ELSEVIER

Statistics & Probability Letters 33 (1997) 379–387

STATISTICS & PROBABILITY LETTERS

# Packing dimension of the image of fractional Brownian motion

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Received 1 April 1996; revised 1 May 1996

**Abstract**

Let $X(t)$ $(t \in \mathbb{R}^N)$ be a fractional Brownian motion of index $\alpha$ in $\mathbb{R}^d$. For any analytic set $E \subseteq \mathbb{R}^N$, we show that

$$\operatorname{Dim} X(E) = \frac{1}{\alpha} \operatorname{Dim}_{\alpha d} E \quad \text{a.s.},$$

where $\operatorname{Dim} E$ is the packing dimension of $E$ and $\operatorname{Dim}_s E$ is the packing dimension profile of $E$ defined by Falconer and Howroyd (1995).

*AMS classification:* Primary 60G15; 60G17

*Keywords:* Packing dimension; Packing dimension profiles; Fractional Brownian motion; Image

## 1. Introduction

Let $X(t) = (X_1(t), \ldots, X_d(t))$ $(t \in \mathbb{R}^N)$ be a fractional Brownian motion of index $\alpha$ $(0 < \alpha < 1)$ in $\mathbb{R}^d$ (see Kahane, 1985, Ch. 18). If $N = 1$, $\alpha = \frac{1}{2}$, then $X(t)$ $(t \in \mathbb{R})$ is the ordinary Brownian motion in $\mathbb{R}^d$. If $N > 1$, $\alpha = \frac{1}{2}$, then $X(t)$ $(t \in \mathbb{R}^N)$ is Levy's Brownian motion with $N$ parameters.

Kahane (1985) proved that for every Borel set $E \subseteq \mathbb{R}^N$,

$$\dim X(E) = \min\left(d, \frac{1}{\alpha} \dim E\right) \quad \text{a.s.}, \tag{1.1}$$

where $\dim E$ is the Hausdorff dimension of $E$. We refer to Falconer (1990) or Mattila (1995) for the definition and properties of Hausdorff measure and Hausdorff dimension.

Packing dimension was introduced in the early 1980s as a dual concept to Hausdorff dimension and since then it has become a very useful tool in analyzing fractal sets (see e.g. Tricot, 1982; Tricot and Taylor, 1985; Saint Raymond and Tricot, 1986; Falconer, 1990; Mattila, 1995). It is natural to ask whether the following

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*PII* S0167-7152(96)00151-4

analogue of (1.1) holds for packing dimension Dim:

$$\operatorname{Dim} X(E) = \min\left(d,\ \frac{1}{\alpha}\operatorname{Dim} E\right) \quad \text{a.s.} \tag{1.2}$$

Since $\dim \leqslant \operatorname{Dim}$ (Tricot, 1982), it is easy to deduce from (1.1) that (1.2) holds for every Borel set $E \subseteq \mathbb{R}^N$ with $\dim E = \operatorname{Dim} E$. When $N \leqslant \alpha d$, Xiao (1993) proved that the following stronger result holds: with probability 1, for every Borel set $E \subseteq \mathbb{R}^N$

$$\operatorname{Dim} X(E) = \frac{1}{\alpha}\operatorname{Dim} E\ .$$

In the case of $N > \alpha d$, Talagrand and Xiao (1996) showed that (1.2) does not hold in general and they also obtained the best lower bound for $\operatorname{Dim} X(E)$: with probability 1,

$$\operatorname{Dim} X(E) \geqslant \max\left\{\min\left\{d,\ \frac{1}{\alpha}\dim E\right\};\ \frac{\operatorname{Dim} E \cdot d}{\alpha d + \operatorname{Dim} E \cdot (N - \alpha d)}\right\}, \tag{1.3}$$

and the equality holds for all sets with $\dim E = \operatorname{Dim} E$ and for some sets with $\dim E = 0$. This raises the obvious question – Can the lower bound in (1.3) be improved by knowing $0 < \dim E < \operatorname{Dim} E$? The following example ($N = 1$) shows that the answer is obviously negative. Let $E_1$ be the compact set constructed in Lemma 3.2 in Talagrand and Xiao (1996) and let $E_2$ be any Borel set of $\mathbb{R}$ with

$$\dim E_2 = \operatorname{Dim} E_2 \leqslant \frac{\operatorname{Dim} E_1 \cdot \alpha d}{\alpha d + \operatorname{Dim} E_1 \cdot (1 - \alpha d)}\,.$$

Let $E = E_1 \cup E_2$, then $\dim E = \dim E_2$, $\operatorname{Dim} E = \operatorname{Dim} E_1$ and by Corollary 4.1 in Talagrand and Xiao (1996) and (1.1), with probability 1,

$$\begin{aligned}\operatorname{Dim} X(E) &= \max\{\operatorname{Dim} X(E_1);\ \operatorname{Dim} X(E_2)\}\\ &= \frac{\operatorname{Dim} E \cdot d}{\alpha d + \operatorname{Dim} E \cdot (1 - \alpha d)}\,.\end{aligned}$$

In particular, we cannot have a general formula for $\operatorname{Dim} X(E)$ in terms of $\dim E$ and $\operatorname{Dim} E$. In fact, no previously known formula for $\operatorname{Dim} X(E)$ is valid for all $E \subseteq \mathbb{R}^N$, even if $X$ is Brownian motion ($N = d = 1, \alpha = \frac{1}{2}$).

Very recently, Falconer and Howroyd (1995) introduced the concept of packing dimension profiles for finite Borel measures and sets in $\mathbb{R}^N$, which carries more information about local and global geometric properties of the measures and sets than packing dimension does. By using packing dimension profiles, Falconer and Howroyd (1995) solved the problem of the packing dimension of projections. The objective of this note is to give a complete solution to the problem of finding a general formula for $\operatorname{Dim} X(E)$. We will show that for any analytic set $E \subseteq \mathbb{R}^N$,

$$\operatorname{Dim} X(E) = \frac{1}{\alpha}\operatorname{Dim}_{\alpha d} E \quad \text{a.s.,} \tag{1.4}$$

where $\operatorname{Dim}_s E$ is the packing dimension profile of $E$ defined by using $s$-dimensional kernel. We remark that when $N \leqslant \alpha d$, $\operatorname{Dim}_{\alpha d} E = \operatorname{Dim} E$, and hence (1.4) reduces to (1.2). If $N > \alpha d$ and $\operatorname{Dim} E = \dim E$, then $\operatorname{Dim}_{\alpha d} E = \min\{\alpha d;\ \operatorname{Dim} E\}$ and (1.4) also reduces to (1.2).

This note is organized as follows. In Section 2, we recall the definitions and some basic properties of packing dimension and packing dimension profiles. In Section 3, we consider the packing dimension of the image measure $\mu_X$ of a finite Borel measure $\mu$ on $\mathbb{R}^N$ under $X(t)$ ($t \in \mathbb{R}^N$). Then in Section 4, we deduce a similar result for the packing dimension of the image of an analytic set under $X(t)$ ($t \in \mathbb{R}^N$) by examining the packing dimension of the finite Borel measures supported on the set.

We will use $K$ to denote an unspecified positive constant which may differ from line to line.

## 2. Preliminaries

In this section we recall briefly the definitions and some basic properties of packing dimension and packing dimension profiles. Let $\Phi$ be the class of functions $\phi : (0,\delta) \to (0,1)$ which are right continuous, increasing with $\phi(0+) = 0$ such that there exists a finite constant $K > 0$ for which

$$\frac{\phi(2s)}{\phi(s)} \leqslant K \quad \text{for } 0 < s < \tfrac{1}{2}\delta .$$

For $\phi \in \Phi$, Taylor and Tricot (1985) defined the set function $\phi\text{-}P(E)$ on $R^N$ by

$$\phi\text{-}P(E) = \lim_{\varepsilon \to 0} \sup \left\{ \sum_i \phi(2r_i) : \overline{B}(x_i, r_i) \text{ are disjoint, } x_i \in E,\ r_i < \varepsilon \right\},$$

where $B(x,r)$ denotes the open ball of radius $r$ centered at $x$. The set function $\phi\text{-}P$ is not an outer measure because it fails to be countably subadditive. But it gives rise to a metric outer measure $\phi\text{-}p$ on $\mathbb{R}^N$ as follows

$$\phi\text{-}p(E) = \inf \left\{ \sum_n \phi\text{-}P(E_n) : E \subseteq \cup_n E_n \right\} .$$

$\phi\text{-}p(E)$ is called the $\phi$-packing measure of $E$. If $\phi(s) = s^\alpha$, $s^\alpha\text{-}p(E)$ is called the $\alpha$-dimensional packing measure of $E$. The packing dimension of $E$ is defined by

$$\operatorname{Dim} E = \inf\{\alpha > 0 : \ s^\alpha\text{-}p(E) = 0\} .$$

The packing dimension of a Borel measure $\mu$ on $\mathbb{R}^N$ (or lower packing dimension as it is sometimes called) is defined by

$$\operatorname{Dim} \mu = \inf\{\operatorname{Dim} E : \mu(E) > 0 \text{ and } E \subseteq \mathbb{R}^N \text{ is a Borel set}\} . \tag{2.1}$$

The upper packing dimension of $\mu$ is defined by

$$\operatorname{Dim}^* \mu = \inf\{\operatorname{Dim} E : \mu(\mathbb{R}^N \backslash E) = 0 \text{ and } E \subseteq \mathbb{R}^N \text{ is a Borel set}\} . \tag{2.2}$$

For a finite Borel measure $\mu$ on $\mathbb{R}^N$ and for any $s > 0$, let

$$F_s^\mu(x,r) = \int_{\mathbb{R}^N} \min\{1,\ r^s |y-x|^{-s}\}\, \mathrm{d}\mu(y),$$

where $|\cdot|$ is the usual Euclidean norm. The following equivalent definitions of $\operatorname{Dim}\mu$ and $\operatorname{Dim}^*\mu$ in terms of the potential $F_N^\mu(x, r)$ are given by Falconer and Howroyd (1995).

$$\operatorname{Dim} \mu = \sup\{t \geqslant 0 :\ \liminf_{r \to 0} r^{-t} F_N^\mu(x,r) = 0 \text{ for } \mu\text{-}a.\ a.\ x \in \mathbb{R}^N\}, \tag{2.3}$$

$$\operatorname{Dim}^* \mu = \inf\{t > 0 :\ \lim\inf\ r^{-t} F_N^\mu(x,r) > 0 \text{ for } \mu\text{-}a.\ a.\ x \in \mathbb{R}^N\} . \tag{2.4}$$

Falconer and Howroyd (1995) also defined the packing dimension profile of $\mu$ by using the $s$-dimensional potential $F_s^\mu(x,r)$ by

$$\operatorname{Dim}_s \mu = \sup\{t \geqslant 0 :\ \liminf_{r \to 0} r^{-t} F_s^\mu(x,r) = 0 \text{ for } \mu\text{-}a.\ a.\ x \in \mathbb{R}^N\}, \tag{2.5}$$

and the upper packing dimension profile of $\mu$ by

$$\mathrm{Dim}_s^* \mu = \inf\left\{t > 0: \liminf_{r\to 0} r^{-t} F_s^{\mu}(x,r) > 0 \text{ for } \mu\text{-}a.\ a.\ x \in \mathbb{R}^N\right\}. \tag{2.6}$$

It is easy to see that

$$0 \leqslant \mathrm{Dim}_s \mu \leqslant \mathrm{Dim}_s^* \mu \leqslant s$$

and if $s \geqslant N$, then

$$\mathrm{Dim}_s \mu = \mathrm{Dim}\, \mu, \qquad \mathrm{Dim}_s^* \mu = \mathrm{Dim}^* \mu\ ,$$

see Falconer and Howroyd (1995).

For any analytic set $E \subseteq \mathbb{R}^N$, we define $M_c^+(E)$ to be the family of finite Borel measures on $E$ with compact support in $E$. Then

$$\mathrm{Dim}\, E = \sup\{\mathrm{Dim}\, \mu : \mu \in M_c^+(E)\}, \tag{2.7}$$

see Hu and Taylor (1994) for a proof. Motivated by this, Falconer and Howroyd (1995) defined the packing dimension profile of $E \subseteq \mathbb{R}^N$ by

$$\mathrm{Dim}_s E = \sup\{\mathrm{Dim}_s \mu : \mu \in M_c^+(E)\}. \tag{2.8}$$

Clearly, $0 \leqslant \mathrm{Dim}_s E \leqslant s$ and for any $s \geqslant N$, $\mathrm{Dim}_s E = \mathrm{Dim}\, E$.

We use $d(s)$ to denote any one of $\mathrm{Dim}_s \mu$, $\mathrm{Dim}_s^* \mu$, $\mathrm{Dim}_s E$.

**Lemma 2.1.** *Let $d$: $\mathbb{R}^+ \to [0,N]$ be as above. Then $d(s)$ is continuous.*

**Proof.** This is a consequence of Proposition 18 in Falconer and Howroyd (1995).

The following lemma contains Theorem 6(a) in Falconer and Howroyd (1995) as a special case.

**Lemma 2.2.** *Let $I$ be any cube in $\mathbb{R}^N$ and let $f : I \to \mathbb{R}^d$ be a continuous function satisfying a uniform Hölder condition of order $\alpha$. Then for any finite Borel measure $\mu$ on $\mathbb{R}^N$ with support contained in $I$, we have*

$$\mathrm{Dim}\, \mu_f \leqslant \frac{1}{\alpha}\ \mathrm{Dim}_{\alpha d}\, \mu, \tag{2.9}$$

$$\mathrm{Dim}^*\, \mu_f \leqslant \frac{1}{\alpha}\ \mathrm{Dim}_{\alpha d}^*\, \mu. \tag{2.10}$$

**Proof.** (a) To prove (2.9), we take any $\gamma < \mathrm{Dim}\, \mu_f$, then by (2.3), for $\mu_f$-a. a. $u \in \mathbb{R}^d$, we have

$$\liminf_{r\to 0} r^{-\gamma} \int_{\mathbb{R}^d} \min\{1,\ r^d |v-u|^{-d}\}\, \mathrm{d}\mu_f(v) = 0,$$

that is, for $\mu$-a. a. $x \in \mathbb{R}^N$,

$$\liminf_{r\to 0} r^{-\gamma} \int_I \min\{1,\ r^d |f(y)-f(x)|^{-d}\}\, \mathrm{d}\mu(y) = 0. \tag{2.11}$$

Since for any $x,\ y \in I$,

$$|f(y)-f(x)| \leqslant K|y-x|^{\alpha},$$

where $K \geqslant 1$ is a constant. We have

$$\min\{1,\ r^d|f(y)-f(x)|^{-d}\} \geqslant K^{-d} \min\{1,\ r^d|y-x|^{-\alpha d}\}. \tag{2.12}$$

It follows from (2.11) and (2.12) that for $\mu$-*a. a.* $x \in \mathbb{R}^N$

$$\liminf_{\rho\to 0}\ \rho^{-\alpha\gamma} \int_{\mathbb{R}^N} \min\{1,\ \rho^{\alpha d}|y-x|^{-\alpha d}\}\, d\mu(y) = 0.$$

This implies $\mathrm{Dim}_{\alpha d}\,\mu \geqslant \alpha\gamma$. Since $\gamma < \mathrm{Dim}\,\mu_f$ is arbitrary, we have (2.9).

(b) For any $\gamma > \mathrm{Dim}^*_{\alpha d}\,\mu$, we have

$$\liminf_{r\to 0} r^{-\gamma} F^{\mu}_{\alpha d}(x,r) > 0 \quad \text{for } \mu\text{-}a.\ a.\ x \in \mathbb{R}^N. \tag{2.13}$$

By (2.12), for $u = f(x)$ with $x \in I$

$$F^{\mu_f}_d(u,r) \geqslant K^{-d} \int_{\mathbb{R}^N} \min\{1, r^d|y-x|^{-\alpha d}\} d\mu(y).$$

Hence, for any $u = f(x)$ with $x \in \mathbb{R}^N$ satisfying (2.13), we have

$$\liminf_{r\to 0} r^{-\frac{\gamma}{\alpha}} F^{\mu_f}_d(u,r) \geqslant K^{-d} \liminf_{r\to 0} r^{-\gamma/\alpha} \int_{\mathbb{R}^N} \min\{1,\ r^d|y-x|^{-\alpha d}\}\, d\mu(y) > 0.$$

This implies $\mathrm{Dim}^*\,\mu_f \leqslant \frac{\gamma}{\alpha}$ and hence (2.10). □

**Corollary 2.3.** *If for any $\varepsilon > 0$, $f : I \to \mathbb{R}^d$ satisfies a uniform Hölder condition of order $\alpha-\varepsilon$, then* (2.9) *and* (2.10) *still hold.*

## 3. Packing dimension of the image measures

Let $X(t) = (X_1(t),\dots,X_d(t))$ $(t \in \mathbb{R}^N)$ be a fractional Brownian motion of index $\alpha$ $(0 < \alpha < 1)$ in $\mathbb{R}^d$, defined on some probability space $(\Omega, \mathscr{F}, \mathscr{P})$. We will use $\mathscr{E}$ to denote the expectation with respect to $\mathscr{P}$. In this section, we consider the packing dimension of the image measure $\mu_X$ of $\mu$ under $X(t)$ $(t \in \mathbb{R}^N)$ defined by

$$\mu_X(B) = \mu\{t \in \mathbb{R}^N : X(t) \in B\} \quad \text{for any Borel set } B \subseteq \mathbb{R}^d.$$

The main result is the following theorem.

**Theorem 3.1.** *Let $\mu$ be any finite Borel measure on $\mathbb{R}^N$. Then with probability* 1,

$$\mathrm{Dim}\,\mu_X = \frac{1}{\alpha}\, \mathrm{Dim}_{\alpha d}\,\mu, \tag{3.1}$$

$$\mathrm{Dim}^*\,\mu_X = \frac{1}{\alpha}\, \mathrm{Dim}^*_{\alpha d}\,\mu. \tag{3.2}$$

**Proof.** We only prove (3.1); (3.2) can be proved similarly. For any positive integer $n$ and any $\varepsilon > 0$, $X(t)$ $(t \in [-n,n]^N)$ a.s. satisfies a uniform Hölder condition of order $\alpha - \varepsilon$ (see Kahane, 1985, Ch. 18). Let $\mu^{(n)}$ be the restriction of $\mu$ on $I_n = [-n, n]^N$, that is, $\mu^{(n)}(B) = \mu(B \cap I_n)$ for any Borel set $B$ in $\mathbb{R}^N$. Then by Corollary 2.3, we have

$$\mathrm{Dim}\,\mu^{(n)}_X \leqslant \frac{1}{\alpha}\, \mathrm{Dim}_{\alpha d}\,\mu^{(n)} \quad \text{a.s.},$$

which implies

$$\operatorname{Dim} \mu_X \leqslant \frac{1}{\alpha} \operatorname{Dim}_{\alpha d} \mu \quad \text{a.s.} \tag{3.3}$$

To prove the reverse inequality, we observe that for any $s \in \mathbb{R}^N$, by Fubini's theorem,

$$\begin{aligned} \mathscr{E} F_d^{\mu_X}(X(s), r) &= \mathscr{E} \int_{\mathbb{R}^d} \min\{1,\ r^d |v - X(s)|^{-d}\} \, \mathrm{d}\mu_X(v) \\ &= \int_{\mathbb{R}^N} \mathscr{E} \min\{1, r^d |X(t) - X(s)|^{-d}\} \, \mathrm{d}\mu(t) . \end{aligned} \tag{3.4}$$

We consider

$$\begin{aligned} &\mathscr{E} \min\{1, r^d |X(t) - X(s)|^{-d}\} \\ &\quad = \mathscr{P}\{|X(t) - X(s)| \leqslant r\} + \mathscr{E}\{r^d |X(t) - X(s)|^{-d} \cdot 1_{\{|X(t)-X(s)| \geqslant r\}}\} \\ &\quad \leqslant \min\left\{1,\ \frac{K\ r^d}{|t-s|^{\alpha d}}\right\} + \left(\frac{1}{2\pi}\right)^{d/2} \int_{|u| \geqslant r} \frac{r^d}{|u|^d} \cdot \frac{1}{|t-s|^{\alpha d}} \exp\left(-\frac{|u|^2}{2|t-s|^{2\alpha}}\right) \mathrm{d}u \\ &\quad \leqslant \min\left\{1,\ \frac{K\ r^d}{|t-s|^{\alpha d}}\right\} + \frac{K\ r^d}{|t-s|^{\alpha d}} \int_{\rho \geqslant \frac{r}{|t-s|^{\alpha}}} \frac{1}{\rho} \exp\left(-\frac{\rho^2}{2}\right) \mathrm{d}\rho . \end{aligned}$$

Hence, for any $0 < \varepsilon < 1$,

$$\mathscr{E} \min\{1,\ r^d |X(t) - X(s)|^{-d}\} \leqslant K \min\left\{1,\ \frac{r^{d-\varepsilon}}{|t-s|^{\alpha(d-\varepsilon)}}\right\} . \tag{3.5}$$

For any $\gamma < \operatorname{Dim}_{\alpha d} \mu$, by Lemma 2.1, there exists $\varepsilon > 0$ such that $\gamma < \operatorname{Dim}_{\alpha(d-\varepsilon)} \mu$. It follows from (2.5) that

$$\liminf_{r \to 0} r^{-\gamma/\alpha} \int_{\mathbb{R}^N} \min\{1,\ r^{d-\varepsilon} |t-s|^{-\alpha(d-\varepsilon)}\} \, \mathrm{d}\mu(t) = 0 \quad \text{for } \mu\text{-}a.a.,\ s \in \mathbb{R}^N. \tag{3.6}$$

By (3.4)–(3.6) we have that for $\mu$-*a.a.*, $s \in \mathbb{R}^N$

$$\begin{aligned} &\mathscr{E}\left(\liminf_{r \to 0} r^{-\gamma/\alpha} F_d^{\mu_X}(X(s), r)\right) \\ &\quad \leqslant K \liminf_{r \to 0} r^{-\gamma/\alpha} \int_{\mathbb{R}^N} \min\{1,\ r^{d-\varepsilon} |t-s|^{-\alpha(d-\varepsilon)}\} \, \mathrm{d}\mu(t) \\ &\quad = 0 . \end{aligned}$$

By using Fubini's argument again, we see that with probability 1,

$$\liminf_{r \to 0} r^{-\gamma/\alpha} F_d^{\mu_X}(X(s), r) = 0 \quad \text{for } \mu\text{-}a.a.,\ s \in \mathbb{R}^N.$$

Hence,

$$\operatorname{Dim} \mu_X \geqslant \frac{\gamma}{\alpha} \quad \text{a.s.}$$

Since, $\gamma$ can be arbitrarily close to $\operatorname{Dim}_{\alpha d} \mu$, we have

$$\operatorname{Dim} \mu_X \geqslant \frac{1}{\alpha} \operatorname{Dim}_{\alpha d} \mu \quad \text{a.s.} \tag{3.7}$$

Combining (3.3) and (3.7), we complete the proof of (3.1). □

If $N \leqslant \alpha d$, Xiao (1993) proved that with probability 1,

$$\operatorname{Dim} X(E) = \frac{1}{\alpha} \operatorname{Dim} E \tag{3.8}$$

for every Borel set $E \subseteq \mathbb{R}^N$. It is easy to deduce from (2.1), (2.2) and (3.8) that a result similar to (3.8) holds for the packing dimension of the image measures.

**Theorem 3.2.** *If $N \leqslant \alpha d$, then with probability* 1,

$$\operatorname{Dim} \mu_X = \frac{1}{\alpha} \operatorname{Dim} \mu,$$

$$\operatorname{Dim}^* \mu_X = \frac{1}{\alpha} \operatorname{Dim}^* \mu$$

*for every finite Borel measure $\mu$ on $\mathbb{R}^N$.*

**Remark.** In Theorem 3.1, the exceptional null probability set depends on $\mu$. But in Theorem 3.2, the exceptional null probability set does not depend on $\mu$.

## 4. Packing dimension of the image set

We see from (2.7) and (2.8) that the packing dimension of an analytic set $E$ can be expressed in terms of the packing dimension of the finite Borel measures supported on $E$. This allows us to deduce from Theorem 3.1 an analogous result for $\operatorname{Dim} X(E)$.

**Theorem 4.1.** *Let $X(t)$ $(t \in \mathbb{R}^N)$ be a fractional Brownian motion of index $\alpha$ in $\mathbb{R}^d$. Then for any analytic set $E \subseteq \mathbb{R}^N$, with probability* 1,

$$\operatorname{Dim} X(E) = \frac{1}{\alpha} \operatorname{Dim}_{\alpha d} E. \tag{4.1}$$

The proof of (4.1) is similar to the proof of Theorem 10 in Falconer and Howroyd (1995). We need several lemmas.

**Lemma 4.1.** *Let $E \subseteq \mathbb{R}^N$ be an analytic set and let $f : \mathbb{R}^N \to \mathbb{R}^d$ be a continuous function. If $0 \leqslant t < \operatorname{Dim} f(E)$, then there exists a compact set $F \subseteq E$ such that $t < \operatorname{Dim} f(F)$.*

**Proof.** The proof is the same as that of Lemma 7 in Falconer and Howroyd (1995) with $f$ replacing the orthogonal projection $P_V$.

The following lemma is Theorem 1.20 in Mattila (1995).

**Lemma 4.2.** *Let $F \subseteq \mathbb{R}^N$ be a compact set and let $f : \mathbb{R}^N \to \mathbb{R}^d$ be a continuous function. If $\nu$ is a finite Borel measure on $\mathbb{R}^d$ with support in $f(F)$, then there exists a finite Borel measure $\mu$ on $\mathbb{R}^N$ such that $\nu = \mu_f$ and the support of $\mu$ is contained in $F$.*

**Lemma 4.3.** *Let $E \subseteq \mathbb{R}^N$ be an analytic set. Then for any continuous function $f : \mathbb{R}^N \to \mathbb{R}^d$*

$$\operatorname{Dim} f(E) = \sup\{\operatorname{Dim} \mu_f : \mu \in M_c^+(E)\}. \tag{4.2}$$

**Proof.** For any $\mu \in M_c^+(E)$, we have $\mu_f \in M_c^+(f(E))$. Then (2.7) implies

$$\operatorname{Dim} f(E) \geqslant \sup\{\operatorname{Dim} \mu_f : \mu \in M_c^+(E)\}. \tag{4.3}$$

To prove the reverse inequality, let $t < \operatorname{Dim} f(E)$. Then by Lemma 4.1 there exists a compact set $F \subseteq E$ such that $\operatorname{Dim} f(F) > t$. Hence, by (2.7) there exists a finite Borel measure $\nu \in M_c^+(f(F))$ such that $\operatorname{Dim} \nu > t$. It follows from Lemma 4.2 that there exists $\mu \in M_c^+(F)$ such that $\nu = \mu_f$, this implies $\sup\{\operatorname{Dim} \mu_f : \mu \in M_c^+(E)\} > t$ . Since $t < \operatorname{Dim} f(E)$ is arbitrary, we have

$$\operatorname{Dim} f(E) \leqslant \sup\{\operatorname{Dim} \mu_f : \mu \in M_c^+(E)\}. \tag{4.4}$$

Eq. (4.2) now follows from (4.3) and (4.4).

**Proof of Theorem 4.1.** Since Dim is $\sigma$-stable, we may and will assume that $E$ is bounded. Hence, there exists a cube $I$ such that $E \subseteq I$. For any $\varepsilon > 0$, $X(t)$ $(t \in I)$ a.s. satisfies a uniform Hölder condition of order $\alpha - \varepsilon$. Then by Lemma 2.2, for any $\mu \in M_c^+(E)$ we have

$$\operatorname{Dim} \mu_X \leqslant \frac{1}{\alpha} \operatorname{Dim}_{\alpha d} \mu \quad \text{a.s.}$$

Hence by (2.8) and (4.2) we have

$$\operatorname{Dim} X(E) \leqslant \frac{1}{\alpha} \operatorname{Dim}_{\alpha d} E. \tag{4.5}$$

On the other hand, for any $t < \frac{1}{\alpha} \operatorname{Dim}_{\alpha d} E$, by (2.8) there exists $\mu \in M_c^+(E)$ such that $\alpha t < \operatorname{Dim}_{\alpha d} \mu$. It follows from (3.1) that

$$\operatorname{Dim} \mu_X > t \quad \text{a.s.}$$

Hence by Lemma 4.3 we have $\operatorname{Dim} X(E) > t$ a.s.. Since $t < \frac{1}{\alpha} \operatorname{Dim}_{\alpha d} E$ is arbitrary, the proof is completed. □

**Remark.** (1) If $N \leqslant \alpha d$, then for any analytic set $E \subseteq \mathbb{R}^N$, $\operatorname{Dim}_{\alpha d} E = \operatorname{Dim} E$. Hence (4.1) reduces to (1.2). If $E \subseteq \mathbb{R}^N$, satisfies $\dim E = \operatorname{Dim} E$, then

$$\operatorname{Dim}_{\alpha d} E = \min\{\alpha d, \operatorname{Dim} E\}$$

and (4.1) also reduces to (1.2).

(2) Let $W(t)$ $(t \in \mathbb{R}_+^N)$ be the Brownian sheet in $\mathbb{R}^d$ (see Orey and Pruitt, 1973). Then with a little modification the above proofs can show that for any analytic set $E \subseteq \mathbb{R}_+^N$,

$$\operatorname{Dim} W(E) = 2 \operatorname{Dim}_{d/2} E \quad \text{a.s.}$$

(3) The Hausdorff dimension of the image set of self-similar processes (including fractional Brownian motion) has been studied by several authors (see Lin and Xiao (1995) and the references therein). It would be interesting to consider the packing dimension of the image set of general self-similar processes, especially self-similar stable processes.

## Acknowledgements

I am very much indebted to Prof. Falconer who sent me a copy of his recent paper with Dr. Howroyd.

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