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CONSTRUCTIVE APPROXIMATION

The Finite Element Method on the Sierpinski Gasket

Michael Gibbons, Arjun Raj, and Robert S. Strichartz

Abstract. For certain classes of fractal differential equations on the Sierpinski gasket, built using the Kigami Laplacian, we describe how to approximate solutions using the finite element method based on piecewise harmonic or piecewise biharmonic splines. We give theoretical error estimates, and compare these with experimental data obtained using a computer implementation of the method (available at the web site http://mathlab.cit.cornell.edu/∼gibbons). We also explain some interesting structure concerning the spectrum of the Laplacian that became apparent from the experimental data.

1. Introduction

Since fractals are not smooth objects, differential equations in the ordinary sense are not defined for functions on fractals. Nevertheless, an interesting theory of what might be called "fractal differential equations" has been developed for a limited class of fractals, including the familiar Sierpinski gasket (SG), based on the construction of an analog of the Laplacian. The definition of a Laplacian on SG by Kigami [Ki1], while not the first, represented a real breakthrough from the point of view of numerical analysis because it is completely explicit. This method was extended to a class of fractals called "post-critically finite" in [Ki2]. A complete exposition will soon be available in the book [Ki3], and an informal survey of the whole field that has developed may be found in [S2].

The definition of a Laplacian on SG as a limit of difference quotients may be described simply by the formula

$$\Delta f(x) = \lim_{m\to\infty} (\tfrac{3}{2})5^m \sum_{y\sim_m x} (f(y) - f(x)). \tag{1.1}$$

Here we regard SG as the limit of graphs Γ_m, consisting of vertices V_m defined inductively by $V_m = \bigcup_{i=1}^3 F_i V_{m-1}$ starting with V_0, the three vertices of a triangle (usually taken to be equilateral), where F_i are the three contractions with ratio $\frac{1}{2}$ having fixed point equal to one of the vertices. The edge relation $x \sim_m y$ of Γ_m is that x and y belong to the image of the original triangle under an m-fold iteration of the IFS $\{F_i\}$. We regard V_0 as the boundary of each of the graphs Γ_m and of SG. Every nonboundary vertex has exactly four neighbors in V_m, so the sum in (1.1) has four terms. We will refer to such vertices as

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junction points. These form a dense set of points in SG, and we will extend the definition of Δf to all points by continuity. Note that (1.1) defines the Laplacian on SG as a limit of graph Laplacians. The analogy with the definition of the usual Laplacian on $\mathbf{R}^n$ via difference quotients is obvious. (The appearance of the factor 5^m is not so obvious, but is essential in order to obtain a nontrivial operator.)

Using (1.1), it is easy to set up the analog of the finite difference method to approximate solutions of various equations involving Δ. This was done in [DSV]. The goal of the present paper is to do the same for the finite element method. As a preliminary step, it is necessary to construct the analog of various spline spaces. This was done in [SU]. Here we will use only the piecewise harmonic spline spaces $S(\mathcal{H}_0, V_m)$ and the piecewise biharmonic spline spaces $S(\mathcal{H}_1, V_m)$, which are the first two of an infinite family constructed in [SU]. These are the analogs of piecewise linear and piecewise cubic splines on an interval. A harmonic function is a solution of $\Delta f = 0$, and a biharmonic function is a solution of $\Delta^2 f = 0$. To get the spline spaces we localize these conditions to each cell $F_w(\mathrm{SG})$ of order m, where $w = (w_1, \ldots, w_m)$ denotes a word and $F_w = F_{w_1} \circ \cdots \circ F_{w_m}$ is the corresponding iterated contraction, and we impose suitable matching conditions at the junction points in V_m. For the harmonic splines the only matching condition is continuity, but for biharmonic splines we also require a matching of normal derivatives (the precise definition is given in Section 2). This is a natural condition, since it is necessary and sufficient on f to glue together local statements $\Delta f = g$ on each cell $F_w(\mathrm{SG})$ to obtain a global statement $\Delta f = g$ on SG. It is easy to see that harmonic splines are uniquely determined by specifying values at vertices in V_m. It is not so obvious, but a result from [SU], that biharmonic splines are uniquely detrmined by specifying both the value and the normal derivative at each vertex in V_m. There are corresponding natural bases for the spline spaces, described in detail in [SU], along with basic formulas to compute inner products and energies for pairs of basis elements.

The first class of equations we consider is the Dirichlet problem

$$-\Delta u + qu = f, \qquad u|_{V_0} = 0, \tag{1.2}$$

where q and f are given continuous functions. The finite element approximations will be projections of the solution onto the spline spaces (with the Dirichlet boundary condition imposed) in a suitable inner product, but the approximations are obtained by an algorithm that does not require knowing the solution. This algorithm is based on the equivalent weak formulation of (1.2), which we now describe. Perhaps the most basic construction in [Ki1] is the Dirichlet form $\mathcal{E}(u, v)$, defined as a limit of Dirichlet forms $\mathcal{E}_m(u, v)$ on the graphs Γ_m. For simplicity, we restrict ourselves to the diagonal $u = v$. Then

$$\mathcal{E}_m(u, u) = (\tfrac{5}{3})^m \sum_{x \sim_m y} (u(x) - u(y))^2, \tag{1.3}$$

and it can be shown that $\mathcal{E}_m(u, u)$ is monotone increasing in m (this explains the factor $(\frac{5}{3})^m$), so that

$$\mathcal{E}(u, u) = \lim_{m \to \infty} \mathcal{E}_m(u, u)$$

is defined for all continuous functions, allowing the value $+\infty$. The domain of the Dirichlet form, denoted $\mathrm{dom}\,\mathcal{E}$, is the set of continuous functions with a finite value for

$\mathcal{E}(u, u)$. It should be noted that points have positive capacity in this theory, as is the case for an interval but not for manifolds of dimension at least 2. This explains why $\operatorname{dom}\mathcal{E}$ is contained in the space of continuous functions. Harmonic functions may be characterized by the property that $\mathcal{E}_m(u, u)$ is independent of m, so that harmonic functions minimize the Dirichlet form among all functions with given boundary values.

The energy may also be computed locally as

$$\mathcal{E}(u, u) = \sum_{|w|=m} (\tfrac{5}{3})^m \mathcal{E}(u \circ F_w, u \circ F_w). \tag{1.4}$$

Let μ denote the standard self-similar probability measure on SG, giving equal weight to all cells of level m:

$$\mu(F_w(\mathrm{SG})) = 3^{-m} \qquad \text{for} \quad |w| = m. \tag{1.5}$$

Then $\Delta f = g$, in the sense of (1.1) (with uniform convergence) for continuous functions f and g, is equivalent to $f \in \operatorname{dom}\mathcal{E}$ and

$$\mathcal{E}(f, v) = -\int g v \, d\mu \qquad \text{for all} \quad v \in \operatorname{dom}_0 \mathcal{E}, \tag{1.6}$$

the subscript 0 denoting functions vanishing on V_0. We say $f \in \operatorname{dom}\Delta$ in this case. We define similarly $f \in \operatorname{dom}_{L^p}\Delta$ if $f \in \operatorname{dom}\mathcal{E}$ and there exists $g \in L^p$ such that (1.6) holds. There are analogous definitions for domains of higher powers of Δ. The Laplacian satisfies the scaling identity

$$\Delta(u \circ F_w) = \rho^m (\Delta u) \circ F_w \qquad \text{for} \quad |w| = m, \tag{1.7}$$

where $\rho = \frac{1}{5}$. The factor ρ^m in (1.7) is the reciprocal of the factor 5^m in (1.1), which is related to the factors $(\frac{5}{3})^m$ in (1.3) and 3^{-m} in (1.5). The weak formulation of (1.2) is then

$$\mathcal{E}(u, v) + \int q u v \, d\mu = \int f \, d\mu \qquad \text{for all} \quad v \in \operatorname{dom}_0 \mathcal{E}, \tag{1.8}$$

where $u \in \operatorname{dom}_0 \mathcal{E}$. The finite element approximations are obtained by restricting (1.8) to the spline spaces $S_0(\mathcal{H}_0, V_m)$ or $S_0(\mathcal{H}_1, V_m)$ instead of $\operatorname{dom}_0 \mathcal{E}$ (same interpretation of the subscript 0). By choosing a basis for the spline spaces this becomes a system of linear equations, and the same conditions on q that guarantee a unique solution to (1.2) also guarantee a unique solution to the system of linear equations. A full description of the algorithm is given in Section 2.

In Section 3 we prove theoretical error estimates, both of a priori and of a posteriori type, based on spline approximation results from [SU]. The estimates are expressed in terms of powers of ρ^m, where m is the level of approximation. The heuristic principle is that each power of the Laplacian should correspond to the same power of ρ^m, since the factor 5^m occurs in (1.1). The power of ρ^m that occurs in an estimate will be called the *exponent*. In comparing with the usual estimates h^α for mesh length h, the exponent corresponds to $\frac{1}{2}\alpha$ because the Laplacian corresponds to the second derivative. Most of our results are for average (L^2) and energy ($\mathcal{E}$) error. The energy error controls the

maximum (L^∞) error, but in practice the maximum error seems to be of the same order of magnitude as the average error. In general, our results show a gain of $\frac{1}{2}$ in the exponent for the average error over the energy error, and a gain of 1 for biharmonic splines over harmonic splines. In Section 4 we discuss some experimental data that confirms the predicted error estimates.

In Section 5 we discuss approximate solutions to the Dirichlet eigenvalue problem

$$-\Delta u + qu = \lambda u, \qquad u|_{V_0} = 0.$$

When $q = 0$ the exact solutions are known [FS], allowing an easy empirical test of the accuracy of the method. We also give some theoretical error estimates. Many of the eigenvalues for the $q = 0$ problem have high multiplicity, so the addition of a small q will split the multiple eigenvalues into clusters. The methods of this paper might be valuable in experimental studies of the nature of these clusters. We present some numerical data obtained in the case $q = 0$. These data reveal an interesting structure to the spectrum of the Laplacian, and we give a proof that this structure continues for the entire spectrum.

In Section 6 we study the heat and wave equations associated with the Laplacian. For simplicity we take $q = 0$, omit forcing terms, and impose Dirichlet boundary conditions. The initial value problem for the heat equation is

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \Delta_x u(x,t), \\ u(x,0) = f(x), \\ u(x,t)|_{x\in V_0} = 0, \end{cases} \tag{1.9}$$

and for the wave equation is

$$\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} = \Delta_x u(x,t), \\ u(x,0) = f(x), \ \dfrac{\partial u(x,0)}{\partial t} = g(x), \\ u(x,t)|_{x\in V_0} = 0. \end{cases} \tag{1.10}$$

We use the finite element method in the x-variable, and either a finite difference method or a matrix exponentiation method in the t-variable. A lot is known about the heat equation; see [HK] for results obtained using probabilistic methods. In [DSV], numerical solutions to these problems were given using the finite difference method. It was pointed out there that the wave equation does not enjoy the finite propagation speed property that is usually expected of wave equations on manifolds. We are able to give a proof of error estimates for the heat equation, but not for the wave equation. Data obtained from trial runs using the method for both equations may be found at the web site.

A complete set of programs implementing the algorithms described in this paper may be found at the web site http://mathlab.cit.cornell.edu/∼gibbons, together with the numerical data for the tests of the algorithms we performed. This paper is written so that it can be read independently, but the web site provides additional information.

The algorithms and error estimates presented here are closely related to the standard finite element method [CL]. The main challenge has been to find the appropriate analogs of the basic function spaces and their approximation properties (mostly taken from [SU]), and then to fashion arguments that use only the limited information at our disposal. The

fundamental idea remains the same: the finite element approximation is an orthogonal projection into a spline space, and so minimizes an appropriate norm while, on the other hand, we know how to obtain certain approximation rates in the norm using interpolation approximation. We present all the details of the arguments in Section 3, while we only give a sketch of the argument in Section 6 because it so closely follows the standard theory. The estimates in Section 5 are a bit ad hoc and perhaps not optimal, so we have kept the dicussion informal.

At present we only have good information about the analogs of L^2 Sobolev spaces of integer order, although we have to treat the even and odd integers separately, using energy for one derivative and the Laplacian for two derivatives. As more is learned about other function spaces it may be possible to obtain more flexible finite element estimates. One of the main obstacles we face is that multiplication by functions is not well behaved in our function spaces (beyond the analog of one derivative in L^2) [BST]. This forces the restriction in Theorem 3.7(b) that q be constant. Our numerical results in Section 4 suggest that this is not the end of the story, however. In particular, squares of harmonic functions do not behave substantially worse than harmonic functions. At present we have no explanation for this.

In this paper we have restricted our attention to spline spaces using uniform grids. An obvious direction for further research is to allow grids of variable size. Instead of the decomposition

$$\mathrm{SG} = \bigcup_{|w|=m} F_w(\mathrm{SG}), \tag{1.11}$$

one should allow

$$\mathrm{SG} = \bigcup_{w\in\mathcal{A}} F_w(\mathrm{SG}), \tag{1.12}$$

where $\mathcal{A}$ is a finite set of words chosen so that the cells on the right side of (1.12) overlap only at single points. It is easy to see that the condition on $\mathcal{A}$ is that every sufficiently long word has a unique prefix in $\mathcal{A}$. One advantage of using such irregular decompositions is that we could obtain good approximations to delta functions within the spline space, without excessively driving up the cardinality of $\mathcal{A}$, simply by taking a very refined grid in a small neighborhood of the point where the delta function is to be supported.

We have also dealt exclusively with SG, although the methods of [SU] can be applied to a wider class of fractals. It would certainly be worthwhile to work out the analogous algorithms on some of these other spaces. There does not seem to be any obstacle in principle to doing this, but the details may be technically demanding. One motivation for doing this is that it would provide a method for experimentally studying the spectrum of the Laplacians on other fractals. The exact methods of [FS] can only be used for a very restricted class of fractals [Sh2].

2. The Basic Algorithm

In this section we describe the finite element algorithm for approximating the solution of the Dirichlet problem

$$-\Delta u + qu = f, \qquad u|_{V_0} = 0, \tag{2.1}$$

on SG, for given continuous functions q and f, using harmonic or biharmonic splines on a uniform grid. Recall that the Laplacian may be defined either using the pointwise formula

$$\Delta u(x) = \lim_{m\to\infty} (\tfrac{3}{2})5^m \left(\sum_{y\sim_m x} (u(y) - u(x)) \right) \tag{2.2}$$

for x an interior vertex point, with uniform convergence, or by the weak formulation

$$-\mathcal{E}(u, v) = \int (\Delta u) v \, d\mu \tag{2.3}$$

for all $v \in \operatorname{dom}\mathcal{E}$ vanishing on V_0, where $u \in \operatorname{dom}\mathcal{E}$ and Δu is continuous. The pointwise formula (2.2) leads to the analog of the finite difference method, which was used in [DSV].

Existence and uniqueness for (2.1) holds under the assumption

$$q(x) \geq -\lambda_0 + \varepsilon \tag{2.4}$$

everywhere for some $\varepsilon > 0$, where $\lambda_0 \approx 16$ is the lowest eigenvalue of $-\Delta$ (see Section 5). While (2.4) is not a necessary condition for existence and uniqueness, the finite element method is likely to fail or perform poorly without it. The choice $q(x) \equiv -\lambda_0$ is an example where existence and uniqueness fails. With (2.4), the inner product

$$\langle u, v\rangle_q = \mathcal{E}(u, v) + \int quv \, d\mu \tag{2.5}$$

dominates both the energy and the L^2 inner products on the space $\operatorname{dom}_0\mathcal{E}$ of functions in $\operatorname{dom}\mathcal{E}$ vanishing on V_0 (see Lemma 3.2).

The finite element method is based on the following weak formulation of (2.1) from [Ki1]–[Ki3]:

Theorem 2.1. *A continuous function u satisfies* (2.1) *if and only if u vanishes on V_0 and*

$$\langle u, v\rangle_q = \int fv \, d\mu \tag{2.6}$$

for all $v \in \operatorname{dom}_0\mathcal{E}$.

For any finite-dimensional subspace $W \subseteq \operatorname{dom}_0\mathcal{E}$, define $P_W u \in W$ as the solution to (2.6) for all $v \in W$. Note that $P_W u$ is exactly the projection onto W of the solution to (2.1) with respect to the inner product (2.5). Thus

$$\langle u - P_W u, v\rangle_q = 0 \tag{2.7}$$

for all $v \in W$, and

$$\langle u - P_W u, u - P_W u\rangle_q \leq \langle u - v, u - v\rangle_q \tag{2.8}$$

for all $v \in W$. Since we usually do not know the true solution u, estimate (2.8) gives us a way of controlling the error in the approximation $P_W u$. Note that (2.6) is just a system

of linear equations that can be solved using linear algebra, provided we have an explicit basis for W and we can compute or approximate all the terms in (2.6) for basis functions.

In this paper we consider only two choices for the space W, either $S_0(\mathcal{H}_0, V_m)$ or $S_0(\mathcal{H}_1, V_m)$. $S(\mathcal{H}_0, V_m)$ is the space of piecewise harmonic continuous functions with nodes at V_m, while $S(\mathcal{H}_1, V_m)$ is the space of piecewise biharmonic functions with nodes at V_m satisfying two consistency conditions at junction points. The precise definition will be given below. The subscript 0 in the spline spaces indicates that we consider only functions vanishing at V_0. We refer to these spaces informally as harmonic and biharmonic splines, and we denote the projection operator simply P_m. Harmonic splines belong to $\mathrm{dom}_0\,\mathcal{E}$ but not to dom Δ. Biharmonic splines are not typically in dom Δ either, but they do belong to $\mathrm{dom}_{L^\infty}\,\Delta$ and the Laplacian may be computed piecewise (it usually has jump discontinuities at the nodes). The additional "smoothness" of biharmonic splines comes from the matching of normal derivatives.

The normal derivatives $\partial_n f(x)$ for $x \in V_0$ are defined by

$$\partial_n f(x) = \lim_{m\to\infty} (\tfrac{5}{3})^m \sum_{y\sim_m x} (f(x) - f(y)), \tag{2.9}$$

if the limit exists. Note that there are only two terms in the sum. In [Ki1] it is shown that the normal derivatives exist for $f \in \mathrm{dom}\,\Delta$, and the following version of the Gauss–Green formula holds for u and v in dom Δ:

$$\int_{\mathrm{SG}} (u\Delta v - v\Delta u)\, d\mu = \sum_{V_0} (u\partial_n v - v\partial_n u). \tag{2.10}$$

We may localize the definition to points in V_m. Each nonboundary point x in V_m can be written $x = F_w v_j = F_{w'} v_{j'}$ for two distinct choices of words with $|w| = |w'| = m$, and v_j and $v_{j'}$ in V_0. For each of these there is a normal derivative; for the first

$$\partial_n u(x) = (\tfrac{5}{3})^m \partial_n (u \circ F_w)(v_j) = \lim_{k\to\infty} (\tfrac{5}{3})^k \sum_{\substack{y\sim_k x \\ y\in F_w(\mathrm{SG})}} (u(x) - u(y)). \tag{2.11}$$

It is easy to see that for $u \in \mathrm{dom}\,\Delta$, the sum of the two normal derivatives at a junction point must vanish. Moreover, if f and g are continuous functions satisfying $\Delta f = g$ on each cell $F_w(\mathrm{SG})$ with $|w| = m$, then $\Delta f = g$ on SG if and only if the sum of the normal derivatives of f vanishes at every nonboundary point in V_m. This explains the importance of imposing the matching condition in the definition of biharmonic splines. There is a more elaborate theory of local derivatives given in [S3], but we only need the normal derivatives here.

Definition 2.2.

(a) $S(\mathcal{H}_0, V_m)$ is defined to be the space of all continuous functions f such that $\Delta(f \circ F_w) = 0$ for all words w with $|w| = m$.

(b) $S(\mathcal{H}_1, V_m)$ is defined to be the space of all continuous functions f such that $\Delta^2(f \circ F_w) = 0$ for all words w with $|w| = m$, satisfying the matching condition for normal derivatives $\partial_n(u \circ F_w)(v_j) = -\partial_n(u \circ F_{w'})(v_{j'})$ at nonboundary points $x = F_w v_j = F_{w'} v_{j'}$ with $|w| = |w'| = m$.

(c) $S_0(\mathcal{H}_j, v_m)$ is the subspace of $S(\mathcal{H}_j, v_m)$ of functions vanishing on the boundary V_0.

It is easy to describe a basis for $S_0(\mathcal{H}_0, V_m)$. This space has dimension $(3^{m+1}-3)/2$. For each vertex $y \in V_m \backslash V_0$ we let φ_y denote the piecewise harmonic function that takes on the value 1 at y and 0 at all other vertices in V_m (this function is denoted $\varphi_{0y}^{(0)}$ in [SU]). The space $S_0(\mathcal{H}_1, V_m)$ has dimension 3^{m+1}, and has a basis consisting of two types of functions, $\widetilde{\varphi}_y$ and $\widetilde{\psi}_y$. For $y \in V_m \backslash V_0$, we let $\widetilde{\varphi}_y$ be the biharmonic spline that has normal derivatives vanishing at all vertices in V_m, and takes the same values as φ_y at vertices. For any $y \in V_m$, the biharmonic spline $\widetilde{\psi}_y$ vanishes at all vertices in V_m and has normal derivatives vanishing at every vertex except y. We make the normal derivative of $\widetilde{\psi}_y$ at y equal 1, but this requires a somewhat arbitrary sign convention at interior vertices as there are actually two normal derivatives on adjacent cells equal to ± 1, and we need to specify which gets the plus sign. (See the web page for the specifics of the choice we used.) The values of all inner products and energies among basis elements are computed explicitly in [SU].

Any function in $S_0(\mathcal{H}_0, V_m)$ has a unique representation

$$\sum_{y \in V_m \backslash V_0} c_y \varphi_y \tag{2.12}$$

where the coefficients c_y are just the values of the function at the points y. The analogous representation for $S_0(\mathcal{H}_1, V_m)$ splines is

$$\sum_{y \in V_m \backslash V_0} \widetilde{c}_y \widetilde{\varphi}_y + \sum_{y \in V_m} \widetilde{d}_y \widetilde{\psi}_y, \tag{2.13}$$

with a similar interpretation for the coefficients. In order to discuss both cases simultaneously we will write

$$\sum c_j \varphi_j \tag{2.14}$$

to stand for either (2.12) or (2.13). We take (2.14) as the form for $P_m u$, with the coefficients determined by substituting in (2.1), multiplying by φ_k, and integrating, to obtain

$$-\sum_j c_j \int (\Delta \varphi_j)\varphi_k \, d\mu + \sum_j c_j \int q \varphi_j \varphi_k \, d\mu = \int f \varphi_k \, d\mu.$$

Since $-\int (\Delta \varphi_j)\varphi_k \, d\mu = \mathcal{E}(\varphi_j, \varphi_k)$ we obtain the matrix equation

$$\sum_j E_{kj} c_j + \sum_j Q_{kj} c_j = F_k \tag{2.15}$$

where

$$\begin{cases} E_{kj} = \mathcal{E}(\varphi_j, \varphi_k), \\ Q_{kj} = \int q \varphi_j \varphi_k \, d\mu, \\ F_k = \int f \varphi_k \, d\mu. \end{cases} \tag{2.16}$$

The energy matrix $\{E_{kj}\}$ is computed theoretically, while the other terms must be approximated using numerical integration, except when q is constant, in which case $Q_{kj} = qG_{kj}$ where

$$G_{kj} = \int \varphi_k \varphi_j \, d\mu \tag{2.17}$$

is the known Gram matrix of inner products for the basis. Under the hypothesis (2.4) the matrix $E + Q$ is invertible, so

$$c = (E + Q)^{-1} F \tag{2.18}$$

gives the coefficients of $P_m u$.

3. Error Estimates

First we derive theoretical error estimates, which we may then compare with actual data. We will need the following basic interpolation result, which is a special case of Theorem 4.8 of [SU]. We will express all estimates in terms of $\rho^{\alpha m}$, where $\rho = \frac{1}{5}$ and α will be called the *exponent*. Many of our estimates are valid for the larger class of fractals considered in [SU], with different values of ρ.

Theorem 3.1.

(a) *For any $u \in \mathrm{dom}_{L^2} \Delta$ vanishing on V_0, let $u_m \in S_0(\mathcal{H}_0, V_m)$ be the interpolating spline taking the same values as u at all vertices in V_m. Then*

$$\mathcal{E}(u - u_m, u - u_m)^{1/2} \le c_0 \|\Delta u\|_2 \rho^{(1/2)m}. \tag{3.1}$$

(b) *For any $u \in \mathrm{dom}_{L^2} \Delta^2$ vanishing on V_0, let $\widetilde{u}_m \in S_0(\mathcal{H}_1, V_m)$ be the interplating spline taking the same values and normal derivatives as u at all vertices of V_m. Then*

$$\mathcal{E}(u - \widetilde{u}_m, u - \widetilde{u}_m)^{1/2} \le c_1 \|\Delta^2 u\|_2 \rho^{(3/2)m}. \tag{3.2}$$

See [SU] for the proof.

Remark. There are also corresponding estimates for the error in the L^2-norm, namely

$$\|u - u_m\|_2 \le c_0' \|\Delta u\|_2 \rho^m \quad \text{in part (a),}$$

and

$$\|u - \widetilde{u}_m\|_2 \le c_1' \|\Delta^2 u\|_2 \rho^{2m} \quad \text{in part (b),}$$

with a gain in exponent of $1/2$. These are not explicitly stated in [SU], but follow by the same argument as given in Corollary 4.8 there (see also Theorem 3.4 below).

Next we prove the basic coercivity estimate for the inner product (2.5). Note that we already know

$$\langle u, u \rangle_q \le (1 + c\|q\|_\infty) \mathcal{E}(u, u) \tag{3.3}$$

(see (4.17) in [SU]).

Lemma 3.2. *Suppose q is continuous and satisfies* (2.4). *Then for any $u \in \mathrm{dom}_0\,\mathcal{E}$, and ε in* (2.4):

$$\|u\|_2^2 \le \varepsilon^{-1}\langle u,u\rangle_q \tag{3.4}$$

and

$$\mathcal{E}(u,u) \le (1+\varepsilon^{-1}\|q\|_\infty)\langle u,u\rangle_q. \tag{3.5}$$

Proof. By the definition of λ_0 we have $\mathcal{E}(u,u) \ge \lambda_0\|u\|_2^2$, hence

$$\langle u,u\rangle_q \ge \int (\lambda_0+q)|u|^2\,d\mu \ge \varepsilon\|u\|_2^2$$

by (2.4), proving (3.4). But then

$$|\mathcal{E}(u,u)| \le \langle u,u\rangle_q + \int |qu^2|\,d\mu \le (1+\varepsilon^{-1}\|q\|_\infty)\langle u,u\rangle_q$$

using (3.4). ∎

We will be concerned with three types of error in a priori estimates. The L^2 *error* (or average error)

$$\|u - P_m u\|_2, \tag{3.6}$$

the L^∞ *error* (or maximum error)

$$\|u - P_m u\|_\infty, \tag{3.7}$$

and the energy error

$$\mathcal{E}(u - P_m u, u - P_m u)^{1/2}. \tag{3.8}$$

Clearly these are of increasing size. We will not be able to give any useful theoretical predictions for the maximum error. In practice, it seems to involve the same exponent as the average error (with perhaps a logarithmic factor, not easily observed from data). Our next goal is a basic comparison result for average and energy errors. In essence, the exponent always increases by $1/2$ for the average error as compared with the energy error, regardless of most details. This is very useful, since the energy error is easier to handle theoretically, and we get a boost in accuracy by considering average error.

Lemma 3.3. *Suppose $f \in L^2$ and q is continuous and satisfies* (2.4). *Then if $u \in \mathrm{dom}_{L^2}(\Delta)$ satisfies* (2.1) *we have*

$$\|\Delta u\|_2 \le c\|f\|_2. \tag{3.9}$$

Proof. The weak formulation of (2.1) (with $v = u$) yields

$$\mathcal{E}(u,u) + \int qu^2\,d\mu = \int fu\,d\mu. \tag{3.10}$$

Apply Cauchy–Schwartz to the right side of (3.10) and (3.4) to the left side, to obtain $\varepsilon\|u\|_2^2 \le \|f\|_2\|u\|_2$, hence $\|u\|_2 \le \varepsilon^{-1}\|f\|_2$. Then from (2.1):

$$\|\Delta u\|_2 \;\le\; \|f\|_2 + \|qu\|_2 \le (1+\varepsilon^{-1}\|q\|_\infty)\|f\|_2.$$ ∎

Theorem 3.4. *Suppose q and f are continuous and q satisfies* (2.4). *Then*

$$\|u - P_m u\|_2 \le c\rho^{(1/2)m}\mathcal{E}(u - P_m u, u - P_m u)^{1/2} \tag{3.11}$$

for either type of spline.

Proof. We have

$$\|u - P_m u\|_2 = \sup\left\{\int (u - P_m u)v\, d\mu : \|v\|_2 \le 1\right\}, \tag{3.12}$$

and we may even require that v be continuous. Let w be the solution of

$$-\Delta w + qw = v, \qquad w|_{V_0} = 0. \tag{3.13}$$

Note that $\|\Delta w\| \le c$ by Lemma 3.3. Now apply Theorem 3.1(a) to w, to obtain $w_m \in S_0(\mathcal{H}_0, V_m)$ with

$$\mathcal{E}(w - w_m, w - w_m)^{1/2} \le c\|\Delta w\|_2\rho^{(1/2)m} \le c'\rho^{(1/2)m}. \tag{3.14}$$

Note that

$$\int (u - P_m u)v\, d\mu = \langle u - P_m u, w\rangle_q$$

by (3.13), and

$$\langle u - P_m u, w\rangle_q = \langle u - P_m u, w - w_m\rangle_q$$

by (2.7) (since $S_0(\mathcal{H}_0, V_m) \subseteq S_0(\mathcal{H}_1, V_m)$). By the Cauchy–Schwartz inequality for the inner product (2.5) we have

$$\left|\int (u - P_m u)v\, d\mu\right| \le \langle u - P_m u, u - P_m u\rangle_q^{1/2}\langle w - w_m, w - w_m\rangle_q^{1/2}.$$

By (3.3) we can replace the inner products by energies, so

$$\left|\int (u - P_m u)v\, d\mu\right| \le c\rho^{(1/2)m}\mathcal{E}(u - P_m u, u - P_m u)^{1/2}$$

by (3.14). Together with (3.12), this proves (3.11). ∎

Lemma 3.5. *Suppose $u \in \operatorname{dom}\Delta$ and $\Delta u \in \operatorname{dom}\mathcal{E}$, and suppose u and its normal derivatives vanish on V_0. Then*

$$\mathcal{E}(u, u) \le c\mathcal{E}(\Delta u, \Delta u). \tag{3.15}$$

Proof. The vanishing of the normal derivative on V_0 implies

$$-\mathcal{E}(u, \Delta u) = \int (\Delta u)^2\, d\mu,$$

while the vanishing of u on V_0 implies

$$\mathcal{E}(u,u) \le c\|\Delta u\|_2^2$$

by Lemma 4.6 of [SU]. Combining the two and using the Cauchy–Schwartz inequality for $\mathcal{E}$ yields

$$\mathcal{E}(u,u) \le c|\mathcal{E}(u,\Delta u)| \le c\mathcal{E}(u,u)^{1/2}\mathcal{E}(\Delta u,\Delta u)^{1/2}$$

which implies (3.15). (Note that although $\mathcal{E}$ fails to be an inner product because of the one-dimensional null space of constants, the usual derivation of the Cauchy–Schwartz inequality is still valid, since the inequality is trivially true if one of the functions is constant.) ∎

Corollary 3.6. *Let $u \in \operatorname{dom}\Delta$ and $\Delta u \in \operatorname{dom}\mathcal{E}$, and let $\widetilde{u}_m$ be the interpolating biharmonic spline, as in Theorem* 3.1(b). *Then*

$$\mathcal{E}(u-\widetilde{u}_m, u-\widetilde{u}_m)^{1/2} \le c\mathcal{E}(\Delta u,\Delta u)^{1/2}\rho^m. \tag{3.16}$$

Proof. For each word w of length m, $(u-\widetilde{u}_m)\circ F_w$ satisfies the hypotheses of Lemma 3.5. Note that since $(\Delta\widetilde{u}_m)\circ F_w$ is harmonic and $(u-\widetilde{u}_m)\circ F_w$ vanishes on V_0, we have

$$\mathcal{E}((u-\widetilde{u}_m)\circ F_w, (\Delta\widetilde{u}_m)\circ F_w) = 0. \tag{3.17}$$

Thus, by the proof of Lemma 3.5:

$$\begin{aligned}\mathcal{E}((u-\widetilde{u}_m)\circ F_w, (u-\widetilde{u}_m)\circ F_w) &\le c|\mathcal{E}((u-\widetilde{u}_m)\circ F_w, \Delta((u-\widetilde{u}_m)\circ F_w))|\\ &= c|\mathcal{E}((u-\widetilde{u}_m)\circ F_w, \Delta(u\circ F_w))|\end{aligned}$$

in view of (3.17). Continuing the argument from the lemma we obtain

$$\begin{aligned}\mathcal{E}((u-\widetilde{u}_m)\circ F_w, (u-\widetilde{u}_m)\circ F_w) &\le c\mathcal{E}(\Delta(u\circ F_w), \Delta(u\circ F_w))\\ &= c\rho^{2m}\mathcal{E}((\Delta u)\circ F_w, (\Delta u)\circ F_w),\end{aligned}$$

using (1.7). Finally, we sum over all words w of length m and take the square root, using (1.4). ∎

Theorem 3.7. *Assume that q and f are continuous, that q satisfies* (2.4), *and that u is the solution to* (2.1).

(a) *Then the harmonic spline approximations $P_m u$ satisfy a priori error estimates with exponent* $1/2$ *for energy error and* 1 *for average error*:

$$\mathcal{E}(u-P_m u, u-P_m u)^{1/2} \le c\rho^{(1/2)m}, \tag{3.18}$$

and

$$\|u-P_m u\|_2 \le c\rho^m. \tag{3.19}$$

(b) *Assume q is constant and $f \in \operatorname{dom}\Delta$. Then the biharmonic spline approximations have exponent* $3/2$ *for energy error and* 2 *for average error*:

$$\mathcal{E}(u - P_m u, u - P_m u)^{1/2} \le c\rho^{(3/2)m}, \tag{3.20}$$

and

$$\|u - P_m u\|_2 \le c\rho^{2m}. \tag{3.21}$$

(c) *Assume q and f are in* $\operatorname{dom}\mathcal{E}$. *Then the biharmonic spline approximations have exponent* 1 *for energy error and* $3/2$ *for average error*:

$$\mathcal{E}(u - P_m u, u - P_m u)^{1/2} \le c\rho^{m}, \tag{3.22}$$

and

$$\|u - P_m u\|_2 \le c\rho^{(3/2)m}. \tag{3.23}$$

Proof. In view of Theorem 3.4 it suffices to establish the energy error estimates. For part (a), since $u \in \operatorname{dom}\Delta$, we apply Theorem 3.1(a) to obtain (3.1) for the interpolating harmonic spline u_m. By (3.3) and (3.5) the energy and the inner product (2.5) are equivalent, and by (2.8) the spline $P_m u$ minimizes the distance to u in the inner product (2.5). Thus (3.1) implies (3.18).

Under the assumptions in (b), $\Delta u = qu - f \in \operatorname{dom}\Delta$ so $u \in \operatorname{dom}\Delta^2$. We can then derive (3.20) from (3.2) in Theorem 3.1(b) as before. Similarly, under the assumptions in (c), $\Delta u \in \operatorname{dom}\mathcal{E}$ (this uses the fact that $\operatorname{dom}\mathcal{E}$ is an algebra under pointwise multiplication). In this case, we use (3.16) from Corollary 3.6 to obtain (3.22). ■

Remark. The constants in (3.18) and (3.19) may be taken to be multiples of $\|\Delta u\|_2$. The constants in (3.20) and (3.21) may be taken to be multiples of $\|\Delta^2 u\|_2$, and in (3.22) and (3.23) multiples of $\mathcal{E}(\Delta u, \Delta u)^{1/2}$. Moreover, the hypotheses on f may be reduced, so that $f \in L^2$ in (a) and $f \in \operatorname{dom}_{L^2}\Delta$ in (b) will suffice. The proofs are essentially the same.

Next we consider a posteriori estimates. Here we see how close $P_m u$ comes to satisfying (2.1), using

$$\|f - (-\Delta + q)P_m u\|_2 \tag{3.24}$$

as a measurement. Note that (3.24) would vanish if $P_m u$ were the exact solution. For simplicity, we restrict ourselves to the case of biharmonic splines, where $P_m u \in \operatorname{dom}_{L^\infty}\Delta \subseteq \operatorname{dom}_{L^2}\Delta$, and $\Delta P_m u$ may be computed separately on each cell of level m. Note that this is, in general, a discontinuous function since the harmonic functions defining $\Delta P_m u$ on the small cells do not satisfy any matching conditions on the vertices of V_m. This explains why the L^2-norm is an appropriate choice in (3.24). On the other hand, for harmonic splines, $P_m u$ will not even belong to $\operatorname{dom}_{L^2}\Delta$, so an appropriate a posteriori errror would have to take into account jumps in the normal derivatives at vertices of V_m.

Lemma 3.8. *Let*

$$\delta_m(u) = \inf\{\|u - v\|_2 : v \in S_0(\mathcal{H}_1, V_m)\} \tag{3.25}$$

denote the L^2 error of the best approximation of u by biharmonic splines. Then

$$\mathcal{E}(u - P_m u, u - P_m u)^{1/2} \le c\delta_m(u)^{1/2} \| f - (-\Delta + q) P_m u \|_2^{1/2}. \tag{3.26}$$

Proof. Note that

$$f - (-\Delta + q) P_m u = (-\Delta + q)(u - P_m u).$$

On the other hand, by (2.7) we have

$$\langle u - P_m u, u - P_m u \rangle_q = \langle u - P_m u, u - v \rangle_q$$

for any $v \in S_0(\mathcal{H}_1, V_m)$, hence

$$\langle u - P_m u, u - P_m u \rangle_q = \int (u - v)(-\Delta + q)(u - P_m u)\, d\mu \le \|u - v\|_2 \| f - (-\Delta + q) P_m u \|_2.$$

Using (3.5), choosing v to attain the infimum of (3.25), and taking the square root, yields (3.26). ∎

Theorem 3.9. *Suppose q and f are continuous and q satisfies* (2.4). *Then*

$$\mathcal{E}(u - P_m u, u - P_m u)^{1/2} \le c\rho^{(1/2)m} \| f - (-\Delta + q) P_m u \|_2. \tag{3.27}$$

In other words, the exponent for the energy error exceeds by $1/2$ *the exponent for the a posteriori error.*

Proof. By taking $v = P_m u$ we obtain $\delta_m(u) \le \|u - P_m u\|_2$. Using (3.11) of Theorem 3.4 with this in (3.26) yields

$$\mathcal{E}(u - P_m u, u - P_m u) \le c\rho^{(1/2)m} \mathcal{E}(u - P_m u, u - P_m u)^{1/2} \| f - (-\Delta + q) P_m u \|_2$$

which yields (3.27). ∎

Unlike the a priori errors (3.6), (3.7), and (3.8), the a posteriori error (3.24) may be computed without knowing the exact solution. Theorem 3.9 shows that the energy error (hence the average error) is controlled by the a posteriori error. Also, if, for any reason, the a posteriori error is smaller than expected, then the same is true for the energy and average errors.

4. Error Testing

To test our implementation of the finite element method we first used problems with a known answer, by choosing u and q, computing f, and then running the method with the

Table 4.1. The exponents for the maximum (E_{Max}), average (E_{Ave}), and energy (E_{En}) error, as estimated by comparison with the known solution for three different triharmonic functions (T_1, T_2, T_3) and one quad-harmonic function (Q). The harmonic splines used data from levels 2, 3, 4, 5, while the biharmonic splines used data from levels 1, 2, 3, 4.

	Harmonic splines			Biharmonic splines		
Function	E_{Max}	E_{Ave}	E_{En}	E_{Max}	E_{Ave}	E_{En}
T_1	0.9965	0.9991	0.5074	1.9132	1.9487	1.4725
T_2	0.9970	0.9999	0.5073	1.8899	1.9396	1.5142
T_3	0.9938	0.9984	0.5050	1.7596	1.8896	1.4835
Q	0.9959	0.9991	0.5075	1.9562	1.9991	1.5369

q and f input. For u we chose functions for which we could compute Δu theoretically rather than numerically. If u is chosen to be harmonic, then both harmonic and biharmonic splines should give the exact answer for any m. Similarly, for biharmonic splines, if u is chosen to be biharmonic. We verified that this was the case. The first interesting test was to choose u to be triharmonic, so that Δu is a known biharmonic function, but the solution is outside the spline spaces so the errors must be nonzero. We found the method performed as expected, with the biharmonic splines producing high accuracy with modest values of m. We computed the average, energy, and maximum errors for harmonic ($2 \le m \le 5$) and biharmonic ($2 \le m \le 4$) splines, found the slope of the least mean square line fitting the log error data, and divided by $\log \rho$ to obtain experimental predictions of the exponents for the three types of error and each of the two spline types. Since we only were able to use modest values for m, we could not expect this method to be highly accurate. Also, we expect that transient and random effects will be more significant for the maximum error than for the other errors which involve averages. Some of the results are reported in Table 4.1. The average and energy error exponents are in close agreement with the values predicted in Section 3, and the maximum error appears to have the same exponent as the average error.

Next we analyzed the same approximate solutions by computing the errors in the differences $P_{m+1}u - P_m u$ of successive approximations. Since this does not involve the solution itself, it is a test that can be applied in examples where the exact solution is not known. We make the assumption that if the error of $P_m u - u$ follows an exponential decay law proportional to ρ^α, then the same should be true of the error of $P_{m+1}u - P_m u$ with the same exponent but with a different proportionality constant. Note that the estimates

$$\|P_m u - u\| \le c_1 \rho^{\alpha m} \qquad \text{for all } m, \tag{4.1}$$

and

$$\|P_{m+1}u - P_m u\| \le c_2 \rho^{\alpha m} \qquad \text{for all } m, \tag{4.2}$$

are equivalent, for any norm. So, in the long run, we should expect the same exponents to be predicted. What is not clear is that the same will be true using only the data from small values of m. In Table 4.2 we show the predictions for error exponents using successive differences on the same data as was used for Table 4.1. Note that the number of data points for fitting straight lines is reduced by one, and this amounts to a drastic decrease (for

Table 4.2. The error exponents for the same functions as in Table 4.1 computed from the same data, but this time using differences from level to level rather than comparison with the exact solution. This table also gives the exponents for the a posteriori error (E_{AP}).

	Harmonic splines			Biharmonic splines			
Function	E_{Max}	E_{Ave}	E_{En}	E_{Max}	E_{Ave}	E_{En}	E_{AP}
T_1	0.9947	0.9996	0.5047	1.9721	1.9838	1.4928	0.9997
T_2	0.9938	0.9996	0.4990	1.9651	1.9832	1.4924	0.9997
T_3	0.9882	0.9994	0.4988	1.9550	1.9678	1.4825	1.0010
Q	0.9937	0.9996	0.4990	1.9755	1.9844	1.4932	0.9996

biharmonic splines we have just two data points). Nevertheless, the predicted exponents do not change very much.

The next set of tests involved systematically varying the nature of the inputs q and f, without knowing the solution in advance. Thus we were limited to measuring the errors between successive approximations. We wanted a range of "smoothness" for the inputs, and we chose three levels, and representative functions for each level. For the "smooth" level, functions in dom Δ, we chose harmonic functions. Although harmonic functions are actually smoother, belonging to the domain of any power of Δ, this additional smoothness is not predicted to yield any improvement at the level of biharmonic or harmonic splines, and no improvement was, in fact, noted. It would, in fact, be quite tricky to produce functions in dom Δ which fail to be in dom Δ^2 except at isolated points, so we did not attempt to test this distinction. For the "semi-rough" level, functions in dom $\mathcal{E}$ but not dom Δ, we chose the square of a harmonic function. It was shown in [BST] that such a function fails to have a Laplacian at every vertex point, but it is easy to see that it belongs to dom $\mathcal{E}$. For the "rough" level, continuous functions not in dom $\mathcal{E}$, we chose the coordinate variable x for the standard embedding of SG in the plane. It is easy to see that the energies $\mathcal{E}_m(x,x)$ grow on the order of $(\frac{5}{4})^m$, and the failure of x to belong to dom $\mathcal{E}$ is a global property of the function. In addition to these smoothness levels, we also considered constant q as a separate choice.

In Table 4.3 we show the estimated error exponents for ten different tests, for different

Table 4.3. Estimates for the error exponents for different choices of f and q, using the same method as in Table 4.2. The functions used are 1 (the constant), h (a harmonic function), h^2, and x (the coordinate function).

		Harmonic splines			Biharmonic splines			
f	q	E_{Max}	E_{Ave}	E_{En}	E_{Max}	E_{Ave}	E_{En}	E_{AP}
h	1	0.9392	0.9987	0.4984	1.8671	1.9561	1.4708	0.8901
x	h	0.9381	0.9913	0.4917	1.4253	1.4270	0.9325	0.4147
h^2	h	0.8790	0.9930	0.4930	1.6443	1.8751	1.3969	0.9208
h	h	0.9266	0.9992	0.4983	1.6470	1.8630	1.3963	0.9383
x	h^2	0.9383	0.9917	0.4918	1.4190	1.4230	0.9292	0.4133
h^2	h^2	0.8488	0.9912	0.4913	1.5608	1.8088	1.3361	0.8887
h	h^2	0.8870	0.9991	0.4978	1.5318	1.7726	1.3074	0.8748
x	x	0.9420	0.9908	0.4917	1.4324	1.4288	0.9339	0.4149
h^2	x	0.8932	0.9936	0.4939	1.6425	1.8760	1.3943	0.9008
h	x	0.9457	0.9984	0.4984	1.4242	1.4752	0.9771	0.4356

types of q and f. The results are in rough agreement with the predictions. The use of the square of a harmonic function in place of a harmonic function does not seem to have as negative an effect on the performance as anticipated. There may be an explanation for this, based on some results in [T2], but this remains to be seen.

5. Eigenfunctions

We may also use the finite element method to approximate solutions of the Dirichlet eigenfunction equation

$$-\Delta u + qu = \lambda u, \qquad u|_{V_0} = 0. \tag{5.1}$$

When $q = 0$, the exact spectrum and eigenvalue multiplicities were determined in [FS] (see also [T1]), and a finite difference-type method for the exact computation of eigenfunctions was given in [DSV]. This presents a good opportunity to test the method, but the actual results are of less interest. However, there are many fractals with great symmetry for which the spectral decimation method of [FS] does not work [Sh2], yet the finite element method offers a viable option for future investigation. And, of course, once a nonconstant potential q is involved, the finite element is at present the only option.

To approximate the solutions of (5.1), using either spline space, we substitute the representation (2.11) into (5.1), multiply by φ_k, and integrate. Instead of (2.12) we obtain

$$\sum_j E_{kj} c_j + \sum_j Q_{kj} c_j = \lambda \sum_j G_{kj} c_j, \tag{5.2}$$

so the coefficients are just the eigenvectors of the matrix $G^{-1}(E + Q)$, and the approximate eigenvalues of (5.1) are the eigenvalues of this matrix. Since this matrix is conjugate to $G^{-1/2}(E + Q)G^{-1/2}$, which is self-adjoint and positive definite, the matrix $G^{-1}(E + Q)$ has a complete set of eigenvectors with positive eigenvalues. In fact, the assumption (2.4) means

$$\int (-\Delta u + qu)u \, d\mu \geq \varepsilon \|u\|_2^2$$

for any u, because λ_0 is the bottom of the spectrum of $-\Delta$. In particular, for u of the form (2.11) this becomes $\langle Ec + Qc, c\rangle \geq \varepsilon \langle Gc, c\rangle$, so ε is a lower bound for the spectrum of $G^{-1}(E + Q)$. In fact, the min–max characterization of eigenvalues ((5.12) and (5.13) below) shows that the approximating eigenvalues are always greater than the true ones, and decrease as m increases.

Let us write $\{\widetilde{u}_j\}$ and $\{\widetilde{\lambda}_j\}$ for an orthonormal basis of eigenfunctions with corresponding eigenvalues, arranged in increasing order (with repetitions in the case of multiplicity) in the spline space for the approximate problem (5.2). Given an honest solution to the true problem (5.1), we would like to be able to say that it is close to one of the approximate solutions on the list (or a linear combination in the case of multiplicity). In order to ensure this we must take m large enough so that u can be well approximated in the spline space. In addition, we need to know that the gaps between distinct approximate eigenvalues are not too small.

Now by Theorem 4.8 and Corollary 4.9 of [SU] we can find a spline $\widetilde{u}$ approximating u so that

$$\begin{cases} \mathcal{E}(u-\widetilde{u}, u-\widetilde{u})^{1/2} \le c\lambda^2\rho^{(3/2)m}, \\ \|u-\widetilde{u}\|_2 \le c\lambda^2\rho^{2m}, \text{ and hence} \\ \langle u-\widetilde{u}, u-\widetilde{u}\rangle_q^{1/2} \le c\lambda^2\rho^{(3/2)m}, \end{cases} \tag{5.3}$$

for biharmonic splines (for harmonic splines, the rates are $\lambda\rho^{(1/2)m}$ and $\lambda\rho^m$, respectively). Let

$$\widetilde{u} = \sum b_j \widetilde{u}_j \tag{5.4}$$

be the expression of $\widetilde{u}$ in terms of the orthonormal basis of spline eigenfunctions. Then $\langle \widetilde{u}, \widetilde{u}_k\rangle_q = \widetilde{\lambda}_k b_k$ while

$$\langle u, \widetilde{u}_k\rangle_q = \lambda \int u\widetilde{u}_k \, d\mu = \lambda \int \widetilde{u}\widetilde{u}_k \, d\mu + \lambda \int (u-\widetilde{u})\widetilde{u}_k \, d\mu = \lambda b_k + \lambda \int (u-\widetilde{u})\widetilde{u}_k \, d\mu.$$

Thus

$$|\widetilde{\lambda}_k - \lambda||b_k| \le |\langle u-\widetilde{u}, \widetilde{u}_k\rangle_q| + \lambda \left| \int (u-\widetilde{u})\widetilde{u}_k \, d\mu \right| \le c\widetilde{\lambda}_k^{1/2}\lambda^2\rho^{(3/2)m} + c\lambda^3\rho^{2m} \tag{5.5}$$

by (5.3). Let $\widetilde{\lambda}_j$ be chosen to be the closest value to λ, and let γ be the smallest value of $|\widetilde{\lambda}_k - \widetilde{\lambda}_j|$ when $\widetilde{\lambda}_k \ne \widetilde{\lambda}_j$ (because of the possibility of multiplicity there may be more than one eigenvalue equal to $\widetilde{\lambda}_j$). Provided that γ is not too small, the estimate (5.5) for $\widetilde{\lambda}_k \ne \widetilde{\lambda}_j$ will tell us that the coefficients b_k are small, so $\widetilde{u}$ is close to a spline eigenfunction. But then (5.5) for $\widetilde{\lambda}_k = \widetilde{\lambda}_j$ will tell us that λ is close to $\widetilde{\lambda}_j$.

Specifically, let

$$\widetilde{v} = \sum_{\widetilde{\lambda}_k = \widetilde{\lambda}_j} b_k \widetilde{u}_k \tag{5.6}$$

so that $\widetilde{v}$ is a spline eigenfunction with eigenvalue $\widetilde{\lambda}_j$. Then

$$\begin{aligned} \|\widetilde{u}-\widetilde{v}\|_2 = \left(\sum_{\widetilde{\lambda}_k \ne \widetilde{\lambda}_j} |b_k|^2 \right)^{1/2} &\le c\lambda^2\rho^{(3/2)m} \left(\sum_{\widetilde{\lambda}_k \ne \widetilde{\lambda}_j} \frac{\widetilde{\lambda}_k}{|\widetilde{\lambda}_k - \lambda|^2} \right)^{1/2} \\ &\quad + c\lambda^3\rho^{2m} \left(\sum_{\widetilde{\lambda}_j \ne \widetilde{\lambda}_j} |\widetilde{\lambda}_k - \lambda|^{-2} \right)^{1/2} \end{aligned} \tag{5.7}$$

by (5.5). We know that for $q = 0$ the true eigenvalues satisfy a Weyl asymptotic law

$$\lambda_k \approx k^{2/\alpha} \qquad \text{for} \quad \alpha = \frac{2\log 3}{\log 5} \approx 1.3652124, \tag{5.8}$$

called the *spectral dimension* [FS], [KL]. Thus it is plausible to assume that the sums on the right side of (5.7) remain uniformly bounded as m increases, and can be dominated

by the largest terms, which occur when $|\widetilde{\lambda}_k - \lambda| \approx \gamma$. Then (5.3) and (5.7) would yield an estimate

$$\|u - \widetilde{v}\|_2 \le c\gamma^{-1}(\lambda^{5/2}\rho^{(3/2)m} + \lambda^3\rho^{2m}). \tag{5.9}$$

Assuming that u is normalized by $\|u\|_2 = 1$ and m is sufficiently large, we can arrange to have $|b_j| \ge \frac{1}{2}$ (if the multiplicity of $\widetilde{\lambda}_j$ is greater than one this may require choosing the eigenfunction $\widetilde{u}_j$ appropriately). Then (5.5) for $k = j$ yields

$$|\widetilde{\lambda}_j - \lambda| \le c(\lambda^{5/2}\rho^{(3/2)m} + \lambda^3\rho^{2m}). \tag{5.10}$$

In other words, if we fix m and fix an acceptable error ε, this determines a value $\lambda(m, \varepsilon)$ such that for every true eigenvalue λ satisfying $\lambda \le \lambda(m, \varepsilon)$ there exists an approximate eigenvalue $\widetilde{\lambda}_j$ satisfying $|\lambda - \widetilde{\lambda}_j| \le \varepsilon$, where $\lambda(m, \varepsilon)$ is determined by

$$c(\lambda(m, \varepsilon)^{5/2}\rho^{(3/2)m} + \lambda(m, \varepsilon)^3\rho^{2m}) = \varepsilon \tag{5.11}$$

(the constant c as in (5.10)). Moreover, if λ has multiplicity μ greater than one, there will be at least μ distinct corresponding approximate eigenvalues. We cannot assert a priori that the multiplicity of the approximate eigenvalues is the same as the multiplicity of λ because the error tolerance ε could result in splitting multiplicities, but our experimental data indicates that such splitting never occurs.

Conversely, we can assert that every approximate eigenvalue $\widetilde{\lambda}_j$ satisfying $|\widetilde{\lambda}_j| \le \lambda(m, \varepsilon)$ is paired with a true eigenvalue, which will be labeled λ_j in increasing order. In other words, the part of the approximate spectrum below $\lambda(m, \varepsilon)$ matches the part of the true spectrum below $\lambda(m, \varepsilon)$ to within the error tolerance ε. To see this we recall the min–max characterization

$$\lambda_j = \min\{\max\{\langle u, u\rangle_q : u \in L_j, \|u\|_2 = 1\} : \dim L_j = j, L_j \subseteq \operatorname{dom}\mathcal{E}_0\} \tag{5.12}$$

for the true eigenvalues, and the corresponding characterization

$$\widetilde{\lambda}_j = \min\{\max\{\langle u, u\rangle_q : u \in L_j, \|u\|_2 = 1\} : \dim L_j = j, L_j \subseteq S_0(\mathcal{H}_1, V_m)\} \tag{5.13}$$

for the approximate eigenvalues. Thus $\lambda_1 \le \widetilde{\lambda}_1$ so $\widetilde{\lambda}_1$ is closest to λ_1 and will be paired with it. Since $\lambda_2 \le \widetilde{\lambda}_2$, the closest approximate eigenvalue to λ_2 must be either $\widetilde{\lambda}_1$ or $\widetilde{\lambda}_2$. However, if it is $\widetilde{\lambda}_1$, then λ_2 must also be close to $\widetilde{\lambda}_2$ because u_2 cannot be close to $\widetilde{u}_1$ (if the spectral separation γ is not too small, this will imply that $\widetilde{\lambda}_1 = \widetilde{\lambda}_2$). Then we can pair λ_2 with $\widetilde{\lambda}_2$ and continue in the same way for the portion of the spectrum below $\lambda(m, \varepsilon)$.

If we fix ε and vary m, then the cutoff value $\lambda(m, \varepsilon)$ will grow on the order of $5^{0.6m}$. This means, using the estimate (5.8), that the number of acceptable approximate eigenvalues will be on the order of $5^{(0.3\alpha)m} \approx (1.933182)^m$, as compared with the number 3^m of approximate eigenvalues. Although this is asymptotically a negligible proportion, it is not feasible to use large values of m in actual computations. Moreover, we might be content if the size of the error is small relative to λ. This would mean demanding that the right side of (5.10) be bounded by $\varepsilon\lambda$ rather than ε. This would put the cutoff value on the order of 5^m and yield on the order of $5^{(\alpha/2)m} = 3^m$ acceptable approximate

Table 5.1. Estimates of the eigenvalues obtained using biharmonic splines at levels 1, 2, and 3. The true eigenvalues and their multiplicities and names are taken from [DSV], with suitable renormalization. At level k there are 3^k eigenvalues counting multiplicity, and $7 \cdot 2^{k-1} - 2$ eigenvalues not counting multiplicity.

Eigenvalue	Name	Multiplicity	Level 1	Level 2	Level 3
16.815999	$\lambda_0^{(2)}$	1	16.816727	16.816006	16.815999
55.885828	$\lambda_0^{(5)}$	2	55.949427	55.886586	55.885834
172.364521	$\lambda_1^{(5)}$	2	175.149206	172.418561	172.365099
240.168595	$\lambda_1^{(2)}$	1	250.117638	240.351470	240.170704
279.429140	$5\lambda_0^{(5)}$	3	297.713415	279.747135	279.432927
677.859	$\lambda_1^{(6)}$	3		683.118556	677.971289
861.822605	$5\lambda_1^{(5)}$	3		875.747103	862.092805
920.619693	$\lambda_2^{(2)}$	1		938.519314	920.964478
1032.035531	$\lambda_2^{(5)}$	2		1059.687917	1032.560608
1262.029498	$\lambda_3^{(5)}$	2		1322.604655	1263.125999
1354.326273	$\lambda_3^{(2)}$	1		1434.788746	1355.716193
1397.1457	$5^2\lambda_0^{(5)}$	6		1488.567073	1398.735676
3389.295	$5\lambda_1^{(6)}$	12			3415.593
4306.130	$5^2\lambda_1^{(5)}$	6			4378.736
4371.676	$\lambda_4^{(2)}$	1			4445.232
4509.405	$\lambda_4^{(5)}$	2			4592.177
4872.030	$\lambda_5^{(5)}$	2			4983.081
5058.588	$\lambda_5^{(2)}$	1			5186.733
5160.178	$5\lambda_2^{(5)}$	3			5298.440
5493.225	$\lambda_2^{(6)}$	3			5669.176
6310.148	$5\lambda_3^{(5)}$	3			6613.023
6399.149	$\lambda_6^{(2)}$	1			6719.463
6557.111	$\lambda_6^{(5)}$	2			6910.399
6842.178	$\lambda_7^{(5)}$	2			7262.035
6942.044	$\lambda_7^{(2)}$	1			7387.533
6985.729	$5^3\lambda_0^{(5)}$	15			7442.835

eigenvalues. Thus a fixed proportion of approximate eigenvalues will have relative error bounded by ε.

In view of (5.9), the approximate eigenfunctions would have an acceptable error (a multiple of ε) for the portion of the spectrum below $\lambda(m, \varepsilon)$, but the eigenfunctions would lose all significance outside the range, even if the relative error in the eigenvalue is small.

We now discuss results obtained in the special case $q = 0$. Table 5.1 records the eigenvalues as estimated using biharmonic splines at the first three levels. The true eigenvalues are computed in [DSV]. Note, however, that a different normalization for the definition of the Laplacian was used there, so the values in [DSV] must be multiplied by 7.5 to obtain the values listed in the first column. (We have also corrected for some roundoff error in the first two eigenvalues as reported in [DSV].)

We follow the notation from [DSV] to label the eigenvalues. There are three series of eigenvalues: $\lambda_n^{(2)}$, $n \geq 0$, occur with multiplicity 1; $5^k\lambda_n^{(5)}$, $n \geq 0$, $k \geq 0$, occur with

multiplicity $(3^k + 3)/2$; $5^k\lambda_n^{(6)}$, $n \geq 0$ and $n \equiv 1$ or 2 (mod 4), $k \geq 0$, occur with multiplicity $(3^{k+2} - 3)/2$. The multiplicities were previously determined in [FS]. The data is consistent with the observation that the estimates are always overestimates and improve as the level increases. According to the data, the relative error is around 6.5% for the largest eigenvalue estimates at each of the levels. If we look only at the eigenvalue estimates at each level that correspond to eigenvalues that appeared at the previous level (that effectively means the first third of the eigenvalues, counting multiplicity) then the relative error drops to around 0.11%.

The data also reveals that the order and multiplicity of the eigenvalues is correctly predicted by the approximation. It is not difficult to present an argument that the approximation using harmonic splines always gives the correct multiplicity. The reason is that the multiplicities are explained by certain symmetries of a combinatorial nature, and these symmetries are also present in the spaces of harmonic splines. It is not clear how to present such an argument in the case of biharmonic splines, but the data strongly suggests that the result is valid.

The data also reveals a striking "octave" pattern to the sequence of eigenvalues, with a periodic repetition of groups of seven consecutive eigenvalues (not counting multiplicity), which has not been previously remarked upon. We will digress a little to explain this pattern, although the argument is unrelated to the finite element approximations.

Theorem 5.1. *The eigenvalues in consecutive order* (*not counting multiplicity*) *occur in "octaves" as follows*:

$$\begin{aligned}(5.14)\qquad \lambda^{(2)}_{2^k(2m+1)-2} &< \lambda^{(5)}_{2^k(2m+1)-2} < \lambda^{(5)}_{2^k(2m+1)-1} < \lambda^{(2)}_{2^k(2m+1)-1} \\ &< 5^k\lambda^{(5)}_{2m} < 5^{k-1}\lambda^{(6)}_{2m+\varepsilon} < 5^k\lambda^{(5)}_{2m+1}\end{aligned}$$

for $m \geq 0$ and $k \geq 1$, where $\varepsilon = 1$ if m is even, and $\varepsilon = 0$ if m is odd. The octaves are ordered according to the size of the even numbers $2^k(2m+1) - 2$, so that the next octave above (5.14) *corresponds to (m', k') satisfying*

$$(5.15)\qquad 2^{k'}(2m'+1) - 2 = 2^k(2m+1).$$

Proof. Let

$$(5.16)\qquad \Phi_\pm(x) = \tfrac{1}{2}(5 \pm \sqrt{25 - 4x})$$

and write $\Phi_w = \Phi_{w_k}\Phi_{w_{k-1}} \cdots \Phi_{w_1}$ where $w = (w_k, \ldots, w_1)$ with each $w_j = \pm$. A complete description of the eigenvalues in terms of these functions is given in [FS] (this is with the other normalization). Following the notation of [DSV]:

$$(5.17)\qquad \lambda_n^{(2)} = \lim_{k\to\infty} 5^k \Phi_{w_{[k]}}(2)$$

where $w_{[k]} = (w_k, \ldots, w_1)$ for

$$(5.18)\qquad w_j = (-1)^{1+\delta_{j-1}+\delta_j}$$

if

$$(5.19)\qquad n = \sum_{j=0}^{\infty} \delta_j 2^j$$

is the binary representation of n. Note that all but a finite number of δ_j are zero, so all but a finite number of Φ_{w_j} are Φ_-. Similarly

$$\lambda_n^{(5)} = \lim_{k\to\infty} 5^k \Phi_{w_{[k]}}(5) \tag{5.20}$$

with the same relationship (5.18) holding. For $\lambda_n^{(6)}$, the formula is slightly different,

$$\lambda_n^{(6)} = \lim_{k\to\infty} 5^{k+1} \Phi_{w_{[k]}}(6). \tag{5.21}$$

The condition that $n \equiv 1$ or 2 mod (4) means that $\Phi_{w_1} = \Phi_+$.

Now we may restrict x to the interval $[0, 6]$ in all these computations. Note that Φ_- is increasing and Φ_+ is decreasing on this interval, and also that $\Phi_-(x) < \Phi_+(x)$. Thus Φ_w is increasing if there are an even number of $w_j = +$, and decreasing if there are an odd number. Now from (5.18) it follows that the parity of the number of $w_j = +$ is the same as the parity of n, so $\Phi_{w_{[k]}}(2) < \Phi_{w_{[k]}}(5)$ for n even and the reverse inequality for n odd. The fact that strict inequality is preserved in the limit as $k \to \infty$ follows from the fact that $5\Phi'_-(x) \to 1$ as $x \to 0$ at a suitable rate. This proves the first and third inequality in (5.14). The second inequality is proved in [DSV].

To prove the fourth inequality in (5.14) it suffices to show

$$\Phi_{w_{[p]}}(2) < \Phi_{w'_{[p-k]}}(5) \tag{5.22}$$

for sufficiently large p, where $w_{[p]}$ is determined by $n = 2^k(2m+1) - 1$ and $w'_{[p-k]}$ by $n = 2m$. For, if we multiply (5.22) by 5^p and take the limit as $p \to \infty$, we obtain $\lambda^{(2)}_{2^k(2m+1)-1} \le 5^k \lambda^{(5)}_{2m}$, and we can remove the equality as before. To prove (5.22) observe that $2^k(2m+1) - 1 = 2^k(2m) + 2^k - 1$ in binary has the digits of $2m$ translated k places to the left and followed by k ones. By (5.18) this means that $\Phi_{w_{[p]}} = \Phi_{w'_{[p-k]}} \Phi_+ \Phi_-^{k-1}$. Recall that $\Phi_{w'_{[p-k]}}$ is an increaisng function because $2m$ is even. Thus (5.22) follows from $\Phi_+(x) < 5$ for x in the interval $[0, 6]$ (here $x = \Phi_-^{k-1}(2)$).

The last two inequalities in (5.14) are quite easy. Note that it suffices to take $k = 1$, and show

$$5\lambda^{(5)}_{4m} < \lambda^{(6)}_{4m+1} < 5\lambda^{(5)}_{4m+1} \qquad \text{and} \qquad 5\lambda^{(5)}_{4m+2} < \lambda^{(6)}_{4m+2} < 5\lambda^{(5)}_{4m+3}. \tag{5.23}$$

The factor of 5 comes from the different powers of 5 in (5.20) and (5.21), so we obtain (5.23) from $\Phi_-(5) < \Phi_+(6) < \Phi_+(5)$ by applying the increasing $\Phi_{w_{[k]}}$ in the $4m$ case and the decreasing $\Phi_{w_{[k]}}$ in the $4m+2$ case.

We have shown that the ordering (5.14) within octaves is valid. Next we need to show that the last term in one octave lies below the first term in the next octave, namely

$$5^k \lambda^{(5)}_{2m+1} < \lambda^{(2)}_{2^k(2m+1)}. \tag{5.24}$$

Note that the binary expansion of $2m+1$ gets shifted to the left k places followed by k zeros to obtain the binary expansion of $2^k(2m+1)$, and there is a one in the place before the added zeros. If we write $w_{[p]}$ to correspond to $n = 2m+1$ and $w'_{[p+k]}$ to correspond to $n = 2^k(2m+1)$, then $\Phi_{w'_{[p+k]}} = \Phi_{w_{[p]}} \Phi_+ \Phi_-^{k-1}$. Now $\Phi_+ \Phi_-^{k-1}(2) < 5$, and since $\Phi_{w_{[p]}}$ is decreasing we obtain

$$\Phi_{w_{[p]}}(5) < \Phi_{w'_{[p+k]}}(2). \tag{5.25}$$

Multiplying (5.25) by 5^{p+k} and taking the limit as $p \to \infty$ proves (5.24).

Finally, since every eigenvalue appears in one of the octaves, we have the complete ordered list. ■

There is considerably more structure to the spectrum than we have described. In particular, the spectral gaps toward the top of the octaves (the last two inequalities in (5.14)) appear to be significantly larger than elsewhere, including between octaves. Undoubtedly, this could be explained by further study of the inequalities in the proof. Also, the largest gaps occur exactly at the breaks between the approximate spectra at different levels. It is not clear what the explanation for this might be (it is not true for harmonic splines). Another interesting question is whether there is any asymptotic statement about the relative eigenvalues within octaves.

6. Space–Time Equations

In this section we consider two basic space–time equations, the heat equation and the wave equation. In both cases we take Dirichlet boundary conditions, and no forcing term. The initial value problem for the heat equation is

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \Delta_x u(x,t), \\ u(x,0) = f(x), \\ u(x,t)|_{x\in V_0} = 0, \end{cases} \tag{6.1}$$

and similarly for the wave equation

$$\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} = \Delta_x u(x,t), \\ u(x,0) = f(x), \ \dfrac{\partial u(x,0)}{\partial t} = g(x), \\ u(x,t)|_{x\in V_0} = 0. \end{cases} \tag{6.2}$$

The data f (or f and g) are assumed to be continuous functions, vanishing on the boundary for consistency. Actually the most interesting solutions are the heat kernel and the wave propagator, where we take f equal to a delta function at a point ($f = 0$ and g a delta function for the wave propagator). Since the delta functions are not continuous functions, we can only hope to approximate these solutions by using approximations to delta functions in our spline spaces.

The solution to the heat equation (6.1) can be written formally

$$u(x,t) = e^{t\Delta} f(x), \tag{6.3}$$

or, more precisely,

$$u(x,t) = \sum a_j e^{-t\lambda_j} u_j(x), \tag{6.4}$$

where $\{u_j\}$ is an orthonormal basis of Dirichlet eigenfunctions for $-\Delta$ with eigenvalues $\{\lambda_j\}$, and

$$a_j = \int f u_j \, d\mu. \tag{6.5}$$

The spline approximation is obtained simply by replacing the eigenfunctions and eigenvalues by the spline approximations $\{\widetilde{u}_k\}$ and $\{\widetilde{\lambda}_k\}$ given by (5.2) (with $Q = 0$). Thus

$$P_m u(x,t) = \sum \widetilde{a}_j e^{-t\widetilde{\lambda}} \widetilde{u}_j(x) \tag{6.6}$$

for

$$\widetilde{a}_j = \int f \widetilde{u}_j \, d\mu, \tag{6.7}$$

where the sum in (6.6) is a finite sum. Note that (6.7) amounts to projecting f into the spline space via the L^2 orthogonal projection. One could also approximate f by an element in the spline space using a different method, and then use this spline in place of f in (6.7).

This is not the way we would normally compute the approximation, however, since we prefer to express functions in terms of our spline basis. So if

$$P_m u(x,t) = \sum c_j(t)\varphi_j(x) \tag{6.8}$$

is this representation, then

$$c'(t) = -G^{-1}Ec(t), \tag{6.9}$$

hence

$$c(t) = e^{-tG^{-1}E}c(0), \tag{6.10}$$

where $\{c_j(0)\}$ are the coefficients of f, if f is already in the spline space or, more generally, an approximation to f in the spline space. We can use any matrix exponentiation algorithm to compute (6.10). If we use a finite difference approximation in (6.9), this amounts to writing

$$e^{-tG^{-1}E} = (e^{-k^{-1}tG^{-1}E})^k$$

and then approximating $e^{-k^{-1}tG^{-1}E}$ by $I - k^{-1}tG^{-1}E$. It is necessary to take k sufficiently large for this approximation to be accurate

$$c(t) \approx (I - k^{-1}tG^{-1}E)^k c(0). \tag{6.11}$$

If $\widetilde{\lambda}_{\max}$ denotes the largest eigenvalue of $G^{-1}E$ (the maximum of the eigenvalues $\widetilde{\lambda}_j$), then $(k^{-1}t\widetilde{\lambda}_{\max})^2$ gives the order of magnitude of the error for this approximation to $P_m u$.

From now on we concentrate on error estimates for $P_m u$ (called the *semidiscrete* approximation). We will assume either that f belongs to the spline space, or that it satisfies the hypotheses of Theorem 3.1 so that it may be approximated by splines with the appropriate accuracy. We will follow closely the arguments in the classical theory, as presented in Theorem 3.1 of [Th]. Since much of the argument is generic in nature, we will only outline the proof.

Theorem 6.1.

(a) *Let* $f \in \mathrm{dom}_{L^2} \Delta$ *with* $f|_{V_0} = 0$. *Then for harmonic splines*

$$\|u - P_m u\|_2 \le c\|\Delta f\|_2 \rho^m \tag{6.12}$$

for all $t > 0$.

(b) *Let* $f \in \mathrm{dom}_{L^2}\,\Delta^2$ *with* $f|_{V_0} = 0$ *and* $\Delta f|_{V_0} = 0$. *Then for biharmonic splines*

$$\|u - P_m u\|_2 \le c\|\Delta^2 f\|_2 \rho^{2m} \tag{6.13}$$

for all $t > 0$.

Proof. We may assume that f belongs to the spline space in view of the remarks following Theorem 3.1 and the fact that $e^{t\Delta}$ contracts L^2-norms. We write $e = P_m u - u$ for the error, and let $(-\Delta)^{-1}$ and $(-\Delta_m)^{-1}$ denote the inverse of $-\Delta$ with Dirichlet boundary conditions and the spline approximation, respectively. Then a simple computation shows that

$$\frac{\partial e}{\partial t} - \Delta_m e = -\Delta u + \Delta_m u$$

hence

$$(-\Delta_m)^{-1}\frac{\partial e}{\partial t} + e = -h \tag{6.14}$$

for

$$h = ((-\Delta_m)^{-1} - (-\Delta)^{-1})\frac{\partial u}{\partial t}. \tag{6.15}$$

We take the inner product of (6.14) with e to obtain

$$\frac{1}{2}\,\frac{\partial}{\partial t}\int e(-\Delta_m)^{-1} e\, d\mu + \|e\|_2^2 = -\int he\, d\mu \le \tfrac{1}{2}\|e\|_2^2 + \tfrac{1}{2}\|h\|_2^2.$$

After integration this yields

$$\int_0^t \|e\|_2^2\, ds \le \int_0^t \|h\|_2^2\, ds \tag{6.16}$$

since $e(0) = 0$ and $(-\Delta_m)^{-1}$ is positive. Also, taking the inner product of (6.14) with $\partial e/\partial t$ and doing similar manipulations yields

$$t\|e(t)\|_2^2 \le 2t\|h\|_2\|e\|_2 + \int_0^t \left(\|e\|_2^2 + 2\|h\|_2\|e\|_2 + 2s\left\|\frac{\partial h}{\partial t}\right\|_2 \|e\|_2\right) ds. \tag{6.17}$$

Combining (6.16) and (6.17) with elementary inequalities yields

$$\|e(t)\|_2 \le c \sup_{o<s\le t}\left(s\left\|\frac{\partial h}{\partial t}(s)\right\|_2 + \|h(s)\|_2\right). \tag{6.18}$$

We will use the estimate (6.18) to prove both parts of the theorem. The arguments used to prove (6.18) did not involve the smoothing properties of the solution of the heat equation, but these will now be used in the estimates for $\|h\|_2$ and $\|\partial h/\partial t\|_2$. For part (a) we use the same part of Theorem 3.7, with $q = 0$, and the Remarks following it. In view of (6.15) we have

$$\|h(t)\|_2 \le c\rho^m \left\|\frac{\partial u}{\partial t}\right\|_2 \tag{6.19}$$

and

$$t\left\|\frac{\partial h}{\partial t}(t)\right\|_2 \le c\rho^m t\left\|\frac{\partial^2 u}{\partial t^2}\right\|_2. \tag{6.20}$$

But

$$\left\|\frac{\partial u}{\partial t}\right\|_2 = \left(\sum |\lambda_j a_j e^{-t\lambda_j}|^2\right)^{1/2} \le \left(\sum |\lambda_j a_j|^2\right)^{1/2}$$

while

$$t\left\|\frac{\partial^2 u}{\partial t^2}\right\|_2 = \left(\sum |t\lambda_j^2 a_j e^{-t\lambda_j}|^2\right)^{1/2} \le e^{-1}\left(\sum |\lambda_j a_j|^2\right)^{1/2},$$

and $\|\Delta f\|_2 = (\sum |\lambda_j a_j|^2)^{1/2}$. This yields (6.12).

The reasoning for part (b) is similar. In place of (6.19) and (6.20) we have

$$\|h(t)\|_2 \le c\rho^{2m}\left\|\frac{\partial^2 u}{\partial t^2}\right\|_2$$

and

$$t\left\|\frac{\partial h}{\partial t}(t)\right\|_2 \le c\rho^{2m} t\left\|\frac{\partial^3 u}{\partial t^3}\right\|_2$$

using part (b) of Theorem 3.7. Now

$$\left\|\frac{\partial^2 u}{\partial t^2}\right\|_2 = \left(\sum |\lambda_j^2 a_j e^{-t\lambda_j}|^2\right)^{1/2} \le \left(\sum |\lambda_j^2 a_j|^2\right)^{1/2}$$

and

$$t\left\|\frac{\partial^3 u}{\partial t^3}\right\|_2 = \left(\sum |t\lambda_j^3 a_j e^{-t\lambda_j}|^2\right)^{1/2} \le e^{-1}\left(\sum |\lambda_j^2 a_j|^2\right)^{1/2}$$

and

$$\|\Delta^2 f\|_2 = \left(\sum |\lambda_j^2 a_j|^2\right)^{1/2}$$

(this uses the assumption that $\Delta u|_{V_0} = 0$). This yields (6.13). ∎

The solution of the wave equation (6.2) can be expressed similarly as

$$u(x,t) = \cos t\sqrt{-\Delta}\, f + \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}} g, \tag{6.21}$$

or, more precisely,

$$u(x,t) = \sum\left(a_j \cos t\sqrt{\lambda_j} + b_j \frac{\sin t\sqrt{\lambda_j}}{\sqrt{\lambda_j}}\right) u_j(x) \tag{6.22}$$

with

$$a_j = \int f u_j \, d\mu, \qquad b_j = \int g u_j \, d\mu. \tag{6.23}$$

The spline approximation is

$$P_m u(x,t) = \sum \left(\widetilde{a}_j \cos t\sqrt{\widetilde{\lambda}_j} + b_j \frac{\sin t\sqrt{\widetilde{\lambda}_j}}{\sqrt{\widetilde{\lambda}_j}} \right) \widetilde{u}_j(x) \tag{6.24}$$

with

$$\widetilde{a}_j = \int f\widetilde{u}_j \, d\mu, \qquad \widetilde{b}_j = \int g\widetilde{u}_j \, d\mu.$$

In the spline basis representation (6.8) we have

$$c(t) = \cos t\sqrt{G^{-1}E}\, c_f(0) + \frac{\sin t\sqrt{G^{-1}E}}{\sqrt{G^{-1}E}} c_g(0)$$

with the coefficients of f and g (or spline approximations to them) in the spline basis denoted $c_f(0)$ and $c_g(0)$. Again there are a variety of options for approximating the matrix computations in (6.26). The numerical implementation is very similar to the case of the heat equation.

We cannot prove the analog of Theorem 6.1 for the wave equation, however, because we do not have the analogous smoothing properties to estimate derivatives of the solution. In other words, we could get as far as (6.18) in the argument, but no further. In fact, it is possible to obtain some results by assuming more smoothness for the data, but these results are essentially uninteresting because the splines themselves do not have this additional smoothness.

The web site contains programs implementing the approximate solutions to both the heat and wave equations, and the graphical output of trial runs. The approximate solutions of the wave equation exhibit quite intricate behavior on a small scale, but it is not clear whether this is attributable to artifacts of the method or actual features of the wave equation. Solutions may be viewed in real time to simulate vibrations propagating through the gasket.

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M. Gibbons
Mathematics Department
Manhattan College
Bronx, NY 10471
USA
mgibbons17@aol.com
Current address:
IBM
North Castle Drive, 2A-70C
Armonk, NY 10504
USA

A. Raj
Mathematics Department
University of California
Berkeley, CA 94720
USA
arhoon@hotmail.com
Current address:
(from September, 2001)
Courant Institute
251 Mercer Street
New York, NY 10012
USA

R. S. Strichartz
Mathematics Department
Malott Hall
Cornell University
Ithaca, NY 14853
USA
str@math.cornell.edu