Q)$ relation* (3.1) *holds if and only if*

$$S - z - \bar{X}(Q - z)^{-1} X = \varphi_0(z)\, H_0 - \varphi_1(z). \tag{3.2}$$

Proof. It is easy to see that, for $z \in \rho(H) \cap \rho(Q)$, the following relation holds

$$J_0^*(H - z)^{-1} J_0(S - z - \bar{X}(Q - z)^{-1} X) = I_0. \quad \blacksquare$$

COROLLARY 3.5. *Let H be spectrally similar to H_0 on $\mathscr{H}_0$. If H_0 is not a constant multiple of I_0, then relation* (3.2) *uniquely defines functions φ_0 and φ_1 on $\rho(Q)$ and these functions are analytic on $\rho(Q)$.*

DEFINITION 3.6. The set $\mathscr{E} = \mathscr{E}(H, H_0) = \{z \in \mathbb{C} : z \notin \rho(Q) \text{ or } \varphi_0(z) = 0\}$ is called the exceptional set for the spectral similarity between operators H and H_0. If $\varphi_0(z) \neq 0$ then $R(z) \overset{\text{def}}{=} \varphi_1(z)/\varphi_0(z)$.

In what follows we assume that any analytic function is extended by continuity to its removable singularities.

Definitions 3.2 and 3.6, and Lemma 3.4 were given, in a sightly different form, in [MT2].

Now we establish the relation between eigenprojectors of spectrally similar operators. Namely, we show how one can find the eigenprojector P_z of H corresponding to an eigenvalue z, if the eigenprojector $P^0_{R(z)}$ of H_0 corresponding to eigenvalue $R(z)$ is known. In [MT2] nothing was said on what happens if $z \in \mathscr{E}$. However this case is the most important in many situations and will be considered here in detail.

We assume that H and H_0 are finite dimensional selfadjoint spectrally similar operators. Notation 3.3 is used here without further notice and we assume that any analytic function is extended by continuity to its removable singularities.

THEOREM 1. (i) *Let* $z \notin \mathscr{E}$. *Then* $R(z) \in \rho(H_0)$ *if and only if* $z \in \rho(H)$. *Moreover, if* $z \notin \rho(H)$ *then* $\operatorname{rank} P_z = \operatorname{rank} P^0_{R(z)}$ *and*

$$R'(z)\, \varphi_0(z)\, P_z = (J_0 - J_1(Q-z)^{-1}\, X)\, P^0_{R(z)}(J_0^* - \bar{X}(Q-z)^{-1}\, J_1^*) \tag{3.3}$$

This implies, in particular, that there is an one-to-one map $f_0 \mapsto f = J_0 f_0 - J_1(Q-z)^{-1}\, X f_0$ *from the eigenspace of* H_0 *corresponding to* $R(z)$ *onto the eigenspace of* H *corresponding to* z.

(ii) *If* $z \in \rho(Q)$, $\varphi_0(z) = \varphi_1(z) = 0$ *and* $R(z)$ *has a removable singularity at* z, *then* $z \notin \rho(H)$ *and* $R(z) \in \rho(H_0)$. *Moreover,* $\operatorname{rank} P_z = \operatorname{rank} P_0$ *and*

$$P_z = (J_0 - J_1(Q-z)^{-1}\, X)(\psi_0(z)\, H_0 - \psi_1(z))^{-1}\, (J_0^* - \bar{X}(Q-z)^{-1}\, J_1^*), \tag{3.4}$$

where $\psi_0(x) = \varphi_0(x)/(z-x)$ *and* $\psi_1(x) = \varphi_1(x)/(z-x)$. *This implies, in particular, that there is an one-to-one map* $f_0 \mapsto f = J_0 f_0 - J_1(Q-z)^{-1}\, X f_0$ *from* $\mathscr{H}_0$ *onto the eigenspace of* H *corresponding to* z.

(iii) *Let* $z \notin \rho(Q)$, z *be an eigenvalue of* Q *with corresponding eigenprojector* P^1_z, *and both* $\varphi_0(z)$ *and* $\varphi_1(z)$ *have poles at* z.

Then the poles of $\varphi_0(z)$ *and* φ_1 *are simple and so* $R(z)$ *has a removable singularity at* z.

If $R'(z) \neq 0$ *then* $P_z P^1_z = P_z$ *and* $P_0 P_z = 0$. *Moreover,*

$$\operatorname{rank} P^1_z - \operatorname{rank} P_z = \operatorname{rank}(\psi_0(z)\, H_0 - \psi_1(z)\, I_0) = \operatorname{corank} P^0_{R(z)},$$

where $\psi_0(x) = \varphi_0(x)(z-x)$ *and* $\psi_1(x) = \varphi_1(x)(z-x)$.

In addition, the following relations hold

$$P_z = J_1\left(P^1_z + \frac{1}{\psi_0(z)}\, P^1_z X (H_0 - R(z))^{-1}\, (I_0 - P^0_{R(z)})\, \bar{X} P^1_z\right) J_1^* \tag{3.5}$$

and $P^1_z X P^0_{R(z)} = 0$. *Note that* $I_0 - P^0_{R(z)}$ *is the projector from* $\mathscr{H}_0$ *onto the space, where* $(Q-z)^{-1}$ *is a well defined bounded operator.*

Notation 3.7. In the case (i) of this theorem we say that an eigenfunction $f = J_0 f_0 - J_1(Q-z)^{-1}\, X f_0$ of H is a *continuation* of the eigenfunction f_0 of H_0.

Proof. First we will prove the key formula for the proof of this theorem. It is not related to spectral similarity and is a known fact. Suppose that operators $Q - x$ and $S - \bar{X}(Q-x)^{-1}\, X$ are invertible. Then $H - x$ is invertible and

$$(H-x)^{-1}=J_1(Q-x)^{-1}J_1^* \\ +(J_0-J_1(Q-x)^{-1}X) \\ \times(S-x-\bar{X}(Q-x)^{-1}X)^{-1}(J_0^*-\bar{X}(Q-x)^{-1}J_1^*). \tag{3.6}$$

It is enough to prove this formula for $x=0$, i.e. to prove

$$H^{-1}=J_1Q^{-1}J_1^*+(J_0-J_1Q^{-1}X)(S-\bar{X}Q^{-1}X)^{-1}(J_0^*-\bar{X}Q^{-1}J_1^*) \tag{3.7}$$

provided Q and $S-\bar{X}Q^{-1}X$ are invertible.

Note that $J_i^*J_i=I_i$ and $J_iJ_i^*=P_i$, $i=0, 1$. In particular this implies $J_0J_0^*+J_1J_1^*=P_0+P_1=I$. Hence

$$HJ_1Q^{-1}J_1^*=P_1+P_0HJ_1Q^{-1}J_1^*$$

and

$$H(J_0-J_1Q^{-1}X)=HJ_0-P_1HJ_0-P_0HJ_1Q^{-1}X=J_0(S-\bar{X}Q^{-1}X).$$

Thus

$$H(J_1Q^{-1}J_1^*+(J_0-J_1Q^{-1}X)(S-\bar{X}Q^{-1}X)^{-1}(J_0^*-\bar{X}Q^{-1}J_1^*)) \\ =P_1+P_0HJ_1Q^{-1}J_1^*+J_0(J_0^*-\bar{X}Q^{-1}J_1^*)=P_1+P_0=I.$$

That is what (3.7) says.

Suppose now conditions of (i) or (ii) are satisfied. Lemma 3.4 and (3.6) imply

$$(H-x)^{-1}=J_1(Q-x)^{-1}J_1^*+(J_0-J_1(Q-x)^{-1}X) \\ \times(\varphi_0(x)H_0-\varphi_1(x))^{-1}(J_0^*-\bar{X}(Q-x)^{-1}J_1^*). \tag{3.8}$$

Then the statements to be proved follow if we pass to the limit as $x\to z$ in this formula.

To prove (iii) first we again pass to the limit as $x\to z$ in formula (3.8). We see that $P_0P_z\neq 0$ if and only if

$$\lim_{x\to z}(x-z)^2(\psi_0(x)H_0-\psi_1(x)I_0)^{-1}\neq 0,$$

that is only possible if $R'(z)=0$. Therefore $P_0P_z=0$ in our case. Relation (3.5) follows from (3.8).

Note that Lemma 3.4 implies

$$\psi_0(z)H_0-\psi_1(z)I_0=-J_0^*HJ_1P_z^1J_1^*HJ_0$$

if $z \in \sigma(Q)$. Hence $rank(\psi_0(z)\, H_0 - \psi_1(z)\, I_0) = rank P_z^1 - dim(P_z^1(\mathscr{H}_1) \cap P_z(\mathscr{H}))$ and in our case we have $rank P_z = dim(P_z^1(\mathscr{H}_1) \cap P_z(\mathscr{H}))$. In addition, we have that $\psi_0(z)\, H_0 - \psi_1(z)\, I_0$ is nonpositive.

Also we see that $J_0^*(H-z)^{-1} J_0$ is a bounded operator on $\mathscr{H}_0$ and so, by (3.2), we have $J_0^*(H-z)^{-1} J_0 = \lim_{x \to z} (z-x)(\psi_0(x)\, H_0 - \psi_1(x)\, I_0)^{-1}$. Hence $J_0^*(H-z)^{-1} J_0 = 0$ if and only if $R(z)$ has a pole at z or $R(z) \in \rho(H_0)$. If $R(z)$ has a removable singularity at z then $\psi_0(z)\, R'(z)\, J_0^*(H-z)^{-1} J_0 = P^0_{R(z)}$. ∎

Remark. 3.8. The statement (i) of this theorem was proved in [MT2], except formula (3.3) for the eigenprojector P_z.

LEMMA 3.9. *For each $n \geqslant 0$ operator Δ_{n+1} is spectrally similar to the operator Δ_n with isometry $J_{n,n+1}$ and functions $\varphi_0(z) = (2z+3)/(2z+1)(4z+5)$ and $\varphi_1(z) = (2z^2+3z)/(2z+1)$.*

Proof. This lemma follows from a general result proved in [MT2], or it can be proved by direct computation. Really, according to Lemma 3.4, it is enough to verify relation (3.2). Taking into account the concrete form of isometries $J_{n,n+1}$, J and operators S, X, $\bar{X}$ and Q, the statement to be proved can be reduced to the following matrix relation:

$$(-1-z)\, I - \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} -1-z & 1/4 & 1/4 \\ 1/4 & -1-z & 1/4 \\ 1/4 & 1/4 & -1-z \end{pmatrix}^{-1} \begin{pmatrix} 1/4 & 0 & 1/4 \\ 1/4 & 1/4 & 0 \\ 0 & 1/4 & 1/4 \end{pmatrix}$$

$$= \frac{2z+3}{(2z+1)(4z+5)} \begin{pmatrix} -1 & 1/2 & 1/2 \\ 1/2 & -1 & 1/2 \\ 1/2 & 1/2 & -1 \end{pmatrix} - \frac{2z^2+3z}{(2z+1)}\, I. \; ∎$$

Remark 3.10. In [MT2] a notion of spectrally self-similar operators was introduced. We say that an operator H is *spectrally self-similar* with functions φ_0 and φ_1 if there exists an isometry J on $\mathscr{H}$ such that

$$J^*(H-z)^{-1} J = (\varphi_0(z)\, H - \varphi_1(z))^{-1} \tag{3.9}$$

for any $z \in \Lambda$ for which the both sides are well defined bounded linear operators.

In particular, if $\varphi_0(z) \neq 0$ and $R(z) = \varphi_1(z)/\varphi_0(z)$, then

$$J^*(H-z)^{-1} J = \frac{1}{\varphi_0(z)} (H - R(z))^{-1} \tag{3.10}$$

It is clear that operator H is spectrally self-similar if and only if H is spectrally similar to an operator H_0 on a closed subspace $\mathscr{H}_0$ such that $\mathscr{H}_0 = J_0(\mathscr{H})$, $H = J_0^* H_0 J_0$ and $J = J_0$.

One can see that the Laplacian on Sierpiński lattice is spectrally self-similar if and only if $\partial V \neq \varnothing$. This is because the restriction of $\Psi^{-1}_{\mathscr{K}, n+2} \Psi_{\mathscr{K}, n+1}$ to V_{-n} is equal to $\Psi^{-1}_{\mathscr{K}, n+1} \Psi_{\mathscr{K}, n}$ if and only if $k_{n+2} = k_{n+1}$. Therefore, by Lemma 2.3, the condition $\partial V \neq \varnothing$ is necessary and sufficient for the existence of an injection $\Psi_{\mathscr{K}, \infty}$ from V to V which is equal to $\Psi^{-1}_{\mathscr{K}, n+1} \Psi_{\mathscr{K}, n}$ on each V_{-n}. This injection induces an isometry J on ℓ^2 such that

$$Jf(x) = \begin{cases} f(y) & \text{if there exist } n \geqslant 0 \text{ and } y \in V_{-n} \\ & \qquad \text{such that } x = \Psi^{-1}_{\mathscr{K}, n+1} \Psi_{\mathscr{K}, n}(y) \\ 0 & \text{otherwise} \end{cases}$$

(see Example 2.6). Then operator Δ is spectrally self-similar with isometry J and the same functions $\varphi_0(z) = (2z+3)/(2z+1)(4z+5)$ and $\varphi_1(z) = (2z^2+3z)/(2z+1)$. Therefore $R(z) = z(4z+5)$ (see Definition 3.6).

Notation 3.11. We denote the dimension of the space ℓ^2_n by $\mathtt{dim}_n$, and the multiplicity of an eigenvalue z in the spectrum of Δ_n by $\mathtt{mult}_n(z)$.

Notation $R_{-n}A$ is used for the preimage of a set A under the nth composition power of the function R. In our situation $R(z) = z(4z+5)$ by Definition 3.6 and Lemma 3.9.

PROPOSITION 3.12. (i) $\sigma(\Delta_0) = \{0, -\frac{3}{2}\}$.

(ii) *For $n \geqslant 1$ we have*

$$\sigma(\Delta_n) = \{-\tfrac{3}{2}\} \cup \left(\bigcup_{m=0}^{n-1} R_{-m}\{0, -\tfrac{3}{4}\} \right).$$

In particular, for $n \geqslant 2$

$$\sigma(\Delta_n) = \{0, -\tfrac{3}{2}\} \cup \left(\bigcup_{m=0}^{n-1} R_{-m}\{-\tfrac{3}{4}\} \right) \cup \left(\bigcup_{m=0}^{n-2} R_{-m}\{-\tfrac{5}{4}\} \right),$$

and for any $n \geqslant 0$

$$\sigma(\Delta_n) \subset \bigcup_{m=0}^{n} R_{-m}\{0, -\tfrac{3}{2}\}.$$

(iii) *For any $n \geqslant 0$, $\mathtt{mult}_n(0) = 1$.*

(iv) *For any $n \geqslant 0$, $\mathtt{mult}_n(-\frac{3}{2}) = (3^n+3)/2$.*

(v) *If* $z \in R_{-m}\{-\frac{3}{4}\}$ *then* $\mathtt{mult}_n(z) = (3^{n-m-1}+3)/2$ *for* $n \geqslant 1$, $0 \leqslant m \leqslant n-1$.

(vi) *If* $z \in R_{-m}\{-\frac{5}{4}\}$ *then* $\mathtt{mult}_n(z) = (3^{n-m-1}-1)/2$ *for* $n \geqslant 2$, $0 \leqslant m \leqslant n-2$.

The spectrum of Δ_n was computed by T. Shima and M. Fukushima. Nevertheless we present one more proof of these known results as an example of how our formulae can be used in this context.

Proof. The proof is a direct application, by induction, of Theorem 1(i), Theorem 1(ii) and Theorem 1(iii). Note that $\sigma(Q) = \{-\frac{1}{2}, -\frac{5}{4}\}$, $\{z : \varphi_0(z) = 0\} = \{-\frac{3}{2}\}$ and so the exceptional set is $\mathscr{E} = \{-\frac{1}{2}, -\frac{5}{4}, -\frac{3}{2}\}$ (see Definition 3.6). Also note $R_{-1}\{0\} = \{-\frac{5}{4}, 0\}$ and $R_{-1}\{-\frac{3}{2}\} = \{-\frac{3}{4}, -\frac{1}{2}\}$.

Statement (i) is evident. Statement (ii) follows from (iii)–(vi) and Theorem 1 by induction. Statement (iii) is evident or can be derived from Theorem 1(i). Statement (iv) follows from Theorem 1(ii) since $(3^n+3)/2 = \mathtt{dim}_{n-1}$. Statement (v) follows from Theorem 1(i) and statement (iv) by induction since $(3^{n-m-1}-1)/2 = \mathtt{mult}_{n-m-1}(-\frac{3}{2})$. Statement (vi) follows from Theorem 1(iii) since $(3^{n-m-1}-1)/2 = (3^{n-m}+3)/2 + 1 + 2 \cdot 3^{n-m-1}$.

We also can check that all the eigenvalues are found by computing the sum of all multiplicities and comparing it with the dimension of ℓ^2_n. Indeed,

$$\sum_{\{z \in \sigma(\Delta_n)\}} \mathtt{mult}_n(z) = 1 + \frac{3^n+3}{2} + \sum_{m=0}^{n-1} \frac{3^{n-m-1}+3}{2} 2^m + \sum_{m=0}^{n-2} \frac{3^{n-m-1}-1}{2} 2^m$$
$$= \frac{3^{n+1}+3}{2} = \mathtt{dim}\, \ell^2_n. \quad \blacksquare$$

Remark 3.13. Note that only one preimage of $-\frac{3}{2}$ under $R(z)$ is in the spectrum of Δ_n. The expression for this spectrum would have looked much simpler if $-\frac{1}{2}$, the excluded preimage, had not been in the spectrum of Q.

Remark 3.14. On Fig. 3 we show some eigenfunctions of the Laplacian on a Sierpiński pre-gasket. These are the eigenfunctions which give rise to all the localized eigenfunctions of the Laplacian on a Sierpiński lattice and an infinite Sierpiński gasket.

The numbers are the values of an eigenfunction at a corresponding site. If a site has no number assigned, it means that the value of the eigenfunction is zero.

The eigenfunctions similar to the second and the third ones on Fig. 3 are localized on any Sierpiński lattice. The first and the fourth eigenfunctions on this figure are also localized if the origin is the boundary point of the Sierpiński lattice. Note that the fourth eigenfunction is a continuation of the first one.

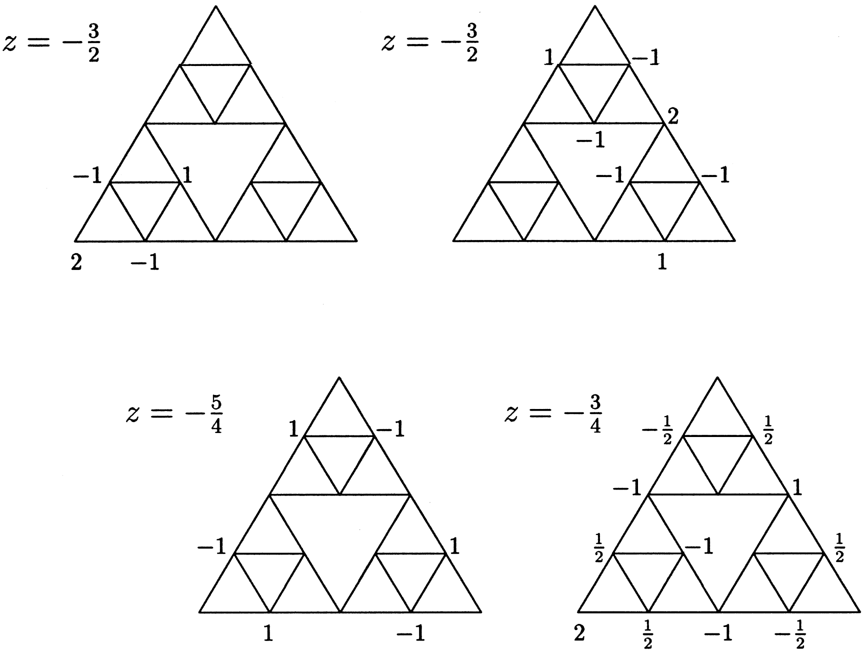

FIG. 3. Examples of eigenfunctions of Δ_2.

4. PURE POINT SPECTRUM OF THE LAPLACIAN ON SIERPIŃSKI LATTICE

THEOREM 2. *The spectrum of* Δ *is pure point and each eigenvalue has infinite multiplicity. The set of eigenvalues of* Δ *is*

$$\Sigma = \{-\tfrac{3}{2}\} \cup \left(\bigcup_{m=0}^{\infty} R_{-m}\{-\tfrac{3}{4}\}\right) \cup \left(\bigcup_{m=0}^{\infty} R_{-m}\{-\tfrac{5}{4}\}\right).$$

Moreover, the set of eigenfunctions of Δ *with finite support is complete in* ℓ^2_n.

The spectrum of Δ *is*

$$\sigma(\Delta) = \mathcal{J} \cup \mathcal{D},$$

where

$$\mathcal{D} \overset{\text{def}}{=} \{-\tfrac{3}{2}\} \cup \left(\bigcup_{m=0}^{\infty} R_{-m}\{-\tfrac{3}{4}\}\right)$$

is the set of isolated eigenvalues and $\mathcal{J}$ is the Julia set of the polynomial R. The set $\mathcal{J}$ is a Cantor set of Lebesgue measure zero and coincides with the set of limit points of $\mathcal{D}$.

The spectrum of Δ as a set was computed by J. Béllissard ([Bé1, Bé2]) and also by T. Shima and M. Fukushima. However the nature of the spectrum and the structure of eigenfunctions have not been known.

This theorem is one of our main results. The proof consists of two parts. First we consider the case when the boundary of V is empty and then the case when it is not. The proof in the first case is simpler and is similar to one given in [MT1]. The difference is that here we have "three point self-similar graph."

Remark 4.1. One can see that all the continuations (possibly of a higher degree) of all the translations of localized eigenfunctions shown on Fig. 3 is a complete set of eigenfunctions of Δ.

Remark 4.2. Note that the spectrum of Δ, i.e., the closed set $\sigma(\Delta)$, does not depend on the sequence $\mathcal{K}$ despite of the fact that the lattices V are not isometric for different sequences $\mathcal{K}$. This independence can be explained by the fact that the localized eigenfunctions are complete, and nonisometric lattices are still *locally* isometric in a sense (except for a corner). Nevertheless, $\sigma(\Delta)$ does not depend on whether V has a corner. Moreover, it is shown in Section 6 that if V does have a corner, the spectrum of the Laplacian with the Dirichlet boundary condition coincides with $\sigma(\Delta)$. However, the spectral decomposition is different, and the localized eigenfunctions are no longer complete.

Notation 4.3. We use standard cycle decomposition notation for S_3, the group of permutations of three elements $\{1, 2, 3\}$. For any $\pi \in S_3$ and $n \geqslant 0$ there exists a unique distance preserving bijection $\pi_n : V_{-n} \to V_{-n}$ such that $\pi_n(\tilde{x}_i) = \tilde{x}_{\pi(i)}$ for each $i = 1, 2, 3$ where $\tilde{x}_i = \Psi^{-1}_{\mathcal{K}, n}(x_i)$ and x_i is the fixed point of Ψ_i (see Definition 2.1). Informally speaking, the isometry π_n permutes the corners of V_{-n} in the same way as π permutes the numbers $\{1, 2, 3\}$ corresponding to these corners. Each π_n induces an isometry $U_{\pi, n}$ on ℓ^2_n, that is $U_{\pi, n} f(x) \stackrel{\text{def}}{=} f(\pi_n(x))$.

LEMMA 4.4. *Let $\ell^2_{a, n} = \{f \in \ell^2_n : U_{\pi, n} f = (-1)^{|\pi|} f, \ \ \pi \in S_3\}$. We can consider this space as a subspace of ℓ^2. Then $\ell^2_{a, n}$ is an invariant subspace of Δ and of any Δ_m, $m \geqslant n$. Any eigenfunction of the restriction $\Delta_n|_{\ell^2_{a, n}}$ is an eigenfunction of Δ and of any Δ_m, $m \geqslant n$. Any eigenvalue of $\Delta_n|_{\ell^2_{a, n}}$ is an eigenvalue of Δ of infinite multiplicity.*

Proof. The proof is evident from the structure of the Sierpiński lattice and the definition of the Laplacian. ∎

Proof of Theorem 2. Case $\partial V = \emptyset$. By Lemma 4.4 we have that $\bigcup_{n \geqslant 0} \ell^2_{a,n}$ is contained in the span of the eigenfunctions of Δ with finite support. Therefore it is enough to show that $\bigcup_{n \geqslant 0} \ell^2_{a,n}$ is complete in ℓ^2.

Let us fix $f \in (\bigcup_{n \geqslant 0} \ell^2_{a,n})^{\perp}$.

Suppose that $k_{n+1} \neq k_{n+2}$. On Fig. 4 we show the location of V_{-n} inside of V_{-n-2} (each small triangle corresponds to a copy of V_{-n}, and the largest triangle corresponds to V_{-n-2}).

Clearly $V_{-n} \cap \partial V_{-n-2} = \emptyset$. Hence $\langle g, U_{\pi, n+3} g \rangle_V = 0$ if support of g is contained in V_{-n} and π is not the identity of S_3. Therefore $\| P_{a,n+3} P_n f \|_V = 1/\sqrt{6}\, \| P_n f \|_V$, where P_n is the orthogonal projector onto the subspace of functions with support in V_{-n} and $P_{a,n}$ is the orthogonal projector onto $\ell^2_{a,n}$, that is $P_n f \overset{\text{def}}{=} f \mathbb{1}_{V_{-n}}$ and $P_{a,n} f \overset{\text{def}}{=} \frac{1}{6} \sum_{\pi \in S_3} (-1)^{|\pi|}\, U_{\pi,n} P_n f$. This implies

$$\begin{aligned} \| P_{a,n+3} f \|_V &= \| P_{a,n+3}(f + P_n f - P_n f) \|_V \\ &\geqslant \| P_{a,n+3} P_n f \|_V - \| f - P_n f \|_V \\ &= \frac{1}{\sqrt{6}} \| P_n f \|_V - \| f - P_n f \|_V. \end{aligned} \tag{4.1}$$

By Lemma 2.3, there exist infinitely many n such that $k_{n+1} \neq k_{n+2}$. Hence (4.1) implies $\limsup_{n \to \infty} \| P_{a,n} f \|_V = 1/\sqrt{6}\, \| f \|_V$. Thus $f = 0$ and the proof is complete in this case. ∎

Proof of Theorem 2. Case $\partial V \neq \emptyset$. Note that according to Lemma 2.3 we can assume, without loss of generality, that $k_j = 1$ for each $j \geqslant 1$. Then we have a simple form of the isometries J_n and J as described in Example 2.6. Also for $\pi = (2, 3) \in S_3$ there is a well defined symmetry map $\pi_\infty : V \to V$ such that for each n we have $\pi_\infty |_{V_{-n}} = \pi_n$ (see Notation 4.3). The map π_∞ is the reflection about a line passing through the origin. It is the only non identity automorphism of V. This symmetry induces a unitary map U_∞ on ℓ^2 such that $U_\infty f = U_{\pi,n} f$ if $\operatorname{supp} f \subset V_{-n}$. Note that no symmetry of V_{-n} can be extended to a symmetry of V in the case when $\partial V = \emptyset$.

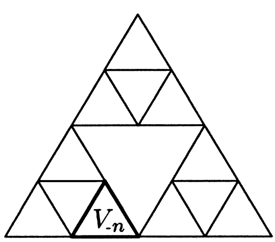

FIG. 4. The location of V_{-n} inside of V_{-n-2} (up to a symmetry).

We denote $\ell^2_S \stackrel{\text{def}}{=} \{f \in \ell^2 : U_\infty f = f\}$ and $\ell^2_{S,n} \stackrel{\text{def}}{=} \ell^2_S \cap \ell^2_n$, the spaces of symmetric functions. Clearly, ℓ^2_S is an invariant subspace of Δ and $(\ell^2_S)^\perp = \{f \in \ell^2 : U_\infty f = -f\}$.

The eigenspace and eigenprojector of Δ_n corresponding to z are denoted by $\mathscr{H}_{n,z}$ and $P_{n,z}$ respectively. Recall that $P_n f \stackrel{\text{def}}{=} f \mathbb{1}_{V_{-n}}$.

Let ℓ^2_{loc} denotes the closed span of the eigenfunctions of Δ with finite support. The theorem is proved if we can show that $\ell^2_{loc} = \ell^2$. We present the proof as a series of several observations.

Observation 1. $(\ell^2_S)^\perp \subseteq \ell^2_{loc}$. This fact is not trivial but the proof of it is very similar to the proof of Theorem 2 in the case $\partial V = \varnothing$.

Observation 2. ℓ^2_{loc} is an invariant subspace of Δ, i.e. $\Delta(\ell^2_{loc}) \subseteq \ell^2_{loc}$.

Observation 3. Let $z \in \mathscr{Z}^n_{loc} \stackrel{\text{def}}{=} \{-\frac{3}{2}\} \cup (\bigcup_{m=0}^{n-2} R_{-m}\{-\frac{3}{4}\}) \cup (\bigcup_{m=0}^{n-2} R_{-m}\{-\frac{5}{4}\})$. If $g \in \mathscr{H}_{n,z}$ then there exists $f \in \mathscr{H}_{n+1,z} \cap \ell^2_{loc}$ such that $P_n f = g$ and $\|f\| \leqslant C_0 \|g\|$ where constant C_0 does not depend on n and z.

Observation 4. $\ell^2_{loc} \subseteq \mathscr{A}^\perp$ where $\mathscr{A} \stackrel{\text{def}}{=} \{f \in \ell^2_S : P_n f \in \bigoplus_{z \in \mathscr{Z}^n_{nonloc}} \mathscr{H}_{n,z}\}$ and $\mathscr{Z}^n_{nonloc} \stackrel{\text{def}}{=} \{0\} \cup R_{-n+1}\{-\frac{3}{4}\}$. This is so because any localized eigenfunction of Δ_n is orthogonal to $\mathscr{H}_{n,z}$ for $z \in \mathscr{Z}^n_{nonloc}$.

Observation 5. $\ell^2_{loc} = \mathscr{A}^\perp$. By Observation 4 it is enough to show that $(\ell^2_{loc})^\perp \cap \mathscr{A}^\perp = \{0\}$. Let $g \in (\ell^2_{loc})^\perp \cap \mathscr{A}^\perp$. Then $P_n g = \sum_{z \in \mathscr{Z}^n_{loc}} P_{n,z} g$. By Observation 3 there exists $f_{n,z}$ such that $\|f_{n,z}\| \leqslant C_0 \|g_{n,z}\|$ and $P_n f_{n,z} = g_{n,z}$. This implies $f \stackrel{\text{def}}{=} \sum_{z \in \mathscr{Z}^n_{loc}} f_{n,z} \in \ell^2_{loc}$, $P_n f = g$ and $\|f\| \leqslant C_0 \|P_n g\|$. Hence $|\langle f, g\rangle| \geqslant |\langle P_n f, P_n g\rangle| - |\langle P_n f - f, P_n g - g\rangle| \geqslant \|P_n g\|^2 - \|P_n f - f\| \, \|P_n g - g\| \geqslant \|P_n g\|^2 - C_0 \|P_n g\| \, \|P_n g - g\| > 0$ for large enough n unless $g = 0$.

Observation 6. ℓ^2_{loc} is an invariant subspace of J, i.e., $J(\ell^2_{loc}) \subseteq \ell^2_{loc}$. This is because $(\ell^2_{loc})^\perp = \mathscr{A}$ and $J^* \mathscr{A} \subseteq \mathscr{A}$ since $J^* \mathscr{H}_{n,z} = \mathscr{H}_{n-1,R(z)}$ for $z \in \mathscr{Z}^n_{nonloc}$.

Observation 7. We have

$$C_1 \stackrel{\text{def}}{=} \max_{z \in \mathscr{I}} \sum_{x \in R_{-1}(z)} \frac{1}{R'(z)\,\varphi_0(z)} = \frac{7}{11}, \tag{4.2}$$

where $\mathscr{I} = [-5/4, -(5+\sqrt{5})/8] \cup [-(5-\sqrt{5})/8, 0]$. Note that the Julia set $\mathscr{J}$ of the polynomial R is a subset of $\mathscr{I}$.

Observation 8. $\mathbb{1}_{\partial V} \in \ell^2_{loc}$. Really, let $z \in \mathscr{Z}^n_{nonloc}$. Then, by (3.3) we have

$$\|P_{n+1,z} \mathbb{1}_{\partial V}\|^2 = \langle P_n P_{n+1,z} P_n \mathbb{1}_{\partial V}, \mathbb{1}_{\partial V}\rangle = \frac{1}{R'(z)\,\varphi_0(z)} \|P_{n,R(z}0\, \mathbb{1}_{\partial V}\|^2.$$

Therefore $\sum_{z \in \mathscr{Z}^{n+1}_{nonloc}} \|P_{n+1,z} \mathbb{1}_{\partial V}\|^2 \leqslant C_1 \sum_{z \in \mathscr{Z}^n_{nonloc}} \|P_{n,z} \mathbb{1}_{\partial V}\|^2$ for large enough n. By induction this implies $P_{\mathscr{A}} \mathbb{1}_{\partial V} = 0$ since, by the definition of $\mathscr{A}$,

$$\|P_{\mathscr{A}} \mathbb{1}_{\partial V}\|^2 = \langle P_n P_{\mathscr{A}} \mathbb{1}_{\partial V}, \mathbb{1}_{\partial V} \rangle = \left\langle \sum_{z \in \mathscr{Z}^n_{nonloc}} P_{n,z} P_{\mathscr{A}} \mathbb{1}_{\partial V}, \mathbb{1}_{\partial V} \right\rangle$$

$$= \left\langle P_{\mathscr{A}} \mathbb{1}_{\partial V}, \sum_{z \in \mathscr{Z}^n_{nonloc}} \mathbb{1}_{\partial V} \right\rangle$$

$$\leqslant \left(2 \sum_{z \in \mathscr{Z}^n_{nonloc}} \|P_{n,z} \mathbb{1}_{\partial V}\|^2 \right)^{1/2} \leqslant const \cdot C_1^{n/2}$$

for any $n \geqslant 1$.

Observation 9. $\ell^2_{loc} = \ell^2$. We give here two different proofs of this observation, which concludes the proof of the theorem. Both proofs use the fact that $\ell^2_0 \subset \ell^2_{loc}$ since, by Observation 2 and Observation 8, $\mathbb{1}_{\partial V} \in \ell^2_{loc}$ and $\Delta \mathbb{1}_{\partial V} \in \ell^2_{loc}$ (Observation 1 is also used here).

The first proof is more geometric. Take any $g \in (\ell^2_{loc})^{\perp} = \mathscr{A}$. Then g is equal to zero on ∂V_{-n} for any $n \geqslant 0$ since $\ell^2_0 \subset \ell^2_{loc} = \mathscr{A}^{\perp}$ and $J\ell^2_{loc} \subset \ell^2_{loc}$. Then there exists a function $f_n \in \ell^2_{loc}$ such that $P_n f_n = P_n g$ and $\|f_n\| = 2\sqrt{2}\, \|P_n g\|$. This implies that $g = 0$ by the same reasoning as in Observation 5.

The construction of f_n is as follows. The set V_{-n-2} is a union of nine copies of V_{-n}. Therefore we can take function $P_n g$ and translate its values to eight out of the nine copies of V_{-n}. Then we change sign to the opposite for every other translated function. We also rotate these copies in such a way that resulting function f_n satisfies our requirements. Informally speaking, we arrange these eight translations in a closed "loop." This construction is shown on Fig. 5. Each small triangle corresponds to a copy of V_{-n}. An arrow "↥" shows where the translation of the origin moves after the rotation. Note that these arrows indicate the axis of symmetry of the values of the function restricted to each small triangle. This function belongs to ℓ^2_{loc} since, for any positive integer k, $\Delta^k f_n \in \ell^2_{n+2}$ because of the form of f_n and the fact that $\mathscr{A}$ is an invariant subspace of Δ.

The second proof of this observation proceeds by induction. Suppose that for some $n \geqslant 0$ we have $\ell^2_n \subset \ell^2_{loc}$. Let $g \in (\ell^2_{loc})^{\perp} = \mathscr{A}$. Then g is zero on V_{-n} and also on all the even (a point $x \in V$ is even if $\frac{1}{2}x \in V$) points of V_{-n-1} since $J\ell^2_n \subset \ell^2_{loc}$ by the induction hypothesis and Observation 6. Hence $P_{n+1} g$ is a linear combination of the eigenfunctions of Q, since $\mathscr{A}$ is an invariant subspace of Δ and any function from $\mathscr{A}$ is zero even points of V_{-n-1}. Let $P_{n+1} g = g_{-1/2} + g_{-5/4}$ where $g_{-1/2}$ and $g_{-5/4}$ are eigenfunctions of Q corresponding to $z = -\frac{1}{2}$ and $z = -\frac{5}{4}$. If x is an even point of V_{-n-1} but not of the boundary of V_{-n-1}, then the sum of the values of

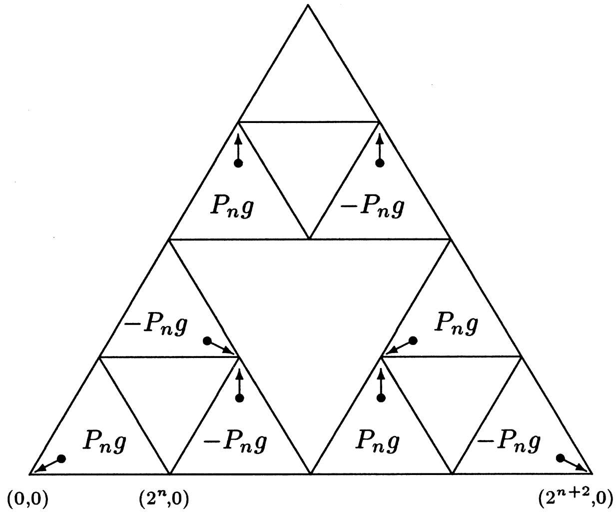

FIG. 5. Construction of the function f_n in Observation 9, Theorem 2.

g at the points adjacent to x is equal to zero, again since $\mathscr{A}$ is invariant. A simple argument (Consider the sequence $\Delta^k g$, $k \geqslant 0$. It is a subset of $\mathscr{A}$ since $\mathscr{A}$ is invariant. It is easy to see that $(-\frac{5}{4})^{-k} \Delta^k g \xrightarrow[k \to \infty]{} g_{-5/4}$ on V_{-n-1}. Then $g_{-1/2}$ is defined as $g - g_{-5/4}$ on V_{-n-1}.) shows that this is true for $g_{-1/2}$ and $g_{-5/4}$ as well. This immediately implies $g_{-1/2} = 0$. Also this implies that, unless $g_{-5/4}$ is zero, it is a localized eigenvalue of Δ. Really, since $g_{-5/4}$ is orthogonal to a constant function, the sum of its values is zero. Thus the sum of the values of $g_{-5/4}$ values is zero, the sum of the values of $g_{-5/4}$ on four points adjacent to any even nonboundary point is zero, and the sum of values of $g_{-5/4}$ on two points adjacent to the origin is zero. Hence the sum of the values of $g_{-5/4}$ at the points adjacent to the boundary of V_{-n-1} is also zero. Therefore $P_{n+1} g = 0$, that is the statement needed for the induction. ∎

5. LAPLACIAN ON AN INFINITE SIERPIŃSKI GASKET

Recall that we defined the Sierpiński gasket as a unique compact set S such that $S = \Psi_1(S) \cup \Psi_2(S) \cup \Psi_3(S)$, where contractions $\Psi_1, \Psi_2, \Psi_3 : \mathbb{R}^2 \to \mathbb{R}^2$ are defined by $\Psi_1(x) = \frac{1}{2}x$, $\Psi_2(x) = \frac{1}{2}x + (\frac{1}{2}, 0)$, $\Psi_3(x) = \frac{1}{2}x + (\frac{1}{4}, \sqrt{3}/4)$. Then we can construct an infinite Sierpiński gasket as follows. Suppose a sequence $\mathscr{K} = \{k_n\}_{n \geqslant 1}$, $k_n \in \{1, 2, 3\}$, is fixed. We define $S_M = \Psi^{-1}_{\mathscr{K}, M}(S)$ where $\Psi_{\mathscr{K}, n} = \Psi_{k_n}, \ldots, \Psi_{k_1}$, $n \geqslant 1$. Then $S_\infty \overset{\text{def}}{=} \bigcup_{M \geqslant 0} S_M$.

The boundary ∂S of S is defined by $\partial S = \{(0,0), (1,0), (1/2, \sqrt{3}/2)\}$, which is the set of fixed points of Ψ_1, Ψ_2 and Ψ_3. Then the boundaries ∂S_M and ∂S_∞ are defined by $\partial S_M \overset{\text{def}}{=} \Psi^{-1}_{\mathcal{K},M}(\partial S)$ and $\partial S_\infty \overset{\text{def}}{=} \bigcap_{N \geqslant 0} \bigcup_{M \geqslant N} \partial S_M$.

Clearly, $S_0 = S$, $S_M \subset S_{M+1}$. It is easy to see that ∂S_M always consists of three points, which are the corners of the largest triangle in S_M, and ∂S_∞ consists of at most one point. Informally, the boundary is the set of all the corners.

The next lemma follows from Lemma 2.3.

Lemma 5.1. (i) *$\partial S_\infty \neq \emptyset$ if and only if there is n_0 such that $k_n = k_{n_0}$ for $n \geqslant n_0$.*

(ii) *Let S'_∞ be an infinite Sierpiński gasket corresponding to a sequence of indices $\mathcal{K}' = \{k'_n\}_{n \geqslant 1}$, $k'_n \in \{1, 2, 3\}$. Then S_∞ is isometric to S'_∞ if and only if there exists a permutation $\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$ and n_0 such that $k_n = k'_{\sigma(n)}$ for $n \geqslant n_0$.*

There is a natural measure μ on S_∞ such that the restriction of μ to S is the standard self-similar probability measure on S (i.e. the normalized Hausdorff measure), and any two isometric sets are of equal measure. Namely, for any small open set $U \subset S_\infty$, $\operatorname{diam} U < \frac{1}{2}$, there exists an open set $\tilde{U} \subset S$ such that U is isometric to $\tilde{U}$. Then $\mu(U) \overset{\text{def}}{=} \mu(\tilde{U})$. Clearly, $\mu(S_M) = 3^M$.

On S there exists a natural (symmetric) Laplacian Δ with Neumann boundary conditions (see [Ki1, Ki2]). This Laplacian is a local operator. Hence it generates a Laplacian Δ_∞ on S_∞, because each point of S_∞ has a neighborhood isometric to a neighborhood of a point in S. There exists a set of continuous functions which are mapped to continuous functions by Δ_∞. It is clear, moreover, that a subset of this set is dense in $L^2(S_\infty, \mu)$. We call operator Δ_∞ the Laplacian on S_∞ (or the Neumann Laplacian if S_∞ has a nonempty boundary).

Theorem 3. *The set of eigenfunctions of Δ_∞ with compact support is complete in $L^2(S_\infty, \mu)$.*

Proof. Let $V_{(n)}$ denote the Sierpiński lattice inside of S_∞ with the distance between adjacent points equal to 2^{-n}. Formally, $V_{(n)} \overset{\text{def}}{=} \bigcup_{M \geqslant 0} \Psi^{-1}_{\mathcal{K},M}(V_{M+n})$, where V_k is defined in Definition 2.1. We denote $V_{(n)}$ and the discrete Laplacian on $V_{(n)}$ by $\Delta_{(n)}$. Then $5^n \Delta_{(n)} f \xrightarrow[n \to \infty]{} \Delta_\infty f$ pointwise for any function f from the domain of Δ_∞ (see [Ki1, Ki2]). Note that $V_{(n)}$ should not be confused with finite graphs V_n defined in Section 4. Also we denote the norm and the scalar product in $\ell^2(V_{(n)})$ by $\|\cdot\|_{(n)}$ and $\langle \cdot, \cdot \rangle_{(n)}$ respectively, and the norm and the scalar product in $L^2(S_\infty, \mu)$ by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ respectively.

Let L^2_{loc} be the closed span of the eigenfunctions of Δ_∞ with compact support. It is enough to prove that any m-harmonic function is in L^2_{loc} (a function is m-harmonic if it is continuous, and is harmonic outside of $V_{(m)}$, see [Ki1, Ki2]). Let $h_{m,p}$, $p \in V_{(m)}$, denote the unique m-harmonic function which is equal to one at p and zero at any other point of $V_{(m)}$. Any m-harmonic function is a linear combination of $\{h_{m,p}, p \in V_{(m)}\}$. Therefore the proof will be complete if we show that $\{h_{m,p}, p \in V_{(m)}\} \subset L^2_{loc}$ for any m.

Let us fix some m and p and denote $h_{m,p}$ by h. The idea of the proof is simple: by Theorem 2 we can represent the restriction of h to $V_{(n)}$ as a linear combination of localized eigenfunctions of the discrete Laplacian on $V_{(n)}$. Then we show that if n is large this representation can "approximate" a representation of h as a linear combination of localized eigenfunctions of Δ_∞. One of the difficulties is that $L^2(S_\infty, \mu)$ and $\ell^2(V_{(n)})$ are two different spaces, neither is a subset of the other. Therefore we can not define a distance between elements of these two spaces.

By direct computation for any $n \geqslant m$

$$\|\Delta_{(n)} h\|^2_{(n)} = C_0\left(\tfrac{9}{25}\right)^{n-m}, \tag{5.1}$$

where C_0 depends on p and m but does not depend on n ($C_0 = \frac{5}{2}$ if $p \in \partial V_{(m)}$, $C_0 = \frac{21}{4}$ if p is adjacent to $\partial V_{(m)}$ and $C_0 = 5$ for any other p).

We know from Theorem 2 that $\Delta_{(n)}$ has pure point spectrum. Let Σ denote the set of eigenvalues of $\Delta_{(n)}$. This set does not depend on n (see Theorem 2). Then for each $\lambda \in \Sigma$ there exists a unique eigenfunction $f_{n,\lambda}$ of $\Delta_{(n)}$, corresponding to this eigenvalue λ, such that

$$h = \sum_{\lambda \in \Sigma} f_{n,\lambda}, \tag{5.2}$$

where the series converges in $\ell^2(V_{(n)})$.

Suppose $\frac{1}{4} > \varepsilon > 0$ is fixed. Then by (5.1) we have

$$\begin{aligned}\Big\| \sum_{\lambda \in \Sigma, \lambda > -\varepsilon} f_{n,\lambda} \Big\|^2_{(n)} &= \sum_{\lambda \in \Sigma, \lambda > -\varepsilon} \langle h, f_{n,\lambda}\rangle_{(n)} \\ &= \sum_{\lambda \in \Sigma, \lambda > -\varepsilon} \langle h, f_{n,\lambda}\rangle^2_{(n)} \|f_{n,\lambda}\|^{-2}_{(n)} \\ &= \sum_{\lambda \in \Sigma, \lambda > -\varepsilon} \frac{1}{\lambda^2} \langle h, \Delta_{(n)} f_{n,\lambda}\rangle^2_{(n)} \|f_{n,\lambda}\|^{-2}_{(n)} \\ &= \sum_{\lambda \in \Sigma, \lambda > -\varepsilon} \frac{1}{\lambda^2} \langle \Delta_{(n)} h, f_{n,\lambda}\rangle^2_{(n)} \|f_{n,\lambda}\|^{-2}_{(n)} \\ &\leqslant \frac{1}{\varepsilon^2} \|\Delta_{(n)} h\|^2_{(n)} = C_0 \frac{1}{\varepsilon^2}\left(\frac{9}{25}\right)^{n-m}.\end{aligned}$$

Denote

$$\tilde{h} = \sum_{\lambda \in \Sigma,\ \lambda > -\varepsilon} f_{n,\lambda}.$$

Then for large enough n (which depends on ε) we have

$$\|\tilde{h} - h\|_{(n)} < \varepsilon. \tag{5.3}$$

We fix $n = n(\varepsilon)$ such that this inequality holds and $n = n(\varepsilon) > k(\varepsilon) + m$. Here $k(\varepsilon) \overset{\text{def}}{=} \min\{l : R_l(-\varepsilon) < -\frac{1}{4}\}$ where R_l is the lth composition power of the polynomial R. Clearly, n tends to infinity as ε tends to zero.

By Theorem 2 eigenfunctions of $\Delta_{(n)}$ with finite support are complete in $\ell^2(V_{(n)})$. Hence there exists a finite set of eigenvalues $\Sigma_n \subset \Sigma \cap [-\varepsilon, 0]$ and corresponding eigenfunctions $\tilde{f}_{n,\lambda}$ with finite support such that

$$\left\| \tilde{h} - \sum_{\lambda \in \Sigma_n} \tilde{f}_{n,\lambda} \right\|_{(n)} < \varepsilon \tag{5.4}$$

and

$$\langle h, \tilde{f}_{n,\lambda} \rangle_{(n)} = \langle f_{n,\lambda}, \tilde{f}_{n,\lambda} \rangle_{(n)} > 0 \tag{5.5}$$

for any $\lambda \in \Sigma_n$.

Let $\delta = -\mathcal{R}_0(-\frac{1}{4})$. Then for any $z \in [-\delta, 0]$ there is a three dimensional space of solutions g of the equation $\Delta_S g = zg$ on the Sierpiński gasket S. Here Δ_S is the Laplacian on the Sierpiński gasket S, that is the same as the "restriction" of Δ_∞ to the interior of S. Each solution is continuous and is uniquely defined by its boundary values. To show this one can either consider the iterations of the continuation procedure (see Notation 3.7), or the fact that the interval $[-\delta, 0]$ is separated from the spectrum of the Dirichlet Laplacian on S. Namely, let G_S be the Green's operator and h_S be a harmonic function on S (see [Ki1, Ki2]). Then $g = \sum_{k=0}^{\infty} z^k G_S^k h_S$ is a unique solution to the equation $\Delta_S g = zg$ which has the same boundary values as h_S. Note that $\delta \|G_S\| < 1$ and so the series $G_{S,z} \overset{\text{def}}{=} \sum_{k=0}^{\infty} z^k G_S^k$ converges absolutely and uniformly on $z \in [-\delta, 0]$. Define a family of these solutions $\mathcal{F} \overset{\text{def}}{=} \{g : \Delta_S g = zg,\ z \in [-\delta, 0],\ \|g\|_{L_S^2} = 1\}$. It is easy to see from the properties of G that the family $\mathcal{F}$ is equicontinuous and so

$$A(l) \overset{\text{def}}{=} \max_{g \in \mathcal{F}} \left| \frac{1}{6 \cdot 3^n} \frac{\|g\|_{(l)}^2}{\|g\|^2} - 1 \right| \xrightarrow[l \to \infty]{} 0. \tag{5.6}$$

Each $\tilde{f}_{n,\lambda}$ has a unique continuation to an eigenfunction $\tilde{f}_{\mathcal{R}_n(\lambda)}$ of Δ_∞ corresponding to an eigenvalue $\mathcal{R}_n(\lambda) \overset{\text{def}}{=} 5^n \lim_{k \to \infty} 5^k R_{-k,1}(\lambda)$. Here $R_{-k,1}(\cdot)$ denotes the branch of the kth inverse composition power of R

that passes through the origin. Let $C(\varepsilon) \stackrel{\text{def}}{=} A(k(\varepsilon)-1)$. Clearly $C(\varepsilon) \to 0$ as $\varepsilon \to 0$. By our assumptions, the restriction of $\tilde{f}_{n,\lambda}$ to $V_{(n-k(\varepsilon)+1)}$ is an eigenfunction of $\Delta_{(n-k(\varepsilon)+1)}$ with the eigenvalue $R_{k(\varepsilon)-1}(\lambda) \in [-\frac{1}{4}, 0]$. Then

$$\left| \frac{1}{6 \cdot 3^n} \frac{\|\tilde{f}_{n,\lambda}\|^2_{(n)}}{\|\tilde{f}_{\mathscr{R}_n(\lambda)}\|^2} - 1 \right| < C(\varepsilon) \qquad \text{and} \qquad \left| \frac{1}{6 \cdot 3^n} \frac{\|h\|^2_{(n)}}{\|h\|^2} - 1 \right| < C(\varepsilon) \tag{5.7}$$

by the definitions of $A(l)$, $k(\varepsilon)$ and $C(\varepsilon)$, and the fact that $\tilde{f}_{n,\lambda}$ is the restriction of $\tilde{f}_{\mathscr{R}_n(\lambda)}$ to $V_{(n)}$.

Also we need another inequality. Direct computations show

$$\langle h, \tilde{f}_{n,\lambda} \rangle_{(n)} = \frac{1}{\lambda} \langle h, \Delta_{(n)} \tilde{f}_{n,\lambda} \rangle_{(n)}$$

$$= \frac{1}{\lambda} \langle \Delta_{(n)} h, \tilde{f}_{n,\lambda} \rangle_{(n)} = \frac{1}{\lambda} \left(\frac{3}{5}\right)^{n-m} \langle \Delta_{(m)} h, \tilde{f}_{n,\lambda} \rangle_{(m)}.$$

In addition, either by taking a limit in this formula or by "Gauss-Green" (integration by parts) formula from Proposition 7.3 of [Ki2] (see also Theorem 1.13 of [KL]), we have

$$\langle h, \tilde{f}_{\mathscr{R}_n(\lambda)} \rangle = \frac{1}{\mathscr{R}_n(\lambda)} \frac{1}{6} \left(\frac{5}{3}\right)^m \langle \Delta_{(m)} h, \tilde{f}_{\mathscr{R}_n(\lambda)} \rangle_{(m)}.$$

Thus

$$\frac{1}{6 \cdot 3^n} \lambda \langle h, \tilde{f}_{n,\lambda} \rangle_{(n)} = \mathscr{R}_0(\lambda) \langle h, \tilde{f}_{\mathscr{R}_n(\lambda)} \rangle. \tag{5.8}$$

It is easy to see that $\mathscr{R}_0(\lambda)/\lambda \to 1$ as $\lambda \to 0$. Denote $B(\varepsilon) \stackrel{\text{def}}{=} \max_{-\varepsilon < \lambda < 0} |\mathscr{R}_0(\lambda)/\lambda - 1|$. Then by (5.8)

$$\left| \frac{1}{6 \cdot 3^n} \frac{\langle h, \tilde{f}_{n,\lambda} \rangle_{(n)}}{\langle h, \tilde{f}_{\mathscr{R}_n(\lambda)} \rangle} - 1 \right| \leqslant B(\varepsilon) \xrightarrow[\varepsilon \to 0]{} 0 \tag{5.9}$$

for any $\lambda \in \Sigma_n$.

Now we can define

$$f_n \stackrel{\text{def}}{=} \sum_{\lambda \in \Sigma_n} \tilde{f}_{\mathscr{R}_n(\lambda)}.$$

Clearly, the eigenfunctions $\operatorname{supp} \tilde{f}_{\mathscr{R}_n(\lambda)}$ have compact support, and so $f_n \in L^2_{loc}$. Moreover, f_n converges to h in $L^2(S_\infty, \mu)$ as ε tends to zero. Indeed, by (5.3), (5.4), (5.5), (5.7) and (5.9) we have that

$$\|h-f_n\|^2$$

$$= \|h\|^2 + \sum_{\lambda\in\Sigma_n} \|\tilde f_{\mathcal{R}_n(\lambda)}\|^2 - 2\sum_{\lambda\in\Sigma_n} \langle h, \tilde f_{\mathcal{R}_n(\lambda)}\rangle$$

$$\leqslant \frac{1}{6\cdot 3^n}\left(\frac{1}{1-C(\varepsilon)}\left(\|h\|^2_{(n)} + \sum_{\lambda\in\Sigma_n}\|\tilde f_{n,\lambda}\|^2_{(n)}\right) - \frac{2}{1+B(\varepsilon)}\sum_{\lambda\in\Sigma_n}\langle h, \tilde f_{n,\lambda}\rangle_{(n)}\right)$$

$$= \frac{1}{6\cdot 3^n}\left(\left\| h - \sum_{\lambda\in\Sigma_n}\tilde f_{n,\lambda}\right\|^2_{(n)} + \frac{C(\varepsilon)}{1-C(\varepsilon)}\left(\|h\|^2_{(n)} + \sum_{\lambda\in\Sigma_n}\|\tilde f_{n,\lambda}\|^2_{(n)}\right)\right.$$

$$\left. + \frac{2B(\varepsilon)}{1+B(\varepsilon)}\sum_{\lambda\in\Sigma_n}\langle h, \tilde f_{n,\lambda}\rangle_{(n)}\right) \xrightarrow[\varepsilon\to 0]{} 0. \quad \blacksquare$$

Remark 5.2. There is another way, proposed by J. Kigami, to describe essentially the same proof. We can define operators $G_{n,z} : \ell^2_{V_{(n)}} \to L^2(S_\infty, \mu) \cap C(S_\infty)$, $z\in[-\delta, 0]$, $n\geqslant 0$, such that $G_{n,z}g(x) = g(x)$ for any $x\in V_{(n)}$ and $\Delta_\infty G_{n,z} g(x) = zg(x)$ outside of $V_{(n)}$. Operators $G_{n,z}$ may be defined in the same way as $G_{S,z}$ above. In the notation of the theorem, $G_{n,0}h = h$ and $G_{n,R_n(\lambda)}\tilde f_{n,\lambda} = \tilde f_{\mathcal{R}_n(\lambda)}$. Recall that $n = n(\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0$. Then one can prove that $\|G_{n,0}h - \sum_{\lambda\in\Sigma_n} G_{n,R_n(\lambda)}\tilde f_{n,\lambda}\| \xrightarrow[\varepsilon\to 0]{} 0$.

Let $\mathcal{R}(z) = \lim_{n\to\infty} 5^n R_{-n,1}(z)$, where $R(z) = z(4z+5)$ and $R_{-k,1}(\cdot)$ denotes the branch of the kth inverse composition power of R that passes through the origin. This limit exists since $R'(0) = 5$. It satisfies $5\mathcal{R}(R_{-n,1}(z)) = \mathcal{R}(z)$.

THEOREM 4. *Operator Δ_∞ is selfadjoint in $L^2(S_\infty, \mu)$. The spectrum of Δ_∞ is pure point and each eigenvalue has infinite multiplicity. The set of eigenfunctions with compact support is complete. The set of eigenvalues is*

$$\Sigma_\infty = \bigcup_{n=0}^{\infty} 5^n\mathcal{R}\{\Sigma\} = \bigcup_{n=-\infty}^{\infty} 5^n\mathcal{R}\{\Sigma\},$$

where

$$\Sigma = \{-\tfrac{3}{2}\} \cup \left(\bigcup_{m=0}^{\infty} R_{-m}\{-\tfrac{3}{4}\}\right) \cup \left(\bigcup_{m=0}^{\infty} R_{-m}\{-\tfrac{5}{4}\}\right)$$

is the set of eigenvalues of operator Δ (see Theorem 2).

$$\sigma(\Delta_\infty) = Closure\left\{\bigcup_{n=0}^{\infty} 5^{-n}\sigma(\Delta)\right\} = Closure\left\{\bigcup_{n=-\infty}^{\infty} 5^n\sigma(\Delta)\right\} = \mathcal{J}_\infty \cup \mathcal{D}_\infty,$$

where

$$\mathcal{D}_\infty \stackrel{def}{=} \bigcup_{n=-\infty}^{\infty} \bigcup_{m=0}^{\infty} 5^n \mathcal{R}\{R_{-m}\{-\tfrac{3}{4}\}\}$$

is the set of isolated eigenvalues and $\mathcal{J}_\infty$ *is the set of limit points of* $\mathcal{D}_\infty$. *Alternatively,*

$$\mathcal{J}_\infty = \bigcup_{n=0}^{\infty} 5^n \mathcal{R}\{\mathcal{J}\} = \bigcup_{n=-\infty}^{\infty} 5^n \mathcal{R}\{\mathcal{J}\},$$

where $\mathcal{J}$ *is the Julia set of the polynomial R. The set* $\mathcal{J}_\infty$ *is a noncompact Cantor set of Lebesgue measure zero.*

Proof. This theorem follows from Theorems 2 and 3. Note that the scaling properties of the operator Δ_∞ easily imply that $\lambda \in \sigma(\Delta_\infty)$ if and only if $5\lambda \in \sigma(\Delta_\infty)$. The argument of Theorem 3 implies that $\lambda \in \sigma(\Delta_\infty)$ if and only if for some m

$$\lambda = 5^m \lim_{n \to \infty} 5^n R_{-n,1}(z_n)$$

were $z_n \in \sigma(\Delta_n)$. ∎

Remark 5.3. The continuations of all the translated and rescaled copies of localized eigenfunctions shown on Fig. 3 is a complete set of eigenfunctions of Δ_∞.

Remark 5.4. Similarly to what was noted in Remark 4.2, the spectrum of Δ_∞, i.e. the closed set $\sigma(\Delta_\infty)$, does not depend on the sequence $\mathcal{K}$ despite of the fact that fractals S_∞ are not isometric for different sequences $\mathcal{K}$. In the absence of a corner this independence can be explained by the fact that the localized eigenfunctions are complete, and all the infinite Sierpiński gaskets are *locally* isometric. Nevertheless, the spectrum does not depend on whether S_∞ has a corner. Moreover, if S_∞ does have a corner, the spectrum does not depend on the boundary condition although the spectral analysis is different. For example, it is shown in the next section that the localized eigenfunctions are not complete.

6. LAPLACIAN WITH ZERO BOUNDARY CONDITION

Notation 6.1. For each $n \geqslant 0$ we define $V_{-n}^{(0)} \stackrel{\text{def}}{=} V_{-n} \backslash \partial V_{-n}$ and also $V^{(0)} \stackrel{\text{def}}{=} V \backslash \partial V$ (see Definition 2.1). Therefore $V_{-n}^{(0)}$ is the finite lattice V_{-n} without the "corner" points.

The Hilbert spaces of complex valued functions on $V^{(0)}$ and $V^{(0)}_{-n}$ are denoted by $\ell^{2(0)}$ and $\ell^{2(0)}_n$ respectively. The norm on these spaces is the same as in ℓ^2 (see Notation 2.4).

The Laplacian on $V^{(0)}$ is defined by

$$\Delta^{(0)} f(x) = \frac{1}{4} \sum_{\{y \in V^{(0)}:\, x \sim y\}} f(y) - f(x).$$

It is a bounded nonpositive selfadjoint operator in $\ell^{2(0)}$. By similar formula, with $V^{(0)}$ replaced by $V^{(0)}_{-n}$, we define Laplacian $\Delta^{(0)}_n$ on $\ell^{2(0)}_n$. Operators $\Delta^{(0)}$ and $\Delta^{(0)}_n$ are called the *Laplacians with zero boundary conditions*, or Dirichlet Laplacians. Note that $\mathtt{deg}_n(x) = 4$ for any $x \in V^{(0)}$ and $\mathtt{deg}_n(x) = 4$ for any $x \in V^{(0)}_{-n}$.

We define an isometry $J^{(0)}_{n,m} : \ell^{2(0)}_n \to \ell^{2(0)}_m$ as the restriction of $J_{n,m}$ to $\ell^{2(0)}_n$ (see Notation 2.5).

LEMMA 6.2. *For each* $n \geqslant 0$ *operator* $\Delta^{(0)}_{n+1}$ *is spectrally similar to the operator* $\Delta^{(0)}_n$ *with isometry* $J^{(0)}_{n,n+1}$ *and functions* $\varphi_0(z) = (2z+3)/(2z+1)(4z+5)$ *and* $\varphi_1(z) = (2z^2+3z)/(2z+1)$.

Proof. Similarly to Lemma 3.9, this lemma follows from a general result proved in [MT2], or it can be proved by direct computation. ∎

Notation 6.3. We denote the dimension of the space $\ell^{2(0)}_n$ by $\mathtt{dim}^{(0)}_n$, and the multiplicity of an eigenvalue z in the spectrum of $\Delta^{(0)}_n$ by $\mathtt{mult}^{(0)}_n(z)$. See Notation 3.11 for the definition of $R_{-n}A$.

The results presented in the following proposition are already known from the works of T. Shima and M. Fukushima.

PROPOSITION 6.4. (i) $\sigma(\Delta^{(0)}_1) = \{-\frac{1}{2}, -\frac{5}{4}\} = \sigma(Q)$.

(ii) *For* $n \geqslant 2$

$$\sigma(\Delta^{(0)}_n) = \{-\tfrac{3}{2}\} \cup R_{-n+1}\{-\tfrac{1}{2}\} \cup \left(\bigcup_{m=0}^{n-1} R_{-m}\{-\tfrac{5}{4}\} \right) \cup \left(\bigcup_{m=0}^{n-3} R_{-m}\{-\tfrac{3}{4}\} \right),$$

(iii) *For any* $n \geqslant 2$, $\mathtt{mult}^{(0)}_n(-\frac{3}{2}) = (3^n - 3)/2$.

(iv) *If* $z \in R_{-n+1}\{-\frac{1}{2}\}$ *then* $\mathtt{mult}^{(0)}_n(z) = 1$ *for* $n \geqslant 1$.

(v) *If* $z \in R_{-m}\{-\frac{5}{4}\}$ *then* $\mathtt{mult}^{(0)}_n(z) = (3^{n-m-1}+3)/2$ *for* $n \geqslant 1$, $0 \leqslant m \leqslant n-1$.

(vi) *If* $z \in R_{-m}\{-\frac{3}{4}\}$ *then* $\mathtt{mult}^{(0)}_n(z) = (3^{n-m-1}-3)/2$ *for* $n \geqslant 3$, $0 \leqslant m \leqslant n-3$.

Proof. One can obtain the proof as a direct application, by induction, of Theorem 1, similarly to the proof of Proposition 3.12. ∎

Throughout this section we assume that $\partial V \neq \emptyset$. Otherwise $V^{(0)} = V$ and $\Delta^{(0)} = \Delta$.

Let $\partial\partial V$ be the set of two points adjacent to the boundary of V. On Fig. 6 we picture the values of the function $\mathbb{1}_{\partial\partial V}$.

We denote by $\mathscr{L}$ the cyclic subspace of $\Delta^{(0)}$ generated by $\mathbb{1}_{\partial\partial V}$, i.e.,

$$\mathscr{L} \stackrel{\text{def}}{=} Span\{(\Delta^{(0)})^n \mathbb{1}_{\partial\partial V}, n \geqslant 0\},$$

and by $\mathscr{L}^{\perp}$ the orthogonal complement to $\mathscr{L}$ in $\ell^{2(0)}$. Note that $\mathscr{L}$ and $\mathscr{L}^{\perp}$ are invariant subspaces of $\Delta^{(0)}$. We denote the restriction of $\Delta^{(0)}$ to $\mathscr{L}$ and $\mathscr{L}^{\perp}$ by $\Delta^{(0)}_{\mathscr{L}}$ and $\Delta^{(0)}_{\mathscr{D}\&\mathscr{N}}$ respectively. Then $\Delta^{(0)}$ is a direct sum of $\Delta^{(0)}_{\mathscr{L}}$ and $\Delta^{(0)}_{\mathscr{D}\&\mathscr{N}}$.

It is easy to see that any function from $\mathscr{L}^{\perp}$ satisfies both Dirichlet and Neumann boundary conditions (in the discrete case Neumann boundary conditions are the same as the absence of boundary conditions). Moreover, $\mathscr{L}^{\perp}$ is the largest invariant subspace of $\Delta^{(0)}$ with this property. Thus the eigenfunctions of $\Delta^{(0)}_{\mathscr{D}\&\mathscr{N}}$ are those eigenfunctions of Δ which satisfy Dirichlet and Neumann boundary conditions at the same time. Thus the operator $\Delta^{(0)}_{\mathscr{D}\&\mathscr{N}}$ may be called the Laplacian with Dirichlet *and* Neumann boundary conditions.

THEOREM 5. (i) *The spectrum of $\Delta^{(0)}_{\mathscr{D}\&\mathscr{N}}$ is pure point and each eigenvalue has infinite multiplicity. Moreover, the set of eigenfunctions of $\Delta^{(0)}_{\mathscr{D}\&\mathscr{N}}$ with finite support is complete in $\mathscr{L}^{\perp}$. The spectrum of $\Delta^{(0)}_{\mathscr{D}\&\mathscr{N}}$ is equal to $\sigma(\Delta)$, and the set of eigenvalues is Σ (see Theorem 2).*

(ii) *The spectrum of $\Delta^{(0)}_{\mathscr{L}}$ is equal to $\mathscr{J}$ (and has multiplicity one). Any element of $\mathscr{L}$ is orthogonal to any localized eigenfunction of $\Delta^{(0)}$. Operator $\Delta^{(0)}_{\mathscr{L}}$ has no eigenfunctions with finite support, and no eigenvalues in Σ.*

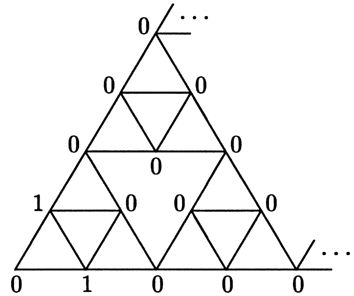

FIG. 6. The values of the function $1_{\partial\partial V}$.

Proof. Let $\mathscr{M}$ be the cyclic subspace of Δ generated by $\mathbb{1}_{\partial V}$, and $\mathscr{M}^{\perp}$ be its orthogonal complement in ℓ^2. It follows from the description of the eigenfunctions of Δ that localized eigenfunctions from $\mathscr{M}^{\perp}$ are complete in $\mathscr{M}^{\perp}$. It is clear that $\mathscr{M}^{\perp} = \mathscr{L}^{\perp}$, and this implies the statement (i).

The absence of the localized eigenfunctions of $\Delta_{\mathscr{L}}^{(0)}$ follows from the examination of the eigenfunctions of $\Delta_n^{(0)}$ which are not orthogonal to $\mathbb{1}_{\partial\partial V}$. The rest of the statement (ii) follows from Proposition 6.4.

Remark 6.5. One can see that all the continuations (possibly of a higher degree) of all the translations of the second and third eigenfunctions shown on Fig. 3 is a complete set of eigenfunctions of $\Delta_{\mathscr{D}\&\mathscr{N}}^{(0)}$.

Remark 6.6. We do not know if the spectrum of $\Delta_{\mathscr{L}}^{(0)}$ is pure point, or singularly continuous, or a mixture of these two types. Although we have some information about a spectral measure of $\Delta_{\mathscr{L}}^{(0)}$ (see Proposition 6.7), we do not know its type. Indeed, $\Delta_{\mathscr{L}}^{(0)}$ does not have absolutely continuous spectrum since $\mathscr{J} = \sigma(\Delta_{\mathscr{L}}^{(0)})$ has Lebesgue measure zero.

Let μ be the spectral measure of $\Delta_{\mathscr{L}}^{(0)}$ associated with the function $\mathbb{1}_{\partial\partial V}$. Then $\operatorname{supp}(\mu) = \mathscr{J}$ and, by the definition of $\mathscr{L}$, μ is the spectral measure of $\Delta^{(0)}$ associated with the function $\mathbb{1}_{\partial\partial V}$.

Proposition 6.7.

$$\frac{d(\mu \circ R_{-1})}{d\mu}(z) = \frac{(2z-1)}{4z(2z+3)}. \tag{6.1}$$

If the polynomial R is one-to-one on an interval $[a, b]$ then on this interval the measure $\mu \circ R$ satisfies the following relation

$$\frac{d(\mu \circ R)}{d\mu}(z) = \frac{(2z+1)(4z+5)(8z+5)}{(2z+3)}. \tag{6.2}$$

Proof. The relation (6.1) follows from (6.2) since

$$\frac{d(\mu \circ R_{-1})}{d\mu}(z) = \sum_{\lambda \in R_{-1}\{z\}} \frac{d\mu}{d(\mu \circ R)}(\lambda)$$

To prove (6.2), let μ_n be the spectral measure of $\Delta_n^{(0)}$ associated with the function $\mathbb{1}_{\partial\partial V}$. Then for $z \notin \mathscr{E}$ relation (3.2), together with some elementary computations, implies

$$\mu_{n+1}\{z\} = \| P_{n+1,z} \mathbb{1}_{\partial\partial V} \|^2$$
$$= \frac{(2z+3)}{(2z+1)(4z+5)(8z+5)} \| P_{n,R(z)} \mathbb{1}_{\partial\partial V} \|^2$$
$$= \frac{(2z+3)}{(2z+1)(4z+5)(8z+5)} \mu_n\{R(z)\}$$

This implies (6.2) since $\mu_n \to \mu$ weakly as $n \to \infty$, all measures μ_n are discrete and $\mu(\mathscr{E}) = 0$. ▮

The next proposition gives certain information about those solutions of the equation $\Delta^{(0)} f = zf$ that are orthogonal to all the localized eigenfunctions of $\Delta^{(0)}$.

PROPOSITION 6.8. *Let points x_n be as shown on Fig. 7, i.e., $x_n = (2^n, 0)$ if V is as in Example* 2.6. *If $f \in \mathscr{L}$ and $\Delta^{(0)} f = zf$ then*

$$f(x_{n+1}) = \frac{1}{\varphi_0(R_n(z))} f(x_n), \tag{6.3}$$

where $\varphi_0(z) = (2z+3)/(2z+1)(4z+5)$ (see Lemma 3.9).

Proof. For $n = 0$ this relation can be verified by an elementary computation. For $n > 0$ the proof follows by induction since if $\Delta^{(0)} f = zf$ then $\Delta^{(0)} J^* f = R(z) J^* f$. Note that $f(x_{n+1}) = J^* f(x_n)$ and, without loss of generality, we can assume that $z \notin \mathscr{E}$. ▮

Remark 6.9. This proposition easily implies that the fixed points of the function $R(z)$, i.e., $z = 0$ and $z = -1$, are not eigenvalues of $\Delta^{(0)}$ since $\varphi_0(0) = \frac{3}{5}$ and $\varphi_0(-1) = -1$. Note that $g(x_n) = g(y_n)$ for any function $g \in \mathscr{L}$.

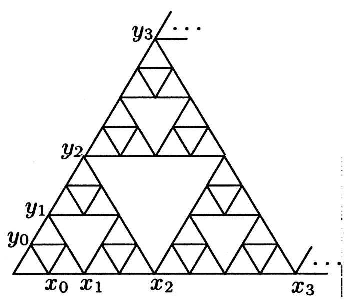

FIG. 7. Notation for Proposition 6.8.

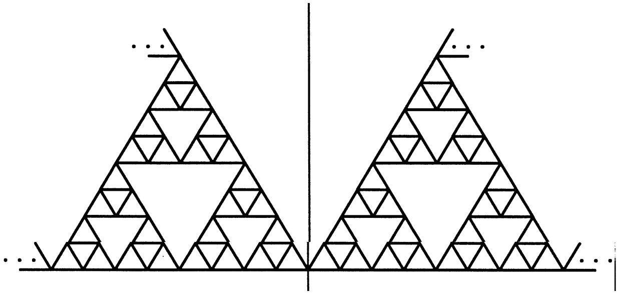

FIG. 8. The structure of the infinite Sierpiński gasket considered by M. T. Barlow and E. A. Perkins in [BP].

The situation is more complicated for the other periodic points of $R(z)$. For example, consider the only orbit of period two, that is $\{-3/4+\sqrt{3}/4, -3/4-\sqrt{3}/4\}$. We have $\varphi_0(-3/4 \pm \sqrt{3}/4) = \pm\sqrt{3}$. Hence if $z = -3/4 \pm \sqrt{3}/4$, $f \in \mathscr{X}$ and $\Delta^{(0)} f = zf$ then the sequence $\{|f(x_n)|\}_{n ¿ 0}$ decreases exponentially fast unless $f = 0$. However Proposition 6.7 implies that these values of z are not eigenvalues of $\Delta^{(0)}$ since $(d(\mu \circ R)/d\mu)(-3/4 \pm \sqrt{3}/4) = 2 \mp \sqrt{3}/3 > 1$.

Remark 6.10. On Fig. 8 we show schematically the structure of the fractal considered in [BP]. Each small triangle represents a copy of the Sierpiński gasket. Although this fractal is not the infinite Sierpiński gasket defined in our paper, it is the union of two copies of an infinite Sierpiński gasket which are joined at the boundary. One can easily see that the Laplacian on this fractal is equivalent to a direct sum of Δ_∞ and $\Delta_\infty^{(0)}$. This direct sum corresponds to the representation of any function as a sum of two functions which are symmetric and skew symmetric with respect to the vertical axis of symmetry.

Remark 6.11. Let $\Delta_\infty^{(0)}$ be the continuous Laplacian on S_∞ with the Dirichlet boundary condition. Then $\sigma(\Delta_\infty^{(0)}) = \sigma(\Delta_\infty)$, and any point of Σ_∞ is an eigenvalue of $\Delta_\infty^{(0)}$ of infinite multiplicity (see Theorem 4). It is easy to show that the set of localized eigenfunctions of $\Delta_\infty^{(0)}$ is a proper subset of the set of localized eigenfunctions of Δ_∞. This fact implies that the set of eigenfunctions of $\Delta_\infty^{(0)}$ with compact support is not complete.

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