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Distributions of Localized Eigenvalues of Laplacians on Post Critically Finite Self-Similar Sets

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In this paper, we study distributions of eigenvalues corresponding to localized eigenfunctions of Laplacians on p.c.f. self-similar sets. Precisely, we divide the eigenvalue counting function $\rho(x)$ of a Laplacian into two parts, $\rho^W(x)$ and $\rho^F(x)$, where $\rho^W(x)$ is the counting function of localized eigenvalues and $\rho^F(x)$ is the counting function of non-localized (global) eigenvalues. We study asymptotic behaviors of $\rho^W(x)$ and $\rho^F(x)$ as $x \to \infty$. It is shown that $\rho^W(x) \asymp x^{d_S/2}$ where d_S is the spectral exponent. On the other hand, for a class of Laplacians, including the standard Laplacian on the Sierpinski gasket, $\rho^F(x) \asymp x^{\kappa_F}$ for some $\kappa_F < d_S/2$. So localized eigenfunctions dominate global eigenfunctions in such cases. © 1998 Academic Press

INTRODUCTION

Analysis on fractals was originated by Kusuoka [Ku1] and Goldstein [G]. They independently constructed a Brownian motion on the Sierpinski gasket. Since then, many works have appeared on diffusion processes, Dirichlet forms, and Laplacians on self-similar sets (in particular, finitely ramified self-similar sets) from both the analytical and probabilistic point of view. For example, Lindstrøm [Li] constructed Brownian motions on nested fractals, which are finitely ramified self-similar sets with strong symmetries. See [BS, Ku2, and Kum] for other examples.

It is known that localized eigenfunctions play a unique role in analysis on self-similar sets. Let Δ be a Laplacian on a space K. (K may be domain of $\mathbb{R}^n$, a smooth manifold, or a self-similar set.) If u is an eigenfunction of Δ and the support of u, denotred by U, is contained in the interior of K,

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then u is called a localized eigenfunction of Δ. If K is a connected domain of $\mathbb{R}^n$ and Δ is the ordinary Laplacian, then obviously there are no such eigenfunctions. In the case of Laplacians on fractals, such eigenfunctions were first observed by physicists Alexander–Orbach [AO], Rammal–Toulouse [RT], and Rammal [R]. Later, from the mathematical point of view, Fukushima–Shima [FS] proved existence of localized eigenfunctions of the standard Laplacian on the Sierpinski gasket defined in [Ki1] by using the eigenvalue decimation method. See Sections 4 and 5.

Existence of localized eigenfunctions leads to "strange" phenomena. For example, let u be a localized eigenfunction, define $f(x, t) = e^{-kt}u(x)$ for $t \geqslant 0$ and $x \in K$, where k is the eigenvalue of $-\Delta : \Delta u = -ku$. Then f becomes a solution of the heat equation $\partial f / \partial t = \Delta f$. The heat corresponding to f will never diffuse outside the support of u, U. Also if $g(x, t) = \cos(\sqrt{k}t)\, u(x)$ for $t \geqslant 0$ and $x \in K$, then g is a solution of the wave equation $\partial g^2 / \partial t^2 = \Delta g$. The energy corresponding to the wave g will remain inside U forever.

In the present paper, we will consider Laplacians on post critically finite self-similar sets (for short, p.c.f. self-similar sets) introduced in [Ki2]. The notion of p.c.f. self-similar sets is a mathematical formulation of "finitely ramified self-similar sets." Nested fractals are a special class of p.c.f. self-similar sets. In Section 1, we will review the definition of p.c.f. self-similar sets and the way of construction of Laplacians on p.c.f. self-similar sets. Also one can find a more detailed review in [Ki5], including the results about localized eigenfunctions. From now on, K is a p.c.f. self-similar set and Δ is a Laplacian on K constructed in [Ki2].

At first, localized eigenfunctions were thought to exist only for a limited class of Laplacians where the eigenvalue decimation method could be applied. Barlow–Kigami [BK] obtained, however, a sufficient condition for existence of localized eigenfunctions without the eigenvalue decimation method. In particular, it was shown that the Laplacian on a nested fractal corresponding to Lindstrøm's Brownian motion has localized eigenfunctions. In fact, they indroduced the notion of a pre-localized eigenfunction, which produces infinitely many localized eigenfunctions and proved the existence of a pre-localized eigenfunction under a kind of weak symmetry condition. See Section 3 for the definition of a pre-localized eigenfunction.

Besides localized eigenfunctions, there exist also ordinary "global" eigenfunctions, whose support are the whole space K. This coexistence of localized and global eigenfunctions gives a unique feature to the spectrum of Δ. It is natural to ask how large the space of localized eigenfunctions is versus global eigenfunctions. This question is the main interest of the present paper. To be more precise, let $\rho(x)$ be the eigenvalue counting function of Δ, that is, $\rho(x) = \sharp\{\text{eigenvalues of} -\Delta \leqslant x\}$. (Of course we need to specify the boundary condition. For now, assume Dirichlet 0-boundary condition.) In [KL1], as one can see in Theorem 2.2, it is shown that there exists $d_S > 0$ such that

$$0 < \liminf_{x \to \infty} \rho(x) x^{-d_S/2} \leqslant \limsup_{x \to \infty} \rho(x) x^{-d_S/2} < \infty.$$

In Section 3, we will divide $\rho(x)$ into two parts $\rho^W(x)$ and $\rho^F(x)$. $\rho^W(x)$ is the counting function of localized eigenvalues, which are eigenvalues corresponding to localized eigenfunctions, and $\rho^F(x)$ is the counting function of global eigenvalues, which are eigenvalues corresponding to global eigenfunctions. (The symbols "W" and "F" stand for "wavelet" and "Fourier", respectively. See Section 3 for an explanation. Of course, $\rho(x) = \rho^W(x) + \rho^F(x)$.

Our main interest in this paper is the asymptotic behavior of $\rho^W(x)$ and $\rho^F(x)$ as $x \to \infty$. If, for example, $\rho^W(x)$ would be going to ∞ faster than $\rho^F(x)$ as $x \to \infty$, we could say that the collection of localized eigenvalues is much larger than that of global eigenvalues. In Theorem 3.5, we will show that if there exists a localized eigenfunction,

$$0 < \liminf_{x \to \infty} \rho^W(x) x^{-d_S/2} \leqslant \limsup_{x \to \infty} \rho^W(x) x^{-d_S/2} < \infty.$$

So the asymptotic order of the counting function of localized eigenvalues is the same as that of all eigenvalues.

The next problem is the asymptotic order of $\rho^F(x)$ as $x \to \infty$. As we will see in Theorem 4.4, for the standard Laplacian on the Sierpinski gaskets, there exists $0 < \kappa_F < d_S/2$ such that

$$0 < \liminf_{x \to \infty} \rho^F(x) x^{-\kappa_F} \leqslant \limsup_{x \to \infty} \rho^F(x) x^{-\kappa_F} < \infty. \tag{I.1}$$

So we may say that localized eigenvalues dominate global eigenvalues in this case. Also, we will see in Theorem 4.5 that the same situation occurs for a Laplacian derived from a strong harmonic structure studied by Shima [Sh2], where the eigenvalue decimation method can be applied. For instance, (I.1) turns out to be true for a class of Laplacians on the Vicsek fractal (Example 4.6) and the modified Koch curve (Example 4.7). Moreover, from these examples, we can observe that κ_W / κ_F, where $\kappa_W = d_S/2$, may be a kind of geometric invariant for a p.c.f. self-similar set. It is conjectured that localized eigenvalues dominate global eigenvalues whenever there is a localized eigenfunction.

There still remain many problems about localized eigenfunctions of Laplacians on p.c.f. self-similar sets. For example, there is no example of a Laplacian which has no localized eigenfunction except for the ordinary

Laplacian on the unit interval. On the other hand, the result about existence of localized eigenfunction in [BK] requires a symmetry condition on both the Laplacian and the self-similar set. This means that at present there is no way to tell whether there exists a localized eigenfunction or not for a non-symmetric case like Laplacians on Hata's tree-like set. (See [KL1, Section 3, Example 4].) Moreover it seems difficult to determine the asymptotic behavior of $\rho^F(x)$ without the eigenvalue decimation method. For instance, we do not know how to evaluate $\rho^F(x)$ for the Laplacian on the Pentakun. (See [Ki5, Example 3.14].) A new method is needed for future study of localized eigenfunctions.

1. LAPLACIANS ON P.C.F. SELF-SIMILAR SETS

In this section, we will briefly review the theory of Laplacians on p.c.f. self-similar sets constructed in [Ki2–Ki4]. For more details, please refer to [Ki5], which is a concise summary of the results in [Ki2–Ki4] and [BK]. The notion of post critically finite (p.c.f.) self-similar sets is a mathematical formulation of so-called finitely ramified self-similar sets.

Definition 1.1. Let K be a compact metrizable topological space and let S be a finite set. In this paper, $S = \{1, 2, ..., N\}$. Also, let F_i, for $i \in S$, be a continuous injection from K to itself. Then, $(K, S, \{F_i\}_{i \in S})$ is called a self-similar structure if there exists a continuous surjection $\pi: \Sigma \to K$ such that $F_i \circ \pi = \pi \circ i$ for every $i \in S$, where $\Sigma = S^{\mathbb{N}}$ is the one-sided shift space and $i: \Sigma \to \Sigma$ is defined by $i(w_1 w_2 w_3 \cdots) = i w_1 w_2 w_3 \cdots$ for each $w_1 w_2 w_3 \cdots \in \Sigma$.

Notation. $W_m = S^m$ is the collection of words with length m. We define $F_w: K \to K$ by $F_w = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m}$ and $K_w = F_w(K)$ for $w = w_1 w_2 \cdots w_m \in W_m$. In particular, $W_0 = \{\emptyset\}$ and $F_\emptyset$ is the identity map. Also we define $W_* = \bigcup_{m \geqslant 0} W_m$.

Definition 1.2. Let $(K, S, \{F_i\}_{i \in S})$ be a self-similar structure. We define the critical set $\mathscr{C} \subset \Sigma$ and the post critical set $\mathscr{P} \subset \Sigma$ by

$$\mathscr{C} = \pi^{-1}\left(\bigcup_{i \neq j} (K_i \cap K_j)\right) \qquad \text{and} \qquad \mathscr{P} = \bigcup_{n \geqslant 1} \sigma^n(\mathscr{C}),$$

where σ is the shift map from Σ to itself defined by $\sigma(\omega_1 \omega_2 \omega_3 \cdots) = \omega_2 \omega_3 \omega_4 \cdots$. A self-similar structure is called post critically finite (p.c.f.) if and only if $\sharp(\mathscr{P})$ is finite.

If $(K, S, \{F_i\}_{i \in S})$ is p.c.f., then K is called a post critically finite self-similar set.

We fix a p.c.f. self-similar structure $(K, S, \{F_i\}_{i \in S})$.

Notation. Let $V_0 = \pi(\mathscr{P})$. For $m \geqslant 1$, set

$$V_m = \bigcup_{w \in W_m} F_w(\pi(\mathscr{P})) \qquad \text{and} \qquad V_* = \bigcup_{m \geqslant 0} V_m.$$

It is easy to see that $V_m \subset V_{m+1}$ and that K is the closure of V_*. In particular, V_0 is thought of as the "boundary" of K.

To construct the theory of harmonic calculus on p.c.f. self-similar sets, we will use some concepts from the theory of electrical networks.

DEFINITION 1.3. For a finite set V, we denote the collection of real-valued functions on V by $\ell(V)$. For a symmetric linear map $H: \ell(V) \to \ell(V)$, we define a symmetric bilinear form $\mathscr{E}_H$ by $\mathscr{E}_H(u, v) = -{}^t uHv$ for $u, v \in \ell(V)$. Then (V, H) is called a resistance network (r-network) if the following three conditions are satisfied.

R.1. $\mathscr{E}_H$ is non-negative definite,

R.2. $\mathscr{E}_H(u, u) = 0$ if and only if u is constant on V,

R.3. $\mathscr{E}_H(u, u) \geqslant \mathscr{E}_H(\bar{u}, \bar{u})$, where $\bar{u}$ is defined by

$$\bar{u}(p) = \begin{cases} 1 & \text{if} \quad u(p) > 1, \\ u(p) & \text{if} \quad 0 \leqslant u(p) \leqslant 1, \\ 0 & \text{if} \quad u(p) < 0. \end{cases}$$

Remark. In previous papers, for example, [Ki4], the condition R.3 of the resistance network was missing. This condition is a kind of Markov property of the form $\mathscr{E}_H$. It is equivalent to $H_{pq} \geqslant 0$ for all $p \neq q \in V$.

DEFINITION 1.4. If (V, H) and (V', H') are r-networks, then write $(V, H) \leqslant (V', H')$ if and only if $V \subset V'$ and, for every $v \in \ell(V)$,

$$\mathscr{E}_H(v, v) = \min\{\mathscr{E}_{H'}(u, u) \colon u \in \ell(V'), u|_V = v\}.$$

Given an r-network (V_0, D) we define a sequence of r-networks $\{(V_m, H_m)\}_{m \geqslant 0}$.

DEFINITION 1.5. Let (V_0, D) be an r-network and let $r = (r_1, r_2, ..., r_N)$ where $r_i > 0$ for $i = 1, 2, ..., N$. We define a symmetric bilinear form $\mathscr{E}^{(m)}$ on $\ell(V_m)$ by $\mathscr{E}^{(m)}(u, v) = \sum_{w \in W_m} r_w^{-1} \mathscr{E}_D(u \circ F_w, v \circ F_w)$ where $r_w = r_{w_1} r_{w_2} \cdots r_{w_m}$ for $w = w_1 w_2 \cdots w_m \in W_m$. The linear map from $\ell(V_m)$ to itself associated with $\mathscr{E}^{(m)}$ is denoted by H_m.

It is easy to see that (V_m, H_m) is an r-network.

DEFINITION 1.6. (D, r) is called a harmonic structure if and only if $(V_0, D) \leqslant (V_1, \lambda H_1)$ for some $\lambda > 0$. Moreover, if $r_i < \lambda$ for all $i = 1, 2, ..., N$, then (D, r) is called a regular harmonic structure.

Replacing $r = (r_1, r_2, ..., r_N)$ by $(r_1/\lambda, r_2/\lambda, ..., r_N/\lambda)$ for a given harmonic structure (D, r), we have $(V_0, D) \leqslant (V_1, H_1)$. Thus we can always renormalize r so that $\lambda = 1$. Note that we then have $(V_m, H_m) \leqslant (V_{m+1}, H_{m+1})$ for all $m \geqslant 0$.

Now we can construct a Dirichlet form on K from a regular harmonic structure. The following results were essentially obtained in [Ki2, Theorem 7.4]. In [Ki3], these results were rephrased in terms of the r-network framework.

THEOREM 1.7. *The following statements* (A) *and* (B) *are equivalent*

(A) *There exists a regular harmonic structure* (D, r) *with* $\lambda = 1$ *and* $(\mathscr{E}, \mathscr{F})$ *is defined by* $\mathscr{F} = \{u \in \ell(V_*) : \lim_{m \to \infty} \mathscr{E}_{H_m}(u|_{V_m}, u|_{V_m}) < \infty\}$ *and* $\mathscr{E}(u, v) = \lim_{m \to \infty} \mathscr{E}_{H_m}(u|_{V_m}, v|_{V_m})$.

(B) *There exists a bilinear form* $\mathscr{E}$ *on* $\mathscr{F} \subset \ell(V_*)$ *that satisfies the following properties.*

B.1. *For any* $u \in \mathscr{F}$, *u has a natural extension to a continuous function of K. In this sense,* $\mathscr{F}$ *is dense in* $C(K)$, *where* $C(K)$ *is the collection of continuous functions on K with the uniform convergent norm.* $\mathscr{E}$ *is symmetric and non-negative.* $\mathscr{F}$ *contains constant functions on K and* $\mathscr{E}(u, u) = 0$ *if and only if u is a constant function on K.*

B.2. *There exists a constant C such that* $C\mathscr{E}(u, u) \geqslant |u(p) - u(q)|^2$ *for any* $p, q \in K$ *and any* $u \in \mathscr{F}$.

B.3. *For all* $f \in \mathscr{F}$, $f \circ F_w \in \mathscr{F}$ *for all* $w \in W_*$. *Moreover, there exists* $0 < r_i < 1$ *for* $i \in \{1, 2, ..., N\}$ *such that, for all* $f, g \in \mathscr{F}$,

$$\mathscr{E}(f, g) = \sum_{i=1}^{N} r_i^{-1} \mathscr{E}(f \circ F_i, g \circ F_i).$$

B.4. *Let* μ *be a finite Borel mearure on K that satisfies* $\mu(O) > 0$ *for any non-empty open set* $O \subset K$. *Then,* $(\mathscr{E}, \mathscr{F})$ *is a regular local Dirichlet form on* $L^2(K, \mu)$.

Remark. The part "(B) implies (A)" is not stated in previous papers. A sketch of a proof is: By B.1, B.2, and B.4, we can see that $(\mathscr{E}, \mathscr{F})$ is a finite resistance form on K. (See [Ki4] for the definition of a finite resistance form.) By the results in [Ki4], there exist a sequence of r-networks $\{(V_m, H_m)\}_{m \geqslant 0}$ satisfying $(V_m, H_m) \leqslant (V_{m+1}, H_{m+1})$ for all $m \geqslant 0$ and $(\mathscr{E}, \mathscr{F})$ is given by the formula in (A). Set $D = H_0$, then by B.3, it follows that (D, r) is a regular harmonic structure.

Let μ be a measure on K satisfying the condition in B.4 of Theorem 1.7. We now give a direct definition of the Laplacian associated with $(\mathscr{E}, \mathscr{F}, \mu)$, as a scaled limit of the discrete Laplacians H_m on V_m.

DEFINITION 1.8. For $p \in V_m$, let $\psi_{m,p}$ be the unique function in $\mathscr{F}$ that attains the following minimum: $\min\{\mathscr{E}(u,u) : u \in \mathscr{F}, u(p)=1, u(q)=0$ for $q \in V_m \setminus \{p\}\}$. For $u \in C(K)$, if there exists $f \in C(K)$ such that

$$\lim_{m \to \infty} \max_{p \in V_m \setminus V_0} |\mu_{m,p}^{-1}(H_m u)(p) - f(p)| = 0,$$

where $\mu_{m,p} = \int_K \psi_{m,p} \, d\mu$, then we define the μ-Laplacian Δ_μ by $\Delta_\mu u = f$. The domain of Δ_μ is denoted by $\mathscr{D}_\mu$.

THEOREM 1.9 (Gauss–Green formula).

(a) *The domain* $\mathscr{D}_\mu \subset \mathscr{F}$, *and the Neumann derivative on the boundary, defined by* $(dv)_p = \lim_{m \to \infty} -(H_m v)(p)$, *exists for* $v \in \mathscr{D}_\mu$, $p \in V_0$.

(b) *For* $u \in \mathscr{F}$ *and* $v \in \mathscr{D}_\mu$,

$$\mathscr{E}(u,v) = \sum_{p \in V_0} u(p)(dv)_p - \int_K u \Delta_\mu v \, d\mu.$$

2. DISTRIBUTIONS OF EIGENVALUES OF LAPLACIANS

In this section, we will introduce results about distributions of eigenvalues of Laplacians on p.c.f. self-similar sets. In this and the next section, $(K, S, \{F_i\}_{i \in S})$ is a p.c.f. self-similar structure with $S = \{1, 2, ..., N\}$ and (D, r) is a regular harmonic structure where $r = (r_1, r_2, ..., r_N)$. Also we only consider a Bernoulli measure μ on K. If $\mu_i > 0$ for all $i \in S$ and $\sum_{i=1}^N \mu_i = 1$, there exists a unique Borel probability measure on K that satisfies $\mu(K_w) = \mu_{w_1}\mu_{w_2} \cdots \mu_{w_m}$ for any $w = w_1 w_2 \cdots w_m \in W_m$ and any $m \geqslant 0$. Such a measure μ is called a Bernoulli measure on K.

DEFINITION 2.1 (Eigenvalues and Eigenfunctions). *For* $k \in \mathbb{R}$, *we define*

$$E_D(k) = \{u : u \in \mathscr{D}_\mu, \Delta_\mu u = -ku, u|_{V_0} = 0\}.$$

If $\dim E_D(k) \geqslant 1$, *then* k *is called a Dirichlet eigenvalue* (D-*eigenvalue for short*) *of* $-\Delta_\mu$ *and* $u \in E_D(k)$ *is said to be a Dirichlet eigenfunction* (D-*eigenfunction for short*) *belonging to the* D-*eigenvalue* k. *Also, for* $k \in \mathbb{R}$, *we define*

$$E_N(k) = \{u : u \in \mathscr{D}_\mu, \Delta_\mu u = -ku, (du)_p = 0 \text{ for all } p \in V_0, \}.$$

If $\dim E_N(k) \geqslant 1$, *then* k *is called a Neumann eigenvalue* (N-*eigenvalue for short*) *of* $-\Delta_\mu$ *and* $u \in E_N(k)$ *is said to be a Neumann eigenfunction* (N-*eigenfunction for short*) *belonging to the* N-*eigenvalue* k.

It is known that the D-eigenvalues (and also N-eigenvalues) are non-negative, of finite multiplicity and the only accumulation point is ∞. Hence if we let $\rho_*(x, \mu) = \sum_{k \leqslant x} \dim E_*(k)$ for $* = D$ or N, $\rho_*(x, \mu)$ is well-defined and $\rho_*(x, \mu) \to \infty$ as $x \to \infty$. $\rho_*(x, \mu)$ is called the eigenvalue counting function.

THEOREM 2.2 ([KL1, THEOREM 2.4]). *Let* d_s *be the unique real number* d *that satisfies* $\sum_{i=1}^N \gamma_i^d = 1$, *where* $\gamma_i = \sqrt{r_i \mu_i}$. *Then*

$$0 < \liminf_{x \to \infty} \rho_*(x, \mu)/x^{d_s/2} \leqslant \limsup_{x \to \infty} \rho_*(x, \mu)/d^{d_s/2} < \infty$$

for $* = D, N$. d_s *is called the spectral exponent of* $(\mathscr{E}, \mathscr{F}, \mu)$. *Moreover*:

(1) *Non-Lattice Case*: *If* $\sum_{i=1}^N \mathbb{Z} \log \gamma_i$ *is a dense subgroup of* $\mathbb{R}$, *then the limit* $\lim_{x \to \infty} \rho_*(x, \mu)/x^{d_s/2}$ *exists.*

(2) *Lattice Case*: *If* $\sum_{i=1}^N \mathbb{Z} \log \gamma_i$ *is a discrete subgroup of* $\mathbb{R}$, *let* $T > 0$ *be its generator. Then*, $\rho_*(x, \mu) = (G(\log x/2) + o(1)) x^{d_s/2}$, *where* G *is a* (*right-continuous*) T-*periodic function with* $0 < \inf G(x) \leqslant \sup G(x) < \infty$ *and* $o(1)$ *is a term which vanishes as* $x \to \infty$.

Remark. More concrete expressions for the value of the limit in the non-lattice case and the function G in the lattice case are obtained in [KL1]. In particular, these limits are independent of boundary conditions.

For the lattice case, we get a more detailed version of the above result.

THEOREM 2.3. *Under the assumptions and the notations of the lattice case of Theorem* 2.2, *set* $Q(z) = (1 - \sum_{i=1}^N (z/p)^{m_i})/(1 - z)$, *where* $p = e^{d_s T}$ *and* $m_i = -\log \gamma_i / T$ *for all* i. *Also define* $\beta = \min\{|z| : Q(z) = 0\}$ *and* $m = \max\{$*multiplicity of* $Q(z) = 0$ *at* $w : |w| = \beta, Q(w) = 0\}$. *Then as* $x \to \infty$,

$$\rho_*(x, \mu) = G(\log x/2) x^{d_s/2} + \begin{cases} O((\log x)^{m-1} x^{(d_s - (\log \beta)/T)/2}) & \text{if} \quad p > \beta, \\ O((\log x)^m) & \text{if} \quad p = \beta, \\ O(1) & \text{if} \quad p < \beta. \end{cases}$$

Remark 1. If $Q(z) \equiv 1$, set $\beta = +\infty$. This happens if, and only if $\gamma_1 = \cdots = \gamma_N$. Then by the above theorem, as $x \to \infty$,

$$\rho_*(x, \mu) = G(\log x/2) x^{d_s/2} + O(1).$$

This includes the case of the standard Laplacian on the Sierpinski gasket, which is considered extensively in the latter half of this paper. Moreover, it covers the case of Laplacians corresponding to Lindstrøm's Brownian motions on nested fractals [Li].

Remark 2. As $p_i^{-m_i} = \gamma_i^{d_S}$, it follows that $1 = \sum_{i=1}^N p^{-m_i}$. Hence $Q(z)$ is a polynomial. Also as $|\sum_{i=1}^N (z/p)^{m_i}| < 1$ on $\{z : |z| \leqslant 1, z \neq 1\}$, we can see that $\beta > 1$.

Theorem 2.3 will be proven in the appendix by using an extended version of the renewal theorem.

3. LOCALIZED EIGENFUNCTIONS

Now we can introduce the main subject of this paper: localized eigenfunctions. In this section, we will introduce a notion of (pre-)localized eigenfunctions and consider eigenvalue counting functions for localized eigenvalues.

It is known that the existence of localized eigenfunctions is related to the continuity of G in Theorem 2.2 for the lattice case.

THEOREM 3.1 [BK, Theorem 4.4]. *$u \in \mathcal{D}_\mu$ is said to be a pre-localized eigenfunction of $-\Delta_\mu$ if u is both a Dirichlet and Neumann eigenfunction for a (Dirichlet and Neumann) eigenvalue. For the lattice case, there exists a pre-localized eigenfunction of $-\Delta_\mu$ if and only if G is discontinuous.*

From a pre-localized eigenfunction, one can produce infinitely many localized eigenfunctions.

PROPOSITION 3.2 [BK, Lemma 4.2]. *For $w \in W_*$ and $f: K \to \mathbb{R}$, $S_w(f)$: $K \to \mathbb{R}$ is defined by*

$$S_w(f)(x) = \begin{cases} f(F_w^{-1}(x)) & \text{if} \quad x \in K_w, \\ 0 & \text{otherwise}. \end{cases}$$

For a pre-localized eigenfunction u, let $u_w = S_w(u)$, then u_w is also a pre-localized eigenfunction belonging to the eigenvalue $k/(r_w \mu_w)$, where $\mu_w = \mu_{w_1} \mu_{w_2} \cdots \mu_{w_m}$ for $w = w_1 w_2 \cdots w_m$.

Note that the support of u_w is contained in K_w. In this sense, u_w is a localized eigenfunction.

DEFINITION 3.3. We define $E^W(k) = E_D(k) \cap E_N(k)$ and $E_*^F(k) = E_*(k) \cap E^W(k)^\perp$ for $* = D, N$. We also define the corresponding eigenvalue counting functions as follows:

$$\rho^W(x, \mu) = \sum_{k \leqslant x} \dim E^W(k) \quad \text{and} \quad \rho_*^F(x, \mu) = \sum_{k \leqslant x} \dim E_*^F(k).$$

$E^W(k)$ is the space of pre-localized eigenfunctions with an eigenvalue k. Also $E_*^F(k)$ can be thought of the space of non-localized (global) eigenfunctions with eigenvalue k.

The "W" letter in $E^W(\cdot)$ and $\rho^W(\cdot, \cdot)$ represents the "W" of wavelets. For a pre-localized eigenfunction u, we get a sequence of (pre-)localized eigenfunctions $\{S_w(u)\}_{w \in W_*}$ which are mutually orthogonal in $L^2(K, \mu)$. Although this sequence is not a complete system of $L^2(K, \mu)$, the way of the construction is exactly same as that of a wavelet basis. Actually, in the following definition, we will construct a complete orthogonal system of a subspace of $L^2(K, \mu)$, E^W, by this method. This is the reason why we use the letter "W" for notations corresponding to localized eigenfunctions.

On the other hand, the "F" letter in $E_*^F(\cdot)$ and $\rho_*^F(\cdot, \cdot)$ represents the "F" of Fourier. The ordinary Fourier basis of $L^2([0, 1], dx)$ is the collection of non-localized eigenfunctions of the Laplacian on $[0, 1]$. By analogy with this fact, we use "F" for notations corresponding to non-localized eigenfunctions.

It is easy to see that $\rho_*(x, \mu) = \rho^W(x, \mu) + \rho_*^F(x, \mu)$.

Note that $S_j(E^W(\mu_j r_j k)) \subset E^W(k)$. The eigenfunctions in $S_j(E^W(\mu_j r_j k))$ are thought to be offsprings of preceding eigenfunctions in $E^W(\mu_j r_j k)$. From such an observation, we can divide $E^W(k)$ into offsprings $E_2^W(k)$ and generators $E_1^W(k)$.

DEFINITION 3.4.

$$E_2^W(k) = \bigoplus_{i=1}^{N} S_i(E^W(k\mu_i r_i)) \qquad \text{and} \qquad E_1^W(k) = (E_2^W(k))^\perp \cap E^W(k).$$

Now we can choose k_i^W and $\varphi_i \in E_1^W(k_i^W)$ for $i \geqslant 1$ so that $k_i^W \leqslant k_{i+1}^W$ and $\{\varphi_i\}_{i=1}^\infty$ is a complete orthonormal system of $E_1^W = \overline{\bigoplus_k E_1^W(k)}$. Then $\{\varphi_{i,w} \mid i \geqslant 1, w \in W_*\}$ is a complet orthonormal system of $E^W = \overline{\bigoplus_k E^W(k)}$, where $\varphi_{i,w} = (\mu_w)^{-1/2} S_w(\varphi_i)$. Note that $\varphi_{i,w} \in E_2^W(k_i^W/(\mu_w r_w))$ if $w \notin W_0$.

The following theorem is one of main results of this paper.

THEOREM 3.5. *Assume that there exists, at least, one pre-localized eigenfunction.*

(1) *For the lattice case, as* $x \to \infty$,

$$\rho^W(x, \mu) = (G^W(\log x/2) + o(1))\, x^{d_s/2}$$
$$\rho^F_*(x, \mu) = (G^F(\log x/2) + o(1))\, x^{d_s/2}$$

where G^W *is discontinuous T-periodic function,* $0 < \inf G^W(x) < \sup G^W(x) < \infty$ *and* G^F *is a non-negative continuous T-periodic function.* (G^F *is independent of boundary conditions* $* = D$ *or* N.) *Of course,* $G = G^W + G^F$.

(2) *For the non-lattice case,* $\lim_{x \to \infty} \rho^W(x, \mu)/x^{d_S/2}$ *and* $\lim_{x \to \infty} \rho^F_*(x, \mu) x^{d_S/2}$ *exist.*

(3) *Define*

$$c_W = \begin{cases} \dfrac{1}{T} \displaystyle\int_0^T G^W(t)\, dt & \text{for the lattice case} \\ \displaystyle\lim_{x \to \infty} \rho^W(x, \mu)/x^{d_S/2} & \text{for the non-lattice case,} \end{cases}$$

then $\sum_{i=1}^{\infty} (k_i^W)^{-d_S/2} < +\infty$ *and*

$$c_W = \left(- \sum_{i=1}^{N} p_i \log p_i \right)^{-1} \sum_{j=1}^{\infty} (k_j^W)^{-d_S/2},$$

where $p_i = \gamma_i^{d_S}$ *for* $i = 1, 2, ..., N$.

Remark. By the definition of d_S in Theorem 2.2, $\sum_{i=1}^N p_i = 1$. Therefore, the value $-\sum_{i=1}^N p_i \log p_i$ is a kind of entropy.

CONJECTURE 3.6. *Assume there exists a pre-localized eigenfunction. Then* $G^F(x) \equiv 0$ *for the lattice case and* $\lim_{x \to \infty} \rho^F_*(x, \mu)/x^{d_S/2} = 0$ *for the non-lattice case. Moreover,*

$$0 < \liminf_{x \to \infty} \rho^F_*(x, \mu)/x^{\kappa_F} \leqslant \limsup_{x \to \infty} \rho^F_*(x, \mu)/x^{\kappa_F} < \infty$$

for some $0 < \kappa_F < d_S/2$.

In the next section, we will show that the above conjecture is true for the standard Laplacians on the Sierpinski gaskets in Theorem 4.4. Also the same method works for Laplacians derived from strong harmonic structures defined by Shima [Sh2]. See Theorem 4.5 for the details. Moreover, through examples in Section 4, we will be able to observe that κ_W/κ_F is a kind of invariant constant where $\kappa_W = d_S/2$.

Remark. In [La], Lapidus defined the notion of a volume measure associated with a Laplacian Δ_μ. In a subsequent paper [KL2], it is shown that the volume measure become a Bernoulli measure if the above conjecture is true.

The rest of this section is devoted to the proof of Theorem 3.5. First, we will discuss the lattice case. Recall the definitions of p and m_i in Theorem 2.3. If $\rho_i(x) = \sharp\{w : k_i^W/(\mu_w r_w) \leqslant x\}$, the it follows that $\sum_i \rho_i(x) = \rho^W(x, \mu)$. We also define $G_n^i(t) = \rho_i(e^{2(t+nT)}) e^{-d_S t} p^{-n}$ and $G_n^W(t) = \rho^W(e^{2(t+nT)}, \mu) e^{-d_S t} p^{-n}$.

Lemma 3.7. *Set $k_i^W = e^{2(t_i + n_i T)}$, where $t_i \in (0, T]$ and $n_i \in \mathbb{Z}$. Then*

$$G_n^i(t) = e^{-d_S t} p^{-n} \begin{cases} \sum_{j=-\infty}^{n-n_i-1} M(j) & (0 \leqslant t < t_i) \\ \sum_{j=-\infty}^{n-n_i} M(j) & (t_i \leqslant t \leqslant T), \end{cases}$$

where $M(n) = \sharp\{w = w_1 w_2 \cdots w_k : \sum_{i=1}^k m_{w_i} = n\}$ for $n \in \mathbb{Z}$.

Remark. In fact, $M(n) = 0$ for $n < 0$ and $M(0) = 1$. Hence $\sum_{-\infty}^n M(j) = \sum_{j=0}^n M(j)$.

By Lemma 4.6 of [BK], using the renewal theorem, we have

$$\lim_{x \to \infty} M(n)/p^n = \left(\sum_{i=1}^N m_i p^{-m_i} \right)^{-1}.$$

Using this fact, we can easily see

Lemma 3.8. *Define $G^i(t)$ for $0 \leqslant t \leqslant T$ by*

$$G^i(t) = e^{-d_s(t-t_i)} L^{-1} (k_i^W)^{-d_S/2} \times \begin{cases} (p-1)^{-1} & (0 \leqslant t < t_i) \\ p(p-1)^{-1} & (t_i \leqslant t \leqslant T)' \end{cases}$$

where $L = \sum_{i=1}^N m_i p^{-m_i}$. Then G_n^i converges uniformly to G^i on $[0, T]$ as $n \to \infty$. Moreover, let $\alpha = \sup_{n \geqslant 0} \sum_{j=0}^n M(j)/p^n$ then $G_n^i(t) \leqslant \alpha (k_i^W)^{-d_S/2}$ for all $n \in \mathbb{N}$, i and $t \in [0, T]$.

By the fact that $\sum_i G^i(t) \leqslant G(t)$, we have $\sum_i (k_i^W)^{-d_S/2} < \infty$. Hence $\sum_i G^i(t)$ converges uniformly on $[0, T]$. Now set $G^W(t) = \sum_i G^i(t)$. As $|G_n^W(t) - G^W(t)| \leqslant \sum_i |G_n^i(t) - G^i(t)|$, we can easily see that G_n^W converges uniformly to G^W on $[0, T]$ as $n \to \infty$. Obviously G^W is discontinuous and $0 < \inf G^W(t) < \sup G^W(t) < \infty$.

Next let $G^F(t) = G(t) - G^W(t)$ then $G^F(t)$ is non-negative. Making use to the same discussion as in the proof of Theorem 4.4 of [BK], we can see that a discontinuous point of G^F implies a pre-localized eigenfunction in

$E_*^F(k)$ for some k. This is a contradiction and hence G^F is continuous. Thus we have completed the proof of (1) of Theorem 3.5.

As we know the concrete forms of the G^W, it is easy to calculate c_W by using the fact that $\int_0^T G^W(t) = \sum_i \int_0^T G^i(t)$. Hence we can confirm (3) of Theorem 3.5 for the lattice case.

Similar discussions imply the results for the non-lattice case of Theorem 3.5.

4. THE STANDARD LAPLACIAN ON THE SIERPINSKI GASKET

In this section, we will consider the case of the standard Laplacian on the Sierpinski gasket, where, by virtue of eigenvalue decimation method, asymptotic behaviors of $\rho^W(x)$ and $\rho_*^F(x)$ can be determined explicitly. In particular, Conjecture 3.6 is verified for this case.

DEFINITION 4.1 (The Sierpinski Gasket). *Let $\{p_1, p_2, ..., p_N\}$ be a set of vertices of an N-simplex in $\mathbb{R}^{N-1}$: $|p_i - p_j| = 1$ for $1 \leqslant i < j \leqslant N$. Set $F_i(x) = (x - p_i)/2 + p_i$ for all i. The N-Sierpinski gasket K is the unique non-empty compact subset of $\mathbb{R}^{N-1}$ that satisfies $K = F_1(K) \cup \cdots \cup F_N(K)$. Then $(K, S, \{F_i\}_{i \in S})$ is a p.c.f. self-similar structure. We can easily see that $V_0 = \{p_1, p_2, ..., p_N\}$.*

For $N=2$, K is an interval $[p_1, p_2]$. For $N=3$, K is the ordinary Sierpinski gasket. The Hausdorff dimension (with respect to the Euclidian metric on $\mathbb{R}^{N-1}$) is known to be log $|N/\log|$ 2.

PROPOSITION 4.2. *If $D_{p_i p_j} = 1$ for $i \neq j$ and $D_{p_i p_i} = -(N-1)$ for all i, then (V_0, D) is an r-network, where $D = (D_{p_i p_j})_{1 \leqslant i, j \leqslant N}$. Furthermore, if $r_i = N/(N+2)$ for all i and $r = (r_1, r_2, ..., r_N)$, then (D, r) is a regular harmonic structure with $\lambda = 1$.*

Let ν be a Bernoulli measure on K that satisfies $\nu_i = 1/N$ for all $i \in S$. Essentially, we will consider the Laplacian Δ_ν on K associated with the regular harmonic structure (D, r) and ν. Combining the constructions in Section 1, we can define $\Delta = (N/2)\Delta_\nu$ directly as follows.

DEFINITION 4.3 (The Standard Laplacian). For $u \in C(K)$, if there exists $f \in C(K)$ such that

$$\lim_{x \to \infty} \max_{p \in V_m \setminus V_0} |(N+2)^m H_{m,p} u - f(p)| = 0$$

where $H_{m,p} u = \sum_{p \in V_{m,p}} (u(q) - u(p))$ and $V_{m,p} = \bigcup_{w \in W_m, p \in F_w(V_0)} F_w(V_0) \setminus \{p\}$ (which is the set of neighboring vertices of p in V_m). Then we define Δ by $\Delta u = f$. The domain of Δ is denoted by $\mathscr{D}$.

Δ is called the standard Laplacian on the N-Sierpinski gasket K. We will use $\rho_*(x)$, $\rho^W(x)$ and $\rho_*^F(x)$ to denote eigenvalue counting functions for the standard Laplacian Δ. The asymptotic behavior of those eigenvalue counting functions as $x \to \infty$ can be determined by using the eigenvalue decimation method.

THEOREM 4.4. *For $N \geqslant 3$, set $T = \log(N+2)/2$, then*

$$\rho^W(x) = G(\log x/2)\, x^{d_S/2} - P(\log x/2)\, x^{\kappa_F} + O(\log x)$$

and

$$\rho_*^F(x) = P(\log x/2)\, x^{\kappa_F} + O(\log x),$$

where G is a discontinuous T-periodic function with $0 < \inf G < \sup G < +\infty$, P is a positive continuous T-periodic function, $d_S = 2\log N/\log(N+2)$ and $\kappa_F = \log 2/\log(N+2)$.

Remark. G and P are independent of the boundary conditions. P is essentially a Cantor-type function for a non-linear Cantor set divided by an exponential function. Please see Section 7, in particular Theorem 7.7, for details. Also a more concrete form of G is given in Theorem 7.10.

The proof of Theorem 4.4 is given in Sections 5–7, where the eigenvalue decimation method plays an essential role.

Furthermore, Shima [Sh2] introduced the notion of strong harmonic structures, where the eigenvalue decimation method works. For strong harmonic structures, we can apply the same method as in Sections 5, 6, and 7 and obtain a similar result as Theorem 4.4. In particular, we may also verify Conjecture 3.6 for these cases.

Let (D, r) be a regular harmonic structure with $\lambda = 1$ on a p.c.f. self-similar set K. Assume that (D, r) is also a strong harmonic structure. (Please see [Sh2] for the definition of strong harmonic structures.) Let ν be a Bernoulli measure on K that satisfies $\nu_i = (r_i)^{-1}/(\sum_{j=1}^N (r_j)^{-1})$. Note that $r_i \nu_i = (\sum_{j=1}^N (r_j)^{-1})^{-1}$ is independent of i. Define $n_0 = (r_i \nu_i)^{-1}$. It is shown in [Sh2] that the eigenvalue decimation method can be applied to study eigenvalues and eigenfunctions of the ν-Laplacian Δ_ν on K associated with (D, r).

THEOREM 4.5. *If $n_R < N$, where n_R is defined in [Sh2], then there exists a pre-localized eigenfunction of Δ_ν. Set $\kappa_F = \log n_R/\log n_0$,*

$$0 < \liminf_{x \to \infty} \rho_*^F(x, \nu)/x^{\kappa_F} \leqslant \limsup_{x \to \infty} \rho_*^F(x, \nu)/x^{\kappa_F} < \infty$$

Moreover, $\kappa_F < d_S/2 = \log N/\log n_0$.

Remark. Define $n_I = \sharp(V_1) - \sharp(V_0)$, then it is shown in [Sh2] that $n_R \leqslant n_I + 1$. Also if you examine the discussions in [Sh2], we can see that $n_R \leqslant N$.

In the case of the N-Sierpinski gasket, $n_0 = N+2$ and $n_R = 2$.

EXAMPLE 4.6 (The Vicsek Set, [Sh2, Section 4.1]). For $1 \leqslant j \leqslant 5$, define $F_j \colon \mathbb{C} \to \mathbb{C}$ by $F_j = (z - p_j)/3 + p_j$, where $p_1 = 1$, $p_2 = i$, $p_3 = -1$, $p_4 = -i$ and $p_5 = 0$. The Vicsek set K is the p.c.f. self-similar set given by $\{F_1, F_2, ..., F_5\}$. See Fig. 1. It is easy to see that $V_0 = \{p_1, p_2, p_3, p_4\}$. If $D_{p_j p_k} = 1$ for $1 \leqslant j \neq k \leqslant 4$ and $D_{p_j p_j} = -3$ for all j, then (V_0, D) is an r-network, where $D = (D_{p_j p_k})_{1 \leqslant j, k \leqslant 4}$. Set $r = (s, s, s, s, t)$ with $2s + t = 1$ and $0 < s < 1/2$, (D, r) is a regular harmonic structure with $\lambda = 1$. Furthermore (D, r) becomes a strong harmonic structure. In this case, $n_0 = (4t + s)/st$, $N = 5$ and $n_R = 3$. Hence $d_S/2 = \log 5/\log n_0$ and $\kappa_F = \log 3/\log n_0$. Observe that $d_S/(2\kappa_F) = \log 5/\log 3$ is independent of s and t.

EXAMPLE 4.7 (The modified Koch Curve, [Sh2, Section 4.2; M]). For $p, q \in \mathbb{C}$, define $f_{p,q} \colon \mathbb{C} \to \mathbb{C}$ by $f_{p,q}(z) = \alpha z + \beta$ where $f_{p,q}(0) = p$ and $f_{p,q}(1) = q$. If $F_1 = f_{0, 1/3}$, $F_2 = f_{2/3, 1}$, $F_3 = f_{1/3, 2/3}$, $F_4 = f_{1/3, c}$ and $F_5 = f_{c, 2/3}$ where $c = \frac{1}{2} + i/(2\sqrt{3})$, the modified Koch curve is the p.c.f. self-similar set given by $\{F_1, F_2, ..., F_5\}$. See Fig. 1. It is easy to see that $V_0 = \{0, 1\}$. Obviously

$$D = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

is an r-network. Set $r = (s, s, t, h, h)$ with $2s + 2ht/(t + 2h) = 1$ for $s, t, h > 0$, (D, r) is a regular harmonic structure with $\lambda = 1$. Furthermore, (D, r) is a strong harmonic structure. In this case, $n_0 = 2s^{-1} + t^{-1} + 2h^{-1}$, $N = 5$ and $n_R = 4$. Hence $d_S/2 = \log 5/\log n_0$ and $\kappa_F = \log 4/\log n_0$. Observe that $d_S/(2\kappa_F) = \log 5/\log 4$ is independent of s, t and h.

It is notable that in the above examples, the ration κ_W/κ_F is a constant where $\kappa_W = d_S/2$. It might be the case that, in general, the ratio κ_W/κ_F depends only on the self-similar structure. Even if this sounds a little optimistic, it should be interesting to consider the following problems. Assume that Conjecture 3.6 is true for the moment.

(1) What determines the ratio κ_W/κ_F is general?

(2) What is the meaning of the value of this ratio? Does it correspond to any kind of geometrical constant of a self-similar set?

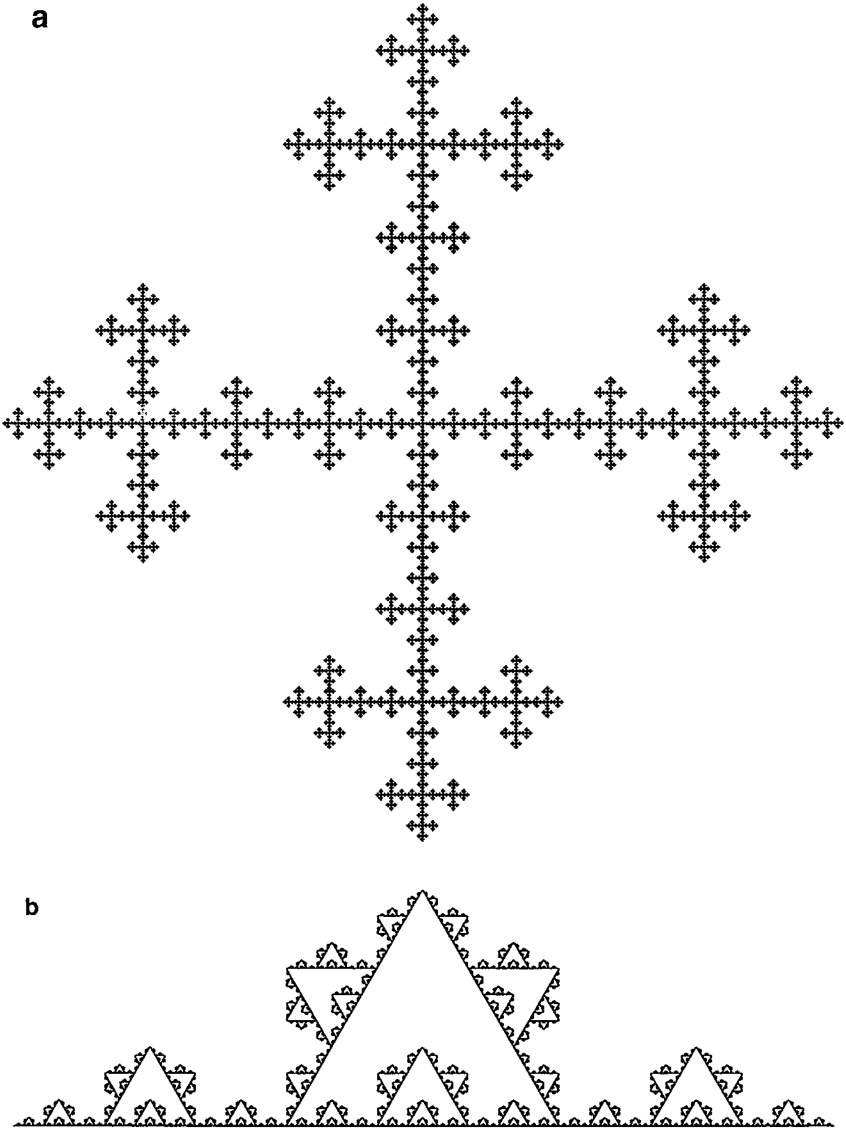

FIG. 1. (a) The Vicsek fractal and (b) the modified Koch curve.

5. EIGENVALUE DECIMATION METHOD

In this section, we will summerize the eigenvalue decimation method for the standard Laplacian Δ on the S.G. in Shima [Sh1] and Fukushima–Shima [FS]. This method is derived from observations by physicists Rammal and Toulouse. In their papers, [R] and [RT], they found a relation between eigenvalues on H_m and H_{m+1} for the ordinary $(N=3)$ Sierpinski gasket by an elementary calculation. Recently Teplyaev [Te] used essentially the same method to study the spectrum of the Laplacians of infinite Sierpinski gaskets.

Let $\Phi(x)=x(N+2-x)$. The branches of Φ^{-1} are $\phi_2(x)=((N+1)+\sqrt{(N+2)^2-4x})/2$ and $\phi_1(x)=((N+2)-\sqrt{(N+2)^2-4x})/2$ defined on $(-\infty, (N+2)^2/4]$. Note that both $\phi_2|_{[0, N+2]}$ and $\phi_1|_{[0, N+2]}$ are contraction mappings from $[0, N+2]$ to itself.

First we treat the Dirichlet case. Set $\ell_0(V_m)=\{f: f\in\ell(V_m), f|_{V_0}=0\}$ and define a linear map $H^D_m: \ell_0(V_m)\to\ell_0(V_m)$ by $(H^D_m f)(p)=H_{m,p}f$ for $p\in V_m\backslash V_0$ and $(H^D_m f)(p)=0$ for $p\in V_0$.

Proposition 5.1 *Define A_m for $m\geqslant 1$ inductively by $A_1=\{2, N+2\}$ and $A_{m+1}=\phi_1(A_m)\cup\phi_2(A_m)\cup\{N, N+2\}$. Then the collection of all of eigenvalues of $-H^D_m$ is equal to $A_m\cup\{2N\}$. Moreover, let $E^m_D(k)=\{f\in\ell_0(V_m): H^D_m f=-kf\}$ and let $E^m_D=\bigoplus_{k\in A_m}E^m_D(k)$, then, for $i=1, 2$, there exists an injective linear map $\mathcal{G}^i_m: E^m_D(k)\to\ell_0(V_{m+1})$ that satisfies $(\mathcal{G}^i_m f)|_{V_m}=f$ and $\mathcal{G}^i_m(E^m_D(k))=E^{m+1}_D(\phi_i(k))$ for all $k\in A_m$. In particular,* $\dim E^m_D(k)=\dim E^{m+1}_D(\phi_i(k))$ *for all $k\in A_m$.*

Proposition 5.2. $\dim E^1_D(2)=1, \dim E^m_D(N+2)=M_m+N$ *for* $m\geqslant 1$ *and* $\dim E^m_D(N)=M_m$ *for $m\geqslant 2$ where $M_m=N(N^{m-1}-2N^{m-2}-2)/2$.*

By the above propositions, if $k\in A_m$,

(1) $k=\phi_{\varepsilon_1\varepsilon_2\cdots\varepsilon_{m-1}}(2)$ and $\dim E^m_D(k)=1$ or

(2) $k=\phi_{\varepsilon_1\varepsilon_2\cdots\varepsilon_j}(N+2)$ for some $0\leqslant j\leqslant m-1$ and $\dim E^m_D(k)=M_{m-j}+N$ or

(3) $k=\phi_{\varepsilon_1\varepsilon_2\cdots\varepsilon_j}(N)$ for some $0\leqslant j\leqslant m-2$ and $\dim E^m_D(k)=M_{m-j}$,

where $\phi_\varepsilon=\phi_{\varepsilon_1}\circ\cdots\circ\phi_{\varepsilon_j}$ for $\varepsilon=\varepsilon_1\varepsilon_2\cdots\varepsilon_j\in\{1, 2\}^j$. Note that $\{1, 2\}^0=\{\varnothing\}$ and $\phi_\varnothing$ is the identity map.

Define $\psi(x)=\lim_{m\to\infty}(N+2)^{m+1}\phi^m_1(x)$ for $x\in(-\infty, (N+2)^2/4)$. It is shown that ψ is a strictly increasing analytic function. Note that $(N+2)\,\psi(\phi_1(x))=\psi(x)$.

Proposition 5.3. *There exists an injective linear map $\mathcal{G}_m: E^m_D\to C(K)$ that satisfies $(\mathcal{G}_m f)|_{V_m}=f$, $\mathcal{G}_{m+1}\circ\mathcal{G}^1_m=\mathcal{G}_m$ and $\mathcal{G}_m(E^m_D(k))=E_D((N+2)^{m-1}\psi(k))$*

for all $k \in A_m$. *Furthermore, if* $-\lambda$ *is a Dirichlet eigenvalue of the standard Laplacian* Δ, *then there exist* m *and* $k \in A_m$ *with* $\lambda = (N+2)^{m-1}\,\psi(k)$.

Combining 5.1, 5.2, and 5.3, we can obtain complete classification of the Dirichlet eigenvalues of $-\Delta$.

THEOREM 5.4. *If* $-\lambda$ *is a Dirichlet eigenvalue of the standard Laplacian* Δ, *then there exists* $\varepsilon = \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n \in \{1, 2\}^n$ *for some* $n \geqslant 0$ *where* $\varepsilon_1 = 2$ *if* $n \geqslant 1$ *such that*

(1) $\lambda = (N+2)^n\,\psi(\phi_\varepsilon(2))$ *and* $\dim E_D(\lambda) = 1$,

(2) $\lambda = (N+2)^{n+m-1}\,\psi(\phi_\varepsilon(N+2))$ *and* $\dim E_D(\lambda) = M_m + N$ *for* $m \geqslant 1$, *or*

(3) $\lambda = (N+2)^{n+m-1}\,\psi(\phi_\varepsilon(N))$ *and* $\dim E_D(\lambda) = M_m$ *for* $m \geqslant 2$.

For Neumann eigenvalues, we have an analogous description. Define $H_m^N\colon \ell(V_m) \to \ell(V_m)$ by $(H_m^N f)(p) = H_{m,p} f$ for $p \in V_m \backslash V_0$ and $(H_m^N f)(p) = 2H_{m,p} f$ for $p \in V_0$.

PROPOSITION 5.5. *Define* B_m *for* $m \geqslant 1$ *inductively by* $B_1 = \{N\}$ *and* $B_{m+1} = \phi_1(B_m) \cup \phi_2(B_m) \cup \{N, N+2\}$. *Then the collection of all of eigenvalues of* $-H_m^N$ *is equal to* $B_m \cup \{0, 2N\}$. *Moreover, let* $E_N^m(k) = \{f \in \ell(V_m) : H_m^N f = -kf\}$ *and let* $E_N^m = \bigoplus_{k \in B_m} E_N^m(k)$. *Then, for* $i = 1, 2$, *there exists an injective linear map* $\widetilde{\mathscr{G}}^i_m\colon E_N^m(k) \to \ell(V_{m+1})$ *that satisfies* $(\widetilde{\mathscr{G}}^i_m f)|_{V_m} = f$ *and* $\widetilde{\mathscr{G}}^i_m(E_N^m(k)) = E_N^{m+1}(\phi_i(k))$ *for all* $k \in B_m$. *In particular,* $\dim E_N^m(k) = \dim E_N^{m+1}(\phi_i(k))$ *for all* $k \in B_m$.

PROPOSITION 5.6. $\dim E_N^m(N+2) = M_m + 1$ *for* $m \geqslant 2$ *and* $\dim E_N^m(N) = M_m + N$ *for* $m \geqslant 1$.

By Propositions 5.5 and 5.6, if $k \in B_m$,

(1) $k = \phi_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_j}(N+2)$ for some $0 \leqslant j \leqslant m-2$ and $\dim E_N^m(k) = M_{m-j} + 1$, or

(2) $k = \phi_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_j}(N)$ for some $0 \leqslant j \leqslant m-1$ and $\dim E_N^m(k) = M_{m-j} + N$.

PROPOSITION 5.7. *There exists an injective linear map* $\widetilde{\mathscr{G}}_m\colon E_N^m \to C(K)$ *that satisfies* $(\widetilde{\mathscr{G}}_m f)|_{V_m} = f$, $\widetilde{\mathscr{G}}_{m+1} \circ \widetilde{\mathscr{G}}^1_m = \widetilde{\mathscr{G}}_m$ *and* $\widetilde{\mathscr{G}}_m(E_N^m(k)) = E_N((N+2)^{m-1}\,\psi(k))$ *for all* $k \in B_m$. *Furthermore, if* $-\lambda$ *is a negative Neumann eigenvalue of the standard Laplacian* Δ, *then there exist* m *and* $k \in B_m$ *with* $\lambda = (N+2)^{m-1}\,\psi(k)$.

THEOREM 5.8. *If* $-\lambda$ *is a Neumann eigenvalue of the standard Laplacian* Δ, *then there exists* $\varepsilon = \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n \in \{1, 2\}^n$ *for some* $n \geqslant 0$ *where* $\varepsilon_1 = 2$ *if* $n \geqslant 1$ *such that*

(1) $\lambda = 0$ *and* $\dim E_N(\lambda) = 1$,

(2) $\lambda = (N+2)^{n+m-1} \psi(\phi_\varepsilon(N+2))$ *and* $\dim E_N(\lambda) = M_m + 1$ *for* $m \geqslant 2$, *or*

(3) $\lambda = (N+2)^{n+m-1} \psi(\phi_\varepsilon(N))$ *and* $\dim E_N(\lambda) = M_m + N$ *for* $m \geqslant 1$.

6. DIMENSIONS OF EIGENSPACES

The main result of this section is Theorem 6.1, which is a complete list of Neumann and Dirichlet eigenvalues of $-\Delta$ with dimensions of eigenspaces $E^W(k)$, $E^F_D(k)$ and $E^F_N(k)$.

THEOREM 6.1. *Define* $\lambda_{n,m}(\varepsilon, k) = (N+2)^{n+m-1} \psi(\phi_\varepsilon(x))$ *where* $n \geqslant 0$, $\varepsilon = \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n \in \{1, 2\}^n$ *and* $\varepsilon_1 = 2$ *if* $n \geqslant 1$. *The following table is the complete list of Dirichlet and Neumann eigenvalues of* $-\Delta$

k	$\dim E^W(k)$	$\dim E^F_D(k)$	$\dim E^F_N(k)$	
0	0	0	1	
$\lambda_{n,1}(\varepsilon, 2)$	0	1	0	$n \geqslant 0$
$\lambda_{n,m}(\varepsilon, N+2)$	$M_m + 1$	$N-1$	0	$n \geqslant 0, m \geqslant 1$
$\lambda_{n,1}(\varepsilon, N)$	0	0	$N-1$	$n \geqslant 0$
$\lambda_{n,m}(\varepsilon, N)$	M_m	0	N	$n \geqslant 0, m \geqslant 2$

The rest of this section is devoted to proving Theorem 6.1.

Set $n_*(k) = \dim E_*(k)$, $n^W(k) = \dim E^W(k)$ and $n^F_*(k) = \dim E^F_*(k)$ for $* = D, N$.

LEMMA 6.2.

$$n_*(k) = n^W(k) + n^F_*(k), \tag{6.1}$$

$$n^F_*(k) \leqslant N, \tag{6.2}$$

$$n^W(k) \leqslant \min\{n_D(k), n_N(k)\}. \tag{6.3}$$

Proof. As $E_*(k) = E^W(k) \oplus E^F_*(k)$, we obtain (6.1) and (6.3). For (6.2), define $\tau_*: E^F_*(k) \to \ell(V_0)$ by $\tau_D(f)(p) = (df)_p$ and $\tau_N(f)(p) = f(p)$ for $p \in V_0$. Then $\ker \tau_* \subset E^W(k) \cap E^F_*(k) = \{0\}$. Hence τ_* is injective and $n^F_*(k) \leqslant \dim \ell(V_0) = N$.

LEMMA 6.3. *Let $f \in \ell(V_{m+1})$. If $H_{m+1,q} f = -\lambda f(q)$ for any $q \in V_{m+1} \setminus V_m$, then, for all $p \in V_m$,*

$$(2N-\lambda) H_{m,p} f = (\lambda-2)(\lambda-(N+2)) H_{m+1,p} f + \lambda(\lambda-(N+2)) \sharp(V_{m,p}) f(p),$$

where $\sharp(V_{m,p}) = 2(N-1)$ if $p \in V_m \setminus V_0$ and $\sharp(V_{m,p}) = N-1$ if $p \in V_0$.

Proof of Lemma 6.3. We define $R_{m+1,p} = \bigcup_{w \in W_m : p \in F_w(V_0)} F_w(V_1 \setminus V_0) \setminus V_{m+1,p}$ for $p \in V_m$. Then, by [Ki1, Lemma 2.2], we have

$$NH_{m,p} f = (N+2) H_{m+1,p} f + 2 \sum_{q \in V_{m+1,p}} H_{m+1,q} f + \sum_{q \in R_{m+1,p}} H_{m+1,q} f. \tag{6.4}$$

Also, by an equality in the proof of [Ki1, Lemma 2.2], we obtain

$$\sum_{q \in V_{m+1,p}} H_{m+1,q} f = -N \sum_{q \in V_{m+1,p}} f(q) + 2 \sum_{q \in R_{m+1,p}} f(q) + \sum_{p \in V_{m,p}} f(q) + \sharp(V_{m,p}) f(p). \tag{6.5}$$

Now, as $H_{m+1,q} f = -\lambda f(q)$ for any $q \in V_{m+1} \setminus V_m$, (6.4) and (6.5) imply

$$NH_{m,p} f = (N+2-2\lambda) H_{m+1,p} f - 2\lambda \sharp(V_{m,p}) f(p) - \lambda \sum_{q \in R_{m+1,p}} f(q)$$

and

$$(N-\lambda) H_{m+1,p} f = 2 \sum_{q \in R_{m+1,p}} f(q) + H_{m,p} f + (2-N+\lambda) \sharp(V_{m,p}) f(p),$$

respectively, where we also use the following facts: $H_{m,p} f + \sharp(V_{m,p}) f(p) = \sum_{q \in V_{m,p}} f(q)$ and $H_{m+1,p} f + \sharp(V_{m,p}) f(p) = \sum_{q \in V_{m+1,p}} f(q)$. Finally, eliminating the term $\sum_{q \in R_{m+1,p}} f(q)$ from the above equalities, we have the required result.

Remark. Lemma 6.3 is an extension of [Sh2, Lemma 2.3]. The original lemma treated the case where $f|_{V_0} \equiv 0$ and $p \in V_m \setminus V_0$.

LEMMA 6.4. *If $g \in E_N^m(N+2)$, then $g|_{V_{m-1}} \equiv 0$.*

Proof. As $H_{m,p} g = -(N+2) g(p)$ for all $p \in V_m \setminus V_{m-1}$, Lemma 6.3 implies that $H_{m-1,q} g = 0$ for all $g \in V_{m-1}$. Hence $g|_{V_{m-1}} \equiv a$ for some constant a.

Now define an inner product $(\cdot, \cdot)_m$ on $\ell(V_m)$ by

$$(u, v)_m = N^{-(m+1)}\Big(2 \sum_{p \in V_m \setminus V_0} u(p)\, v(p) + \sum_{p \in V_0} u(p)\, v(p)\Big).$$

Then H_m^N is a symmetric operator with respect to $(\cdot, \cdot)_m$. So, if $h(p) = 1$ for all $p \in V_m$, we can verify that $(g, h)_m = 0$ because g and h are eigenfunctions of H_m^N belonging to different eigenvalues $-(N+2)$ and 0. Set $g_w = g \circ F_w$ for $w \in W_{m-1}$. Note that $g_w \in \ell(V_1), H_{1,p} g_w = -(N+2)\, g_w(p)$ for all $p \in V_1 \setminus V_0$ and $g_w|_{V_0} \equiv 1$. Elementary enumeration as in [Ki1, Sections 1, 2] shows that

$$\sum_{p \in V_1 \setminus V_0} H_{1,p} f = (N-1) \sum_{p \in V_0} f(p) - 2 \sum_{p \in V_1 \setminus V_0} f(p).$$

for all $f \in \ell(V_1)$. By this formula, we can see that $\sum_{p \in V_1 \setminus V_0} g_w(p) = -(N-1)a$. Hence $\sum_{p \in V_1 \setminus V_0} 2g(F_w(p)) + \sum_{p \in V_0} g(F_w(p)) = -2(N-1)a + Na = (2-N)a$. As

$$(g, h)_m = N^{-(m+1)} \sum_{w \in W_{m-1}} \Big(2 \sum_{p \in V_1 \setminus V_0} g(F_w(p)) + \sum_{p \in V_0} g(F_w(p))\Big),$$

we obtain $(g, h)_m = N^{-2}(2-N)a$. By the fact that $(g, h)_m = 0$, $a = 0$ and hence $g|_{V_{m-1}} \equiv 0$.

Proof of Theorem 6.1. If $k = 0$ or $k = \lambda_{n,1}(\varepsilon, 2)$, using (6.3), we can see that $n^W(k) = 0$.

If $k = \lambda_{n,1}(\varepsilon, N)$, we know that $n_D(k) = 0$. Hence (6.3) implies $n_W(k) = 0$ and $n_N^F(k) = M_1 + N = N - 1$.

If $k = \lambda_{n,m}(\varepsilon, N)$ for $m \geqslant 2$, we can see that $n^W(k) \leqslant M_m$ by (6.3). On the other hand, (6.1) along with (6.2) implies $n_N(k) = M_m + N = n^W(K) + n_N^F(k) \leqslant n^W(k) + N$. Hence, $n^W(k) = M_m$ and $n_N^F(k) = N$.

If $k = \lambda_{n,1}(\varepsilon, N+2)$, then $n_D(k) = N - 1$ and $n_N(k) = 0$. Hence by (6.3), we obtain $n^W(k) = 0$ and $n_D^F(k) = N - 1$.

If $k = \lambda_{n,m}(\varepsilon, N+2)$ for $m \geqslant 2$, let $f \in E_N(k)$. Using Propositions 5.5 and 5.7, there exists $g \in E_N^m(N+2)$ such that $f = \tilde{\mathscr{G}}_{m+n} \circ \tilde{\mathscr{G}}^{\varepsilon_1}_{m+n-1} \circ \cdots \circ \tilde{\mathscr{G}}^{\varepsilon_{n-1}}_{m+1} \circ \tilde{\mathscr{G}}^{\varepsilon_n}_m g$. Note that $f|_{V_0} = g|_{V_0}$. Now by Lemma 6.4, we can see that $g|_{V_0} \equiv 0$. Therefore, $f \in E^W(k)$. This implies $n_N(k) = n^W(k)$. As $n_N(k) = M_m + 1$ and $n_D(k) = M_m + N$, we have $n^W(k) = M_m + 1$ and $n_D^F(k) = N - 1$.

7. ASYMPTOTIC BEHAVIOR OF EIGENVALUE COUNTING FUNCTIONS

In this section, we will determine the asymptotic behavior of the eigenvalue counting functions $\rho_*^F(x)$ and $\rho^W(x)$ for the standard Laplacian Δ on the N-Sierpinski gasket. Theorem 4.4 will, finally, be proven by using the facts obtained in Sections 5 and 6. Also we will observe the real nature of the periodic functions G and P appearing in Theorem 4.4.

Recall that both $\phi_2|_{[0, N+2]}$ and $\phi_1|_{[0, N+2]}$ are contraction mappings from $[0, N+2]$ to itself. Hence there exists a unique non-empty compact subset $J \subset [0, N+2]$ that satisfies $J = \phi_2(J) \cup \phi_1(J)$. J is a non-linear Cantor set and, in fact, it coincides with the Julia set J_Φ of Φ. Let $\tilde{\nu}$ be the self-similar measure on J that satisfies $\tilde{\nu}(\phi_2(J)) = 1/2$ and $\tilde{\nu}(\phi_1(J)) = 1/2$. We also use $\tilde{\nu}$ to denote the measure on $[0, N+2]$ defined by $\tilde{\nu}(A) = \tilde{\nu}(A \cap J)$.

DEFINITION 7.1. Define

$$\mathscr{M} = \{\mu : \mu \text{ is a finite Borel measure on } [0, N+2]\}$$

$$X = \{f : f : [0, N+2] \to [0, \infty), \text{ non-decreasing, right continuous}\}.$$

For $f \in X$, $f_{-}(a) = \lim_{x \to a-0} f(x)$ and we set $f_{-}(0) = 0$. Also for $a > 0$, $X_a = \{f \in X : f(N+2) = a\}$. For $f, g \in X$, let $|f - g| = \sum_{x \in [0, N+2]} |f(x) - g(x)|$. Then $|f - g|$ is a metric on X. X (and also X_a) is complete under this metric.

LEMMA 7.2. *Define T: $\mathscr{M} \to \mathscr{M}$ by $T(\mu) = (\phi_2^*(\mu) + \phi_1^*(\mu))/2$ for $\mu \in \mathscr{M}$. (For a Borel set $A \subset [0, N+2]$, $T(\mu)(A) = (\mu(\phi_2^{-1}(A)) + \mu(\phi_1^{-1}(A)))/2$.) Also define T_X: $X \to X$ by*

$$T_X(f)(x) = \begin{cases} f(\Phi(x))/2 & \text{for} \quad 0 \leqslant x < \alpha \\ f(N+2)/2 & \text{for} \quad \alpha \leqslant x < \beta, \\ f(N+2) - f_{-}(\Phi(x))/2 & \text{for} \quad \beta \leqslant x \leqslant N+2 \end{cases}$$

where $\alpha = \phi_1(N+2)$ and $\beta = \phi_2(N+2)$ Set $f(x) = \int_0^x d\mu$ for $\mu \in \mathscr{M}$. Then $T_X(f)(x) = \int_0^x d(T(\mu))$.

We write T instead of T_X hereafter. Note that $T(X_a) \subset X_a$. Also we can easily see that $T|_{X_a}$ is a contraction map.

LEMMA 7.3. *For $f, g \in X_a$, $|T(f) - T(g)| = |f - g|/2$.*

By using the contraction mapping theorem, we have

THEOREM 7.4. *Define $\varphi(x) = \int_0^x d\tilde{\nu}$. Then $a\varphi$ is the unique fixed point of $T|_{X_a}$. Moreover, for any $f \in X_a$, $|T^m(f) - a\varphi(x)| = 2^{-m} |f - a\varphi|$.*

Now we calculate the asymptotic behavior of $\rho_D^F(x)$. We define $\nu_m \in \mathscr{M}$ inductively by

$$\begin{aligned} \nu_1 &= \delta_2 + (N-1)\delta_{N+2} \\ \nu_{m+1} &= 2T(\nu_m) + (N-1)\delta_{n+2} \qquad \text{for} \quad m \geqslant 1. \end{aligned} \tag{7.1}$$

Also set $\psi_m(x) = (N+2)^{m-1}\psi(x)$ and $\mu_m = \psi_m^*(\nu_m)$. Note that $\psi_{m+1} \circ \phi_1 = \psi_m$.

PROPOSITION 7.5. *For* $x \in [0, (N+2)^{m-1}\psi(N+2)]$, $\rho_D^F(x) = \int_0^x d\mu_m$.

Proof. Set $C_1 = \{2, N+2\}$ and $C_{m+1} = \phi_1(C_m) \cup \phi_2(C_m) \cup \{N+2\}$. If $\mathscr{A} = \bigcup_{m \geqslant 1} \mathscr{A}_m$ where $\mathscr{A}_m = \psi_m(C_m)$, then we can deduce from Theorem 6.1 that $\mathscr{A} = \{k : n_D^F(k) > 0\}$. Also, note that $\phi_1(C_m) \subset [0, \alpha]$ and $\phi_2(C_m) \subset [\beta, N+2]$. It follows that $\mathscr{A}_{m+1} \cap [0, (N+2)^{m-1}\psi(N+2)] = \mathscr{A}_m$. This implies $\mathscr{A} \cap [0, (N+2)^{m-1}\psi(N+2)] = \mathscr{A}_m$.

Now by (7.1) and the definition of C_m, we can see that $\nu_m = \sum_{k \in C_m} n(k)\delta_k$, where $n(k) = 1$ if $k = \phi_{\varepsilon_1\varepsilon_2\cdots\varepsilon_j}(2)$ and $n(k) = N-1$ if $k = \phi_{\varepsilon_1\varepsilon_2\cdots\varepsilon_j}(N+2)$. Hence it follows that $\mu_m = \sum_{k \in \mathscr{A}_m} n_D^F(k)\delta_k$. This implies the statement of the proposition.

By the above proposition, if we define a Borel measure $\tilde{\mu}$ on $[0, \infty)$ by $\tilde{\mu} = \mu_m$ on $[0, (N+2)^{m-1}\psi(N+2)]$, we have $\rho_D^F(x) = \int_0^x d\tilde{\mu}$.

LEMMA 7.6. *Let* $\rho_m(x) = \rho_D^F((N+2)^{m-1}x)/2^{m-1}$ *for* $x \in [0, \psi(N+2)]$. *Then for all* $m > 0$,

$$|\rho_m - (2N-1)\varphi \circ \psi^{-1}| \leqslant \frac{(2N-1)m}{2^{m-1}},$$

where $|\cdot|$ *is the* L^∞*-norm for the bounded functions on* $[0, \psi(N+2)]$.

Proof. Let $f_m(x) = \int_0^x d(\nu_m/2^{m-1})$. Then $\rho_m(x) = f_m(\psi^{-1}(x))$. As $\nu_m/2^{m-1} = T(\nu_{m-1}/2^{m-2}) + (N-1)\delta_{N+2}/2^{m-1}$, using Lemma 7.2, we have

$$f_m = T(f_{m-1}) + (N-1)\frac{\tilde{H}_{N+2}}{2^{m-1}},$$

where $\tilde{H}_a$ is the Heviside function at a; $\tilde{H}_a(x) = \delta_a((-\infty, x])$. Therefore $f_m = T^{m-1}(\tilde{H}_2) + (N-1)\sum_{k=1}^m T^{m-k}(\tilde{H}_{N+2})/2^{k-1}$. Making use of Theorem 7.4,

$$
\begin{aligned}
|f_m-(2N-1)\varphi| &\leqslant |T^{m-1}(H_2)-\varphi|+(N-1) \\
&\quad \times \sum_{k=1}^{m} \frac{|T^{m-k}(\widetilde{H}_{N+2})-\varphi|}{2^{k-1}}+(N-1)\,\frac{|\varphi|}{2^{m-1}} \\
&\leqslant \frac{|\widetilde{H}_2-\varphi|}{2^{m-1}}+(N-1)\sum_{k=1}^{m}\frac{|\widetilde{H}_{N+2}-\varphi|}{2^{m-1}}+(N-1)\,\frac{|\varphi|}{2^{m-1}} \\
&\leqslant \frac{(N+(N-1)m)}{2^{m-1}}.
\end{aligned}
$$

This implies the claim of the lemma.

THEOREM 7.7. *Define a continuous* $\log(N+2)/2$*-periodic function* P *by*

$$P(t)=2(2N-1)\,\varphi\circ\psi^{-1}\left(\frac{e^{2t}}{N+2}\right)e^{-2\kappa_F t}$$

for $t\in[\log\psi(N+2)/2, \log\psi(N+2)/2+\log(N+2)/2]$, *where* $\kappa_F=\log 2/\log(N+2)$. *Then, for* $*=D$, N, *as* $x\to\infty$,

$$\rho_*^F(x)=P(\log x/2)\,x^{\kappa_F}+O(\log x).$$

Proof. By Lemma 7.6, for $x\in[(N+2)^{m-2}\,\psi(N+2), (N+2)^{m-1}\,\psi(N+2)]$,

$$|\rho_D^F(x)-2^{m-1}g(x/(N+2)^{m-1})|\leqslant(2N-1)m,$$

where $g=(2N-1)\,\varphi\circ\psi^{-1}$. It follows that $2^{m-1}g(x/(N+2)^{m-1})=P(\log x/2)\,x^{\kappa_F}$. Also, there exists $C>0$ such that $(2N-1)m\leqslant C\log x$ for all m and x. The same estimate follows for the Neumann case because $|\rho_D^F(x)-\rho_N^F(x)|\leqslant N$. Thus we have completed the proof of this theorem.

Remark. As $\tilde{\nu}$ is a self-similar measure on the non-linear Cantor set J_Φ, φ is a Cantor-type function. So P is essentially a Cantor-type function divided by an exponential function.

Next we study the asymptotic behavior of $\rho^W(x)$. By using the same method as in the proof of Proposition 7.5, we have

PROPOSITION 7.8. *For* $x\in[(N+2)^{m-2}\,\psi(N+2), (N+2)^{m-1}\,\psi(N+2)]$,

$$\rho^W(x)=h_m\circ\psi^{-1}(x/(N+2)^{m-1}),$$

where $h_m \in X$ *is defined inductively by* $h_2 = (M_2+1)\tilde{H}_{N+2} + M_2\tilde{H}_N$ *and, for* $m \geqslant 3$,

$$h_m = 2T(h_{m-1}) + (M_m+1)\tilde{H}_{N+2} + M_m\tilde{H}_N.$$

Now, for $m \geqslant 2$, we define θ_m and $\omega_m \in X$ inductively by $\theta_2 = Nh$, $\omega_2 = \omega$,

$$\theta_m = 2T(\theta_{m-1}) + N^{m-1}h \quad \text{and} \quad \omega_m = 2T(\omega_{m-1}) + \omega$$

for $m \geqslant 3$, where $h = ((N-2)/2)(\tilde{H}_N + \tilde{H}_{N+2})$ and $\omega = ((N-2)/2)\tilde{H}_{n+2} + (N/2)\tilde{H}_N$. Note that $h_m = \theta_m - \omega_m$.

LEMMA 7.9.

$$\theta_m = N^{m-1}\theta - 2^{m-1}N\varphi + O(1),$$

where $\theta \in X$ *is defined by* $\theta = \sum_{k=0}^{\infty} (2/N)^k\, T^k(h)$.

Proof. It follows from the definition that $\theta_m = N^{m-1}\sum_{k=0}^{m-2} (2/N)^k\, T^k(h)$. Hence we have

$$N^{m-1}\theta - \theta_m = 2^{m-1}N\varphi + N^{m-1}\sum_{k \geqslant m-1}\left(\frac{2}{N}\right)^k (T^k(h) - (N-2)\varphi).$$

By Lemma 7.3,

$$\begin{aligned}\left|N^{m-1}\sum_{k \geqslant m-1}\left(\frac{2}{N}\right)^k (T^k(h) - (N-2)\varphi)\right| &\leqslant N^{m-1}\sum_{k \geqslant m-1}\left(\frac{2}{N}\right)^k |T^k(h) - (N-2)\varphi| \\ &= N^{m-1}\sum_{k \geqslant m-1} N^{-k}\,|h - (N-2)\varphi| \leqslant \frac{N(N-2)}{N-1}.\end{aligned}$$

By a similar discussion as in Lemma 7.6, we also obtain

$$\omega_m = 2^{m-1}(N-1)\varphi + O(m).$$

Combining this with Lemma 7.9,

$$h_m = N^{m-1}\theta - 2^{m-1}(2N-1)\varphi + O(m).$$

So the same discussion as in Theorem 7.7. implies the following asymptotic expansion of $\rho^W(x)$.

THEOREM 7.10. *Define a* $\log(N+2)/2$*-periodic function G by*

$$G(t)=N\cdot\theta\circ\psi^{-1}\left(\frac{e^{2t}}{N+2}\right)e^{-td_s}$$

for $t\in[\log\psi(N+2)/2, \log\psi(N+2)/2+\log(N+2)/2]$, *where* $d_S=2\log N/\log N+2$. *Then, as* $x\to\infty$,

$$\rho^W(x)=G(\log x/2)\,x^{d_S/2}-P(\log x/2)\,x^{\kappa_F}+O(\log x).$$

Combining Theorem 7.7 and Theorem 7.10, we obtain Theorem 4.4.

APPENDIX

In this appendix, we will give a proof of Theorem 2.3. The proof will depend essentially on the following extended version of the renewal theorem for the arithmetic case.

THEOREM A.1. *Let f be a measurable function on* $\mathbb{R}$ *with* $f(t)\to 0$ *as* $t\to-\infty$. *Suppose f satisfies a renewal equation*

$$f(t)=\sum_{j=1}^{N} f(t-m_jT)p_j+u(t), \tag{A.1}$$

where $m_1, m_2, ..., m_N$ *are positive integers whose greatest common divider is* 1, $\sum_{j=1}^N p_j=1$ *and* $p_j>0$ *for all j. Also assume that* $\sum_{j=-\infty}^{+\infty}|u_j(t)|$ *converges uniformly on* $[0, T]$, *where* $u_j(t)=u(t+jT)$ *for* $t\in[0, T]$. *Set* $f_n(t)=f(t+nT)$ *for* $n\in\mathbb{Z}$ *and* $G(t)=(\sum_{j=1}^N m_jp_j)^{-1}\sum_{j=-\infty}^{\infty}u_j(t)$. *Then as* $n\to\infty$, f_n *converges to G uniformly on* $[0, T]$.

Moreover, set $Q(z)=(1-\sum_{j=1}^N p_jz^{m_j})/(1-z)$ *and define* $\beta=\min\{|z|: Q(z)=0\}$ *and* $m=\max\{$*multiplicity of* $Q(z)=0$ *at w*: $|w|=\beta$, $Q(w)=0\}$. *If there exist* $C>0$ *and* $\alpha>1$ *such that* $|u(t)|\leqslant C\alpha^{-t}$ *for all t, then, as* $t\to\infty$,

$$|G(t)-f(t)|=\begin{cases} O(t^{m-1}\beta^{-t/T}) & \text{if } \alpha^T>\beta,\\ O(t^m\alpha^{-t}) & \text{if } \alpha^T=\beta,\\ O(\alpha^{-t}) & \text{if } \alpha^T<\beta.\end{cases}$$

Remark 1. If $Q(z)=1$, then we set $\beta=+\infty$. As $\sum_{j=1}^N p_i=1$, $Q(z)$ is a polynomial. Furthermore, $|\sum_{j=1}^N p_jz^{m_j}|<1$ for $\{z: |z|\leqslant 1, z\neq 1\}$. Hence we see that $\beta>1$.

Remark 2. See Feller's book [Fe] for the statement and the proof of the classical renewal theorem.

LEMMA A.2. *Set* $F(z)=\prod_{j=1}^{k}(1-e^{i\theta_j}z)^{-1}$ *where* $0\leqslant\theta_j<2\pi$ *for* $j=1, 2, ..., k$. *If* $F(z)=\sum_{n=0}^{\infty}a_n z^n$, *then* $|a_n|=O(n^{m-1})$ *as* $n\to\infty$ *where* $m=\max\{\sharp\{j:\theta=\theta_j\}:0\leqslant\theta<2\pi\}$.

Proof. We use induction on k. The conclusion is obvious when $k=1$. Assume that the conclusion holds for k. For $F(z)=\prod_{j=1}^{k+1}(1-e^{i\theta_j}z)^{-1}=\sum_{n=0}^{\infty}a_n z^n$, if $\theta_j=\theta$ for all j then it is easy to see that $|a_n|=O(n^k)$. If $\theta_p\neq\theta_q$ for some $p\neq q$, then $(1-e^{i\theta_p}z)^{-1}(1-e^{i\theta_q}z)^{-1}=a(1-e^{i\theta_p}z)^{-1}+b(1-e^{i\theta_q}z)^{-1}$ for some a, b. Hence the statement follows from the induction assumption.

LEMMA A.3. *For* $w=w_1w_2\cdots w_k\in W_k$, *set* $m(w)=\sum_{j=1}^{k}m_{w_j}$ *and* $p_w=p_{w_1}\cdots p_{w_k}$. *Define* $\tilde{M}(k)=\sum_{w\in W_*:m(w)=k}p_w$. *Then* $|(\sum_{j=1}^{N}m_jp_j)^{-1}-\tilde{M}(n)|=O(n^{m-1}\beta^{-n})$ *as* $n\to\infty$ *where* β *and* m *are the same as in Theorem* A.1.

Proof. By using the fact that $\tilde{M}(k)=\sum_{j=1}^{N}\tilde{M}(k-m_j)p_j$, it follows that

$$\sum_{n=0}^{\infty}\tilde{M}(n)z^n=\left(1-\sum_{j=1}^{N}p_jz^{m_j}\right)^{-1}.$$

Note that $(\sum_{i=1}^{N}m_ip_i)=Q(1)$, we have $\sum_{n=0}^{\infty}(\tilde{M}-\tilde{M}(n))z^n=R(z)/Q(z)$ where $\tilde{M}=Q(1)^{-1}$ and $R(z)$ is a polynomial defined by $R(z)=(Q(1)\,Q(z)-1)/(1-z)$. If $Q(z)=a\prod_{j=1}^{\deg(Q)}(z-z_j)$, it is shown that

$$\sum_{n=0}^{\infty}(\tilde{M}-\tilde{M}(n))z^n=R(z)/Q(z)=\left(\sum_{j=0}^{\infty}c_nz^n\right)\prod_{j:|z_j|=\beta}(z-z_j)^{-1},$$

where the radius of convergence of $\sum_{n=0}^{\infty}c_nz^n$ is greater than β. Applying Lemma A.2 to $\prod_{j:|z_j|=\beta}(z-z_j)^{-1}$, we can obtain the required estimate of $|\tilde{M}-\tilde{M}(n)|$.

Proof of Theorem A.1. By the renewal equation (A.1), we have

$$f(t)=\sum_{w\in W_n}f(t-m(w)T)p_w+\sum_{k=0}^{n}\sum_{w\in W_k}u(t-m(w)T)p_w.$$

As $\lim_{t\to-\infty}f(t)=0$ and $\sum_{n=0}^{\infty}u(t-nT)$ is absolutely convergent, we have

$$f(t)=\sum_{n=0}^{\infty}u(t-nT)\,\tilde{M}(n)$$

Hence we obtain

$$G(t)-f(t)=\tilde{M}\sum_{k>0}u(t+kT)+\sum_{k=0}^{\infty}u(t-kT)(\tilde{M}-\tilde{M}(k)).\qquad\text{(A.2)}$$

As $\tilde{M} \leqslant 1$ and $\tilde{M}(k) \leqslant 1$,

$$|G(t) - f_n(t)| \leqslant 2 \sum_{k > n-m} |u_k(t)| + \sum_{k=m}^{\infty} |u_{n-k}(t)|\ |\tilde{M}(k) - \tilde{M}|.$$

For $\varepsilon > 0$, choose m so that $|\tilde{M}(k) - \tilde{M}| < \varepsilon$ for $k \geqslant m$. Then for sufficiently large n, we have $\sum_{k>n-m} |u_k(t)| \leqslant \varepsilon$. Therefore $|f_n(t) - G(t)| \leqslant (2 + A)\varepsilon$, where $A = \sup_{0 \leqslant t \leqslant T} \sum_{k=-\infty}^{\infty} |u_k(t)|$. Hence f_n is uniformly convergent to G as $n \to \infty$ on $[0, T]$.

Now suppose $|u(t)| \leqslant C\alpha^{-t}$, then by (A.2) and Lemma A.3, it follows that

$$\begin{aligned}|G(t) - f(t)| &\leqslant c_1 \alpha^{-t} + c_2 \alpha^{-t} \sum_{0 \leqslant k \leqslant t/T} k^{m-1} (\alpha^T/\beta)^k + c_3 \sum_{k > t/T} k^{m-1} \beta^{-k} \\ &\leqslant c_1 \alpha^{-t} + c_2 \alpha^{-t} \sum_{0 \leqslant k \leqslant t/T} k^{m-1} (\alpha^T/\beta)^k + c_4 (t/T)^{m-1}\, \beta^{-t/T},\end{aligned}$$

where c_1, c_2, c_3, and c_4 are some positive constants. From this inequality, it is easy to deduce the required estimate.

Proof of Theorem 2.3. By [KL1, Lemma 2.8], if $\tilde{R}(x) = \rho_D(x, \mu) - \sum_{i=1}^{N} \rho_D(\gamma_i^2 x, \mu)$, then $\tilde{R}(x)$ is a non-negative bounded function. Note that $\rho_D(x, \mu) = 0$ on $[0, c]$ for some $c > 0$. Set $f(t) = e^{-d_S t} \rho_D(e^{2t}, \mu)$, $p_i = p^{-m_i} = \gamma_i^{d_S}$ for $i = 1, 2, ..., N$ and $u(t) = e^{-d_S t} \tilde{R}(e^{2t})$. Then we obtain the renewal equation (A.1). It is routine to verify the conditions of Theorem A.1. Moreover, as $|u(t)| \leqslant Ce^{-d_S t}$ where $C = \sup_{x \geqslant 0} \tilde{R}(x)$, we can set $\alpha = e^{-d_S}$. So Theorem A.1 implies Theorem 2.3 immediately.

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