An introduction to the theory of selfsimilar stochastic processes*

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Dedicated to Professor Hans Bühlmann
on the occasion of his 70th birthday

Abstract

Selfsimilar processes such as fractional Brownian motion are stochastic processes that are invariant in distribution under suitable scaling of time and space. These processes can typically be used to model random phenomena with long-range dependence. Naturally, these processes are closely related to the notion of renormalization in statistical and high energy physics. They are also increasingly important in many other fields of application, as there are economics and finance. This paper starts with some basic aspects on selfsimilar processes and discusses several topics from the point of view of probability theory.

Contents and key words

1 Selfsimilarity and long-range dependence — definition of selfsimilarity, existence of exponent of selfsimilarity, long-range dependence.
2 Brownian motion and fractional Brownian motions — Brownian motion, fractional Brownian motion, fixed point of renormalization group transformations.

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3 From central limit theorem to noncentral limit theorem — fundamental limit theorem for selfsimilar process, Rosenblatt process, noncentral limit theorem, multiple Wiener-Itô integral process.

4 Selfsimilar stable-integral processes with stationary increments — stable distribution, stable-integral process, linear fractional stable motion, log-fractional stable motion, random walk in random scenery.

5 Selfsimilar processes with independent increments — selfdecomposable distribution, several examples.

6 Semi-selfsimilar processes — semi-stable Lévy process, diffusions on Sierpinski gaskets, checking selfsimilarity.

1 Selfsimilarity and long-range dependence

Brownian motion is a very important example of a stochastic process. It is a Gaussian process, a diffusion process, a Lévy process, a Markov process, a martingale and a selfsimilar process. Each property above of Brownian motion was a starting point of a new subfield of the theory of stochastic processes. By now, Gaussian processes, diffusion processes, Lévy processes, Markov processes and martingales constitute themselves major areas of research in the modern theory of stochastic processes. The notion of selfsimilarity did not immediately reach the same fundamental level; many more recent applications have however called for a deeper understanding.

Selfsimilar processes are stochastic processes that are invariant in distribution under suitable scaling of time and space. These processes also enter naturally in the analysis of random phenomena (in time) exhibiting certain forms of long-range dependence.

Fractional Brownian motion, which is a Gaussian selfsimilar process with stationary increments, was first discussed by Kolmogorov [Kol40]. The first paper giving a rigorous probabilistic treatment of general selfsimilar processes is due to Lamperti [Lam62]. Later, the study of non-Gaussian selfsimilar processes with stationary increments was initiated by Taqqu [Taq75], who further developed a non-Gaussian limit theorem by Rosenblatt [Ros61].

On the other hand, the works of Sinai [Sin76] and Dobrushin [Dob80] for instance, in the field of statistical physics, appeared around 1976. It seems that similar problems were attacked independently in the fields of probability theory and statistical physics (see [Dob80]). The connection between these developments was pointed out by Dobrushin.

Most stochastic processes discussed in this paper are real-valued. They are defined on a common probability space $(\Omega, \mathcal{F}, P)$. In the following, by $\{X(t)\} \stackrel{d}{=} \{Y(t)\}$, we mean the equality of all finite-dimensional distributions. Occasionally we simply write $X(t) \stackrel{d}{=} Y(t)$. $X_1 \stackrel{d}{\sim} X_2$ means the equality in law of X_1 and X_2. By $X_n(t) \stackrel{d}{\Rightarrow} Y(t)$, we mean the convergence of all finite-dimensional distributions of $\{X_n(t)\}$ to $\{Y(t)\}$ as $n \to \infty$, and by $\xi_n \stackrel{d}{\to} \xi$, the convergence in law of random variables $\{\xi_n\}$ to ξ. $\mathcal{L}(X)$ stands for the law of a random variable X and the characteristic function of X with $\mathcal{L}(X) = \mu$ is denoted by $\widehat{\mu}(\theta) = E[e^{i\theta X}]$, $\theta \in \mathbf{R}$.

Definition 1.1 *A stochastic process $\{X(t), t \geq 0\}$ is said to be "selfsimilar" if for any $a > 0$, there exists $b > 0$ such that*

$$\{X(at)\} \stackrel{d}{=} \{bX(t)\}\,. \tag{1.1}$$

We say that $\{X(t), t \geq 0\}$ is *stochastically continuous* at t, if for any $\varepsilon > 0$,
$\lim_{h\to 0} P\{|X(t+h) - X(t)| > \varepsilon\} = 0$. We also say that $\{X(t), t \geq 0\}$ is *trivial*, if $\mathcal{L}(X(t))$ is a delta measure for every $t > 0$.

Theorem 1.1 ([Lam62]) *If $\{X(t), t \geq 0\}$ is nontrivial, stochastically continuous at $t = 0$ and selfsimilar, then there exists a unique exponent $H \geq 0$ such that b in* (1.1) *can be expressed as $b = a^H$. Moreover, $H > 0$ if and only if $X(0) = 0$ a.s.*

In the more recent literature, selfsimilar processes are usually defined in the following way: A stochastic process $\{X(t), t \geq 0\}$ is selfsimilar, if there exists $H > 0$ such that for any $a > 0, \{X(at)\} \stackrel{d}{=} \{a^H X(t)\}$. In this case, it follows that $X(0) = 0$ a.s. However, the uniqueness of the exponent is not obvious form this definition, although it is unique by Theorem 1.1. There seems to be some confusion about this fact in the more applied literature.

A stochastic process $\{X(t)\}$ is said to have stationary increments, if the distributions of $\{X(h+t) - X(h)\}$ are independent of h. In the following, we discuss some properties of selfsimilar processes with stationary increments. When $\{X(t), t \geq 0\}$ is selfsimilar with stationary increments and its exponent is H, then we call it H-ss, si, for short.

Theorem 1.2 *Let $\{X(t)\}$ be nontrivial and H-ss, si, and suppose $E\left[|X(1)|^2\right] < \infty$. Then*

$$E[X(t)X(s)] = \frac{1}{2}\left\{t^{2H} + s^{2H} - |t-s|^{2H}\right\} E\left[|X(1)|^2\right] .$$

Proof. ([Taq81]) By H-ss, si,

$$\begin{aligned} E[X(t)X(s)] &= \frac{1}{2}\left\{E\left[X(t)^2\right] + E\left[X(s)^2\right] - E\left[(X(t)-X(s))^2\right]\right\} \\ &= \frac{1}{2}\left\{E\left[X(t)^2\right] + E\left[X(s)^2\right] - E\left[X(|t-s|)^2\right]\right\} \\ &= \frac{1}{2}\left\{t^{2H} + s^{2H} - |t-s|^{2H}\right\} E\left[|X(1)|^2\right] . \quad \square \end{aligned}$$

Theorem 1.3 *Let $\{X(t)\}$ be nontrivial and H-ss, si, $H > 0$.*
(i) ([Mae86]) *If $E\left[|X(1)|^\gamma\right] < \infty$ for some $\gamma < 1$, then $H < 1/\gamma$.*
(ii) *If $E[|X(1)|] < \infty$, then $H \leq 1$.*
(iii) ([Kon84]) *If $E[|X(1)|] < \infty$ and $0 < H < 1$, then $E[X(t)] = 0$.*
(iv) ([Ver85]) *If $E[|X(1)|] < \infty$ and $H = 1$, then $X(t) = tX(1)$ a.s.*

((ii) is easily seen from (i).) Because of (ii) and (iv) above, when the process has finite first moment, we always consider the case $0 < H < 1$.

Let $\{X(t), t \geq 0\}$ be nontrivial, H-ss, si, $0 < H < 1$, and $E[|X(1)|^2] < \infty$, and define

$$\begin{aligned} \xi(n) &= X(n+1) - X(n)\,, \qquad && n = 0, 1, 2, \cdots , \\ r(n) &= E[\xi(0)\xi(n)]\,, && n = 0, 1, 2, \cdots . \end{aligned}$$

Then

$$(1.2) \qquad r(n) \begin{cases} \sim H(2H-1)n^{2H-2}E[|X(1)|^2], & \text{as } n \to \infty, \quad \text{if } H \neq \frac{1}{2}\,, \\ = 0\,, \quad n \geq 1\,, & \text{if } H = \frac{1}{2}\,, \end{cases}$$

where $a_n \sim b_n$, as $n \to \infty$, means $\lim_{n\to\infty} a_n/b_n = 1$. This can be shown as follows. Noticing that $X(0) = 0$ a.s. (Theorem 1.1) and using Theorem 1.2, we have for $n \geq 1$,

$$\begin{aligned} r(n) &= E[\xi(0)\xi(n)] = E[X(1)\{X(n+1) - X(n)\}] \\ &= E[X(1)X(n+1)] - E[X(1)X(n)] \\ &= \frac{1}{2}\left\{(n+1)^{2H} - 2n^{2H} + (n-1)^{2H}\right\} E[|X(1)|^2]\,, \end{aligned}$$

which implies (1.2). Hence,
(1) if $0 < H < \frac{1}{2}$, $\sum_{n=0}^{\infty} |r(n)| < \infty$,
(2) if $H = \frac{1}{2}$, $\{\xi(n)\}$ is uncorrelated,
(3) if $\frac{1}{2} < H < 1$, $\sum_{n=0}^{\infty} |r(n)| = \infty$.
Actually, if $0 < H < \frac{1}{2}$, $r(n) < 0$, for $n \geq 1$ (negative correlation), if $\frac{1}{2} < H < 1$, $r(n) > 0$ for $n \geq 1$ (positive correlation). The property $\sum |r(n)| = \infty$ is called long-range dependence and especially of interest in statistics (see [Ber94] and [Cox84]).

2 Brownian motion and fractional Brownian motions

A stochastic process $\{X(t), t \geq 0\}$ is said to have *independent increments*, if for any $m \geq 1$ and for any partition $0 \leq t_0 < t_1 < \cdots < t_m$, $X(t_1) - X(t_0), \ldots, X(t_m) - X(t_{m-1})$ are independent.

Definition 2.1 *If a stochastic process $\{B(t), t \geq 0\}$ satisfies*
(i) $B(0) = 0$ *a.s.,*
(ii) *it has independent and stationary increments,*
(iii) *for each $t > 0$, $B(t)$ has a Gaussian distribution with mean zero and variance t, and*
(iv) *its sample paths are continuous a.s.,*
then it is called (*standard*) *Brownian motion.*

Theorem 2.1 *Brownian motion $\{B(t)\}$ is $\frac{1}{2}$-ss.*

Proof. It is enough to show that for every $a > 0$, $\{a^{-1/2}B(at)\}$ is also Brownian motion. Conditions (i), (ii) and (iv) follow from the same conditions for $\{B(t)\}$. As to (iii), Gaussianity and mean-zero property also follow from the properties of $\{B(t)\}$. As to the variance, $E[(a^{-1/2}B(at))^2] = t$. Thus $\{a^{-1/2}B(at)\}$ is a Brownian motion. □

Theorem 2.2 $E[B(t)B(s)] = \min\{t, s\}$.

Proof. Brownian motion is $\frac{1}{2}$-ss, si. Thus by Theorem 1.2, $E[B(t)B(s)] = \frac{1}{2}\{t + s - |t - s|\} = \min\{t, s\}$. □

Remark 2.1 It is known that the distribution of a Gaussian process is determined by its mean and covariance structure. For, the distribution of a process is determined by all its finite-dimensional distributions and the density of a multidimensional Gaussian distribution is explicitly given through its mean and covariance. Thus, a mean-zero Gaussian process with covariance as in Theorem 2.2 must be Brownian motion.

Definition 2.2 *Let* $0 < H < 1$. *A mean-zero Gaussian process* $\{B_H(t), t \geq 0\}$ *is called "fractional Brownian motion", if*

$$E\left[B_H(t)B_H(s)\right] = \frac{1}{2}\left\{t^{2H} + s^{2H} - |t-s|^{2H}\right\} E\left[B_H(1)^2\right]. \tag{2.1}$$

Theorem 2.3 $\{B_{1/2}(t)\}$ *is the same as Brownian motion up to a multiplicative constant.*

Proof. (2.1) with $H = \frac{1}{2}$ determines the covariance structure of Brownian motion as mentioned in Remark 2.1. □

Theorem 2.4 ([ManVNe68]) *Fractional Brownian motion* $\{B_H(t), t \geq 0\}$ *is* H*-ss, si, and it has a stochastic integral representation*

$$\left\{ \int_{-\infty}^{0} \left((t-u)^{H-1/2} - (-u)^{H-1/2}\right) dB(u) + \int_{0}^{t} (t-u)^{H-1/2}\, dB(u) \right\} E\left[B_H(1)^2\right]. \tag{2.2}$$

Fractional Brownian motion is unique in the sense that the class of all fractional Brownian motions coincides with that of all Gaussian selfsimilar processes with stationary increments. $\{B_H(t)\}$ *has independent increments if and only if* $H = \frac{1}{2}$.

Sample path properties of Brownian motion have been well studied. As Brownian motion, fractional Brownian motion is also sample continuous, nowhere differentiable and of unbounded variation almost surely. For sample path properties of general selfsimilar process with stationary increments, see [Ver85], and for that of selfsimilar stable processes with stationary increments, see [KonMae91].

Several properties of trajectories of multidimensional fractional Brownian motion with multiparameter have also been studied. Let $\{B_H(t), t \in \mathbf{R}^N\}$ be a mean-zero Gaussian process with covariance

$$E[B_H(t)B_H(s)] = |t|^{2H} + |s|^{2H} - |t-s|^{2H},$$

where $|t|$ is the Euclidean norm of $t \in \mathbf{R}^N$. Consider independent copies $\{B_H^{(j)}(t)\}, j = 1, ..., d$, of $\{B_H(t)\}$ and the process $\{\mathbf{B}_H(t) = (B_H^{(1)}(t), ..., B_H^{(d)}(t))\}$. This is an $\mathbf{R}^d$-valued fractional Brownian motion with multiparameter $t \in \mathbf{R}^N$. For the Hausdorff measure and multiple point properties of the trajectories of $\{\mathbf{B}_H(t)\}$, see [Tal95, Tal98], [Xia97, Xia98] and the references therein.

Selfsimilar processes are related to the notion of a renormalization group. The following result is due to Sinai [Sin76]. Let $H > 0$ and let $Y = \{Y_j, j = 0, 1, 2, ...\}$ be a sequence of random variables. Define, for each $N \geq 1$, the transformation

$$T(N,H) : Y \to T(N,H)Y = \left\{ (T(N,H)Y)_j \,, j = 0, 1, 2, ... \right\} ,$$

where

$$(T(N,H)Y)_j = \frac{1}{N^H} \sum_{k=jN}^{(j+1)N-1} Y_k, \quad j = 0, 1, 2, \dots .$$

Because $T(N,H)T(M,H) = T(NM,H)$, the sequence of transformations $\{T(N,H), N \geq 1\}$ forms a multiplicative semi-group. It is called the *renormalization group* of index H. Suppose $Y = \{Y_j, j = 0, 1, 2, ...\}$ is a stationary sequence.

Definition 2.3 *A stationary sequence $Y = \{Y_j, j = 0, 1, 2, ...\}$ is called H-selfsimilar, if Y is a fixed point of the renormalization group $\{T(N,H), N \geq 1\}$ with index H, namely for all $N \geq 1$,*

$$\left\{ (T(N,H)Y)_j \,, j = 0, 1, 2, ... \right\} \stackrel{d}{=} \{Y_j, j = 0, 1, 2, ...\} .$$

Since fractional Brownian motion $\{B_H(t)\}$ has stationary increments, the random variables

$$Y_j = B_H(j+1) - B_H(j) \,, \quad j = 0, 1, 2, ...$$

form a stationary sequence. This sequence $\{Y_j, j = 0, 1, 2, ...\}$ is called *fractional Gaussian noise* with exponent H. The following is a discrete analogue of the statement on the uniqueness of fractional Brownian motion in Theorem 2.4.

Theorem 2.5 *Let $0 < H < 1$. Within the class of stationary sequences, fractional Gaussian noise with exponent H is the only Gaussian fixed point of the renormalization group $\{T(N, H), N \geq 1\}$.*

Proof. For any $\theta_0, \dots, \theta_k$, $k \geq 0$ and $N \geq 1$,

$$\begin{aligned}
\sum_{j=0}^{k} \theta_j \left(T(N,H)Y\right)_j &= \sum_{j=0}^{k} \theta_j \frac{1}{N^H} \sum_{k=jN}^{(j+1)N-1} Y_k \\
&= \sum_{j=0}^{k} \theta_j \frac{1}{N^H} \left\{B_H((j+1)N) - B_H(jN)\right\} \\
&\overset{d}{=} \sum_{j=0}^{k} \theta_j \left\{B_H(j+1) - B_H(j)\right\} \\
&= \sum_{j=0}^{k} \theta_j Y_j \,,
\end{aligned}$$

and thus fractional Gaussian noise is a fixed point of $\{T(N, H), N \geq 1\}$. Since fractional Brownian motion is the unique Gaussian H-selfsimilar process with stationary increments (Theorem 2.4), fractional Gaussian noise is the unique Gaussian fixed point. □

Remark 2.2 In general, suppose that $\{X(t), t \geq 0\}$ is H-ss, si with $H > 0$. (Recall that $X(0) = 0$ a.s. by Theorem 1.1.) Then the increment process

$$Y_j = X(j+1) - X(j), \quad j = 0, 1, 2, ...$$

is a fixed point of the renormalization group transformation $\{T(N, H), N \geq 1\}$, since the proof of Theorem 2.5 also works in this general case.

3 From central limit theorem to noncentral limit theorem

Let $X_1, X_2, \dots$ be a sequence of independent and identically distributed random variables with $E[X_1] = 0$ and $E[X_1^2] = 1$. Then

$$\frac{1}{n^{1/2}} \sum_{j=1}^{[nt]} X_j \overset{d}{\Rightarrow} B(t). \tag{3.1}$$

Remark 3.1 In probability theory, several notions of convergence of stochastic processes, stronger than convergence of finite-dimensional distributions, exist. In the case of (3.1), the so-called Donsker invariance principle assures that a measure of the left hand side, defined on the function space $D[0,\infty)$ consisting of all functions being right continuous and having left limits, converges to a Gaussian measure on $D[0,\infty)$. The same comment will be true for Theorem 4.5 in Section 4, but in this paper we do not discuss such a weak convergence concept, because selfsimilarity is determined only by finite-dimensional distributions of stochastic processes.

The convergence (3.1) is one way of constructing a Brownian motion. Actually, any selfsimilar process arises in this way as the following fundamental limit theorem by Lamperti [Lam62] shows.

Theorem 3.1 ([Lam62]) *Suppose $\{X(t), t \geq 0\}$ is stochastically continuous at $t = 0$ and $\mathcal{L}(X(t))$ is nondegenerate for each $t > 0$. If there exist a stochastic process $\{Y(t), t \geq 0\}$ and real numbers $\{a(\lambda), \lambda \geq 0\}$ with $a(\lambda) > 0$, $\lim_{\lambda\to\infty} a(\lambda) = \infty$ such that as $\lambda \to \infty$,*

$$\frac{1}{a(\lambda)} Y(\lambda t) \overset{d}{\Rightarrow} X(t)\,, \tag{3.2}$$

then for some $H > 0$, $\{X(t), t \geq 0\}$ is H-ss. Moreover, $a(\lambda)$ is of the form $a(\lambda) = \lambda^H L(\lambda)$, L being a slowly varying function.

In the above, $L(\lambda)$ is said to be *slowly varying*, if $\lim_{\lambda\to\infty} L(c\lambda)/L(\lambda) = 1$ for any $c > 0$. For more information about slowly varying functions, see [BinGolTeu87].

If we fix $t > 0$ in (3.1), we have the classical central limit theorem. Historically, the next question was how we can relax the assumption on independence of $\{X_j\}$ by keeping the validity of the central limit theorem (3.1). Rosenblatt [Ros56] introduced a mixing condition which is a kind of weak dependence condition for stationary sequences of random variables. Numerous extensions to other mixing conditions have been introduced. The next problem addressed by Rosenblatt was as follows: Suppose that a stationary sequence has a stronger dependence violating the validity of the central limit theorem, then what type of limiting distributions are expected to appear. He answered this question in [Ros61] laying the foundation of so-called noncentral limit theorems.

Theorem 3.2 ([Ros61]) *Let $\{\xi_n\}$ be a stationary Gaussian sequence such that $E[\xi_1] = 0, E[\xi_1^2] = 1$ and $E[\xi_1 \xi_{n+1}] \sim n^{H-1} L(n)$ as $n \to \infty$ for some $H \in (\frac{1}{2}, 1)$ and some slowly varying function L. Define another stationary sequence $\{X_j\}$ by*

$$X_j = \xi_j^2 - 1. \tag{3.3}$$

Then

$$\frac{1}{n^H} \sum_{j=1}^{n} X_j \xrightarrow{d} Z, \tag{3.4}$$

where Z is a non-Gaussian random variable and its characteristic function is given by

$$E[e^{i\theta Z}] = \exp\left\{ \sum_{p=2}^{\infty} \frac{(2i\theta)^p}{2p} \int_{x \in [0,1]^p} |x_1 - x_p|^{2(H-1)} \prod_{j=2}^{p} |x_j - x_{j-1}|^{2(H-1)} dx \right\}, \quad \theta \in \mathbf{R}.$$

Later, Taqqu [Taq75] considered a "process version" of (3.3) and obtained the limiting process of $n^{-H} \sum_{j=1}^{[nt]} X_j$. This limiting process is H-ss by Theorem 3.1 and the first example of non-Gaussian selfsimilar processes having strongly dependent increment structure. It is referred to as the *Rosenblatt* *process* and it is expressed by the multiple integral (3.4) below with $k=2$, as we will see.

A point we want to emphasize is that the functional $f(x)=x^2-1$ considered in (3.2) is the 2nd order Hermite polynomial. Dobrushin and Major [DobMaj79] and Taqqu [Taq79] extended this idea to general nonlinear functionals of strongly dependent Gaussian sequences to get noncentral limit theorems, by expanding nonlinear functionals in terms of Hermite polynomials.

Let $\{\xi_n\}$ be a sequence of stationary Gaussian random variables with $E[\xi_1]=0$, $E[\xi_1^2]=1$, and further assume that the covariances satisfy

$$r(n)=E[\xi_1\xi_{n+1}]\sim|n|^{-q}L(|n|), \quad n\to\infty, \tag{3.5}$$

where $0<q<1$ and L is a slowly varying function. Let G be the spectral measure of $\{\xi_n\}$ such that $r(n)=\int_{-\pi}^{\pi}e^{inx}G(dx)$.

Lemma 3.1 ([DobMaj79]) *Define a set of measures* $\{G_n, n=1,2,...\}$ *by*

$$G_n(A)=\frac{n^q}{L(n)}\,G\left(\frac{A}{n}\cap[\lambda-\pi,\pi)\right), \quad A\in\mathfrak{B}(\mathbf{R}).$$

Then there exists a locally finite measure G_0 *such that* $G_n\to G_0$ *(vaguely) and for any* $c>0$, $A\in\mathfrak{B}(\mathbf{R})$, $G_0(cA)=c^qG_0(A)$.

In the above, for a definition of vague convergence, see [EmbKluMik97], p.563.

Let Z_{G_0} be a random spectral measure corresponding to G_0, namely a mean-zero, complex-valued Gaussian random measure such that $E[Z_{G_0}(A)\overline{Z_{G_0}(B)}]=G_0(A\cap B)$, and put

$$X_0(t)=\int_{\mathbf{R^k}}^{\prime\prime}\frac{e^{it(x_1+\cdots+x_k)}-1}{i(x_1+\cdots+x_k)}Z_{G_0}(dx_1)\cdots Z_{G_0}(dx_k), \tag{3.6}$$

where $\int_{\mathbf{R^k}}^{\prime\prime}$ is the integral over $\mathbf{R}^k$ except the hyperplanes $x_i=\pm x_j, i\neq j$, and the integral is a so-called multiple Wiener-Itô integral. (For multiple Wiener-Itô integrals, see [Maj81].) Let f be a function satisfying $E[f(\xi_1)]=0$, $E\left[f(\xi_1)^2\right]<\infty$, and expand f in terms of Hermite polynomials as

$$f(x)=\sum_{p=0}^{\infty}c_p\,H_p(x),$$

where $H_p(x)$ is the p-th order Hermite polynomial defined by $H_p(x) = (-1)^p e^{x^2/2} \frac{d^p}{dx^p} e^{-x^2/2}, c_p = \frac{1}{p!} E\left[f\left(\xi_1\right) H_p\left(\xi_1\right)\right]$, and the convergence is taken in the sense of mean square. For example, $H_0(x) = 1, H_1(x) = x$ and $H_2(x) = x^2 - 1$, which is considered in (3.2). Define

$$k = \min\left\{p \mid c_p \neq 0\right\} .$$

This k is referred to as the Hermite rank of f. By the assumption $E\left[f\left(\xi_1\right)\right] = 0$, $c_0 = 0$ so that $k \geq 1$.

Theorem 3.3 (Noncentral limit theorem, [DobMaj79], [Taq79]). *Let k be the Hermite rank of f and $\{\xi_n\}$ a sequence of stationary Gaussian random variables with $E[\xi_1] = 0$ and $E[\xi_1^2] = 1$, and assume that* (3.5) *holds for some q with $0 < q < 1/k$. (We define G_0 as in Lemma* 3.1 *by using this q and further $X_0(t)$ by* (3.4).) *If $a_n = n^{1-kq/2} L(n)^{k/2}$, then*

$$\frac{1}{a_n} \sum_{j=1}^{[nt]} f\left(\xi_j\right) \stackrel{d}{\Rightarrow} c_k \, X_0(t) \, .$$

Notice that the multiplicity k of the integral of the limiting selfsimilar process $X_0(t)$ is identical to the Hermite rank of f. The idea of the proof is the following.
(1) Consider $f(x) = c_k \, H_k(x) + f_k^*(x)$, where $f_k^*(x) = \sum_{p=k+1}^{\infty} c_p H_p(x)$.
(2) Verify under our assumptions that

$$E\left[\left|\frac{1}{a_n} \sum_{j=1}^{[nt]} f_k^*\left(\xi_j\right)\right|^2\right] \to 0 \, .$$

Here, the condition $q < 1/k$ is essential. Hence, it is enough to show the assertion when $f(x) = H_k(x)$.
(3) Prove

$$\frac{1}{a_n} \sum_{j=1}^{[nt]} H_k(\xi_j) \stackrel{d}{\Rightarrow} X_0(t).$$

(Theorem 3.2 is the case $k = 2$.)

Remark 3.2 As mentioned above, the condition $q < 1/k$ is essential for the validity of the noncentral limit theorem in Theorem 3.3. This condition assures that the order of Var $\left(\sum_{j=1}^{n} f(\xi_j)\right)$, which is the same order as Var $\left(\sum_{j=1}^{n} H_k(\xi_j)\right)$, is greater than n, implying that the random variables $\{f(\xi_j), j = 1, 2, \dots\}$ are strongly dependent. This is the reason why non-Gaussian limits appear and why the theorem is called the noncentral limit theorem. What will happen, if the order of Var $\left(\sum_{j=1}^{n} f(\xi_j)\right)$ is n or $n\ell(n)$, $\ell(\cdot)$ being slowly varying? This corresponds to the case $q \geq 1/k$, and it is known that the central limit theorem again holds ([BreMaj87], [GirSur85], [Mar76], [Mar80]).

4 Selfsimilar stable-integral processes with stationary increments

A probability distribution μ is said to be *strictly α-stable*, $0 < \alpha \leq 2$, if it is not a delta measure, $\widehat{\mu}(\theta)$ does not vanish and for any $a > 0$,

$$\widehat{\mu}(\theta)^a = \widehat{\mu}(a^{1/\alpha}\theta), \quad \forall \theta \in \mathbf{R}.$$

In the following, we call it just α-stable. For $\alpha = 2$, we have the Gaussian case. A stochastic process $\{X(t), t \geq 0\}$ is said to be a Lévy process if it has independent and stationary increments, it is stochastically continuous at any $t \geq 0$, its sample paths are right continuous and have left limits, and $X(0) = 0$ a.s. If $\{X(t), t \geq 0\}$ is a Lévy process and $\mathcal{L}(X(1))$ is α-stable, then it is called an α-stable Lévy process and denoted by $\{Z_\alpha(t), t \geq 0\}$. $\{Z_2(t)\}$ is Brownian motion.

Non-Gaussian stable distributions are, sometimes by physicists, called Lévy distributions (see [Tsa97]). The special case $\alpha = 1$ is called Cauchy distribution (or Lorentz distribution by physicists). A significant difference between Gaussian distributions and non-Gaussian stable ones like the Cauchy is that the latter have heavy tails, namely their variances are infinite. Such models were for a long time not accepted by physicists. More recently, the importance of modelling stochastic phenomena with heavy-tailed processes is dramatically increasing in many fields. See, for instance, [EmbKluMik97].

One important such heavy tail property is the following.

Theorem 4.1 *If Z_α is a random variable with α-stable distribution, $0 < \alpha < 2$, then for any $\gamma < \alpha$, $E[|Z_\alpha|^\gamma] < \infty$, but $E[|Z_\alpha|^\alpha] = \infty$.*

Proof. See [SamTaq94], for instance. □

Selfsimilar processes with independent and stationary increments are the only stable Lévy processes as the following theorem shows.

Theorem 4.2 *Suppose $\{X(t), t \geq 0\}$ is a Lévy process and let $0 < \alpha \leq 2$. Then $\mathcal{L}(X(1))$ is α-stable if and only if $\{X(t)\}$ is $\frac{1}{\alpha}$-ss.*

Proof. Denote $\mu_t = \mathcal{L}(X(t))$ and $\mu = \mu_1$. Since $\{X(t)\}$ is a Lévy process, for each $t \geq 0$, $\widehat{\mu}_t$ satisfies $\widehat{\mu}_t(\theta) = \widehat{\mu}(\theta)^t$. Indeed, for any n and m,

$$X\left(\frac{m}{n}\right) = \left\{X\left(\frac{m}{n}\right) - X\left(\frac{m-1}{n}\right)\right\} + \cdots + \left\{X\left(\frac{1}{n}\right) - X(0)\right\}, \tag{4.1}$$

where $X\left(\frac{k}{n}\right) - X\left(\frac{k-1}{n}\right)$, $k = 1, \ldots, m$, are independent and identically distributed. It follows from (4.1) that $\widehat{\mu}_{m/n}(\theta) = \widehat{\mu}_{1/n}(\theta)^m$ and in particular $\widehat{\mu}_{1/n}(\theta) = \widehat{\mu}(\theta)^{1/n}$. Thus

$$\widehat{\mu}_{m/n}(\theta) = \widehat{\mu}_{1/n}(\theta)^m = \widehat{\mu}(\theta)^{m/n} .$$

This, with the stochastic continuity of $\{X(t)\}$, implies that $\widehat{\mu}_t(\theta) = \widehat{\mu}(\theta)^t$ for any $t > 0$.

We now prove the "if" part of the theorem. By $\frac{1}{\alpha}$-ss, $X(a) \overset{d}{\sim} a^{1/\alpha}X(1)$ $\forall a > 0$, hence $\widehat{\mu}(\theta)^a = \widehat{\mu}(a^{1/\alpha}\theta)$, $\forall a > 0$, $\forall \theta \in \mathbf{R}^d$, implying that μ is stable, and $\alpha = \frac{1}{H}$. As a result, necessarily $\frac{1}{2} \leq H < \infty$.

For the "only if" part, suppose μ is α-stable. Since $\{X(t)\}$ has independent and stationary increments, it is enough to show that for any $a > 0$,

$$X(at) \overset{d}{\sim} a^{1/\alpha}X(t).$$

However,

$$\begin{aligned} E[\exp\{i\theta X(at)\}] &= \widehat{\mu}_{at}(\theta) = \widehat{\mu}(\theta)^{at} = \widehat{\mu}\left(a^{1/\alpha}\theta\right)^t = \widehat{\mu}_t\left(a^{1/\alpha}\theta\right) \\ &= E\left[\exp\left\{i\theta a^{1/\alpha}X(t)\right\}\right] . \end{aligned}$$

This completes the proof. □

We extend the definition of $\{Z_\alpha(t), t \geq 0\}$ to the case to the whole of $\mathbf{R}$ in the following way. Let $\{Z_\alpha^{(-)}(t), t \geq 0\}$ be an independent copy of $\{Z_\alpha(t), t \geq 0\}$ and define for $t < 0$, $Z_\alpha(t) = -Z_\alpha^{(-)}(-t)$.

Let $f : \mathbf{R} \to \mathbf{R}$ be a nonrandom function. We consider the integral of f with respect to $\{Z_\alpha(t), t \in \mathbf{R}\}$. From now on, for simplicity, we assume symmetry in the sense that $\mathcal{L}(Z_\alpha(t)) = \mathcal{L}(-Z_\alpha(t))$ for every t. Then $E[e^{i\theta Z_\alpha(t)}] = e^{-ct|\theta|^\alpha}$ for some $c > 0$. For simplicity we assume $c = 1$ in the following.

Theorem 4.3 *If $f \in L^\alpha(\mathbf{R})$, then*

$$X_\alpha = \int_{-\infty}^{\infty} f(u) dZ_\alpha(u)$$

can be defined in the sense of convergence in probability, and X_α is also symmetric α-stable with

$$E\left[e^{i\theta X_\alpha}\right] = \exp\left\{-|\theta|^\alpha \int_{-\infty}^{\infty} |f(u)|^\alpha \, du\right\}. \tag{4.2}$$

For the proof, see, for instance, [SamTaq94]. We define stable-integral processes by

$$X_\alpha(t) = \int_{-\infty}^{\infty} f_t(u) dZ_\alpha(u), \quad t \geq 0,$$

where $f_t : \mathbf{R} \to \mathbf{R}$, and $f_t \in L^\alpha(\mathbf{R})$ for each $t \geq 0$.

We consider here two stable-integral processes of moving average type, represented as

$$X_1(t) = \int_{-\infty}^{\infty} \left(|t-u|^{H-1/\alpha} - |u|^{H-1/\alpha}\right) dZ_\alpha(u), \quad t \geq 0, 0 < H < 1, H \neq \frac{1}{\alpha},$$

and

$$X_2(t) = \int_{-\infty}^{\infty} \log\left|\frac{t-u}{u}\right| dZ_\alpha(u), \quad t \geq 0, 1 < \alpha \leq 2.$$

Both integrals are well defined because the integrands are L^α-integrable in the respective cases. The process $\{X_1(t), t \geq 0\}$ is H-ss, si, $\{X_1(t)\}$ with $\alpha = 2$ is a fractional Brownian motion and $\{X_1(t)\}$ with $0 < \alpha < 2$ is an extension of $\{B_H(t)\}$ to infinite variance processes. It is called the *linear* *fractional stable motion* ([TaqWol83], [Mae83]). Finally $\{X_2(t)\}$ is $\frac{1}{\alpha}$-ss, si. This is called the *log-fractional stable motion* ([KasMaeVer88]). Note that $\{X_2(t)\}$ with $\alpha = 2$ is no more than Brownian motion. This can be verified by calculating its covariance; clearly it has independent increments. However, if $1 < \alpha < 2$, $\{X_2(t)\}$ does not have independent increments, in contrast to α-stable Lévy process $\{Z_\alpha(t)\}$ which is also $\frac{1}{\alpha}$-ss, si, but has independent increments.

We are going to give limit theorems on convergence to $\{X_k(t)\}, k = 1, 2$. Suppose $\{X_j, j \in \mathbf{Z}\}$ are independent and identically distributed symmetric random variables satisfying

$$\frac{1}{n^{1/\alpha}} \sum_{j=1}^{n} X_j \stackrel{d}{\to} Z_\alpha(1). \tag{4.3}$$

Take δ such that $\frac{1}{\alpha} - 1 < \delta < \frac{1}{\alpha}$, and define a stationary sequence

$$Y_k = \sum_{j \in \mathbf{Z}} c_j X_{k-j}, \quad k = 1, 2, ...,$$

where

$$c_j = \begin{cases} 0, & \text{if } j = 0 \\ j^{-\delta-1}, & \text{if } j > 0 \\ -|j|^{-\delta-1}, & \text{if } j < 0. \end{cases}$$

We can easily see that the infinite series Y_k is well defined for each k and Y_k does not have finite variance unless $\alpha = 2$. Define further for $H = \frac{1}{\alpha} - \delta$,

$$W_n(t) = \frac{1}{n^H} \sum_{k=1}^{[nt]} Y_k. \tag{4.4}$$

Theorem 4.4

$$W_n(t) \stackrel{d}{\Rightarrow} \begin{cases} \frac{1}{|\delta|} X_1(t) & \textit{when } \delta \neq 0 \\ X_2(t) & \textit{when } \delta = 0. \end{cases}$$

Remark 4.1 If $\delta < 0$ (necessarily $\alpha > 1$), then $H = \frac{1}{\alpha} - \delta > \frac{1}{\alpha}$. Thus the normalization n^H in (4.4) grows much faster than $n^{1/\alpha}$ in (4.3), the case of partial sums of independent random variables. This explains why $\{Y_k\}$ exhibits long range dependence.

We give an outline of the proof of Theorem 4.4.

Step 1. For $m \in \mathbf{Z}$ and $t \geq 0$, define

$$c_m(t) = \sum_{j=1-m}^{[t]-m} c_j,$$

where $\sum_{j=1-m}^{-m}$ means 0. Then we have

$$W_n(t) = n^{-H} \sum_{m \in \mathbf{Z}} c_m(nt) X_m.$$

Step 2. For any $t_1, ..., t_p \geq 0$ and $\theta_1, ..., \theta_p \in \mathbf{R}$,

$$\sum_{m \in \mathbf{Z}} \left| n^{-H} \sum_{j=1}^{p} \theta_j c_m(nt_j) \right|^{\alpha}$$

$$\to \begin{cases} \displaystyle\int_{-\infty}^{\infty} \left| \frac{1}{|\delta|} \sum_{j=1}^{p} \theta_j \left(|t_j - u|^{-\delta} - |u|^{-\delta} \right) \right|^{\alpha} du & \text{when } \delta \neq 0 \\ \displaystyle\int_{-\infty}^{\infty} \left| \sum_{j=1}^{p} \theta_j \log \frac{|t_j - u|}{|u|} \right|^{\alpha} du & \text{when } \delta = 0. \end{cases}$$

Step 3. Denote the characteristic function of X_1 by $\lambda(\theta), \theta \in \mathbf{R}$. Then we have that

$$\log \lambda(\theta) \sim -|\theta|^{\alpha} \quad \text{as } \theta \to 0$$

([MaeMas94]). Also

$$\lim_{n \to \infty} n^{-H} \sup_{m} c_m(n) = 0$$

([Mae83]).

Step 4. We have

$$\begin{aligned} I_n :=& E\left[\exp\left\{ n^{-H} \sum_{j=1}^{p} \theta_j W_n(t) \right\} \right] \\ =& E\left[\exp\left\{ n^{-H} \sum_{j=1}^{p} \theta_j \sum_{m \in \mathbf{Z}} c_m(nt_j) X_m \right\} \right] \\ =& E\left[\prod_{m \in \mathbf{Z}} \lambda\left(n^{-H} \sum_{j=1}^{p} \theta_j c_m(nt_j) \right) \right] \end{aligned}$$

and, by Steps 2 and 3,

$$\begin{aligned}
\lim_{n\to\infty} I_n &= \lim_{n\to\infty} E\left[\prod_{m\in\mathbf{Z}} \lambda\left(n^{-H}\sum_{j=1}^{p}\theta_j c_m(nt_j)\right)\right] \\
&= \lim_{n\to\infty} E\left[\exp\left\{\sum_{m\in\mathbf{Z}} \log\lambda\left(n^{-H}\sum_{j=1}^{p}\theta_j c_m(nt_j)\right)\right\}\right] \\
&= \begin{cases}
E\left[\exp\left\{-\int_{-\infty}^{\infty}\left|\frac{1}{|\delta|}\sum_{j=1}^{p}\theta_j\left(|t_j-u|^{-\delta}-|u|^{-\delta}\right)\right|^{\alpha} du\right\}\right] & \text{when } \delta\neq 0 \\
E\left[\exp\left\{-\int_{-\infty}^{\infty}\left|\sum_{j=1}^{p}\theta_j \log\frac{|t_j-u|}{|u|}\right|^{\alpha} du\right\}\right] & \text{when } \delta = 0
\end{cases} \\
&= \begin{cases}
E\left[\exp\left\{i\frac{1}{|\delta|}\sum_{j=1}^{p}\theta_j X_1(t_j)\right\}\right] & \text{when } \delta\neq 0 \\
E\left[\exp\left\{i\sum_{j=1}^{p}\theta_j X_2(t_j)\right\}\right] & \text{when } \delta = 0,
\end{cases}
\end{aligned}$$

where we have used (4.2) at the last stage.

The above Step 4 gives us the conclusion. □

Kesten and Spitzer [KesSpi79] constructed an interesting class of ss, si processes as a limit of *random walks in random scenery*, where the limiting process is expressed as a stable-integral process with a random integrand. Let $\{Z_\alpha(t), t\in\mathbf{R}\}$ be a symmetric α-stable Lévy process $(0<\alpha\le 2)$ and $\{Z_\beta(t), t\in\mathbf{R}\}$ a symmetric β-stable Lévy process $(1<\beta\le 2)$ independent of $\{Z_\alpha(t)\}$. Let $L_t(x)$ be the local time of $\{Z_\beta(t)\}$, that is

$$L_t(x) = \lim_{\varepsilon\downarrow 0}\frac{1}{4\varepsilon}\int_0^t I\left[Z_\beta(s)\in(x-\varepsilon, x+\varepsilon)\right] ds\,,$$

which is known to exist as an almost sure limit, if $1<\beta\le 2$. Then we can define

$$X(t) = \int_{-\infty}^{\infty} L_t(x)\, dZ_\alpha(x)$$

and $\{X(t), t \geq 0\}$ is H-ss, si, with $H = 1 - \frac{1}{\beta} + \frac{1}{\alpha\beta} \left(> \frac{1}{2}\right)$.

A limit theorem for this process $\{X(t)\}$ is given as follows. Let $\{S_n, n \geq 0\}$ be an integer-valued random walk with mean 0 and $\{\xi(j), j \in \mathbf{Z}\}$ be a sequence of symmetric independent and identically distributed random variables, independent of $\{S_n\}$ such that

$$\frac{1}{n^{1/\alpha}} \sum_{j=1}^{n} \xi(j) \stackrel{d}{\to} Z_\alpha(1) \quad \text{and} \quad \frac{1}{n^{1/\beta}} S_n \stackrel{d}{\to} Z_\beta(1)\,.$$

The new stationary sequence $\{\xi(S_k)\}$, which is a random walk in random scenery, is strongly dependent.

Theorem 4.5 ([KesSpi79]). *Under the above assumptions, we have*

$$\frac{1}{n^H} \sum_{k=1}^{[nt]} \xi\left(S_k\right) \stackrel{d}{\Rightarrow} \int_{-\infty}^{\infty} L_t(x)\, dZ_\alpha(x)\,.$$

5 Selfsimilar processes with independent increments

We are now going to discuss selfsimilar processes with independent increments but not necessarily having stationary increments. We call $\{X(t), t \geq 0\}$ which is H-selfsimilar with independent increments as H-ss, ii.

As already seen in Theorem 4.2, if selfsimilar processes have independent and stationary increments, then their distributions are completely determined. However, this is not the case any more for selfsimilar processes without either independent or stationary increments. Actually, as mentioned in [BarnPer98], there is no simple characterization of the possible families of marginal distributions of selfsimilar processes with only stationary increments. Several authors have looked at this problem. For instance, O'Brien and Vervaat [OBrVer83] studied the concentration function of $\log X(1)^+$ and the support of $X(1)$, gave some lower bounds for the tail of the distribution of $X(1)$ in the case $H > 1$, and showed that $X(1)$ cannot have atoms except in certain trivial cases. Also Maejima [Mae86] studied the relationship between the existence of moments and exponent of selfsimilarity, as mentioned in Theorem 1.3. One of interesting questions is: Is the distribution of $X(1)$ outside 0 absolutely continuous, if $H \neq 1$? This question was raised by O'Brien and Vervaat [OBriVer83], but as far as we know it is still open.

For selfsimilar processes with independent increments, the situation is better. To state the main theorem in this chapter (due to Sato [Sat91]), we start with the notion of *selfdecomposability.*

Definition 5.1 *A probability distribution μ is called "selfdecomposable" if for any $b \in (0,1)$, there exists a probability distribution ρ_b such that*

$$\widehat{\mu}(\theta) = \widehat{\mu}(b\theta)\ \widehat{\rho}_b(\theta)\,, \quad \forall \theta \in \mathbf{R}^d\,. \tag{5.1}$$

Remark 5.1 Selfdecomposable distributions are infinitely divisible.

Theorem 5.1 (e.g. [Sat80]) *Suppose that there exists a sequence of independent random variables $\{X_j\}$, real sequences $\{a_n\}$ with $a_n > 0, \uparrow \infty$ and $\{b_n\}$ such that for some probability distribution μ*

$$\mathcal{L}\left(\frac{1}{a_n}\sum_{j=1}^{n} X_j + b_n\right) \to \mu$$

and the following asymptotic negligibility condition holds:

$$\lim_{n\to\infty} P\left\{\max_{1\le j\le n}\left|\frac{X_j}{a_n}\right| > \varepsilon\right\} = 0\,, \quad \forall \varepsilon > 0.$$

Then μ is selfdecomposable. Conversely, any selfdecomposable distribution can be obtained as such a limit.

Many distributions are known to be selfdecomposable, and their importance has been increasing, for instance, in mathematical finance, turbulence theory and other fields; see, e.g. [Barn98] and [Jur97]. The following result links selfsimilarity to selfdecomposability.

Theorem 5.2 *If $\{X(t), t \geq 0\}$ is stochastically continuous and H-ss, ii, then for each t, $\mathcal{L}(X(t))$ is selfdecomposable.*

Proof. ([Sat91]) Let $\mu_t = \mathcal{L}(X(t))$ and $\mu_{s,t} = \mathcal{L}(X(t) - X(s))$. We have by H-ss, ii that, for any $b \in (0,1)$,

$$\widehat{\mu}_t(\theta) = \widehat{\mu}_{bt}(\theta)\ \widehat{\mu}_{bt,t}(\theta) = \widehat{\mu}_t\left(b^H\theta\right)\ \widehat{\mu}_{bt,t}(\theta)\,.$$

This shows that μ_t is selfdecomposable. $\square$

Sato [Sat91] also showed that for any given $H > 0$ and a selfdecomposable distribution μ, there exists a uniquely in law H-ss process $\{X(t)\}$ with independent increments such that $\mathcal{L}(X(1)) = \mu$.

In the following we give several examples of ss, ii processes.

Example 5.1 ([Get79]) Assume $d \geq 3$ and let $\{B(t)\}$ be an $\mathbf{R}^d$-valued Brownian motion. For $t > 0$, define

$$L(t) = \sup\{u > 0 : \|B(u)\| \leq t\}.$$

Since $\|B(u)\| \to \infty$ a.s. as $u \to \infty$ when $d \geq 3$, $L(t)$ is finite a.s. Then the process $\{L(t)\}$ is 2-ss, ii. Moreover, $\{L(t)\}$ has si if and only if $d = 3$.

Proof. Selfsimilarity can be easily obtained:

$$\begin{aligned} L(at) &= \sup\{u > 0 : \|B(u)\| \leq at\} \\ &= \sup\left\{u > 0 : a^{-1}\|B(u)\| \leq t\right\} \\ &\stackrel{d}{=} \sup\left\{u > 0 : \left\|B\left(a^{-2}u\right)\right\| \leq t\right\} \\ &= a^2 L(t). \end{aligned}$$

As to the other parts of the proof, see [Get79]. $\square$

Example 5.2 (Due to Kawazu, see [Sat91].) Let $\{B(t)\}$ be a real-valued Brownian motion. Define

$$\begin{aligned} M(t) &= \inf\left\{u > 0 : B(u) - \min_{s \leq u} B(s) \geq t\right\}, \\ V(t) &= -\min_{s \leq M(t)} B(s) \end{aligned}$$

and

$$N(t) = \inf\{u > 0 : B(u) = -V(t)\}.$$

These processes appear in limit theorems of diffusions in random environment. Then the processes $\{M(t)\}, \{V(t)\}$ and $\{N(t)\}$ have *independent increments*, but none of them has stationary increments. The three processes are also selfsimilar, actually $\{M(t)\}$ is 2-ss, $\{V(t)\}$ is 1-ss and $\{N(t)\}$ is 2-ss, which can be seen as follows.

$$
\begin{aligned}
M(at) &= \inf\left\{u>0 : B(u) - \min_{s\le u} B(s) \ge at\right\} \\
&= \inf\left\{u>0 : a^{-1}\left(B(u) - \min_{s\le u} B(s)\right) \ge t\right\} \\
&\stackrel{d}{=} \inf\left\{u>0 : \left(B\left(a^{-2}u\right) - \min_{s\le u} B(a^{-2}s)\right) \ge t\right\} \\
&= a^2 M(t)\,, \\
V(at) &= -\min_{s\le M(at)} B(s) \stackrel{d}{=} -\min_{s\le a^2 M(t)} B(s) = -\min_{s\le M(t)} B\left(a^2 s\right) \stackrel{d}{=} aV(t)
\end{aligned}
$$

and

$$
\begin{aligned}
N(at) &= \inf\{u>0 : B(u) = -V(at)\} \\
&\stackrel{d}{=} \inf\{u>0 : B(u) = -aV(t)\} \\
&= \inf\left\{u>0 : a^{-1}B(u) = -V(t)\right\} \\
&\stackrel{d}{=} \inf\left\{u>0 : B\left(a^{-2}u\right) = -V(t)\right\} \\
&= a^2 N(t)\,.
\end{aligned}
$$

Example 5.3 ([NorValVir96]) Let $\{B_H(t), t\ge 0\}$ be a fractional Brownian motion with $1/2 < H < 1$. Define $\{M(t), t \ge 0\}$ by

$$M(t) = \int_0^t u^{1/2-H}(t-u)^{1/2-H} dB_H(u).$$

Then $\{M(t)\}$ is Gaussian, $(1-H)$-ss and it has independent increments (but not stationary increments).

6 Semi-selfsimilar processes

There exist various generalizations of the notion of selfsimilarity, one of which is the so-called semi-selfsimilarity. We motivate the definition (see Definition 6.2 below) with two examples, one based on the notion of semi-stable distributions and the second via processes defined on Sierpinski gaskets.

Definition 6.1 *A probability distribution μ is said to be strictly (a, α)-semi-stable, if for some $a \in (0,1) \cup (1, \infty)$ and $\alpha \in (0,2)$, $\widehat{\mu}(\theta)^a = \widehat{\mu}\left(a^{1/\alpha}\theta\right)$.*

Theorem 6.1 *Let $\{Z_\alpha(t), t \geq 0\}$ be a Lévy process such that $\mathcal{L}(Z_\alpha(1))$ is strictly (a, α)-semi-stable. Then we have*

$$\{Z_\alpha(at)\} \stackrel{d}{=} \left\{a^{1/\alpha} Z_\alpha(t)\right\} . \tag{6.1}$$

Proof. Easy. □

Once (6.1) is true for some $a \neq 1$, then it is also true for a^n, $n \in \mathbf{N}$. However, unless $\mathcal{L}(Z_\alpha(1))$ is stable, (6.1) does not hold for some $a > 0$, and thus it is not selfsimilar.

Kusuoka [Kus87], Goldstein [Gol87] and Barlow and Perkins [BarlPer88] constructed diffusions on Sierpinski gaskets in the following way. On $\mathbf{R}^2$ let $a_0 = (0,0)$, $a_1 = (1,0)$ and $a_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, and let $F_0 = \{a_0, a_1, a_2\}$. Define inductively

$$F_{n+1} = F_0 \cup \{2^n a_1 + F_n\} \cup \{2^n a_2 + F_n\} , \quad n = 0, 1, 2, \dots ,$$

where $y + A = \{y + x; x \in A\}$. Let

$$G_0' = \bigcup_{n=0}^{\infty} F_n$$

and let G_0 be G_0' together with its reflection around the y-axis. Let

$$G_n = 2^{-n} G_0 , \quad n \in \mathbf{Z} , \quad G_\infty = \bigcup_{n=0}^{\infty} G_n , \quad G = \overline{G_\infty} .$$

The resulting set G is the Sierpinski gasket. Define a simple random walk on G_n as a G_n-valued Markov chain $\{Y_r, r = 0, 1, 2, \dots\}$ with transition probabilities

$$P\{Y_{r+1} = y \mid Y_r = x\} = \begin{cases} \frac{1}{4}, & \text{if } y \in N_n(x) , \\ 0, & \text{otherwise} , \end{cases}$$

where $N_n(x)$ are the four nearest points of G_n. Consider

$$X^{(n)}(t) = 2^{-n} Y_{[5^n t]} , \quad t \geq 0 , \quad n = 0, 1, 2, \dots .$$

Theorem 6.2 *The* $\mathbf{R}^2$*-valued process* $\left\{X^{(n)}(t)\right\}$ *converges in distribution to an* $\mathbf{R}^2$*-valued process* $\{X(t)\}$*, and*

$$\{X(5^n t)\} \stackrel{d}{=} \{2^n X(t)\}, \quad \forall n \in \mathbf{Z}. \tag{6.2}$$

Motivated by (6.1) and (6.2) above, a notion of *semi-selfsimilarity* which extends selfsimilarity was introduced in [MaeSat99].

Definition 6.2 *A stochastic process* $\{X(t), t \geq 0\}$ *is said to be "semi -selfsimilar" if there exist* $a \in (0,1) \cup (1,\infty)$ *and* $b > 0$ *such that*

$$\{X(at)\} \stackrel{d}{=} \{bX(t)\}. \tag{6.3}$$

We say that $\{X(t)\}$ is *proper*, if $\mathcal{L}(X(t))$ is nondegenerate for every $t > 0$.

Theorem 6.3 ([MaeSat99]) *Let* $\{X(t), t \geq 0\}$ *be a proper, stochastically continuous at any* $t \geq 0$*, semi-selfsimilar process. Then the following statements hold.*
(i) *Let* Γ *be the set of* $a > 0$ *such that there is* $b > 0$ *satisfying* (6.3). *Then* $\Gamma \cap (1,\infty)$ *is nonempty. Denote the infimum of* $\Gamma \cap (1,\infty)$ *by* a_0.
(a) *If* $a_0 > 1$*, then* $\Gamma = \{a_0^n : n \in \mathbf{Z}\}$*, and* $\{X(t)\}$ *is not selfsimilar.*
(b) *If* $a_0 = 1$*, then* $\Gamma = (0,\infty)$*, and* $\{X(t)\}$ *is selfsimilar.*
(ii) *There exists a unique* $H \geq 0$ *such that, if* $a > 0$ *and* $b > 0$ *satisfy* (6.3)*, then* $b = a^H$.
(iii) $H > 0$ *if and only if* $X(0) = 0$ *a.s.,* $H = 0$ *if and only if* $X(t) = X(0)$ *a.s.*

An important application of Theorem 6.3 (i) is the following. Suppose one wants to check the selfsimilarity of a process. Following the definition of selfsimilarity, one has to check (1.1) for all $a > 0$. However, suppose one could show the relationship (1.1) only for $a = 2$ and 3. Then, by Theorem 6.3 (i), the fact that $2, 3 \in \Gamma$ implies that $\Gamma = (0,\infty)$, since $\log 2/\log 3$ is irrational, and thus one concludes that $\{X(t)\}$ is selfsimilar. This yields an easy way to check selfsimilarity of a given process. Namely, we have

Theorem 6.4 ([MaeSatWat99]) *Suppose that* $\{X(t)\}$ *is stochastically continuous at any* t*. If* $\{X(t)\}$ *satisfies* (1.1) *for some* a_1 *and* a_2 *such that* $\log a_1/\log a_2$ *is irrational, then it is selfsimilar.*

Remark 6.1 The condition $\log a_1 / \log a_2$ is irrational comes from an application of Kronecker's theorem, see [HarWri79], XXIII, Theorem 438.

Remark 6.2 The reader is referred to [Ber94] for the statistical analysis of selfsimilar processes.

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