Local Times of Fractional Brownian Sheet

Yimin Xiao *
Michigan State University

Tusheng Zhang
University of Manchester

August 22, 2001

Abstract

Let $B_0^H = \{B_0^H(t),\ t \in \mathbb{R}_+^N\}$ be a real-valued fractional Brownian sheet. Consider the (N, d) Gaussian random field B^H defined by

$$B^H(t) = (B_1^H(t), \ldots, B_d^H(t)) \quad (t \in \mathbb{R}_+^N),$$

where $B_1^H, \ldots, B_d^H$ are independent copies of B_0^H. In this paper, the existence and joint continuity of the local times of B^H are established.

Running Title: Local Times of Fractional Brownian Sheet

Key Words: Fractional Brownian sheet, local times, joint continuity.

AMS Subject Classification: 60G15, 60G17.

1 Introduction

For a given vector $H = (H_1, \ldots, H_N)$ $(0 < H_\ell < 1$ for $\ell = 1, \ldots, N)$, a real valued fractional Brownian sheet $B_0^H = \{B_0^H(t), t \in \mathbb{R}_+^N\}$ with Hurst index H is a real-valued, centered Gaussian random field with covariance function given by

$$\mathbb{E}[B_0^H(s)B_0^H(t)] = \prod_{\ell=1}^N \frac{1}{2}\Big(|s_\ell|^{2H_\ell} + |t_\ell|^{2H_\ell} - |s_\ell - t_\ell|^{2H_\ell}\Big), \quad s, t \in \mathbb{R}_+^N. \tag{1.1}$$

It follows from (1.1) that $B_0^H(t) = 0$ a.s. for every $t \in \partial\mathbb{R}_+^N$, where $\partial\mathbb{R}_+^N$ is the boundary of $\mathbb{R}_+^N$.

Fractional Brownian sheet has the following stochastic integral representation

$$B_0^H(t) = \kappa_H^{-1} \int_{-\infty}^{t_1} \cdots \int_{-\infty}^{t_N} \prod_{\ell=1}^N g_{H_\ell}(t_\ell, s_\ell) W(ds), \tag{1.2}$$

*Research partially supported by grants from NSF and NATO.

where $W = \{W(s), s \in \mathbb{R}_+^N\}$ is a standard Brownian sheet and

$$g_H(t,s) = ((t-s)_+)^{H-1/2} - ((-s)_+)^{H-1/2},$$

with $s_+ = \max\{s, 0\}$, and κ_H is the normalizing constant given by

$$\kappa_H^2 = \int_{-\infty}^1 \cdots \int_{-\infty}^1 \Big[\prod_{\ell=1}^N g_{H_\ell}(1, s_\ell)\Big]^2 ds.$$

We associate with B_0^H a Gaussian random field $B^H = \{B^H(t) : \ t \in \mathbb{R}_+^N\}$ with values in $\mathbb{R}^d$ by

$$B^H(t) = (B_1^H(t), \ldots, B_d^H(t)), \tag{1.3}$$

where $B_1^H, \ldots, B_d^H$ are independent copies of B_0^H. We call B^H the (N,d)-fractional Brownian sheet with Hurst index $H = (H_1, \ldots, H_N)$.

Note that if $N = 1$, then B^H is a fractional Brownian motion in $\mathbb{R}^d$ with Hurst index $H_1 \in (0,1)$; if $N > 1$ and $H_1 = \cdots = H_N = 1/2$, then B^H is the (N,d)-Brownian sheet. Hence B^H can be regarded as a natural generalization of one parameter fractional Brownian motion in $\mathbb{R}^d$ to Gaussian random fields in $\mathbb{R}^d$, as well as a generalization of the Brownian sheet. Another well known generalization is the multiparameter fractional Brownian motion $X = \{X(t), \ t \in \mathbb{R}^N\}$, which is a centered Gaussian random field with covariance function

$$\mathbb{E}[X_i(s)X_j(t)] = \frac{1}{2}\delta_{ij}\Big(|s|^{2H_1} + |t|^{2H_1} - |s-t|^{2H_1}\Big), \tag{1.4}$$

where $0 < H_1 < 1$ is a constant and $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$, and where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$. Fractional Brownian sheets arise naturally in many areas, including in stochastic partial differential equations (cf. Øksendal and Zhang (2000), Hu, Øksendal and Zhang (2000)) and in studies of most visited sites of symmetric Markov processes (cf. Eisenbaum and Khoshnevisan (2001)).

Many authors have studied the sample path properties of the Brownian sheet and fractional Brownian motion. See Orey and Pruitt (1973), Adler (1978, 1981), Pitt (1978), Ehm (1981), Rosen (1984), Talagrand (1995), Xiao (1997), just mention a few. In all these papers, the independent increment property of the Brownian sheet and the local nondeterminism of fractional Brownian motion have played crucial rôles. Since, in general, the fractional Brownian sheet B_H has neither the property of independent increments nor local nondeterminism, it seems quite difficult to investigate fine properties of its sample paths. Recently, Dunker (2000) has studied the small ball probability of fractional Brownian sheet. For certain special class of fractional Brownian sheets, Mason and Shi (2001) have obtained the exact rate for small ball probability and have computed the Hausdorff dimension of some exceptional sets related to the oscillation of their sample paths. Stochastic partial differential equations driven by fractional Brownian sheet have been studied by Øksendal and Zhang (2000), Hu, Øksendal and Zhang (2000) and Duncan et al (2000).

The main objective of this paper is to study the existence and joint continuity of the local times of fractional Brownian sheet B^H. Our existence theorem is sharp and its proof is quite different from the proofs for the Brownian sheet and fractional Brownian motion. However, for the joint continuity, we can only establish a sufficient condition; see Theorem 4.1. It is still an open problem to find the best possible condition for the joint continuity of the local times.

The rest of the paper is organized as follows. In Section 2, we collect some definitions and basic facts about fractional Brownian motion and local times that will be useful to our arguments. In Section 3, we prove the existence theorem (Theorem 3.7) and in Section 4, we give a sufficient condition for the joint continuity of the local times of B^H and list some open problems.

2 Preliminaries

In this section, we present some notations, and collect basic facts about Gaussian processes as well as local times.

2.1 General Notations

The underlying parameter space is $\mathbb{R}^N_+ = [0, \infty)^N$, throughout. A typical parameter, $t \in \mathbb{R}^N$ is written as $t = (t_1, \ldots, t_N)$, coordinatewise. There is a natural partial order, "$\preccurlyeq$", on $\mathbb{R}^N$. Namely, $s \preccurlyeq t$ if and only if $s_j \le t_j$ for all $j = 1, \ldots, N$. When $s \preccurlyeq t$, we define the *closed interval*,

$$[s, t] = \prod_{\ell=1}^{N} [s_\ell, t_\ell].$$

Throughout, we will let $\mathcal{A}$ denote the class of all N-dimensional closed intervals $I \subset (0, \infty)^N$ that are parallel to the axes. That is $I \in \mathcal{A}$ is of the form $I = [s, t]$, where $s \preccurlyeq t$ are both in $(0, \infty)^N$. We always write λ_m for Lebesgue's measure on $\mathbb{R}^m$, no matter the value of the integer m. We use $\langle \cdot, \cdot \rangle$ and $|\cdot|$ to denote the ordinary scalar product and the Euclidean norm in $\mathbb{R}^m$ respectively.

We will use K, K_1, $K_2, \ldots$, to denote unspecified positive and finite constants which may not be the same in each occurrence.

2.2 Fractional Brownian Motion

Given a constant $\alpha \in (0,1)$, a fractional Brownian motion in $\mathbb{R}$ with index α is a real-valued, centered Gaussian process $X^\alpha = \{X^\alpha(t),\ t \in \mathbb{R}\}$, with the covariance function

$$\mathbb{E}[X^\alpha(s)X^\alpha(t)] = \frac{1}{2}\Big(|s|^{2\alpha} + |t|^{2\alpha} - |s-t|^{2\alpha}\Big).$$

Fractional Brownian motion was introduced by Mandelbbrot and van Ness (1968) as a moving average Gaussian process

$$X^\alpha(t) = \kappa_\alpha \int_{-\infty}^{t} [((t-s)_+)^{\alpha-1/2} - ((-s)_+)^{\alpha-1/2}]dB(s),$$

where B is the ordinary Brownian motion and κ_α is a normalizing constant.

From the convariance function, it is easy to verify that X^α is a self-similar process with self-similarity index α and it has stationary increments. Except in the Brownian motion case (i. e. $\alpha = 1/2$), X^α does not have the independent increment property. Instead, when $\alpha > 1/2$, X^α is a processs with long range dependence. As such, fractional Brownian motion has become a popular model in many areas from telecommunication networks to mathematical finance. We refer to Samorodnitsky and Taqqu (1994) for more information on fractional Brownian motion and related properties.

The following property of strong local nondeterminism of fractional Brownian motion was discovered by Pitt (1978): there exists a constant $0 < K_1 < \infty$, depending on α only, such that for all $t \in \mathbb{R}$ and $0 \le r \le |t|$,

$$\mathrm{Var}\big(X^\alpha(t)|X^\alpha(s) : |s-t| \ge r\big) = K_1 r^{2\alpha}. \tag{2.1}$$

This property has been very useful in studying sample path properties of fractional Brownian motion. See, for example, Talagrand (1995) and Xiao (1997).

Let $\mathrm{detCov}(Z_1, \cdots, Z_n)$ denote the determinant of the covariance matrix of a Gaussian random vector $(Z_1, \ldots, Z_n)$. It is well known that

$$\mathrm{detCov}(Z_1, \ldots, Z_n) = \mathrm{Var}(Z_1)\prod_{j=2}^{n} \mathrm{Var}(Z_j|Z_1, \ldots, Z_{j-1}). \tag{2.2}$$

Applying this fact, together with (2.1), we see that for all integer $n \ge 2$ and distinct $t_1, \ldots, t_n \in \mathbb{R}$,

$$\mathrm{detCov}(X^\alpha(t_1), \cdots, X^\alpha(t_n)) \ge K_1^{n-1} \prod_{j=1}^{n} \min\{|t_j - t_i|^{2\alpha} : 0 \le i \le j-1\}, \tag{2.3}$$

where $t_0 := 0$.

2.3 Local Times

We end this section by briefly recalling aspects of the theory of local times. More information on local times of random, as well as non-random, functions can be found in Ref.'s (Geman and Horowitz 1980; Geman et al 1984; Xiao 1997).

Let $X(t)$ be a Borel vector field on $\mathbb{R}^N$ with values in $\mathbb{R}^d$. For any Borel set $T \subseteq \mathbb{R}^N$, the occupation measure of X on T is defined as the following measure on $\mathbb{R}^d$:

$$\mu_T(\bullet) = \lambda_N\big\{t \in T : X(t) \in \bullet\big\}.$$

If μ_T is absolutely continuous with respect to λ_d, we say that $X(t)$ has *local times* on T, and define its local times, $L(\bullet, T)$, as the Radon–Nikodým derivative of μ_T with respect to λ_d, i.e.,

$$L(x,T) = \frac{d\mu_T}{d\lambda_d}(x), \qquad \forall x \in \mathbb{R}^d.$$

In the above, x is the so-called *space variable*, and T is the *time* variable. Sometimes, we write $L(x,t)$ in place of $L(x,[0,t])$. It is clear that if X has local times on T, then for every Borel set $I \subseteq T$, $L(x,I)$ also exists.

By standard martingale and monotone class arguments, one can deduce that the local times have a measurable modification that satisfies the following *occupation density formula*: for every Borel set $T \subseteq \mathbb{R}^N$, and for every measurable function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\int_T f(X(t))\,dt = \int_{\mathbb{R}^d} f(x)L(x,T)\,dx. \tag{2.4}$$

Suppose we fix a rectangle $T = \prod_{i=1}^N [a_i, a_i + h_i]$ in $\mathcal{A}$. Then, whenever we can choose a version of the local time, still denoted by $L(x, \prod_{i=1}^N [a_i, a_i + t_i])$, such that it is a continuous function of $(x, t_1, \cdots, t_N) \in \mathbb{R}^d \times \prod_{i=1}^N [0, h_i]$, X is said to have *jointly continuous local times* on T. When a local time is jointly continuous, $L(x, \bullet)$ can be extended to be a finite Borel measure supported on the level set

$$X_T^{-1}(x) = \{t \in T : X(t) = x\}; \tag{2.5}$$

see Adler (1981) for details. In other words, local times often act as a natural measure on the level sets of X. As such, they are useful in studying the various fractal properties of level sets and inverse images of the vector field X. In this regard, we refer to Berman (1972), Adler (1978), Ehm (1981), Monrad and Pitt (1986) Rosen (1984), and Xiao (1997).

It follows from (25.5) and (25.7) in Geman and Horowitz (1980) (see also Geman, Horowitz and Rosen (1984), Pitt (1978)) that for any $x, y \in \mathbb{R}^d$, $T \in \mathcal{A}$ and any integer $n \geq 1$,

$$\begin{aligned}\mathbb{E}\Big[L(x,T)^n\Big] &= (2\pi)^{-nd} \int_{T^n} \int_{\mathbb{R}^{nd}} \exp\Big(-i \sum_{j=1}^n \langle u^j, x\rangle\Big) \\ &\qquad \times \mathbb{E}\exp\Big(i \sum_{j=1}^n \langle u^j, X(t^j)\rangle\Big) d\overline{u}\ d\overline{t} \qquad (2.6)\end{aligned}$$

and for any even integer $n \geq 2$

$$\begin{aligned}&\mathbb{E}\Big[(L(x,T) - L(y,T))^n\Big] \\ &= (2\pi)^{-nd} \int_{T^n} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n \Big[\exp(-i\langle u^j, x\rangle) - \exp(-i\langle u^j, y\rangle)\Big] \\ &\qquad \cdot \mathbb{E}\exp\Big(i \sum_{j=1}^n \langle u^j, X(t^j)\rangle\Big) d\overline{u}\ d\overline{t}, \qquad (2.7)\end{aligned}$$

where $\overline{u} = (u^1, \ldots, u^n)$, $\overline{t} = (t^1, \ldots, t^n)$, and each $u^j \in \mathbb{R}^d$, $t^j \in (0, \infty)^N$. In the coordinate notation we then write $u^j = (u_1^j, \ldots, u_d^j)$.

3 Existence of the Local Times

In this section, we prove a sufficient condition for the existence of the local times of an (N, d)-fractional Brownian sheet on any rectangle $I \in \mathcal{A}$. Because of the complicated covariance, the proof of the existence is quite involved. Therefore, we split the proof into several lemmas, which are also of their own interest.

Lemma 3.1 *Let $0 < \alpha \leq 1$, $a \geq b \geq 0$. Then*

$$(a+b)^\alpha \leq a^\alpha + \alpha b^\alpha. \qquad (3.1)$$

Remark 3.2 The above inequality is stronger than the well known inequality : $(a+b)^\alpha \leq a^\alpha + b^\alpha$. □

Lemma 3.3 *Let $-1 < \alpha \leq 1$ be fixed. Then for all $u \geq 0$, it holds that*

$$(u+1)^\alpha \leq \frac{u^{\alpha+1} - 1}{u - 1}, \qquad (3.2)$$

where for $u = 1$ the right hand side of (3.2) is defined to be $\alpha + 1$, the limiting value as $u \to 1$.

Proof First we prove (3.2) for $0 \le \alpha \le 1$ and $u \ge 1$. Define

$$f(u) = \frac{(u-1)(u+1)^\alpha}{u^{\alpha+1}-1}.$$

Then, by Lemma 3.1

$$\lim_{u\to 1} f(u) = \frac{2^\alpha}{1+\alpha} \le 1, \quad \lim_{u\to\infty} f(u) = 1.$$

To prove (3.2), it suffices to show that $f'(u) \ge 0$ for $u \ge 1$. By a simple calculation, we get that

$$f'(u) = \frac{(u+1)^{\alpha-1}}{(u^{\alpha+1}-1)^2}\Big[(1+\alpha)u^\alpha + (1-\alpha)u^{\alpha+1} - ((1+\alpha)u + (1-\alpha))\Big].$$

Thus, $f'(u) \ge 0$ if and only if

$$f_1(u) := (1+\alpha)u^\alpha + (1-\alpha)u^{\alpha+1} - [(1+\alpha)u + (1-\alpha)] \ge 0.$$

Observe that $f_1(1) = 0$ and

$$f_1'(u) = (1+\alpha)[\alpha u^{\alpha-1} + (1-\alpha)u^\alpha - 1].$$

To show that $f_1(u) \ge 0$, it is sufficient to prove that $f_2(u) := \alpha u^{\alpha-1} + (1-\alpha)u^\alpha - 1 \ge 0$ for $u \ge 1$. This follows from the fact that $f_2(1) = 0$ and

$$f_2'(u) = \alpha(1-\alpha)u^{\alpha-2}(u-1) \ge 0, \quad \text{for all} \quad u \ge 1.$$

This completes the proof of (3.2) for $0 \le \alpha \le 1$ and $u \ge 1$.

Inequality (3.2) clearly holds for $u = 0$. The case of $0 \le \alpha \le 1$ and $0 < u \le 1$ follows by applying the first part to $\hat{u} = \frac{1}{u}$.

Finally, for $-1 < \alpha < 0$, we have

$$\lim_{u\to 1} f(u) = \frac{2^\alpha}{1+\alpha} > 1 \quad \text{and} \quad \lim_{u\to\infty} f(u) = 1.$$

Hence to prove (3.2) for $u \ge 1$, we only need to show $f'(u) \le 0$ for all $u \ge 1$. This, as well as the proof for the remaining cases, is similar to the case of $0 \le \alpha \le 1$. we omit the details. $\square$

Fix a constant $0 < h < 1$, we consider the function

$$\begin{aligned} F(u) &= \frac{1}{2}(1+u^{-1})^{2h} - \frac{1}{4}(1+u^{-1})^{2h}(1+u)^{2h} - \frac{1}{4}u^{-2h} \\ &\quad -\frac{1}{4}u^{2h} + \frac{1}{2}(1+u)^{2h} + \frac{1}{2}, \quad (0 < u < \infty). \end{aligned} \tag{3.3}$$

Lemma 3.4 *Let $K_2 = 4^h(1-4^{h-1})$. Then,*

$$F(u) \ge K_2 \quad \textit{for all} \quad 0 < u < \infty.$$

Proof As

$$F(1) = K_2 \le 1 \quad \text{and} \quad \lim_{u\to 0} F(u) = \lim_{u\to\infty} F(u) = 1,$$

it suffices to show that $F(\cdot)$ reaches its global minimum at $u = 1$. To this end, we study the sign change of $F'(u)$. Differentiating F gives

$$\begin{aligned} F'(u) &= -h(1+u^{-1})^{2h-1}u^{-2} - \frac{h}{2}(1+u)^{2h-1}(1+u^{-1})^{2h} \\ &\quad +\frac{h}{2}(1+u)^{2h}(1+u^{-1})^{2h-1}u^{-2} + \frac{h}{2}u^{-2h-1} \\ &\quad -\frac{h}{2}u^{2h-1} + h(1+u)^{2h-1}. \end{aligned} \tag{3.4}$$

Clearly, $F'(1) = 0$. By rewriting and rearranging the terms in (3.4), we have

$$\begin{aligned} F'(u) &= -h(1+u)^{2h-1}u^{-2h-1} - \frac{h}{2}(1+u)^{4h-1}u^{-2h} \\ &\quad +\frac{h}{2}(1+u)^{4h-1}u^{-2h-1} + \frac{h}{2}u^{-2h-1} - \frac{h}{2}u^{2h-1} + h(1+u)^{2h-1} \\ &= \frac{h}{2}u^{-2h-1}\Big[1 + (1+u)^{4h-1} - 2(1+u)^{2h-1}\Big] \\ &\quad -\frac{h}{2}u^{2h-1}\Big[1 + (1+u)^{4h-1}u^{-4h+1} - 2(1+u)^{2h-1}u^{-2h+1}\Big] \\ &= \frac{h}{2}u^{-2h-1}\Big[((1+u)^{2h-1} - 1)^2 + (1+u)^{4h-1} - (1+u)^{4h-2}\Big] \\ &\quad -\frac{h}{2}u^{2h-1}\Big[((1+u)^{2h-1}u^{-2h+1} - 1)^2 + (1+u)^{4h-1}u^{-4h+1} - (1+u)^{4h-2}u^{-4h+2}\Big] \\ &= \frac{h}{2}u^{-2h-1}\Big((1+u)^{2h-1} - 1\Big)^2 - \frac{h}{2}u^{2h-1}\Big((1+u)^{2h-1}u^{-2h+1} - 1\Big)^2. \end{aligned}$$

Thus $F'(u) \ge 0$ if and only if

$$\Big((1+u)^{2h-1} - 1\Big)u^{-h-\frac{1}{2}} \ge \Big((1+u^{-1})^{2h-1} - 1\Big)u^{h-\frac{1}{2}}.$$

This holds if and only if

$$g(u) := u^{2h} + (1+u)^{2h-1} - u(1+u)^{2h-1} - 1 \ge 0. \tag{3.5}$$

Applying Lemma 3.3 with $\alpha = 2h - 1 \in (-1, 1)$, we see that (3.5) holds if and only if $u \ge 1$. Therefore, we have showed that $F'(u) \ge 0$ if and only if $u \ge 1$. This together with $F'(1) = 0$ implies that $u = 1$ is the global minimum point of F. $\square$

Corollary 3.5 *For $0 < h < 1$, $s \ge r > 0$, define*

$$G(s,r) = s^{2h}r^{2h} - \frac{1}{4}\big[s^{2h} + r^{2h} - (s-r)^{2h}\big]^2. \tag{3.6}$$

Then $G(s,r) \ge K_2\, r^{2h}(s-r)^{2h}$, where K_2 is defined as in the previous lemma.

Proof Set $u = \frac{s}{r} - 1$. We have

$$\begin{aligned} G(s,r) &= r^{2h}(s-r)^{2h}\Big[\frac{1}{2}\big(\frac{s}{s-r}\big)^{2h} - \frac{1}{4}\big(\frac{s}{r}\big)^{2h}\big(\frac{s}{s-r}\big)^{2h} \\ &\quad -\frac{1}{4}\big(\frac{r}{s-r}\big)^{2h} - \frac{1}{4}\big(\frac{s-r}{r}\big)^{2h} + \frac{1}{2}\big(\frac{s}{r}\big)^{2h} + \frac{1}{2}\Big] \\ &= r^{2h}(s-r)^{2h}F(u), \end{aligned}$$

where $F(u)$ is the function defined in (3.3). Thus Corollary 3.5 follows. □

Lemma 3.6 *Let $-1 < \alpha \leq 1$. Then, for all $u \geq 1$,*

$$u^{\alpha+1} \leq 1 + (u-1)^\alpha (u+1). \tag{3.7}$$

Proof Define for $u > 1$

$$f(u) = \frac{(u+1)(u-1)^\alpha}{u^{\alpha+1} - 1}.$$

Then,

$$\lim_{u\to\infty} f(u) = 1.$$

To prove the lemma, it suffices to show that $f'(u) \leq 0$ for all $u > 1$.

Indeed, as $-1 < \alpha \leq 1$, $u > 1$, elementary computation shows

$$f'(u) = \frac{(u-1)^{\alpha-1}}{(u^{\alpha+1}-1)^2}\Big[(1+\alpha)(u^\alpha - u) + (1-\alpha)(1-u^{\alpha+1})\Big] \leq 0.$$

This proves Lemma 3.6. □

Theorem 3.7 *Let $B^H = \{B^H(t),\ t \in \mathbb{R}_+^N\}$ be an (N,d) fractional Brownian sheet with Hurst index $H = (H_1, \ldots, H_N)$. If $d < \sum_{\ell=1}^N \frac{1}{H_\ell}$, then for all $I \in \mathcal{A}$, B^H has local times $\{L(x,I),\ x \in \mathbb{R}^d\}$ on I; and $L(x,I)$ admits the following L^2 representation:*

$$L(x,I) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle y,x\rangle} \int_I e^{i\langle y, B^H(s)\rangle} ds\, dy, \tag{3.8}$$

where, $y = (y_1, y_2, ..., y_d)$.

Remark 3.8 The result in Theorem 3.7 is sharp. When $H_1 = \cdots = H_N = \frac{1}{2}$, it recovers the corresponding result for the Brownian sheet. See Ehm (1981) for additional information about local times of the Brownian sheet. □

Proof Let $I=[s,t]\in\mathcal{A}$ be fixed. Without loss of generality, we assume $I=[\epsilon,1]^N$. By the method in Orey (1970) and Pitt (1978), (see also (21.3) in Geman and Horowitz (1980)), it is sufficient to prove that

$$\mathcal{J}(I):=\int_I ds\int_I dr\int_{\mathbb{R}^d} dy\int_{\mathbb{R}^d}\Big|E\exp\Big(i\langle y,B^H(s)\rangle+i\langle z,B^H(r)\rangle\Big)\Big|dz<\infty. \tag{3.9}$$

For this purpose, we first establish the following estimate: let $0<h<1$ be a constant, then for any $\delta>2h$, $M>0$ and $p>0$, there exist positive and finite constants K_3 and K_4, depending on $\delta,\ \epsilon,\ M$ only, such that for $0<a\le M$,

$$\begin{aligned}\mathcal{I}(a) &:= \int_\epsilon^1 dr\int_\epsilon^1 ds\Big[a+|s-r|^{2h}\Big]^{-p}\\ &\le K_3(a^{-p+\frac{1}{\delta}}+K_4).\end{aligned} \tag{3.10}$$

Indeed, by the symmetry of the integrand, we see that

$$\mathcal{I}(a)=2\int_\epsilon^1 r^{-2ph}dr\int_r^1\Big[\frac{a}{r^{2h}}+(\frac{s}{r}-1)^{2h}\Big]^{-p}ds.$$

Putting $u=s/r$ and using the fact that $r\ge\varepsilon,\ s\ge\varepsilon$, we see that the above integral is bounded by

$$\begin{aligned}K\int_\epsilon^1 r dr\int_1^{1/r}\Big[a+(u-1)^{2h}\Big]^{-p}du &\le K\int_1^{1/\epsilon}\Big[a+(u-1)^{2h}\Big]^{-p}du\\ &= K\int_a^{a+(\frac{1}{\epsilon}-1)^{2h}} v^{-p}(v-a)^{\frac{1-2h}{2h}}dv,\end{aligned} \tag{3.11}$$

where we have used the substitution $v:=a+(u-1)^{2h}$ and $K>0$ is a finite constant depending on ϵ, h and p.

Let $b_\epsilon=a+(\frac{1}{\epsilon}-1)^{2h}$. We prove (3.10) for $h>\frac12$ and $0<h\le\frac12$ separately. If $h>\frac12$, then for $\delta>2h$, (3.11) and Hölder's inequality imply

$$\begin{aligned}\mathcal{I}(a) &\le K\Big(\int_a^{b_\epsilon}v^{-p\delta}dv\Big)^{\frac{1}{\delta}}\Big(\int_a^{b_\epsilon}(v-a)^{(-\frac{2h-1}{2h}\cdot\frac{\delta}{\delta-1})}dv\Big)^{\frac{\delta-1}{\delta}}\\ &\le K_3(a^{-p+\frac{1}{\delta}}+K_4),\end{aligned}$$

where we have used the fact that $\frac{2h-1}{2h}\cdot\frac{\delta}{\delta-1}<1$. This proves (3.10) for $h>1/2$.

If $0<h\le 1/2$, elementary calculation shows that for all $0<a\le M$

$$\begin{aligned}\mathcal{I}(a) &\le K\int_a^{b_\epsilon}v^{-p}(v-a)^{\frac{1-2h}{2h}}dv\\ &\le K\int_a^{b_\epsilon}v^{-p-1+\frac{1}{2h}}dv\\ &\le K_3(a^{-p+\frac{1}{\delta}}+K_4)\end{aligned}$$

for some positive and finite constants K_3 and K_4 depending on $\delta,\ \epsilon,\ M$ only.

Let us now go back to show (3.9). As B^H is Gaussian, using the expression for the characteristic functions of Gaussian random variables it turns out that

$$\mathcal{J}(I) = (2\pi)^d \int_I ds \int_I \Big[\prod_{\ell=1}^N s_\ell^{2H_\ell} r_\ell^{2H_\ell} - \prod_{\ell=1}^N \frac{1}{4}(s_\ell^{2H_\ell} + r_\ell^{2H_\ell} - |s_\ell - r_\ell|^{2H_\ell})^2\Big]^{-\frac{d}{2}} dr. \tag{3.12}$$

We claim that there exists a constant $K_5 = K_{\epsilon,H} > 0$ such that for all $s,\ r \in I = [\epsilon, 1]^N$,

$$\prod_{\ell=1}^N s_\ell^{2H_\ell} r_\ell^{2H_\ell} - \prod_{\ell=1}^N \frac{1}{4}(s_\ell^{2H_\ell} + r_\ell^{2H_\ell} - |s_\ell - r_\ell|^{2H_\ell})^2 \geq K_5 \sum_{\ell=1}^N |s_\ell - r_\ell|^{2H_\ell} \tag{3.13}$$

To see (3.13), we observe that

$$\begin{aligned}
&\prod_{\ell=1}^N s_\ell^{2H_\ell} r_\ell^{2H_\ell} - \prod_{\ell=1}^N \frac{1}{4}\big[s_\ell^{2H_\ell} + r_\ell^{2H_\ell} - |s_\ell - r_\ell|^{2H_\ell}\big]^2 \\
&= \Big(\prod_{\ell=2}^N s_\ell^{2H_\ell} r_\ell^{2H_\ell}\Big)\Big(s_1^{2H_1} r_1^{2H_1} - \frac{1}{4}(s_1^{2H_1} + r_1^{2H_1} - |s_1 - r_1|^{2H_1})^2\Big) \\
&\qquad + \frac{1}{4}\big[s_1^{2H_1} + r_1^{2H_1} - |s_1 - r_1|^{2H_1}\big]^2 \times \Big[\prod_{\ell=2}^N s_\ell^{2H_\ell} r_\ell^{2H_\ell} \\
&\qquad - \prod_{\ell=2}^N \frac{1}{4}(s_\ell^{2H_\ell} + r_\ell^{2H_\ell} - |s_\ell - r_\ell|^{2H_\ell})^2\Big] \\
&\geq K_2\, \epsilon^{\sum_{\ell=1}^N 4H_\ell} |s_1 - r_1|^{2H_1} + \\
&\qquad \frac{1}{4}\epsilon^{2H_1}\left[\prod_{\ell=2}^N s_\ell^{2H_\ell} r_\ell^{2H_\ell} - \prod_{\ell=2}^N \frac{1}{4}(s_\ell^{2H_\ell} + r_\ell^{2H_\ell} - |s_\ell - r_\ell|^{2H_\ell})^2\right]
\end{aligned}$$

where we have used Corollary 3.5 to obtain the last inequality.

Applying the above procedure repeatedly we finally can find a positive constant K_5 depending on ϵ, N and H only such that (3.13) holds.

Choose $\delta_2, ..., \delta_N$ such that $\delta_\ell > 2H_\ell$ and

$$\frac{d}{2} < \Big(\frac{1}{2H_1} + \frac{1}{\delta_2} + \cdots + \frac{1}{\delta_N}\Big).$$

This is possible because $d < (\frac{1}{H_1} + \cdots + \frac{1}{H_N})$. Applying the estimate (3.10) for

$$a = \sum_{\ell=1}^{N-1} |s_\ell - r_\ell|^{2H_\ell} \quad \text{and} \quad p = d/2,$$

we obtain that $\mathcal{J}(I)$ is at most

$$
\begin{aligned}
& K_6 \int_\epsilon^1 ds_{_1} \int_\epsilon^1 dr_{_1} \cdots \int_\epsilon^1 ds_{_{N-1}} \int_\epsilon^1 dr_{_{N-1}} \Big[\int_\epsilon^1 ds_{_N} \int_\epsilon^1 dr_{_N} \Big(\sum_{\ell=1}^N |s_\ell - r_\ell|^{2H_\ell} \Big)^{-d/2} \Big] \\
& \le K_7 \int_\epsilon^1 ds_{_1} \int_\epsilon^1 dr_{_1} \cdots \int_\epsilon^1 ds_{_{N-1}} \int_\epsilon^1 \Big[\Big(\sum_{\ell=1}^{N-1} |s_\ell - r_\ell|^{2H_\ell} \Big)^{-d/2 + \frac{1}{\delta_N}} + K_4 \Big] dr_{_{N-1}} \\
& \le K_8 + K_9 \int_\epsilon^1 ds_1 \int_\epsilon^1 dr_1 \cdots \int_\epsilon^1 ds_{_{N-1}} \int_\epsilon^1 dr_{_{N-1}} \Big[\sum_{\ell=1}^{N-1} |s_\ell - r_\ell|^{2H_\ell} \Big]^{-\frac{d}{2} + \frac{1}{\delta_N}}, \qquad (3.14)
\end{aligned}
$$

where K_8 and K_9 are positive and finite constants depending on ϵ N, δ_N and H only. By repeatedly using the estimate (3.10) as above, after $N-1$ steps, we obtain that

$$
\mathcal{J}(I) \le K_{10} + K_{11} \int_\epsilon^1 ds_1 \int_\epsilon^1 dr_1 \Big[|s_1 - r_1|^{2H_1} \Big]^{-\frac{d}{2} + \frac{1}{\delta_N} + \cdots + \frac{1}{\delta_2}} \qquad (3.15)
$$

This is finite because $(\frac{d}{2} - \frac{1}{\delta_N} - \cdots - \frac{1}{\delta_2}) 2H_1 < 1$ by the choices of δ_ℓ. The proof is completed. $\square$

4 Joint Continuity

When the local times of B^H exist, it is natural to ask whether there exists a version of the local times which is jointly continuous in both space and time variables. The answers are affirmative for the Brownian sheet and (N, d)-fractional Brownian motion; see Pitt (1978), Ehm (1981) for details. The question for (N, d)-fractional Brownian sheets is significantly harder due to the fact that B^H does not have the independent increment property nor local nondeterminism. In this section, we prove a sufficient condition for the joint continuity of the local times of B^H.

Theorem 4.1 *Let $B^H = \{B^H(t),\ t \in \mathbb{R}_+^N\}$ be a fractional Brownian sheet in $\mathbb{R}^d$ with Hurst index $H = (H_1, \ldots, H_N)$. If $H_\ell d < 1$ for all $\ell = 1, \ldots, d$, then for all closed intervals $I \in \mathcal{A}$, B^H has a jointly continuous local time on I. In particular, a real-valued fractional Brownian sheet has jointly continuous local times.*

Our proof of Theorem 4.1 is based on an moment argument and the continuity lemma of Garsia (1971). The basic estimates that are required for the proof are contained in Lemmas 4.8 and 4.9. For their proofs, we also need some results from Cuzick and Du Peez (1982) and Xiao (1997).

Lemma 4.2 *Let $Z_1, \ldots, Z_n$ be the mean zero Gaussian variables which are linearly independent and assume that*

$$\int_{-\infty}^{\infty} g(v) e^{-\epsilon v^2} dv < \infty$$

for all $\epsilon > 0$. Then

$$\int_{\mathbb{R}^n} g(v_1) \exp\Big[-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^{n} v_j Z_j\Big)\Big] dv_1 \cdots dv_n$$
$$= \frac{(2\pi)^{(n-1)/2}}{(\mathrm{detCov}(Z_1, \cdots, Z_n))^{1/2}} \int_{-\infty}^{\infty} g\Big(\frac{v}{\sigma_1}\Big) e^{-v^2/2} dv \ ,$$

where $\sigma_1^2 = \mathrm{Var}(Z_1 | Z_2, \ldots, Z_n)$ is the conditional variance of Z_1 given $Z_2, \ldots, Z_n$.

Lemma 4.3 *Assume $p(y)$ is positive and non-decreasing on $(0, \infty)$, $p(0) = 0$, $y^n/p^n(y)$ is non-decreasing on $[0,1]$, and $\int_1^\infty p^{-2}(y) dy < \infty$. Then there exists a constant K_{12} such that for all $n \geq 1$*

$$\int_0^\infty \frac{|\exp(ivy) - 1|^n}{p^n(y)} dy \leq K_{12}^n \, p_+^{-n}\Big(\frac{1}{v}\Big),$$

where $p_+(y) = \min\{1, p(y)\}$.

Lemma 4.4 *For $\alpha \geq e^2/2$,*

$$\int_1^\infty (\log x)^\alpha \exp(-x^2/2) dx \leq \sqrt{\pi}\, (\log \alpha)^\alpha.$$

In the above, Lemmas 4.2 and 4.4 are due to Cuzick and Du Peez (1982), and Lemma 4.3 is a slight modification of their Lemma 3.

Lemma 4.5 below will be important for our purpose. It connects the determinants of the covariance matrices of B^H with those of fractional Brownian motions, hence makes it possible for us to use the arguments in Xiao (1997) to prove the joint continuity of the local times of fractional Brownian sheet.

Lemma 4.5 *For any integer $n \geq 2$ and $t^1, \ldots, t^n \in \mathbb{R}_+^N$, we have*

$$\mathrm{detCov}(B_0^H(t^1), \cdots, B_0^H(t^n)) \geq \prod_{\ell=1}^{N} \mathrm{detCov}(X^{H_\ell}(t_\ell^1), \cdots, X^{H_\ell}(t_\ell^n)), \tag{4.1}$$

where for $\ell = 1, \ldots, N$, X^{H_ℓ} is the one parameter fractional Brownian motion in $\mathbb{R}$ with Hurst index H_ℓ.

Proof Recall that the Hadamard product of two $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ is an $n \times n$ matrix defined as $A \circ B = (a_{ij} b_{ij})$. A classical theorem of Oppenheim (cf. Horn and Johnson (1999, p.480) or Bapat and Raghavan (1996, p.137)) asserts that if A and B are positive semidefinite Hermitian matrices, then

$$\det(A \circ B) \geq \det(A) \cdot \det(B), \tag{4.2}$$

where $\det(A)$ denotes the determinant of A.

By Eq. (1.1), we see that the covariance matrix

$$\mathrm{Cov}(B_0^H(t^1), \cdots, B_0^H(t^n))$$

is the Hadamard product of the covariance matrices

$$\mathrm{Cov}(X^{H_\ell}(t_\ell^1), \cdots, X^{H_\ell}(t_\ell^n)), \quad (\ell = 1, \ldots, N).$$

Hence Inequality (4.1) follows from (4.2) and induction. □

Lemma 4.6 *Let $0 < \gamma < 1$. Then for all $r > 0$, $a \in \mathbb{R}$, all integers $n \geq 1$ and all distinct $t_1, \cdots, t_n \in [a, a+r]$, we have*

$$\int_{[a,a+r]} \frac{dt}{\min\Big\{|t - t_j|^\gamma,\ j = 1, \cdots, n\Big\}} \leq K_{13}\, r^{1-\gamma} n^\gamma.$$

where $K_{13} > 0$ is a finite constant depending on γ only.

Proof This is a special case of Lemma 2.3 in Xiao (1997). □

For completeness, we also state the following basic result of Garsia (1971).

Lemma 4.7 *Assume that $p(u)$ and $\Psi(u)$ are two positive increasing functions on $[0, \infty)$, $p(u) \downarrow 0$ as $u \downarrow 0$, $\Psi(u)$ is convex and $\Psi(u) \uparrow \infty$ as $u \uparrow \infty$. Let D denote an open hypercube in $\mathbb{R}^d$. If the function $f(x)$ is measurable in D and*

$$A := A(D, f) = \int_D \int_D \Psi\Big(\frac{|f(x) - f(y)|}{p(|x - y|/\sqrt{d})}\Big) dx dy < \infty, \tag{4.3}$$

then after modifying $f(x)$ on a set of Lebesgue measure 0, we have

$$|f(x) - f(y)| \leq 8 \int_0^{|x-y|} \Psi^{-1}\Big(\frac{A}{u^{2d}}\Big)\, dp(u), \qquad \textit{for all } x, y \in D.$$

As we mentioned earlier, the following Lemmas 4.8 and 4.9 give the required moment estimates for proving the joint continuity of local times.

Lemma 4.8 *Let $T \in \mathcal{A}$ be a hypercube with edge length $r > 0$. Then for all $x \in \mathbb{R}^d$ and all integers $n \geq 1$,*

$$\mathbb{E}\big[L(x,T)^n\big] \leq K_{14}^n \, r^{(N-\sum_{\ell=1}^N H_\ell d)n} (n!)^{\sum_{\ell=1}^N H_\ell d}. \tag{4.4}$$

Proof Similar to the argument in Xiao (1997, pp 137–138), we apply (2.6) and the independence of $B_1^H, \ldots, B_d^H$ to deduce

$$\begin{aligned}\mathbb{E}\big[L(x,T)^n\big] &\leq (2\pi)^{-nd} \int_{T^n} \prod_{k=1}^d \Big\{ \int_{\mathbb{R}^n} \exp\Big[-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^n u_k^j B_0^H(t^j)\Big)\Big] dU^k \Big\} d\bar{t} \\ &= (2\pi)^{-nd/2} \int_{T^n} \Big[\mathrm{detCov}(B_0^H(t^1), \ldots, B_0^H(t^n))\Big]^{-d/2} \, d\bar{t}. \end{aligned} \tag{4.5}$$

where $U^k = (u_k^1, \ldots, u_k^n) \in \mathbb{R}^n$, and we have used the fact that

$$\int_{\mathbb{R}^n} \exp\Big[-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^n u_k^j B_0^H(t^j)\Big)\Big] dU^k = \frac{(2\pi)^{n/2}}{\Big[\mathrm{detCov}(B_0^H(t^1), \ldots, B_0^H(t^n))\Big]^{1/2}}$$

for λ_{Nn}-almost all $(t^1, \ldots, t^n) \in T^n$.

It follows from Lemma 4.5 and (2.3) that

$$\begin{aligned}\mathrm{detCov}(B_0^H(t^1), \ldots, B_0^H(t_n)) &\geq \prod_{\ell=1}^N \mathrm{detCov}(X^{H_\ell}(t_\ell^1), \cdots, X^{H_\ell}(t_\ell^n)) \\ &\geq K_{15}^n \prod_{\ell=1}^N \prod_{j=1}^n \Big[\min\{|t_\ell^j - t_\ell^i|^{2H_\ell}, 0 \leq i \leq j-1\}\Big]. \end{aligned} \tag{4.6}$$

Putting together (4.5), (4.6) and applying Lemma 4.6 n times, we obtain

$$\begin{aligned}\mathbb{E}\big[L(x,T)^n\big] &\leq K_{16}^n \int_{T^n} \prod_{\ell=1}^N \prod_{j=1}^n \Big[\min\{|t_\ell^j - t_\ell^i|^{2H_\ell}, 0 \leq i \leq j-1\}\Big]^{-d/2} \, d\bar{t} \\ &\leq K_{16}^n \prod_{\ell=1}^N \int_{[\epsilon, \epsilon+r]^n} \prod_{j=1}^n \Big[\min\{|t_\ell^j - t_\ell^i|^{2H_\ell}, 0 \leq i \leq j-1\}\Big]^{-d/2} \, dt_\ell^1 \ldots dt_\ell^n \\ &\leq K_{14}^n \, r^{(N-\sum_{\ell=1}^N H_\ell d)n} (n!)^{\sum_{\ell=1}^N H_\ell d}, \end{aligned}$$

for some positive and finite constant K_{14}. This proves (4.4). □

Lemma 4.8 implies that for all $p \geq 1$, $L(x,T) \in L^p(R^d)$ a.s. (see Geman and Horowitz (1980, page 42). Let $\gamma > 0$ be a constant whose value will be determined later. Define

$$p(y) = \begin{cases} 0, & \text{if } y = 0 \\ \log^{-\gamma}\big(e/|y|\big), & \text{if } 0 < |y| \leq 1 \\ \gamma|y| - \gamma + 1, & \text{if } |y| > 1. \end{cases}$$

Lemma 4.9 *Let T be the same as in Lemma 4.8. Then for all even integers $n \geq 2$ and all hypercubes $D \subset \mathbb{R}^d$, we have*

$$\mathbb{E}\int_D\int_D \Big(\frac{L(x,T) - L(y,T)}{p(|x-y|/\sqrt{d})}\Big)^n dxdy \leq K_{17}^n (n!)^N (\log n)^{n\gamma}\, r^{n(N - \sum_{\ell=1}^N H_\ell d)} \log_+^{nN\gamma}\Big(\frac{e}{r}\Big). \quad (4.7)$$

where $K_{17} > 0$ is a finite constant depending on N, d, H and D.

Proof First we note that for $u^1, \ldots, u^n,\ y \in \mathbb{R}^d$,

$$\begin{aligned} \prod_{j=1}^n \Big|\exp(-i\langle u^j, y\rangle) - 1\Big| &= \prod_{j=1}^n \Big|\exp\Big(-i\sum_{\ell=1}^d u_\ell^j y_\ell\Big) - 1\Big| \\ &\leq \prod_{j=1}^n \Big|\sum_{k=1}^d \Big[\exp\Big(-i\sum_{\ell=0}^k u_\ell^j y_\ell\Big) - \exp\Big(-i\sum_{\ell=0}^{k-1} u_\ell^j y_\ell\Big)\Big]\Big| \qquad (y_0 = 0,\ u_0^j = 0) \\ &\leq \prod_{j=1}^n \Big[\sum_{k=1}^d \Big|\exp(-iu_k^j y_k) - 1\Big|\Big] \\ &= \sum{}' \prod_{j=1}^n \Big|\exp(-iu_{k_j}^j y_{k_j}) - 1\Big|, \end{aligned} \quad (4.8)$$

where the summation $\sum{}'$ is taken over all sequences $(k_1, \cdots, k_n) \in \{1, \ldots, d\}^n$.

Secondly, without loss of generality, we may and will assume the edge length of T is sufficiently small such that for all $s, t \in T$

$$\mathbb{E}\Big(|B_0(s) - B_0(t)|^2\Big) \leq \min\{1, [e\,\mathrm{Var}(B_0^H(u))]^{-1}, u \in T\}. \quad (4.9)$$

Otherwise, we can divide T into small pices so that (4.9) holds and (4.7) will only be affected by a factor $n2^{n-1}$.

It follows from (2.7) and (4.8) that

$$\begin{aligned}
\mathbb{E}\Big[(L(x,T)-L(y,T))^n\Big] &\le (2\pi)^{-nd}\int_{T^n}\int_{\mathbb{R}^{nd}}\prod_{j=1}^n\Big|\exp(-i\langle u^j, y-x\rangle)-1\Big| \\
&\qquad \cdot\exp\Big[-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^n\langle u^j,\ B^H(t^j)\rangle\Big)\Big]d\overline{u}\ d\overline{t} \\
&\le (2\pi)^{-nd}\sum{}'\int_{T^n}\int_{\mathbb{R}^{nd}}\prod_{j=1}^n\Big|\exp(iu^j_{k_j}(y_{k_j}-x_{k_j}))-1\Big| \\
&\qquad \cdot\exp\Big[-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^n\langle u^j,\ B^H(t^j)\rangle\Big)\Big]d\overline{u}\ d\overline{t}.
\end{aligned} \tag{4.10}$$

Hence for any fixed hypercube $D\subset\mathbb{R}^d$ and any even integer $n\ge 2$, we have

$$\begin{aligned}
&\mathbb{E}\int_D\int_D\Big[\Big(\frac{L(x,T)-L(y,T)}{p(|x-y|/\sqrt{d})}\Big)^n\Big]dxdy \le (2\pi)^{-nd}\sum{}'\int_D\int_D\int_{T^n}\int_{\mathbb{R}^{nd}} \\
&\quad\prod_{j=1}^n\frac{\big|\exp(iu^j_{k_j}(y_{k_j}-x_{k_j}))-1\big|}{p(|y-x|/\sqrt{d})}\cdot\exp\Big[-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^n\langle u^j,\ B^H(t^j)\rangle\Big)\Big]d\overline{u}\ d\overline{t}\,dx\,dy \\
&\le (2\pi)^{-nd}\lambda_d(D)\sum{}'\int_{D\ominus D}\int_{T^n}\int_{\mathbb{R}^{nd}}\prod_{j=1}^n\frac{\big|\exp(iu^j_{k_j}y_{k_j})-1\big|}{p(|y_{k_j}|/\sqrt{d})} \\
&\qquad\cdot\exp\Big[-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^n\langle u^j,\ B^H(t^j)\rangle\Big)\Big]d\overline{u}\ d\overline{t}\,dy.
\end{aligned} \tag{4.11}$$

In the above, $D\ominus D=\{x-y: x,y\in D\}$ and we have made a change of variables and have used the fact that $p(|y|/\sqrt{d})\ge p(|y_j|/\sqrt{d})$ for all $j=1,\ldots,d$.

Now we fix a sequence $(k_1,\ldots,k_n)\in\{1,\ldots,d\}^n$ and consider the integral

$$\mathcal{M}=\int_{D\ominus D}\int_{\mathbb{R}^{nd}}\prod_{j=1}^n\frac{\big|\exp(iu^j_{k_j}y_{k_j})-1\big|}{p(|y_{k_j}|/\sqrt{d})}\ \exp\Big[-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^n\langle u^j,\ B^H(t^j)\rangle\Big)\Big]d\overline{u}\,dy.$$

For any fixed n points $t^1,\ldots,t^n\in T$ such that $t^1_\ell,\ldots,t^n_\ell$ are all distinct for $1\le\ell\le N$ (the set of such points has full Lebesgue measure), Lemma 4.5 implies that $B^H_\ell(t^j)$ ($\ell=1,\ldots,d,\ j=1,\ldots,n$) are linearly independent. Hence by applying the generalized Hölder's inequality, Lemma 4.2 and Lemma 4.3, we have

$$\begin{aligned}
\mathcal{M} &\le K\prod_{j=1}^{n}\Big\{\int_{\mathbb{R}}\int_{\mathbb{R}^{nd}}\Big[\frac{|\exp(iu^j_{k_j}y_{k_j})-1|}{p(|y_{k_j}|/\sqrt{d})}\Big]^n \exp\Big[-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^{n}\sum_{\ell=1}^{d}u^j_\ell B^H_\ell(t^j)\Big)\Big]\, d\overline{u}\, dy_{k_j}\Big\}^{1/n}\\
&= \frac{K(2\pi)^{n(d-1)/2}}{[\det\mathrm{Cov}(B^H_0(t^1),\dots,B^H_0(t^n))]^{d/2}}\\
&\qquad\cdot\prod_{j=1}^{n}\Big\{\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\big|\exp\Big(iu^j_{k_j}y_{k_j}/\sigma_j\Big)-1\big|^n}{p^n(y_{k_j}/\sqrt{d})}\exp\Big(-\frac{(u^j_{k_j})^2}{2}\Big)dy_{k_j}\,du^j_{k_j}\Big\}^{1/n}\\
&\le \frac{K^n}{[\det\mathrm{Cov}(B^H_0(t^1),\dots,B^H_0(t^n))]^{d/2}}\prod_{j=1}^{n}\Big[\int_{\mathbb{R}}p_+^{-n}\Big(\frac{\sigma_j}{v}\Big)\exp\Big(-\frac{v^2}{2}\Big)dv\Big]^{1/n} \qquad (4.12)
\end{aligned}$$

where $K>0$ is a constant depending on D and K_{12} in Lemma 4.3, and σ_j^2 is the conditional variance of $B^H_{k_j}(t^j)$ given $B^H_\ell(t^i)$ ($\ell\neq k_j$ or $\ell=k_j,\ i\neq j$).

Since

$$p_+^{-n}(x)=\begin{cases}\log^{n\gamma}\big(e/x\big), & \text{if } 0<x<1\\ 1 & \text{if } x\ge 1.\end{cases}$$

and $\log_+^\alpha(xy)\le 2^\alpha(\log_+^\alpha x+\log_+^\alpha y)$ for all $\alpha\ge 0$, where $\log_+ x=\max\{1,\log x\}$, we deduce

$$\begin{aligned}
\int_{\mathbb{R}}p_+^{-n}\Big(\frac{\sigma_j}{v}\Big)\exp\Big(-\frac{v^2}{2}\Big)dv &\le \int_{|\sigma_j/v|\ge 1}\exp(-v^2/2)dv+2^{n\gamma}\int_{|\sigma_j/v|<1}\log_+^{n\gamma}(v)\exp\Big(-\frac{v^2}{2}\Big)dv\\
&\quad +2^{n\gamma}\int_{|\sigma_j/v|<1}\log_+^{n\gamma}\Big(\frac{e}{\sigma_j}\Big)\exp\Big(-\frac{v^2}{2}\Big)dv. \qquad (4.13)
\end{aligned}$$

By Lemma 4.4, for n large, the above is bounded by

$$K^n\Big[\log_+^{n\gamma}\Big(\frac{e}{\sigma_j}\Big)+(\log(n\gamma))^{n\gamma}\Big]\le K^n\Big[\log_+^{n\gamma}\Big(\frac{e}{\sigma_j}\Big)\Big][\log n]^{n\gamma}. \qquad (4.14)$$

It follows from (4.11), (4.12), (4.13) and (4.14) that

$$\begin{aligned}
&\mathbb{E}\int_D\int_D\Big(\frac{L(x,T)-L(y,T)}{p(|x-y|/\sqrt{d})}\Big)^n dxdy\le K^n\lambda_d(D)[\log n]^{n\gamma}\\
&\qquad\cdot\int_{T^n}\frac{1}{(\det\mathrm{Cov}(B^H_0(t^1),\dots,B^H_0(t^n)))^{d/2}}\prod_{j=1}^{n}\log_+^{\gamma}\Big(\frac{e}{\sigma_j}\Big)d\overline{t}. \qquad (4.15)
\end{aligned}$$

By the independence of $B^H_1,\dots,B^H_d$, we deduce that

$$\begin{aligned}
\sigma_j^2 &= \mathrm{Var}\Big(B^H_{k_j}(t^j)\Big|B^H_\ell(t^i),\ \ell\neq k_j \text{ or } \ell=k_j,\ i\neq j\Big)\\
&= \mathrm{Var}\Big(B^H_{k_j}(t^j)\Big|B^H_{k_j}(t^i),\ i\neq j\Big)\\
&= \frac{\det\mathrm{Cov}(B^H_0(t^1),\dots,B^H_0(t^n))}{\det\mathrm{Cov}(B^H_0(t^i),i\neq j)}\\
&\ge e\det\mathrm{Cov}(B^H_0(t^1),\dots,B^H_0(t^n)). \qquad (4.16)
\end{aligned}$$

Notice that in obtaining the last inequality in (4.16), we have used (2.2) and the assumption that the edge length of T is very small (so that (4.9) holds) to derive that for any $s^1, \ldots, s^n \in T$

$$\begin{aligned}
&\det\mathrm{Cov}(B_0^H(s^1), \ldots, B_0^H(s^n)) \\
&= \mathrm{Var}(B_0^H(s^1)) \prod_{j=2}^{n} \mathrm{Var}\Big(B_0^H(s^j) \big| B_0^H(s^i), i = 1, \ldots, j-1\Big) \\
&\le \mathrm{Var}(B_0^H(s^1)) \prod_{j=2}^{n} \mathbb{E}\Big(|B_0^H(s^j) - B_0^H(s^{j-1})|^2\Big) \le e^{-1}.
\end{aligned}$$

By (4.16), we have

$$\prod_{j=1}^{n} \log_+^{\gamma}\Big(\frac{e}{\sigma_j}\Big) \le 2^{-n\gamma} \log_+^{n\gamma} \frac{e}{\det\mathrm{Cov}(B_0^H(t^1), \ldots, B_0^H(t^n))}. \tag{4.17}$$

It follows from (4.15), (4.17), Lemma 4.5 and (2.2) that

$$\begin{aligned}
&\mathbb{E}\int_D \int_D \Big(\frac{L(x,T) - L(y,T)}{p(|x-y|/\sqrt{d})}\Big)^n \, dx dy \\
&\le K^n \lambda_d(D) [\log n]^{n\gamma} \int_{T^n} \frac{1}{(\det\mathrm{Cov}(B_0^H(t^1), \ldots, B_0^H(t^n)))^{d/2}} \\
&\qquad \cdot \log_+^{n\gamma} \frac{e}{\det\mathrm{Cov}(B_0^H(t^1), \ldots, B_0^H(t^n))} \, d\bar{t} \\
&\le K^n \lambda_d(D) [\log n]^{n\gamma} \sum_{k=1}^{N} \int_{T^n} \prod_{\ell=1}^{N} \frac{1}{(\det\mathrm{Cov}(X^{H_\ell}(t_\ell^1), \ldots, X^{H_\ell}(t_\ell^n)))^{d/2}} \\
&\qquad \cdot \log_+^{n\gamma} \frac{e}{\det\mathrm{Cov}(X^{H_k}(t_k^1), \ldots, X^{H_k}(t_k^n))} \, d\bar{t} \\
&\le K^n \lambda_d(D) [\log n]^{n\gamma} \prod_{\ell=1}^{N} \int_{[\epsilon, \epsilon+r]^n} \frac{1}{(\det\mathrm{Cov}(X^{H_\ell}(t_\ell^1), \ldots, X^{H_\ell}(t_\ell^n)))^{d/2}} \\
&\qquad \cdot \log_+^{n\gamma} \frac{e}{\det\mathrm{Cov}(X^{H_\ell}(t_\ell^1), \ldots, X^{H_\ell}(t_\ell^n))} \, d\bar{t}_\ell,
\end{aligned} \tag{4.18}$$

where we have written $d\bar{t}_\ell$ for $dt_\ell^1 \cdots dt_\ell^n$.

For similicity, we fix an $\ell \in \{1, \ldots, N\}$ and use $\mathcal{N}_\ell$ to denote the integral

$$\int_{[\epsilon, \epsilon+r]^n} \frac{1}{(\det\mathrm{Cov}(X^{H_\ell}(t_\ell^1), \ldots, X^{H_\ell}(t_\ell^n)))^{d/2}} \log_+^{n\gamma} \frac{e}{\det\mathrm{Cov}(X^{H_\ell}(t_\ell^1), \ldots, X^{H_\ell}(t_\ell^n))} \, d\bar{t}_\ell.$$

Since the integrand is invariant under permulations of the t_ℓ^j, we have

$$\begin{aligned}
\mathcal{N}_\ell &= n! \int_{\{\epsilon \le t_\ell^1 \le \cdots \le t_\ell^n \le \epsilon + r\}} \frac{1}{(\mathrm{detCov}(X^{H_\ell}(t_\ell^1), \ldots, X^{H_\ell}(t_\ell^n)))^{d/2}} \\
&\qquad \cdot \log_+^{n\gamma} \frac{e}{\mathrm{detCov}(X^{H_\ell}(t_\ell^1), \ldots, X^{H_\ell}(t_\ell^n))} d\overline{t_\ell} \\
&\le K^n n! \sum_{k=1}^n \int_{\{\epsilon \le t_\ell^1 \le \cdots \le t_\ell^n \le \epsilon + r\}} \prod_{j=1}^n \frac{1}{|t_\ell^j - t_\ell^{j-1}|^{H_\ell d}} \cdot \log_+^{n\gamma} \frac{e}{|t_\ell^k - t_\ell^{k-1}|} \; d\bar{t}_\ell \\
&\le K^n n! \, r^{n(1-H_\ell d)} \log_+^{n\gamma} \Big(\frac{e}{r}\Big). \qquad (4.19)
\end{aligned}$$

The last inequality follows from the fact that $H_\ell d < 1$ and

$$\int_0^r \frac{1}{s^{H_\ell d}} \log^{n\gamma} \Big(\frac{e}{s}\Big) ds \le K 2^{n\gamma - 1} \, r^{1-H_\ell d} \log^{n\gamma} \Big(\frac{e}{r}\Big)$$

for all $\gamma \ge 0$.

Combining (4.18) and (4.19), we have

$$\mathbb{E} \int_D \int_D \Big(\frac{L(x,T) - L(y,T)}{p(|x-y|/\sqrt{d})}\Big)^n dxdy \le K^n (n!)^N (\log n)^{n\gamma} \, r^{n(N - \sum_{\ell=1}^N H_\ell d)} \log_+^{nN\gamma} \Big(\frac{e}{r}\Big).$$

This finishes the proof of Lemma 4.9. □

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1 Let $\Psi(u) = u \exp(u^\theta)$, where $\theta \in (0, 1/N)$ is a constant. Then Ψ is increasing and convex on $(0, \infty)$. It follows from Lemma 4.9 and Jensen's inequality (we will ignore the first few terms in the series) that for all closed hypercubes $D \subset \mathbb{R}^d$

$$\begin{aligned}
&\mathbb{E} \int_D \int_D \Psi\Big(\frac{|L(x,T) - L(y,T)|}{p(|x-y|/\sqrt{d})}\Big) \, dxdy \\
&\le \sum_{n=0}^\infty \frac{1}{n!} \Big\{ \mathbb{E} \int_D \int_D \Big(\frac{|L(x,T) - L(y,T)|}{p(|x-y|/\sqrt{d})}\Big)^n dxdy \Big\}^{\theta + \frac{1}{n}} < \infty. \qquad (4.20)
\end{aligned}$$

Hence Garsia's lemma implies that there are positive and finite random variables A_1 and A_2 such that for almost all $x,\, y \in D$ with $|x-y| \le e^{-1}$

$$\begin{aligned}
|L(x,T) - L(y,T)| &\le \int_0^{|x-y|} \Psi^{-1}\Big(\frac{A_1}{u^{2d}}\Big) \, dp(u) \\
&\le A_2 \Big[\log\big(1/|x-y|\big)\Big]^{-(\gamma - 1/\theta)}.
\end{aligned}$$

By chosing $\gamma > 1/\theta$, we see that B^H has almost surely a local time $L(x,T)$ that is continuous for all $x \in D$. By taking an increasing sequence of closed hypercubes $\{D_n,\ n \ge 1\}$ such that $\mathbb{R}^d = \cup_{n=1}^\infty D_n$, we have proved that almost surely $L(x,T)$ is continuous for all $x \in \mathbb{R}^d$.

The proof of the joint continuity is very similar. It follows from Lemmas 4.8, 4.9 and the easily verifiable inequality $\Psi(u+v) \le \Psi(2u) + \Psi(2v)$ that

$$\begin{aligned}
&\mathbb{E}\int_T\int_T\int_D\int_D \Psi\Big(\frac{|L(x,[\epsilon,t]) - L(y,[\epsilon,s])|}{p(\max\{|s-t|,|x-y|\}/\sqrt{N+d})}\Big)\,dxdy\,dsdt \\
&\le \mathbb{E}\int_T\int_T\int_D\int_D \Psi\Big(\frac{2|L(x,[\epsilon,t]) - L(x,[\epsilon,s])|}{p(|s-t|/\sqrt{N+d})}\Big)\,dxdy\,dsdt \\
&\qquad +\mathbb{E}\int_T\int_T\int_D\int_D \Psi\Big(\frac{2|L(x,[\epsilon,s]) - L(y,[\epsilon,s]|)}{p(|x-y|/\sqrt{N+d})}\Big)\,dxdy\,dsdt < \infty.
\end{aligned}$$

Therefore the joint continuity of the local times follows again from Garsia's lemma. This finishes the proof of Theorem 4.1. □

Remark 4.10 We conjecture that B^H has jointly continuous local times whenever the condition $d < \sum_{\ell=1}^N \frac{1}{H_\ell}$ is satisfied. However, we can not prove it by using the method in the present paper. □

Furthermore, it would be interesting to investigate the local behavior of the Borel measure $L(x, \bullet)$. More specifically, we state the following

Problem: *Let $L^*(I) = \sup_{x\in\mathbb{R}^d} L(x, I)$ be the maximum local time of B^H on I. Find Hausdorff measure functions $\varphi_1(r)$ and $\varphi_2(r)$ such that*

(i). for every $t \in (0,\infty)^N$ almost surely

$$\limsup_{r\to 0} \frac{L^*(I(t,r))}{\varphi_1(r)} \le K,$$

where $I(t,r)$ is the (open) ball (or cube) centered at t with radius (edge length) r and $K > 0$ is a constant; and

(ii). for any rectangle $T \in \mathcal{A}$, there exists a positive finite constant K such that almost surely

$$\limsup_{r\to 0}\ \sup_{t\in T} \frac{L^*(I(t,r))}{\varphi_2(r)} \le K.$$

Solutions to this problem will have implications on the fractal properties of the level sets of B^H. For results on the Brownian sheet and fractional Brownian motion, we refer to Ehm (1981) and Xiao (1997) respectively. Similar results were obtained by Cuzick (1982) for a class of sationary Gaussian processes using a different approach.

Acknowledgment Yimin Xiao would like to thank Davar Khoshnevisan for very stimulating conversations on fractional Brownian sheet.

References

[1] R. J. Adler, The uniform dimension of the level sets of a Brownian sheet. *Ann. of Probab.* **6** (1978) 509 – 515.

[2] R. J. Adler, *The Geometry of Random Fields.* Wiley, New York, 1981.

[3] R. B. Bapat and T. E. S. Raghavan, *Non-negative Matrics and Applications.* Cambridge University Press, 1996.

[4] S. M. Berman, Local times and sample function properties of stationary Gaussian processes. *Trans. Amer. Math. Soc* **137** (1969) 277 – 299.

[5] S. M. Berman, Gaussian sample function: uniform dimension and Hölder conditions nowhere. *Nagoya Math. J.* **46** (1972) 63 – 86.

[6] S. M. Berman, Local nondeterminism and local times of Gaussian processes. *Indiana Univ. Math. J.* **23** (1973) 69 – 94.

[7] J. Cuzick, Continuity of Gaussian local times. *Ann. of Probab.* **10** (1982) 818 – 823.

[8] J. Cuzick and J. Du Peez, Joint continuity of Gaussian local times. *Ann. of Probab.* **10** (1982) 810 – 817.

[9] T. E. Duncan, B. Maslowski and B. Pasik-Duncan, Fractional Brownian motion and stochastic equations in Hilbert spaces, *Preprint*, 2000.

[10] T. Dunker, Estimates for the small ball probabilities of the fractional Brownian sheet. *J. Theoret. Probab.* **13** (2000), 357 – 382.

[11] W. Ehm, Sample function properties of multi-parameter stable processes. *Z. Wahrsch. verw Gebiete* **56** (1981), 195 – 228.

[12] N. Eisenbaum and D. Khoshnevisan, On the most visited sites of symmetric Markov processes. *Preprint*, (2001).

[13] A. M. Garsia, Continuity properties of Gaussian processes with multidimensional time parameter. *Proc. 6th Berkeley Symp. Math. Statist. Probability* **II**, pp. 369 – 374, (1971).

[14] D. Geman and J. Horowitz, Occupation densities. *Ann. of Probab.* **8** (1980), 1 – 67.

[15] D. Geman, J. Horowitz and J. Rosen, A local time analysis of intersections of Brownian paths in the plane. *Ann. of Probab.* **12** (1984), 86 – 107.

[16] R. A. Horn and C. R. Johnson, *Matrix Analysis.* Cambridge University Press, 1999.

[17] Yaozhong Hu, B. Øksendal and Tusheng Zhang, Stochastic partial differential equations driven by multiparameter fractional white noise. *Stochastic Processes, Physics and Geometry: new interplays, II (Leipzig, 1999)*, 327–337, Amer. Math. Soc., Providence, RI, 2000.

[18] B. B. Mandelbbrot and J. W. Van Ness, Fractional Brownian motions, fractional noises and applications. *SIAM Rev.* **10** (1968), 422–437.

[19] D. M. Mason and Zhan Shi, Small deviations for some multi-parameter Gaussian processes. *J. Theoret. Probab.* **14** (2001), 213 – 239.

[20] D. Monrad and L. D. Pitt, Local nondeterminism and Hausdorff dimension. *Prog. in Probab. and Statist., Seminar on Stochastic Processes (1986)*, (E, Cinlar, K. L. Chung, R. K. Getoor, Editors), pp163 – 189, Birkhauser, 1987.

[21] B. Øksendal and Tusheng Zhang, Multiparameter fractional Brownian motion and quasi-linear stochastic partial differential equations. *Stochastics and Stochastics Reports*, **71** (2000), 141-163.

[22] S. Orey, Gaussian sample functions and the Hausdorff dimension of level crossing. *Z. Wahrsch. verw Gebiete* **15** (1970), 249 – 256.

[23] S. Orey and W. E. Pruitt, Sample functions of the N-parameter Wiener process. *Ann. of Probab.* **1** (1973), 138 – 163.

[24] L. D. Pitt, Local times for Gaussian vector fields. *Indiana Univ. Math. J.* **27** (1978), 309 – 330.

[25] J. Rosen, Self-intersections of random fields. *Ann. of Probab.* **12** (1984), 108 – 119.

[26] G. Samorodnitsky and M. S. Taqqu, *Stable non-Gaussian Random Processes: Stochastic models with infinite variance.* Chapman & Hall, New York, 1994.

[27] M. Talagrand, Hausdorff measure of trajectories of multiparameter fractional Brownian motion. *Ann. of Probab.* **23** (1995), 767–775.

[28] Yimin Xiao, Hölder conditions for the local times and the Hausdorff measure of the level sets of Gaussian random fields. *Probab. Th. Rel. Fields* **109** (1997), 129 – 157.

Yimin Xiao
Department of Statistics and Probability
A-413 Wells Hall
Michigan State University
East Lansing, MI 48824
E-mail: `xiao@stt.msu.edu`

Tusheng Zhang
Department of Mathematics
University of Manchester
Oxford Road
Manchester M13 9PL, UK
E-mail: `tzhang@ma.man.ac.uk`