Large deviation and self-similarity analysis of graphs: DAX stock prices

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Abstract – Two methods for analyzing graphs such as those occurring in the stock market, geographical profiles and rough surfaces, are investigated. They are based on different scaling laws for the distributions of jumps as a function of the lag. The first is a large deviation analysis, and the second is based on the concept of a self-similar process introduced by Mandelbrot and van Ness. We show that large deviation analysis does not apply to either the stock market nor fractional Brownian motion ($H \neq 0.5$). Instead the analysis based on self-similarity is applicable to both, and does indicate that especially the negative log-price fluctuations have a large degree of self-similarity. The latter analysis allows one to probe the degree of self-similarity of a process, beyond what is possible with the exponent H typically used to describe self-affine graphs.

Introduction

Sums of random variables (random walks) and their properties play a key role in the modeling and explanation of a wide variety of phenomena including the stock markets[1, 2] and many problems in statistical mechanics[3, 4]. Their use in the modeling of prices[5] goes back to Bachelier and is closely related to the theory of Brownian motion. Basically, one assumes that the logarithm of tomorrow's price $\ln z(t+1)$ is that of today's price $\ln z(t)$ plus a random price increment. Part of economic research focuses on the estimation of the probability distribution of these log-price changes and the determination of possible dependencies or deterministic components[6].

When the log-price $\ln z(t)$ at time t is interpreted as a height $h(x)$ at position x, the situation becomes typical for that encountered in profiles of rough surfaces[3, 4, 7] without overhangs. Like the log-price record shown in Figure 1, many rough surfaces fit well in the framework of affine fractal geometries. Typically[7] such geometries are described by an exponent H related to the surface fluctuations $\langle |h(x+\tau) - h(x)| \rangle \sim \tau^H$. Theoretical efforts aim at understanding the ubiquity of certain values of H. However, this exponent gives limited information about the underlying distribution of height differences. The distribution and it's scaling properties as a function of the displacement (or lag) τ can provide a much more accurate quantitative characterization as well as an instrument for a more detailed comparison between surfaces, or the effects of various treatments.

This paper focuses on two methods of analyzing such distributions reflecting two distinct scaling laws: the first is based on large deviation analysis, which is closely related to multifractal analysis. The second is an application of the concept of *self-similarity* (S-S) defined by Mandelbrot and van Ness[8] in connection with fractional Brownian motion (fBm). Fractional Brownian motions have been applied as

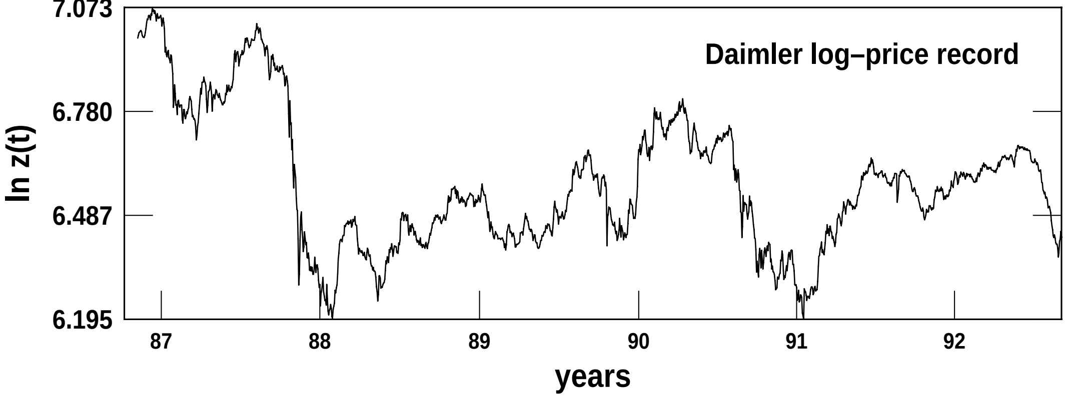

Figure 1: Natural logarithm of the Daimler price record

crude models of stock records and have been extensively used as models for topographical profiles and other processes[8, 9, 10, 11, 12, 13, 14, 15].

We shall see that in some cases, like the Gaussian random walk, both large deviation and self-similarity analysis apply. For fBm (with $H \neq 0.5$) only the self-similarity analysis provides a description. For the German stock market discussed here, each of the two analysis approximately describe a different part of the data. The applications show that very different distributions can give rise to the same exponent H.

We now first discuss large deviation analysis and then the analysis of self-similarity. The role played by the former in the theory of multifractal[16, 17] measures was discussed in Refs. [18, 19]

Large deviation analysis

One assumes either the height differences $D(\tau) = h(x+\tau) - h(x)$ or the log-price differences $D(\tau) = \ln z(t+\tau) - \ln z(t)$ to be the sample values of sums of random variables $\mathbf{L}_i$, i.e, sample values of the random variable

$$\mathbf{S}_L(k) = \sum_{i=1}^{k} \mathbf{L}_i.$$

Here, we consider the most simplest cases where k is a linear function of the lag τ. The validity of such a stochastic modeling for different phenomena can be found in the corresponding literature. The Gaussian central limit theorem applies when the random variables $\mathbf{L}_i$ are independent and identically distributed (i.i.d.) with finite expectation and variance. Then, the variance of $\mathbf{S}_L(k)$ and thus of $D(\tau)$ grows linearly with the number k of addends and with the lag τ. Therefore the size of the fluctuations grows proportional to $\sqrt{\tau}$, and the exponent H equals $\frac{1}{2}$. This illustrates the degree of degeneracy in H: whether the distribution of each of the $\mathbf{L}_i$ is a Gaussian, or a double exponential, or discrete such as 1 or -1 with equal probability $\frac{1}{2}$; in each of these cases $H = \frac{1}{2}$.

An analysis of the large deviations[20, 19] can distinguish between these different distributions of the single addends. For i.i.d. random variables $\mathbf{L}_i$ with a finite moment generating function $M(\theta) = \mathrm{E}e^{\theta\mathbf{L}}$ for all $\theta \in \Re$, one can show[20, 19] that the distributions of the sums $\mathbf{S}_L(k) = \sum_{i=1}^{k} \mathbf{L}_i$ satisfy a large deviation principle. This asserts that the probability for deviations ($\mathbf{R}_k$) of the *sample averages* from the expected value, i.e.,

$$\mathbf{R}_k = \frac{\mathbf{S}_L(k)}{k} - \mathrm{E}\mathbf{L}$$

decays exponentially in the sense that

$$\mathrm{Prob}\{\mathbf{R}_k \geq \rho\} \sim e^{kC(\rho)}, \tag{1}$$

moment generating function $M(\theta) = \mathrm{E}e^{\theta\mathbf{X}}$ using the following relationship

$$C(\rho) = \sup_{\theta} \left[\theta\rho - \ln M(\theta)\right]. \tag{2}$$

Appendix I shows an application to a double exponential random variable. This relation is essentially[19] the Legendre transform used to compute the $f(\alpha)$ curve in the method of moments [17].

About the notation: when price jumps $z(t+1) - z(t)$ are small compared to the price $z(t)$, one finds that $\ln z(t+1) - \ln z(t) \approx (z(t+1) - z(t))/z(t)$. The latter is the return. The letters "$\mathbf{R}$" and "ρ" have been used to resonate with the fact that the quantity $\mathbf{R}_k = (1/k)\mathbf{S}_L(k) - \mathrm{E}\mathbf{L} \approx (1/k)\mathbf{S}_L(k)$ is the average daily return rate in a k day trading interval. Note that we used the empirical result that for most markets the average rate of return $\mathrm{E}\mathbf{L} \approx 0$. For surfaces $\mathbf{R}_k$ would be the rate of change of the height (the slope) with respect to the expected rate of change $\mathrm{E}\mathbf{L}$. The latter can be interpreted as the surface tilt.

The method of estimation

There are (at least) two methods to estimate the large deviation rate function $C(\rho)$ in equation 1 from real data. The first estimates $C(\rho)$ by first estimating the left-hand side of equation 1 from the empirical data. The second first estimates the moment generating function $M(\theta)$ from the raw data, and subsequently uses Equation 2 to compute an estimate of $C(\rho)$. – For multifractals measures these methods correspond to the histogram method and the method of moments[16, 17, 18, 19]. The first method will be applied on empirical data, and the second in case the distribution of the single addends $\mathbf{L}_i$ are known (e.g. Appendix I).

When dealing with empirical data we will not make any assumptions on either the distribution or the dependencies of the random variables. Instead, knowing that a large deviation rate function $C(\rho)$ has been proven to exists under a wide range of conditions[20] on the random variables $\mathbf{L}$, we will interpret Equation 1 as a possible scaling law for the probability distributions for different lags. There are cases where such a relationship does not hold, such as for Lévy stable processes; then other rules of collapse apply (see e.g. ref. [21, 22, 23].)

From Equation 1 it follows that $C(\rho)$ could be obtained by plotting $(1/k)\ln \mathrm{Prob}\{\mathbf{R}_k \geq \rho\}$ versus ρ for $\rho > \mathrm{E}\mathbf{L} \approx 0$ in the limit $k \to \infty$. The $\rho < 0$ part of $C(\rho)$ is obtained by plotting $(1/k)\ln \mathrm{Prob}\{\mathbf{R}_k \leq \rho\}$ versus ρ. When $C(\rho)$ is strictly convex, this procedure can be replaced (p. 951 in Ref [19]) by plotting

$$C_k(\rho) \equiv \frac{\ln \mathrm{P}_k(\rho)}{k} \quad \text{versus} \quad \rho, \tag{3}$$

where the probability density $\mathrm{P}_k(\rho)d\rho$ is the probability to have a deviation lying between ρ and $\rho + d\rho$. So to say, this plotting procedure provides a way to "collapse" all the $\mathrm{P}_k(\rho)$ for different lags k.

In order to assess the speed of convergence to $C(\rho)$ let us look at a simple case: Namely we assume that the $\mathbf{L}_i$ (e.g, the daily log-price changes, or the height differences for points a unit distance from each other) are independent and identically distributed (i.i.d.) with a Gaussian distribution with zero mean, i.e.,

$$\mathrm{P}^G(\rho)d\rho = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\rho}{\sigma}\right)^2\right\} d\rho, \tag{4}$$

where σ^2 is the variance and the superscript ^{G} stands for "Gaussian." The probability distribution of the sum of k such random variables is known exactly. For the average (ρ) over k such r.v. one finds $\mathrm{P}_k^G(\rho)d\rho = \sqrt{k}/(\sigma\sqrt{2\pi}) \exp\left\{-\frac{1}{2}\left(\sqrt{k}\rho/\sigma\right)^2\right\} d\rho$. Applying the collapse rule Equation 3 we find that

$$C_k(\rho) = \frac{(1/2)\ln k - \ln(\sigma\sqrt{2\pi})}{k} - \frac{1}{2}\left(\frac{\rho}{\sigma}\right)^2. \tag{5}$$

$$C(\rho) = -\frac{1}{2}\left(\frac{\rho}{\sigma}\right)^2, \tag{6}$$

which is well-known. The reason for this exercise is that Equation 5 exposes the finite-lag corrections $\varepsilon(k) = (1/k)[(1/2)\ln k - \ln(\sigma\sqrt{2\pi})]$, related to the convergence of Equation 3 to a single curve[24]. These corrections are of importance in practical cases where it is impossible to take $k \to \infty$. Therefore, by including the finite-lag corrections as follows in Equation 3

$$\frac{\ln \mathrm{P}_k(\rho)}{k} - \varepsilon(k) \;\text{ versus }\; \rho. \tag{7}$$

one gets immediate convergence. We used the following slight generalization

$$\varepsilon'(k) = (1/k)[(1/2)\ln k - \ln(A\sigma\sqrt{2\pi})], \tag{8}$$

where the tuning parameter A is chosen such that the correction yields the fasted convergence (see Appendix I).

Large deviation analysis of fractional Brownian motions

Fractional Brownian motions (fBm) are "Gaussian random functions defined as follows: $\mathbf{B}(t)$ being ordinary Brownian motion, and H a parameter satisfying $0 < H < 1$, fBm of exponent H is the moving average of $d\mathbf{B}(t)$, in which past increments of $B(t)$ are weighted by the kernel $(t-s)^{H-1/2}$[8]." It is important to note that the increments $\mathbf{Y}_k = \mathbf{X}(t+k) - \mathbf{X}(t)$ of a fBm $\mathbf{X}(t)$ of type H are stationary and have a Gaussian distribution with variance depending on the lag k. Furthermore the motion (or: graph, function) is *self-similar*[8] in the sense that

$$\mathbf{Y}_l \;\text{ i.d. }\; k^{-H}\mathbf{Y}_{kl}, \tag{9}$$

where i.d. stand for identically distributed. With a choice of appropriate units so that $l = 1$, the above property implies that the rescaled jumps $\mathbf{Z}_k = k^{-H}\mathbf{Y}_k$ for lag k have the same distribution as $\mathbf{Y}_1$ for lag 1. We refer to the latter distribution as the *basic distribution*. In the case of fBm this is a Gaussian distribution. In general, let $P_{Z,k}(z)\,dz$ be the probability density of $\mathbf{Z}_k$ for a stationary self-similar function, then Equation 9 implies that

$$P_{Z,k}(z)\,dz = Q(z)\,dz \tag{10}$$

where $Q(z)\,dz \equiv P_{Z,1}(z)\,dz$ is the density of the basic distribution, which can differ from the Gaussian.

To find out whether such a stationary self-similar function has a non-degenerate large deviation rate function we write $\mathbf{Y}_k = \mathbf{X}(t+k) - \mathbf{X}(t)$ as a sum random variables $\mathbf{Y}_k = \sum_{i=1}^{k} \mathbf{L}_i$. For fBm it is clear that the $\mathbf{L}_i$ are dependent. Therefore, even though the jumps have a Gaussian distribution, it is not obvious that a rate function exists. To study the large deviation rate function for the deviations of $\frac{1}{k}\mathbf{Y}_k$ from the expectation, which is zero for fBm, it is convenient to introduce the random variable $\mathbf{R}_k = \frac{1}{k}\mathbf{Y}_k$, whose sample values are denoted by ρ. So $\mathbf{Z}_k = k^{1-H}\mathbf{R}_k$, and a change of variable in Equation 10 yields

$$P_{R,k}(\rho)\,d\rho = k^{1-H}Q(k^{1-H}\rho)\,d\rho. \tag{11}$$

The existence of the large deviation rate function C depends on the existence of

$$C(\rho) = \lim_{k\to\infty}\frac{1}{k}\ln P_{R,k}(\rho) = \lim_{k\to\infty}\frac{1}{k}\ln\left[k^{1-H}Q(k^{1-H}\rho)\right] = \lim_{k\to\infty}\frac{1}{k}\ln Q(k^{1-H}\rho). \tag{12}$$

The right hand term is independent of k, only when $\ln Q(k^{1-H}\rho) \sim k\rho^{\xi}$ for k large, where $\xi = 1/(1-H)$. Therefore, a stationary self-similar process with parameter H will only have a non-degenerate any type H, the basic density is the Gaussian $Q^G(z)\,dz$ in Equation 4, so the above condition is only satisfied for fBm with parameter H such that $\xi = 2$, i.e., plain Brownian motion $H = \frac{1}{2}$. Moreover, starting from the Gaussian density P^G, and using Equation 11 we find for fBm with parameter H that $P_{R,k}(\rho)\,d\rho = \left(\sigma\sqrt{2\pi}k^{H-1}\right)^{-1} \exp\left(-\frac{1}{2}[\rho/(\sigma k^{(H-1)})]^2\right)$. Equation 12 yields $C(0) = 0$ for all H and for $\rho \neq 0$

$$C(\rho) = \begin{cases} -\infty & H < \frac{1}{2} \\ -\frac{1}{2}\left(\frac{\rho}{\sigma}\right)^2 & H = \frac{1}{2} \\ 0 & H > \frac{1}{2}. \end{cases} \tag{13}$$

So only the case $H = \frac{1}{2}$ is non degenerate. As is shown for H=0.8 in Figure 2, the behaviors described by the above equations are clearly seen when applying a large deviation analysis to numerical simulations for different values of H. These simulations used a mid-point displacement algorithm[11].

Self-similarity analysis: alternative estimation of H

Except for the case $H = \frac{1}{2}$, large deviation analysis does not yield a rate function for the family of fBm. The dependencies turn out to be too strong. On the other hand self-similarity (Equation 9,) implies the relation Equation 11 between the densities for various values of the lags k. Therefore, for an arbitrary stationary self-similar process the densities $P_{R,k}(\rho)$ can be collapsed onto a single curve by plotting $k^{H-1}P_{R,k}(\rho)$ versus $k^{1-H}\rho$. For processes attracted by the Gaussian which do not have fat tails, this plotting procedure can give reasonable estimates of H. When the tails of the distributions are fat, which is the case for the log-price records discussed later, it is better to plot the log of the distributions, i.e.,

$$(H - 1 + A)\ln k + \ln P_{R,k}(\rho) \quad \text{versus} \quad k^{1-H}\rho. \tag{14}$$

Here, A, is a parameter which, when chosen appropriately, gets rid of spurious vertical shifts of the distributions which are due to under sampling for large k. This parameter is harmless in the sense that one is interested in whether the *shapes* of the distributions are the same on different scales: the heights are not important.

In the case of fBm of any type H, such a plot will collapse the densities for the different values of the lag k onto the curve of shape $C(\rho) = -\frac{1}{2}\left(\frac{\rho}{\sigma}\right)^2$ for some σ. Such a collapse is shown in the left part of Figure 2 for fBm with $H = 0.54$

Conversely, the exponent H can be estimated as that value of H yielding the best collapse, However, this is not the most convenient procedure for determining H. When H is not known a priori, which is the case for the stocks, we first estimated it by studying the scaling behavior of the width of the distributions as a function of k. In the cases considered here we determined H through $< D(k) >_t = < |\ln z(t+k) - \ln z(t)| >_t \sim k^H$.

Application to DAX stocks

The DAX stocks are those 30 stocks on the Frankfurt stock exchange that are used to evaluate what is known as the DAX index.

The daily price record of stocks $\sigma = 1, 2, \ldots, 30$ are denoted by $z^\sigma(t)$

$$\{z^\sigma(1), z^\sigma(2), \ldots, z^\sigma(T)\} = \{z^\sigma(t)\}_{t=1}^T$$

where t counts business days and $T(= 1452)$ is the total number of business days studied. This is from November 11, 1986 until September 7, 1992. The symbols for the 30 DAX stocks are *alv, bas, basf, bay,*

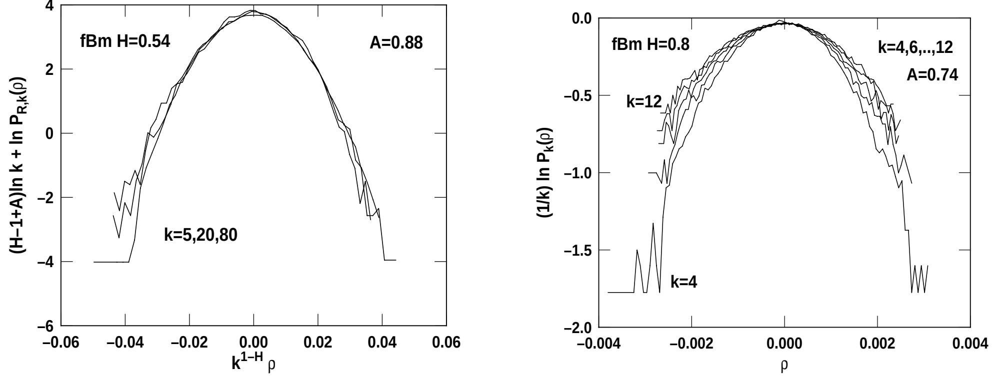

Figure 2: LEFT) Self-similarity analysis for fBm with $H = 0.54$. The correction parameter $A = 0.88$ was used to put the maxima of the distributions at equal heights. RIGHT) Large deviation analysis for fBm $H = 0.8$. The convergence to 0 is in agreement with Equation 13. The $H = 0.54$ case behaves similarly, but much slower with increasing k.

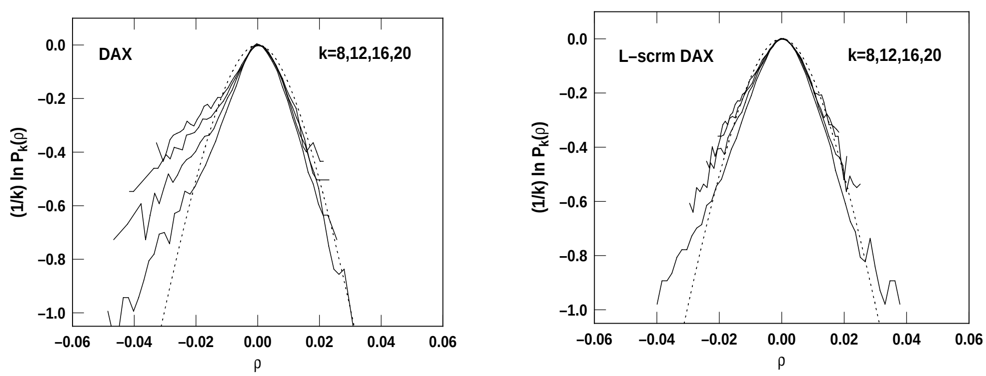

Figure 3: Fig. 2. LEFT) Large deviation analysis for the log-price records of 30 DAX stocks. The lags shown are $k = 2, 4, \ldots, 12$ business days. In both the left and right figure, the value of the parameters in correction term $\varepsilon'(k)$ are $\nu = 0.5$, $A = 0.39$. The dotted curve is the large deviation rate function for Laplace random variables with the same standard deviation STD $= 0.082$ as the price records. RIGHT) Large deviation analysis of the time-scrambled DAX log-price records. Also here the corresponding Laplacian rate function is included for comparison.

bhw, bmw, bvm, cbk, cnt, dai, dgs, dbc, dbk, drb, hen, hfa, kar, kfh, lha, lin, man, mmw, met, prs, rwe, sch, sie, thy, veb, via, vow.

The left portion of Figure 3 shows the sequence of curves that results when applying the collapse rule 7 to the empirical distributions of returns rates ρ

$$\rho \in \{ \ (1/k)(\ln z^\sigma(t+k) - \ln z^\sigma(t)) \ \}_{t=1,\ldots,T-k; \ \sigma=1,\ldots,30}$$

for the both the DAX and the time-scrambled DAX stocks. Here the time-scrambling is done by randomly permuting the daily jumps in the logarithm of the price-record and then integrating to a new scrambled log-price record. This removes all time correlations in the records, and allows a quick (rough) analysis of the role of such correlations. Included in both figures is the large deviation rate function (Equation 17) for a double exponential random variable with the same variance as those of the stocks. For comparison we chose this random variable because the center portion of the distribution of daily log-price changes is very close to a double exponential.

We are sure that these tail fluctuations are not due to poor sampling, because they are absent in a similar analysis of 30 double exponential random walks of the same size as the stock samples. The situation is less clear for the right-hand side of the distribution. Our findings very much resemble those found in DLA [22, 23]. The relations between the methods of analysis used in this paper and those in references [22, 23], will be discussed elsewhere.

The symmetry of the rescaled distributions of the scrambled stocks shows that the excessive fatness of negative log-price fluctuations are a particular property of the stock market. From the plots resulting from these analyses one concludes that the time correlations in stock prices are such as to strongly moderate positive returns, slightly enhance the negative fluctuations. Correlations in the DAX stock records have been measured previously in Ref. [26].

As was discussed in the previous section, we estimated the values of H from $\ln < D(k) >_t$ versus $\ln k$ plots where the average runs over all 30 stocks and all time. The plots yielded remarkably straight lines between 1 and 85 business days. The value of the exponents were $H = 0.54$ for stocks and $H = 0.52$ for scrambled stocks. By means of comparison, the typical values for currencies fall between 0.56 and 0.60[2, 25]. In order to test for self-similarity of the underlying distributions over this range of lags, we used Equation 14 with the values of H estimated above and $A \approx 1$. The results in Figure 4 show that the densities of the log-price changes for scrambled stocks have a high degree of self-similarity over a range of 5 to 80 days. On the other hand the right tails of the distributions of the unscrambled log-prices are becoming narrower, while the maximum shifts to the right. That the maximum shifts to the right, is probably due to an overall increase of the stock prices in the period considered here, and as such is not of much significance. The narrowing of the right tail shows that large positive fluctuations become relatively less probable the more one waits (large k).

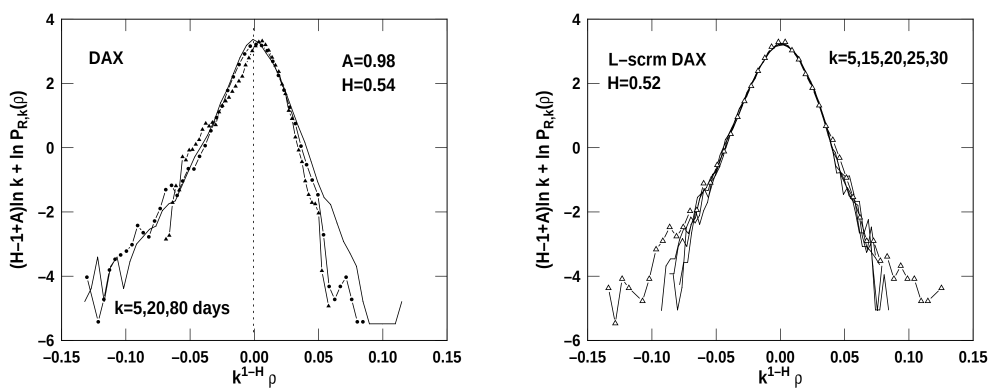

Figure 4: Fig. 3. LEFT) Analysis of self-similarity in the sense of Equation 9 for the log-price records of 30 DAX stocks. The values $k = 5, 20, 80$ were chosen so as to span the range over which the log-log plot used to estimate H was straight. In this way the behavior of the underlying distribution is exposed. RIGHT) Same analysis done for scrambled DAX stocks for $k = 5, 15, 20, 25, 30$.

For both the DAX and the scrambled DAX the plots of $\ln < D(k) >_t$ versus $\ln k$ yield straight lines and thus a seemingly well defined exponent H, not too far from the the Gaussian exponent $\frac{1}{2}$. However, a large deviation analysis, clearly shows that the process is not Gaussian, since there is no large deviation rate function. By investigating the self-similarity of the distributions in the sense of Equation 9, we find approximate self-similarity in the scrambled case with a symmetric basic distribution, and an asymmetric basic distribution for the actual log-price records. For the log-price records the negative fluctuations show a larger degree of self-similarity than the positive ones. The relations between the self-similarity found in the DAX records and the long-tailed Lévy distributions found in References [1, 27] is discussed in Ref. [25].

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Appendix I: Correction term in Large deviation analysis

The following is a heuristic argument for a generalization Equation 8. The finite size corrections $\varepsilon(k)$ are clearly linked to the normalization factor in the Gaussian density. In the general case, the width of the distribution of the sum of k random variables $\mathbf{X}$ will grow as $2k^H \mathrm{E}|\mathbf{X}|$, so the width of the distribution of ρ will grow as $2k^{H-1}\mathrm{E}|\mathbf{X}|$. Since the density should normalize to 1, the height at the $\rho = 0$ should scale like $\mathrm{P}_k(0) \approx A(2k^{H-1}\mathrm{E}|\mathbf{X}|)^{-1}$. In case the law of large numbers applies, one expects in the limit $k \to \infty$ that $\frac{1}{k}\ln \mathrm{P}_k(0) = 0$. For finite k one would like a correction term so that $\frac{1}{k}\ln \mathrm{P}_k(0) - \varepsilon''(k) = 0$. Substituting the above estimate for $P_k(0)$, one finds

$$\varepsilon''(k) = (1/k)[\nu \ln k - \ln(2A\mathrm{E}|\mathbf{X}|)]$$

with

$$\nu = 1 - H.$$

When dealing with random variables *attracted* to the Gaussian, the term $\mathrm{E}|\mathbf{X}|$ could replaced by the variance. This above suggest that one could further improve the convergence in such cases by empirically determining a value for the tuning parameters A and H, such that

$$\frac{\ln \mathrm{P}_k(\rho)}{k} - \varepsilon''(k) \ \text{ versus } \ \rho$$

yields the fastest convergence. We will restrict ourselves to the Gaussian case $\nu = \frac{1}{2}$

$$\begin{aligned} \frac{\ln \mathrm{P}_k(\rho)}{k} - \quad & \varepsilon'(k) \quad \text{versus } \ \rho \\ & \varepsilon'(k) \ \ = (1/k)[\frac{1}{2}\ln k - \ln(A\sigma\sqrt{2\pi})], \end{aligned} \tag{15}$$

which is a slight generalization of the exact Gaussian correction term.

Appendix II: Laplacian random variable

The large deviation rate function $C(x)$ for a Laplacian random variable $\mathbf{X}$ will be computed. First we will compute the moment generating function $M(\theta) = \mathrm{E}e^{\theta\mathbf{X}}$ and then use Equation 2

$$C(x) = \sup_{\theta}\left[\theta x - \ln M(\theta)\right] \tag{16}$$

to find the rate function.

The probability density for the double exponential or Laplacian random variable $\mathbf{X}$ is

$$p(x)dx = \frac{1}{2\beta}e^{-\left|\frac{x-\alpha}{\beta}\right|}dx$$

with $-\infty < \alpha < \infty$ and $0 < \beta < \infty$. The expectation $\mathrm{E}\mathbf{X} = \alpha$ and the variance $\mathrm{VAR}\{\mathbf{X}\} = 2\beta^2$ are both finite. The moment generating function is thus

$$M(\theta) = \int_{-\infty}^{\infty} \frac{1}{2\beta}e^{-\left|\frac{x-\alpha}{\beta}\right|}e^{\theta x}dx.$$

this interval one finds

$$M(\theta) = \frac{e^{\alpha\theta}}{1-\theta^2\beta^2} \qquad -\frac{1}{\beta} < \theta < \frac{1}{\beta}.$$

The supremum is found by computing the first derivative with respect to θ of the expression within square brackets in Equation 2. Equating this derivative to 0, yields two roots

$$\theta_{\pm}(x) = \frac{-1 \mp \sqrt{1 + \left(\frac{x-\alpha}{\beta}\right)^2}}{x-\alpha}.$$

The correct branch is easily identified as being $\theta(x) = \theta_{-}(x)$ because of the condition $\theta(\alpha) = 0$. Substituting this in Equation 2 yields

$$C(x) = -1 + \sqrt{1+y^2} \; + \ln[1 - y^{-2}(-1 + \sqrt{1+y^2})^2], \tag{17}$$

where we defined $y(x) = (x-\alpha)/\beta$. For small values of y, i.e., small deviation around the expectation α one finds

$$C(x) \approx \frac{1}{2}\frac{x^2}{2\beta^2} \quad (x-\alpha)/\beta \ll 1.$$

Since $2\beta^2$ is the variance, one thus recovers the Gaussian result for $\sigma^2 = 2\beta^2$ given in Equation 6.

References

[1] B.B. Mandelbrot, The variation of certain speculative prices. *The Journal of Business of the University of Chicago* **36**, 394-419 (1963)

[2] U.A. Müller, M.M. Dacorogna, R.B. Olsen, O.V. Pictet, M. Schwarz, C. Morgenegg, Statistical study of foreign exchange rates, empirical evidence of price change scaling law, and intraday analysis, *Journal of Banking and Finance* **14**, 1189-1208 (1990)

[3] J. Feder and A. Aharony (Eds.), *Fractals in Physics: Essays in honour of B.B. Mandelbrot* North-Holland (1990), also *Phycica D* **38**, 1-396 (1989)

[4] A. Bunde and S. Havlin (Eds.), *Fractals and Disordered Systems* Springer-Verlag, (1991)

[5] B.B. Mandelbrot, *The fractal geometry of nature* W.H.Freeman, San Francisco, 1982.

[6] E.E. Peters, *Chaos and order in the capital markets* John Wiley & Sons, (1991)

[7] F. Family and T. Vicsek(Eds.), *Dynamics of fractal surfaces* World Scientific, (1991)

[8] B.B. Mandelbrot and J.W. van Ness, Fractional Brownian motions, fractional noises and applications, *SIAM Review* **10**, 4, 422-437 (1968)

[9] B.B. Mandelbrot and J.R. Wallis, Computer experiments with fractional Gaussian noises I, II, III, *Water resources research* **5**, 1,228-267 (1969)

[10] R.F. Voss, 1/f noise and fractals in Economic time series In: *Fractal geometry and computer graphics* J.L. Encarnação, H.-O. Peitgen, G. Sakas, G. Englert (Eds.) Springer-Verlag, 45-52 (1992).

[11] J. Feder, *Fractal* Plenum Press (1988)

[12] H.-O. Peitgen and D. Saupe (Eds), *The science of fractal images* Springer Verlag (1988)

[13] D.L. Turcotte, *Fractals and chaos in geology and geophysics* Cambridge (1992)

[14] K. Falconer, *Fractal geometry* John Wiley & Sons (1990)

[15] H.-O. Peitgen, H. Jürgens, D. Saupe, *Chaos and Fractals* Springer-Verlag, New York, 1992.

of Geophysical Flows and Climate Dynamics, Proc. of the International School of Physics "Enrico Fermi,", Course LXXXVIII, Varenna 9083, edited by M. Ghil, R. Benzi and G. Parisi, North-Holland, New York, p. 84 (1985)

[17] T.C. Halsey, M.H. Jensen, L.P. Kadanoff, I. Procaccia and B.I. Shraiman, Fractal measures and their singularities: The characterization of strange sets *Phys. Rev. A* **33** 1141 (1986)

[18] B.B. Mandelbrot: An introduction to multifractal distribution functions, in *Random fluctuations and pattern growth* H.E. Stanley and N. Ostrowsky, Kluwer Academic Publishers, Dordrecht, 279-291 (1988).

[19] C.J.G. Evertsz and B.B. Mandelbrot: Multifractal measures in Ref [15], 921-953 (1992).

[20] J.A. Buckley, *Large deviation techniques in decision, simulation and estimation* Wiley (1990)

[21] B.B. Mandelbrot, New "anomalous" multiplicative multifractals: left-sided $f(\alpha)$ and the modeling of DLA, *Physica* A**168** (1990) 95-111

[22] B.B. Mandelbrot and C.J.G. Evertsz, Multifractality of the harmonic measure on fractal aggregates, and extended self-similarity, *Physica* **A 177**, 386-393 (1991)

[23] C.J.G. Evertsz and B.B. Mandelbrot, Self-similarity of harmonic measure on DLA *Physica* **A 185**, 77-86 (1992)

[24] C. Meneveau, K.R. Sreenivasan, A method for the direct measurement of $f(\alpha)$ of multifractals, and its applications to dynamical systems and fully developed turbulence, *Phys. Lett.* A 137 (1989) 103

[25] C.J.G. Evertsz, Self-similarity of high-frequency USD-DEM exhange rates, submitted (1995)

[26] R.A. Hanssen and W. Reiß, Autokorrelationsanalyse und das Problem der Datenfehler *Zeitschrift für Nationalökonomie* **36** 153-172 (1976)

[27] E.F. Fama, Mandelbrot and the stable Paretian hypothesis. *The Journal of Business of the University of Chicago* **36**, 420-426 (1963)