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Hausdorff-type measures of the sample paths of fractional Brownian motion

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Abstract

Let ϕ be a Hausdorff measure function and let Λ be an infinite increasing sequence of positive integers. The Hausdorff-type measure ϕ-m_Λ associated to ϕ and Λ is studied. Let $X(t)$ $(t \in \mathbb{R}^N)$ be fractional Brownian motion of index α in $\mathbb{R}^d$. We evaluate the exact ϕ-m_Λ measure of the image and graph set of $X(t)$. A necessary and sufficient condition on the sequence Λ is given so that the usual Hausdorff measure functions for $X([0,1]^N)$ and $\mathrm{Gr}\,X([0,1]^N)$ are still the correct measure functions. If the sequence Λ increases faster, then some smaller measure functions will give positive and finite (ϕ,Λ)-Hausdorff measure for $X([0,1]^N)$ and $\mathrm{Gr}\,X([0,1]^N)$

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1. Introduction

Let Φ be the class of functions ϕ: $(0,\delta) \to (0,1)$ which are right continuous, monotone increasing with $\phi(0+)=0$ and such that there exists a finite constant $K>0$ for which

$$\frac{\phi(2s)}{\phi(s)} \leqslant K, \quad \text{for } 0<s<\frac{1}{2}\delta. \tag{1.1}$$

Functions in Φ are usually called Hausdorff measure functions. For $\phi \in \Phi$, the ϕ-Hausdorff measure of $E \subseteq \mathbb{R}^d$ is defined by

$$\phi\text{-}m(E) = \lim_{\varepsilon \to 0} \inf \left\{ \sum_i \phi(2r_i) \colon E \subseteq \bigcup_{i=1}^{\infty} B(x_i, r_i),\ r_i < \varepsilon \right\}, \tag{1.2}$$

where $B(x,r)$ denotes the open ball of radius r centered at x. It is known that ϕ-m is a metric outer measure and every Borel set in $\mathbb{R}^d$ is ϕ-m measurable. The Hausdorff dimension of E is defined by

$$\dim E = \inf\{\alpha > 0 \colon s^\alpha\text{-}m(E) = 0\} = \sup\{\alpha > 0 \colon s^\alpha\text{-}m(E) = \infty\}.$$

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The packing measure is defined in a dual way by Taylor and Tricot (1985). For $\phi \in \Phi$ and $E \subset \mathbb{R}^d$, the ϕ-packing measure of E is defined by

$$\phi\text{-}p(E) = \inf\left\{\sum_n \phi\text{-}P(E_n): E \subseteq \bigcup_n E_n\right\}. \tag{1.3}$$

where for any $E \subset \mathbb{R}^d$,

$$\phi\text{-}P(E) = \lim_{\varepsilon \to 0} \sup\left\{\sum_i \phi(2r_i): \overline{B}(x_i, r_i) \text{ are disjoint, } x_i \in E,\ r_i < \varepsilon\right\}. \tag{1.4}$$

A sequence of closed balls satisfying the conditions in the right-hand side of Eq. (1.4) is called an ε-packing of E. The packing dimension of E is defined by

$$\operatorname{Dim} E = \inf\{\alpha > 0: s^\alpha\text{-}p(E) = 0\} = \sup\{\alpha > 0: s^\alpha\text{-}p(E) = +\infty\}.$$

We refer to Falconer (1990), Mattila (1995) for more properties of Hausdorff measure, packing measure and related dimensions, and to Taylor (1986) and Xiao (1997c) for applications of Hausdorff measure and packing measure in the studying of sample path properties of stochastic processes.

Let $B(t)$ $(t \in \mathbb{R}_+)$ be a Brownian motion in $\mathbb{R}^d$ $(d \geqslant 2)$. The exact Hausdorff measure for the image set of $B(t)$ was considered by Lévy (1953), who showed that if $d \geqslant 3$ and $\phi(s) = s^2 \log\log 1/s$, then $\phi\text{-}m(B([0,1])) < \infty$. Later, Ciesielski and Taylor (1962) proved that $\phi\text{-}m(B([0,1])) > 0$. For planar Brownian motion, the Hausdorff measure problem is more difficult due to the neighborhood recurrence of the process. It was proved by Ray (1964) and Taylor (1964) that the correct Hausdorff measure function is

$$\phi(s) = s^2 \log 1/s \log\log\log 1/s.$$

The Hausdorff measure of the graph of Brownian motion and Lévy stable processes were calculated by Jain and Pruitt (1968) in the transient case, by Pruitt and Taylor (1969) in the recurrent cases.

One natural generalization of Brownian motion is fractional Brownian motion of index α, i.e. the centered, real-valued Gaussian random field $Y(t)$ $(t \in \mathbb{R}^N)$ with covariance function

$$E(Y(t)Y(s)) = \tfrac{1}{2}(|t|^{2\alpha} + |s|^{2\alpha} - |t-s|^{2\alpha}).$$

Associated with $Y(t)$ $(t \in \mathbb{R}^N)$, one can define a Gaussian random field $X(t)$ $(t \in \mathbb{R}^N)$ in $\mathbb{R}^d$ by

$$X(t) = (X_1(t), \ldots, X_d(t)),$$

where $X_1, \ldots, X_d$ are independent copies of Y. The Gaussian random field $X(t)$ is called d-dimensional fractional Brownian motion of index α or the (N, d, α) Gaussian process (see Kahane, 1985). When $N = 1$, $\alpha = \frac{1}{2}$, $X(t)$ is the ordinary d-dimensional Brownian motion. If $\alpha = \frac{1}{2}$, $d = 1$, it is the multiparameter Lévy Brownian motion. It is easy to see that X is a self-similar process of exponent α, i.e. for any $a > 0$,

$$X(a\cdot) \overset{\mathrm{d}}{=} a^\alpha X(\cdot),$$

where $X \overset{d}{=} Y$ means that the two processes X and Y have the same finite dimensional distributions.

When $\alpha = \frac{1}{2}$, $N > 1$ and $N < d/2$, Goldman (1988) obtained the exact Hausdorff measure of the image set $X([0,1]^N)$. Talagrand (1995) extended the result to the general case of $0 < \alpha < 1$ and the proof is much simpler. They proved that if $N < \alpha d$ then with probability 1,

$$0 < \phi_1\text{-}m(X([0,1]^N)) < \infty, \tag{1.5}$$

where

$$\phi_1(s) = s^{N/\alpha} \log\log \frac{1}{s}. \tag{1.6}$$

The Hausdorff measure of the graph set

$$\mathrm{Gr}\, X([0,1]^N) = \{(t, X(t)) \colon t \in [0,1]^N\}$$

of fractional Brownian motion was calculated by Xiao (1997b) for the cases of $N < \alpha d$ and $N > \alpha d$. It is proved that if $N < \alpha d$, then almost surely

$$K_1 \leqslant \phi_1\text{-}m(\mathrm{Gr}\, X([0,1]^N)) \leqslant K_2, \tag{1.7}$$

If $N > \alpha d$, then almost surely

$$K_3 \leqslant \phi_2\text{-}m(\mathrm{Gr}\, X([0,1]^N)) \leqslant K_4, \tag{1.8}$$

where K_1, K_2, K_3, K_4 are positive finite constants depending on N, d and α only and where

$$\phi_2(s) = s^{N+(1-\alpha d)} \left(\log\log \frac{1}{s}\right)^{\alpha d/N}. \tag{1.9}$$

These results have also been extended to a large class of strongly locally nondeterministic Gaussian random fields by Xiao (1996, 1997a). In the case of $N = \alpha d$, the problem of finding $\phi\text{-}m(X([0,1]^N))$ seems more difficult. Recently Talagrand (1996b) proved that with probability 1

$$s^d \log 1/s \log\log\log 1/s\text{-}m(X([0,1]^N)) < \infty.$$

This is also true for the Hausdorff measure of the graph set of $X(t)$. But in both cases, the lower bound problems remain open.

The purpose of this paper is to study properties of some Hausdorff-type measures and generalize the results about the Hausdorff measure of the image and graph of fractional Brownian motion. In Section 2, we study the properties of Hausdorff-type measures associated to an increasing sequence of positive integers. In Section 3, we prove some lemmas which will be useful in proving the main results. In Section 4, we study the exact Hausdorff-type measure of the image and graph of fractional Brownian motion. A necessary and sufficient condition on Λ is proved so that the usual Hausdorff measure functions are still the correct measure functions for the image and graph. If the condition is not satisfied, then some smaller measure functions will give positive and finite (ϕ,Λ)-Hausdorff measure for $X([0,1]^N)$ and $\mathrm{Gr}\,X([0,1]^N)$. These results extend those of Lévy (1953), Ciesielski and Taylor (1962), Goldman (1988) and Talagrand (1995). The proof of the results depend mainly on the refinements of the argument of Talagrand (1995) and these methods can be applied to more general strongly locally nondeterministic Gaussian random fields (see Xiao, 1996).

We will use $K, K_1, K_2, \ldots$ to denote unspecified positive and finite constants, they may be different in each appearance.

2. Hausdorff-type measures

Let $\Lambda = \{\lambda_k\}$ be an increasing sequence of positive integers with $\lambda_k \to \infty$. For each $k \geq 1$, let $\mathscr{F}_k$ be the family of (closed) balls of radius $2^{-\lambda_k}$ in $\mathbb{R}^d$ and denote $\mathscr{F}_\Lambda = \bigcup_{k=1}^{\infty} \mathscr{F}_k$. Then $\mathscr{F}_\Lambda$ is a covering family of $\mathbb{R}^d$. For any $\phi \in \Phi$ and $E \subseteq \mathbb{R}^d$, we define

$$\phi\text{-}m_\Lambda(E) = \lim_{k_0 \to \infty} \inf \left\{ \sum_i \phi(|F_i|) \colon E \subseteq \bigcup_{i=1}^{\infty} F_i,\ F_i \in \bigcup_{k=k_0}^{\infty} \mathscr{F}_k \right\}, \tag{2.1}$$

where $|F|$ denotes the diameter of F. Then $\phi\text{-}m_\Lambda$ is a metric outer measure in the sense of Carathéodory and hence every Borel set in $\mathbb{R}^d$ is $\phi\text{-}m_\Lambda$ measurable (see Rogers, 1970). We will call $\phi\text{-}m_\Lambda(E)$ the (ϕ,Λ)-Hausdorff measure of E. If $\Lambda = \mathbb{N}$, the set of all positive integers, then $\phi\text{-}m_\Lambda$ is equivalent to $\phi\text{-}m$, $\phi\text{-}m^*$ and $\phi\text{-}m^{**}$, where $\phi\text{-}m^*$ and $\phi\text{-}m^{**}$ are Hausdorff-type measures defined by Taylor and Tricot (1985) using the families of dyadic cubes and semidyadic cubes, respectively.

We first summarize some elementary properties of $\phi\text{-}m_\Lambda$ in the following lemma.

Lemma 2.1. (1) *Let $\Lambda \subseteq \mathbb{N}$ and $\phi, \psi \in \Phi$ with*

$$\lim_{s \to 0} \frac{\psi(s)}{\phi(s)} = 0.$$

Then for any $E \subseteq \mathbb{R}^d$, $\phi\text{-}m_\Lambda(E) < \infty$ implies $\psi\text{-}m_\Lambda(E) = 0$.

(2) *Let $\phi \in \Phi$. If $\Lambda_1 \subseteq \Lambda_2$, then for any $E \subset \mathbb{R}^d$,*

$$\phi\text{-}m_{\Lambda_2}(E) \leq \phi\text{-}m_{\Lambda_1}(E).$$

In particular, for any set $E \subseteq \mathbb{R}^d$, $\phi\text{-}m(E) \leq \phi\text{-}m_\Lambda(E)$.

(3) *For any $\Lambda \subseteq \mathbb{N}$ and any $E \subseteq \mathbb{R}^d$, $\phi\text{-}m_\Lambda(E) \leq \phi\text{-}p(E)$.*

Proof. (1) and (2) follow directly from definition (2.1). In order to prove (3), by Eq. (1.3) and the Borel regularities of both measures, it suffices to prove for any bounded Borel set E,

$$\phi\text{-}m_\Lambda(E) \leq \phi\text{-}P(E). \tag{2.2}$$

For this, we may further assume $\phi\text{-}P(E)<\infty$. Then by Eq. (1.4), for any $\varepsilon>0$, there is $r_0>0$ such that for every $0<r\leqslant r_0$ and every r-packing $\{\overline{B}(x_i,r_i)\}$ of E,

$$\sum_{i=1}^{\infty}\phi(2r_i)\leqslant\phi\text{-}P(E)+\varepsilon. \tag{2.3}$$

Now the family $\mathscr{F}_E$ of closed balls $\overline{B}(x,2^{-\lambda_k})$ with $x\in E$ and $\lambda_k\in\Lambda$ such that $2^{-\lambda_k}\leqslant r_0$ is a Vitali covering of E, we can find an r_0-packing $\{\overline{B}_i\}$ in $\mathscr{F}_E$ of E such that for any $n\geqslant 1$

$$E\Big\backslash\bigcup_{i=1}^{n}\overline{B}_i\subseteq\bigcup_{i=n+1}^{\infty}\overline{B}_i^*$$

where B_i^* is the ball with the same center as B_i and with radius multiplied by 5. Hence

$$E\subseteq\bigcup_{i=1}^{n}\overline{B}_i\cup\bigcup_{i=n+1}^{\infty}\overline{B}_i^*\subseteq\bigcup_{i=1}^{n}\overline{B}_i\cup\bigcup_{i=n+1}^{\infty}\bigcup_{j=1}^{J}\overline{B}_{ij}$$

where $\overline{B}_{ij}\in\mathscr{F}_\Lambda$ with $|B_{ij}|=|B_i|$ and J is a constant depending on d only. It follows from Eq. (2.3) that

$$\sum_{i=n+}^{\infty}\sum_{j=1}^{J}\phi(|\overline{B}_{ij}|)=J\sum_{i=n+1}^{\infty}\phi(|\overline{B}_i|)\to 0\quad\text{as } n\to\infty.$$

Hence for n large enough,

$$\sum_{i=1}^{n}\phi(|\overline{B}_i|)+\sum_{i=n+1}^{\infty}\sum_{j=1}^{J}\phi(|\overline{B}_{ij}|)\leqslant\phi\text{-}P(E)+2\varepsilon,$$

which implies Eq. (2.2).

The Hausdorff dimension, associated to $\phi\text{-}m_\Lambda$, of E is defined by

$$\dim_\Lambda E=\inf\{\alpha>0\colon s^\alpha\text{-}m_\Lambda(E)=0\}=\sup\{\alpha>0\colon s^\alpha\text{-}m_\Lambda(E)=\infty\}.$$

It is easy to verify that $\dim_\Lambda$ is σ-stable, that is,

$$\dim_\Lambda\left(\bigcup_n E_n\right)=\sup_n\dim_\Lambda E_n.$$

By (3) of Lemma 2.1, we see that for any $\Lambda\subseteq\mathbb{N}$ and any $E\subset\mathbb{R}^d$,

$$0\leqslant\dim E\leqslant\dim_\Lambda E\leqslant\operatorname{Dim}E\leqslant d. \tag{2.4}$$

Inequalities (2.4) can be strict. The following example shows that for any $0<\beta<1$, there exist $\Lambda\subseteq\mathbb{N}$ and a Cantor type compact set $E\subset[0,1]$ such that

$$0=\dim E<\dim_\Lambda E=\operatorname{Dim}E=\beta.$$

The construction is a special case of Lemmas 3.1 and 3.2 in Talagrand and Xiao (1996). Let $\Lambda=\{\lambda_k\}$ be a sequence of positive integers satisfying $\lambda_1=1$, and

$$\lambda_{k+1}\geqslant\frac{\beta}{1-\beta}2^{k+1}(\lambda_1+\cdots+\lambda_k).$$

We take

$$\eta_k = 2^{-\lambda_k}, \quad m_k = [2^{\beta\lambda_k}], \quad \delta_k = 2 \cdot 2^{-(1-\beta)\lambda_{k+1}}.$$

We construct a generalized Cantor set $E \subseteq [0,1]$ in the following way.

Let $E_0 = [0,1]$ and let E_1 be the union of m_1 closed subintervals of $[0,1]$ of length δ_1, which are arranged in such a way that the distance (or gap) between any two of them is equal to η_1. This is possible since $m_1(\eta_1 + \delta_1) < 1$. At the second stage, each interval I_{i_1} of E_1 contains m_2 closed subintervals $I_{i_1 i_2}$ of length δ_2 with gaps equal to η_2. This is possible since $m_2(\eta_2 + \delta_2) \leqslant \delta_1$. Let $E_2 = \bigcup_{i_1=1}^{m_1} \bigcup_{i_2=1}^{m_2} I_{i_1 i_2}$. Suppose now that E_{n-1} has been constructed, $E_{n-1} = \bigcup_{i_1=1}^{m_1} \cdots \bigcup_{i_{n-1}=1}^{m_{n-1}} I_{i_1 \cdots i_{n-1}}$. Since $|I_{i_1 \cdots i_{n-1}}| = \delta_{n-1}$ and by the choice of δ_{n-1}, we have $m_n(\eta_n + \delta_n) \leqslant \delta_{n-1}$, so we can construct m_n closed subintervals $I_{i_1 \cdots i_n}$ $(i_n = 1, \ldots, m_n)$ of length δ_n and gaps equal to η_n in $I_{i_1 \cdots i_{n-1}}$. We set $E_n = \bigcup_{i_1=1}^{m_1} \cdots \bigcup_{i_n=1}^{m_n} I_{i_1 \cdots i_n}$. Continuing this process, we obtain a decreasing sequence $\{E_n\}$ of compact subsets of $[0,1]$. Let $E = \bigcap_{n=1}^{\infty} E_n$; then $E \subset [0,1]$ is compact. By Lemma 3.2 of Talagrand and Xiao (1996), $\dim E = 0$, $\operatorname{Dim} E = \beta$. The fact that $\dim_\Lambda E = \beta$ is a consequence of the definition of the Borel measure in the proof of Lemma 3.2 of Talagrand and Xiao (1996) and the following density theorem (see also the proof of Theorem 2.1 below).

Lemma 2.2. *Let $\phi \in \Phi$ and let $\Lambda = \{\lambda_k\}$ be an increasing sequence of positive integers with $\lambda_k \to \infty$. Then there exist positive constants K_1 and K_2 such that for any Borel measure μ on $\mathbb{R}^d$ and every Borel set $E \subset \mathbb{R}^d$, we have*

$$K_1 \mu(E) \inf_{x\in E} \{\overline{D}_\mu^{\phi,\Lambda}(x)\}^{-1} \leqslant \phi\text{-}m_\Lambda(E) \leqslant K_2 \mu(\mathbb{R}^d) \sup_{x\in E} \{\overline{D}_\mu^{\phi,\Lambda}(x)\}^{-1},$$

where

$$\overline{D}_\mu^{\phi,\Lambda}(x) = \limsup_{k\to\infty} \frac{\mu(B(x, 2^{-\lambda_k}))}{\phi(2^{-\lambda_k - 1})}.$$

Proof. It is proved in the same way as in Rogers and Taylor (1961) (see also Saint Raymond and Tricot, 1988).

For any measure function $\phi \in \Phi$ with $\limsup_{s\to 0} \phi(s)/s^d < \infty$, it is easy to verify that there exists a positive and finite constant K such that for any infinite subsequence $\Lambda \subseteq \mathbb{N}$ we have $\phi\text{-}m_\Lambda(E) \leqslant K L_d(E)$ for any Borel set $E \subset \mathbb{R}^d$, where L_d is the Lebesgue measure in $\mathbb{R}^d$. On the other hand, given any two infinite sequences of positive integers Λ and Γ, it is possible to define a piecewisely linear function $\phi \in \Phi$ (depending on Λ and Γ) with

$$\limsup_{s\to 0} \frac{\phi(s)}{s} = \infty,$$

but $\phi\text{-}m_\Gamma(E) \leqslant \phi\text{-}m_\Lambda(E)$ for every $E \subseteq \mathbb{R}$. Hence the assumptions on the measure function ϕ in the following theorem is reasonable.

Theorem 2.1. *Let $\phi \in \Phi$ with $\phi(s)=s^{\alpha}L(s)$, where $0<\alpha<d$ and $L(s)$ is slowly varying at the origin. Then*

$$\phi\text{-}m_{\Lambda}(E)<\infty \Rightarrow \phi\text{-}m_{\Gamma}(E)<\infty \quad \textit{for every } E\subset \mathbb{R}^d$$

if and only if there exists $p\geqslant 1$ such that $\Lambda \subseteq \Gamma+[-p,p]$.

Proof. Assume that there exists $p\geqslant 1$ such that $\Lambda \subseteq \Gamma+[-p,p]$. We will prove the following stronger result: there exists a finite constant $K>0$ depending on ϕ, p and d only such that for every $E\subseteq \mathbb{R}^d$

$$\phi\text{-}m_{\Gamma}(E)\leqslant K\phi\text{-}m_{\Lambda}(E). \tag{2.5}$$

We may assume $\phi\text{-}m_{\Lambda}(E)<\infty$. For any $\eta>\phi\text{-}m_{\Lambda}(E)$, we can choose a covering of E, say, $\{B(x_i,2^{-\lambda_{k_i}}),\ \lambda_{k_i}\in\Lambda\}$, such that

$$E\subseteq \bigcup_{i=1}^{\infty} B(x_i,2^{-\lambda_{k_i}})$$

and

$$\sum_{i=1}^{\infty}\phi(2^{-\lambda_{k_i}+1})\leqslant \eta. \tag{2.6}$$

For each λ_{k_i}, there is $\gamma_{k_i}\in\Gamma$ such that $\gamma_{k_i}-p\leqslant\lambda_{k_i}\leqslant\gamma_{k_i}+p$. Hence each $B(x_i,2^{-\lambda_{k_i}})$ can be covered by at most $J=J(p,d)$ balls of radius $2^{-\gamma_{k_i}}$, that is

$$E\subseteq \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{J} B(y_{ij},2^{-\gamma_{k_i}}).$$

It follows from Eqs. (2.6) and (1.1) that

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{j=1}^{J}\phi(2^{-\gamma_{k_i}+1}) &\leqslant K\sum_{i=1}^{\infty}\phi(2^{-(\lambda_{k_i}-p)+1})\\ &\leqslant K\sum_{i=1}^{\infty}\phi(2^{-\lambda_{k_i}+1})\leqslant K\eta.\end{aligned}$$

Since $\eta>\phi\text{-}m_{\Lambda}(E)$ is arbitrary, this proves Eq. (2.5).

To prove the necessity, we assume that for each positive integer p, there is $\lambda_p\in\Lambda$ such that

$$\inf_{\gamma\in\Gamma} \gamma-\lambda_p|>p. \tag{2.7}$$

Thus we can choose a subsequence, still denoted by $\{\lambda_p\}$, from Λ such that Eq. (2.7) and

$$\frac{2^{-\lambda_1 d}}{\phi(2^{-\lambda_1})}\leqslant\frac{1}{2},\quad \frac{2^{-\lambda_p d}}{\phi(2^{-\lambda_p})}\leqslant\frac{2^{-\lambda_{p-1}d}}{2\phi(2^{-\lambda_{p-1}})}\ (p\geqslant 1) \tag{2.8}$$

hold. We now proceed to construct a compact set $E\subset[0,1]^d$ such that

$$\phi\text{-}m_{\Lambda}(E)<\infty \quad \text{but } \phi\text{-}m_{\Gamma}(E)=\infty. \tag{2.9}$$

For simplicity of the notations, we assume $d=1$. The construction for the case of $d>1$ is very similar. Let $E_0=[0,1]$. At the first step, let

$$M_1=\left[\frac{1}{\phi(2^{-\lambda_1})}\right],$$

where $[x]$ is the integer part of x. Let E_1 be the union of M_1 disjoint closed subintervals I_{i_1} $(i_1=1,\dots,M_1)$ of length $2^{-\lambda_1}$ with distance from each other at least $\frac{1}{2}\phi(2^{-\lambda_1})$. This is possible since by Eq. (2.8)

$$M_1(2^{-\lambda_1}+\tfrac{1}{2}\phi(2^{-\lambda_1}))<1.$$

Suppose that E_p has been constructed as a union of $M_1\cdots M_{p-1}$ disjoint closed subintervals $I_{i_1\cdots i_{p-1}}$ of length $2^{-\lambda_{p-1}}$ with distance from each other at least

$$2^{-\lambda_{p-2}}\frac{\phi(2^{-\lambda_{p-1}})}{2\phi(2^{-\lambda_{p-2}})}.$$

At the pth stage, let

$$M_p=\left[\frac{\phi(2^{-\lambda_{p-1}})}{\phi(2^{-\lambda_p})}\right].$$

In each closed interval $I_{i_1\cdots i_{p-1}}$ of E_{p-1}, we construct M_p disjoint closed subintervals $I_{i_1\cdots i_p}$ $(i_p=1,2,\dots,M_p)$ of length $2^{-\lambda_p}$ with distance from each other at least

$$2^{-\lambda_{p-1}}\frac{\phi(2^{-\lambda_p})}{2\phi(2^{-\lambda_{p-1}})}.$$

This is possible since by Eq. (2.8)

$$M_p\left(2^{-\lambda_p}+2^{-\lambda_{p-1}}\frac{\phi(2^{-\lambda_p})}{2\phi(2^{-\lambda_{p-1}})}\right)<2^{-\lambda_{p-1}}.$$

Continuing the process, we obtain a decreasing sequence $\{E_p\}$ of compact subsets of $[0,1]$. Let $E=\bigcap_{p=1}^{\infty}E_p$. Then $E\subset[0,1]$ is compact and clearly $\phi\text{-}m_A(E)\leqslant 1$.

In order to prove the second conclusion in Eq. (2.9), we first define a Borel measure σ on E by distributing mass to E_n and then apply Lemma 2.2. For each I_{i_1}, define $\sigma(I_{i_1})=M_1^{-1}$. In general, for each $I_{i_1\cdots i_p}$ in E_p, we define $\sigma(I_{i_1\cdots i_p})=(M_1\cdots M_p)^{-1}$ and $\sigma(\mathbb{R}^d\backslash E_p)=0$ for $p\geqslant 1$. Then by the mass distribution principle (see Falconer, 1990), σ can be extended to a Borel measure on $\mathbb{R}^d$ with $\sigma(E)=1$. For any $x\in E$, there exists an infinite sequence $\boldsymbol{i}=i_1i_2\cdots i_p\cdots$ such that

$$\{x\}=\bigcap_{p=1}^{\infty}I_{i_1\cdots i_p}.$$

Now for any $\gamma\in\Gamma$, there exists $p\geqslant 1$ such that $\lambda_{p-1}<\gamma<\lambda_p$. Since $B(x,2^{\gamma})$ can intersect at most

$$\frac{2^{\gamma}4\phi(2^{-\lambda_{p-1}})}{2^{-\lambda_{p-1}}\phi(2^{-\lambda_p})}$$

intervals $I_{i_1 \cdots i_p}$, we have

$$\mu(B(x, 2^{\gamma})) \leqslant \frac{2^{\gamma} \cdot 4\phi(2^{-\lambda_{p-1}})}{2^{-\lambda_{p-1}}\phi(2^{-\lambda_p})}\phi(2^{-\lambda_p})$$
$$= 4 \times 2^{-\gamma+\lambda_{p-1}}\phi(2^{-\lambda_{p-1}}). \tag{2.10}$$

It follows from Eqs. (2.7), (2.10) and the assumptions on $\phi(s)$ that

$$\limsup_{\gamma \in \Gamma,\, \gamma \to \infty} \frac{\mu(B(x, 2^{\gamma}))}{\phi(2 \times 2^{-\gamma})} \leqslant \limsup_{\gamma \in \Gamma, \gamma \to \infty} \frac{2 \times 2^{-\gamma+\lambda_{p-1}}\phi(2^{-\lambda_{p-1}})}{\phi(2 \times 2^{-\gamma})}$$
$$\leqslant \limsup_{\gamma \in \Gamma, \gamma \to \infty} 2^{-(1-\alpha)(\gamma-\lambda_{p-1})}\frac{L(2^{-\lambda_{p-1}})}{L(2^{-\gamma})}$$
$$= 0.$$

Hence by Lemma 2.2, we have $\phi\text{-}m_\Gamma(E) = \infty$. This finishes the proof of Theorem 2.1. □

Remark. If $\phi(s) = s^d L(s)$ with $L(s)$ slowly varying at the origin and $\lim_{s \to 0} L(s) = \infty$, it is not known whether the condition in Theorem 2.1 is still necessary.

3. Some estimates

Given $\phi \in \Phi$ and an infinite increasing sequence $\Lambda \subseteq \mathbb{N}$, in order to study the exact (ϕ, Λ)-Hausdorff measure of the sample paths of fractional Brownian motion, we need to generalize the main estimate of Talagrand (1995). We start with the following elementary lemma.

Lemma 3.1. *Let $\Lambda = \{\lambda_k\}$ be an increasing sequence of positive integers and*

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k^{\gamma}} = \infty$$

for some $\gamma > 0$. For any fixed constant $c > 0$, we define a sequence $\{i_k\}$ of positive integers by

$$i_1 = 1,\ i_k = \inf\{i\colon \lambda_i \geqslant \lambda_{i_{k-1}} + c \log \lambda_{i_{k-1}}\}\ (k \geqslant 2)$$

and denote $\eta_k = \lambda_{i_k}$. Then for every $n \geqslant 2$,

$$\eta_n - \eta_{n-1} \geqslant c \log \eta_{n-1} \tag{3.1}$$

and for every $0 < \varepsilon < \gamma$,

$$\sum_{k=1}^{\infty} \frac{1}{\eta_k^{\varepsilon}} = \infty. \tag{3.2}$$

Proof. Inequality (3.1) is obvious. For any integer $n_0 \geqslant 1$, let n_1 be the largest integer with $\eta_{n_1} \leqslant \lambda_{n_0}$. Then for any $N > n_0$ and any $0 < \varepsilon < \gamma$,

$$\sum_{k=n_0}^{N} \frac{1}{\lambda_k^{\gamma}} \leqslant \sum_{k=n_1}^{N} \sum_{j=0}^{c \log \eta_k} \frac{1}{(\eta_k + j)^{\gamma}} \leqslant \sum_{k=n_1}^{N} \frac{c \log \eta_k + 1}{\eta_k^{\gamma}} \leqslant \sum_{k=n_1}^{N} \frac{1}{\eta_k^{\varepsilon}}$$

at least for n_0 large enough. This proves Eq. (3.2).

Lemma 3.2. *Let $\Lambda = \{\lambda_k\}$ be an increasing sequence of positive integers and*

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k^{\gamma}} = \infty$$

for some $\gamma > 0$. Then for any fixed $0 < \varepsilon < \gamma$, $c > 0$ and for any $k_0 \geqslant 1$, there exist integers $k_2 \geqslant k_1 \geqslant k_0$ such that

$$\lambda_{k_2} \leqslant \lambda_{k_1}^2 \quad \text{and} \quad \sum_{k=k_1}^{k_2} \frac{1}{\lambda_k^{\varepsilon}} \geqslant c \log \lambda_{k_2}. \tag{3.3}$$

Proof. Suppose the conclusion is not true. Then for some $0 < \varepsilon < \gamma$, there exist $k_0 \geqslant 1$ and $c > 0$ such that for any $k_2 \geqslant k_1 \geqslant k_0$ with $\lambda_{k_2} \leqslant \lambda_{k_1}^2$ (such λ_{k_2} exists for infinitely many k_1's), we have

$$\sum_{k=k_1}^{k_2} \frac{1}{\lambda_k^{\varepsilon}} < c \log \lambda_{k_2}. \tag{3.4}$$

Then we also have

$$\sum_{k=k_1}^{k_2+1} \frac{1}{\lambda_k^{\varepsilon}} < (c+1) \log \lambda_{k_2+1}. \tag{3.5}$$

Denote

$$S_{k_1} = \frac{1}{\lambda_{k_1}^{\varepsilon}}, \quad S_k = \sum_{j=k_1}^{k} \frac{1}{\lambda_j^{\varepsilon}} \qquad (k = k_1 + 1, \dots, k_2 + 1).$$

Then it follows from Eqs. (3.4) and (3.5) that for any $k_2 \geqslant k_1$ with $\lambda_{k_2} \leqslant \lambda_{k_1}^2$,

$$\sum_{k=k_1}^{k_2+1} \frac{1}{\lambda_k^{\gamma}} = \sum_{k=k_1}^{k_2+1} \frac{1}{\lambda_k^{\gamma-\varepsilon}} \frac{1}{\lambda_k^{\varepsilon}} = \sum_{k=k_1}^{k_2+1} \frac{1}{\lambda_k^{\gamma-\varepsilon}} (S_k - S_{k-1}) \quad (S_{k_1-1} \hat{=} 0)$$

$$= \sum_{k=k_1}^{k_2} S_k \left(\frac{1}{\lambda_k^{\gamma-\varepsilon}} - \frac{1}{\lambda_{k+1}^{\gamma-\varepsilon}} \right) + \frac{S_{k_2+1}}{\lambda_{k_2+1}^{\gamma-\varepsilon}}$$

$$\leqslant c \sum_{k=k_1}^{k_2} \log \lambda_k \left(\frac{1}{\lambda_k^{\gamma-\varepsilon}} - \frac{1}{\lambda_{k+1}^{\gamma-\varepsilon}} \right) + \frac{(c+1)\log \lambda_{k_2+1}}{\lambda_{k_2+1}^{\gamma-\varepsilon}}. \tag{3.6}$$

Now we let $n_0 = k_0$ and $n_k = \inf\{n\colon\ \lambda_n \geqslant \lambda_{n_{k-1}}^2\}$ for $k \geqslant 1$. Then $\lambda_{n_k-1} \leqslant \lambda_{n_{k-1}}^2$. It follows from Eq. (3.6) that for any $m \geqslant 1$, we can write

$$\sum_{k=n_0}^{n_m} \frac{1}{\lambda_k^{\gamma}} = \sum_{k=n_0}^{n_1} \frac{1}{\lambda_k^{\gamma}} + \sum_{k=n_1+1}^{n_2} \frac{1}{\lambda_k^{\gamma}} + \cdots + \sum_{k=n_{m-1}+1}^{n_m} \frac{1}{\lambda_k^{\gamma}}$$

$$\leqslant c \sum_{k=n_0}^{n_1-1} \log \lambda_k \left(\frac{1}{\lambda_k^{\gamma-\varepsilon}} - \frac{1}{\lambda_{k+1}^{\gamma-\varepsilon}} \right) + \frac{(c+1)\log \lambda_{n_1}}{\lambda_{n_1}^{\gamma-\varepsilon}} + \cdots$$

$$+ c \sum_{k=n_{m-1}+1}^{n_m-1} \log \lambda_k \left(\frac{1}{\lambda_k^{\gamma-\varepsilon}} - \frac{1}{\lambda_{k+1}^{\gamma-\varepsilon}} \right) + \frac{(c+1)\log \lambda_{n_m}}{\lambda_{n_m}^{\gamma-\varepsilon}}$$

$$\leqslant c \sum_{k=n_0}^{n_m-1} \log \lambda_k \left(\frac{1}{\lambda_k^{\gamma-\varepsilon}} - \frac{1}{\lambda_{k+1}^{\gamma-\varepsilon}} \right) + (c+1) \sum_{j=1}^{m} \frac{\log \lambda_{n_j}}{\lambda_{n_j}^{\gamma-\varepsilon}}. \tag{3.7}$$

We will show

$$\sum_{k=n_0}^{\infty} \log \lambda_k \left(\frac{1}{\lambda_k^{\gamma-\varepsilon}} - \frac{1}{\lambda_{k+1}^{\gamma-\varepsilon}} \right) < \infty \tag{3.8}$$

$$\sum_{j=1}^{\infty} \frac{\log \lambda_{n_j}}{\lambda_{n_j}^{\gamma-\varepsilon}} < \infty. \tag{3.9}$$

Then Eqs. (3.7)–(3.9) imply $\sum_{k=1}^{\infty} 1/\lambda_k^{\gamma} < \infty$, a contradiction.

Notice that

$$\frac{1}{\lambda_k^{\gamma-\varepsilon}} - \frac{1}{\lambda_{k+1}^{\gamma-\varepsilon}} = \sum_{j=\lambda_k}^{\lambda_{k+1}-1} \left(\frac{1}{j^{\gamma-\varepsilon}} - \frac{1}{(j+1)^{\gamma-\varepsilon}} \right)$$

we have

$$\log \lambda_k \left(\frac{1}{\lambda_k^{\gamma-\varepsilon}} - \frac{1}{\lambda_{k+1}^{\gamma-\varepsilon}} \right) \leqslant \sum_{j=\lambda_k}^{\lambda_{k+1}-1} \log j \left(\frac{1}{j^{\gamma-\varepsilon}} - \frac{1}{(j+1)^{\gamma-\varepsilon}} \right).$$

Hence

$$\sum_{k=n_0}^{n} \log \lambda_k \left(\frac{1}{\lambda_k^{\gamma-\varepsilon}} - \frac{1}{\lambda_{k+1}^{\gamma-\varepsilon}} \right)$$
$$\leqslant \sum_{k=n_0}^{n} \sum_{j=\lambda_k}^{\lambda_{k+1}-1} \log j \left(\frac{1}{j^{\gamma-\varepsilon}} - \frac{1}{(j+1)^{\gamma-\varepsilon}} \right)$$
$$= \sum_{j=\lambda_{n_0}}^{\lambda_{n+1}-1} \log j \left(\frac{1}{j^{\gamma-\varepsilon}} - \frac{1}{(j+1)^{\gamma-\varepsilon}} \right),$$

which is convergent as $n \to \infty$. This proves Eq. (3.8). The convergence of Eq. (3.9) follows easily from the definition of λ_{n_j}. We have completed the proof of Lemma 3.2. □

Let $Y(t)$ $(t \in \mathbb{R}^N)$ be a real-valued fractional Brownian motion of index α $(0<\alpha<1)$. We will make use of the fact that there exist two independent scattered Gaussian random measures on $\mathbb{R}^N$, with

$$E(m(A)^2) = E(m'(A)^2) = L_N(A)$$

for all $A \subset \mathbb{R}^N$, where L_N is the Lebesgue measure in $\mathbb{R}^N$, such that

$$Y(t) = c(\alpha,N) \int_{\mathbb{R}^N} (1 - \cos\langle t,x\rangle) \frac{\mathrm{d}m(x)}{|x|^{\alpha+N/2}} + c(\alpha,N) \int_{\mathbb{R}^N} \sin\langle t,x\rangle \frac{\mathrm{d}m'(x)}{|x|^{\alpha-N/2}},$$

where $c(\alpha,N)>0$ is a normalizing constant. For any $0<a<b$, let

$$Y(a,b;t) = c(\alpha,N) \int_{a\leqslant|x|\leqslant b} (1 - \cos\langle t,x\rangle) \frac{\mathrm{d}m(x)}{|x|^{\alpha+N/2}} + \sin\langle t,x\rangle \frac{\mathrm{d}m'(x)}{|x|^{\alpha+N/2}}.$$

Then for any $0<a<b<a'<b'<\infty$, the processes $Y(a,b;t)$ and $Y(a',b';t)$ are independent. We denote

$$X(a,b;t) = (X_1(a,b;t),\ldots,X_d(a,b;t)),$$

where the components are independent copies of $Y(a,b;t)$.

The following two lemmas were proved in Talagrand (1995). For more general results about the small ball probabilities of Gaussian processes, see Talagrand (1993) and Monrad and Rootzén (1995).

Lemma 3.3. *If* $0<\varepsilon\leqslant r^\alpha$. *Then for any* $0<a<b$ *we have*

$$P\left\{ \sup_{|t|\leqslant r} |X(a,b,t)| \leqslant \varepsilon \right\} \geqslant \exp\left(-K\left(\frac{r}{\varepsilon^{1/\alpha}}\right)^N \right). \tag{3.10}$$

Lemma 3.4. *Consider* $1<a<b$ *and* $0<r<1$. *Let*

$$A = r^2 a^{2-2\alpha} + b^{-2\alpha}.$$

If $A \leqslant \frac{1}{2} r^{2\alpha}$. *Then for any*

$$u \geqslant K\left(A \log \frac{K r^{2\alpha}}{A}\right)^{1/2}$$

we have

$$P\left\{\sup_{|t|\leqslant r}|X(t)-X(a,b,t)|\geqslant u\right\}\leqslant \exp\left(-\frac{u^2}{KA}\right). \tag{3.11}$$

Now consider an increasing sequence $\Lambda=\{\lambda_k\}$ of positive integers. Let

$$\rho_k=\sup\left\{\rho=2^{-m}:\rho^{\alpha}\left(\log\log\frac{1}{\rho}\right)^{-\alpha/N}\leqslant 2^{-\lambda_k}\right\}.$$

Then there exist positive and finite constants K_3, K_4 such that for k large enough

$$K_3 2^{-\lambda_k/\alpha}(\log\log 2^{\lambda_k})^{1/N}\leqslant\rho_k\leqslant K_4 2^{-\lambda_k/\alpha}(\log\log 2^{\lambda_k})^{1/N}. \tag{3.12}$$

The following estimate, which generalizes Proposition 4.1 of Talagrand (1995), is essential to our purpose.

Proposition 3.1. *Let us be given an increasing sequence $\Lambda=\{\lambda_k\}$ of positive integers with*

$$\sum_{k=1}^{\infty}\frac{1}{\lambda_k^{\gamma}}=\infty$$

for some $\gamma>0$. Then for any fixed constant $c>0$, there exists a constant $K>0$ with the following property: for any integer $k_0\geqslant 1$ there exist integers $k_2\geqslant k_1\geqslant k_0$ with $\lambda_{k_2}\leqslant\lambda_{k_1}^2$ such that

$$P\left\{\exists k \text{ such that } k_1\leqslant k\leqslant k_2 \text{ and } \sup_{|t|\leqslant\sqrt{N}\rho_k}|X(t)|\leqslant K\rho_k^{\alpha}\left(\log\log\frac{1}{\rho_k}\right)^{-\alpha/N}\right\}$$
$$\geqslant 1-\exp(-c\log\lambda_{k_2}). \tag{3.13}$$

Proof. By Eq. (3.1), we may and will assume that

$$\lambda_{k+1}-\lambda_k\geqslant c_1\log\lambda_k, \tag{3.14}$$

where $c_1>0$ is a constant which will be chosen later, and by Eq. (3.2) and Lemma 3.2, we will further assume that for any constant $c_2>0$ and any integer $k_0\geqslant 1$, there exist integers $k_2\geqslant k_1\geqslant k_0$ such that

$$\lambda_{k_2}\leqslant\lambda_{k_1}^2 \quad\text{and}\quad \sum_{k=k_1}^{k_2}\frac{1}{\lambda_k^{\gamma}}\geqslant c_2\log\lambda_{k_2}. \tag{3.15}$$

Now fix such a pair (k_1,k_2). Let $\beta=\min(1-\alpha,\alpha)$ and let

$$U=\left(\log\frac{1}{\rho_{k_1}}\right)^{1/\beta}. \tag{3.16}$$

It follows from Eq. (3.12) that $K_3\lambda_{k_1}^{1/\beta} \leqslant U \leqslant K_4\lambda_{k_1}^{1/\beta}$. Let

$$a_k = \frac{1}{\rho_k U} \quad \text{and} \quad b_k = \frac{U}{\rho_k}.$$

Elementary calculations and Eq. (3.14) show that we can take $c_1 > 0$ (depending only on α and β) such that $b_k \leqslant a_{k+1}$ for every $k \geqslant 1$. We denote $\tilde{X}_k(t) = X(a_k, b_k; t)$ and notice that $X(t)$ can be written as

$$X(t) = \tilde{X}_k(t) + (X(t) - \tilde{X}_k(t)).$$

Then the processes $\tilde{X}_k$ $(k = 1, 2, \ldots)$ are independent. By Lemma 3.3, we can choose $K > 0$, depending on γ, such that

$$\begin{aligned} P&\left\{ \sup_{|t| \leqslant \sqrt{N}\rho_k} |\tilde{X}_k(t)| \leqslant K\rho_k^{\alpha} \left(\log\log \frac{1}{\rho_k}\right)^{-\alpha/N} \right\} \\ &\geqslant \exp\left(-\gamma \log\log \frac{1}{\rho_k}\right) \\ &\geqslant \frac{K}{\lambda_k^{\gamma}}. \end{aligned}$$

Thus, by independence of $\tilde{X}_k$ $(k = 1, 2, \ldots)$ and Eq. (3.15), we have

$$\begin{aligned} P&\left\{ \exists k \text{ such that } k_1 \leqslant k \leqslant k_2 \text{ and } \sup_{|t| \leqslant \sqrt{N}\rho_k} |\tilde{X}_k(t)| \leqslant K\rho_k^{\alpha} \left(\log\log \frac{1}{\rho_k}\right)^{-\alpha/N} \right\} \\ &\geqslant 1 - \prod_{k=k_1}^{k_2} \left(1 - \frac{K}{\lambda_k^{\gamma}}\right) \\ &\geqslant 1 - \exp\left(-K \sum_{k=k_1}^{k_2} \frac{1}{\lambda_k^{\gamma}}\right) \\ &\geqslant 1 - \exp(-Kc_2 \log \lambda_{k_2}). \end{aligned} \tag{3.17}$$

Let $A_k = \rho_k^2 a_k^{2-2\alpha} + b_k^{-2\alpha}$. Then

$$A_k \rho_k^{-2\alpha} = (\rho_k a_k)^{2-2\alpha} + (\rho_k b_k)^{-2\alpha} \leqslant 2U^{-2\beta}. \tag{3.18}$$

By Lemma 3.4 and Eq. (3.18), for any

$$u \geqslant K\left(A_k \log \frac{K\rho_k^{2\alpha}}{A_k}\right)^{1/2}$$

we have

$$P\left(\sup_{|t| \leqslant \sqrt{N}\rho_k} |X(t) - \tilde{X}_k(t)| \geqslant u\right) \leqslant \exp\left(-\frac{u^2 U^{2\beta}}{K\rho_k^{2\alpha}}\right). \tag{3.19}$$

By Eq. (3.16) and the first inequality in relation (3.15) we see that for $k_1 \leqslant k \leqslant k_2$

$$U^{\beta}(\log U)^{-1/2} \geqslant \left(\log\log \frac{1}{\rho_k}\right)^{\alpha/N}.$$

Hence we can take

$$u = K\rho_k^{\alpha}\left(\log\log\frac{1}{\rho_k}\right)^{-\alpha/N}$$

and by Eq. (3.19) we obtain

$$P\left(\sup_{|t|\leqslant\sqrt{N}\rho_k}|X(t)-\tilde{X}_k(t)|\geqslant K\rho_k^{\alpha}\left(\log\log\frac{1}{\rho_k}\right)^{-\alpha/N}\right)$$

$$\leqslant \exp\left(-\frac{U^{2\beta}}{K(\log\log 1/\rho_k)^{2\alpha/N}}\right). \tag{3.20}$$

Combining Eqs. (3.17) and (3.20), we have

$$P\left\{\exists k \text{ such that } k_1\leqslant k\leqslant k_2 \text{ and } \sup_{|t|\leqslant\sqrt{N}\rho_k}|X(t)|\leqslant K\rho_k^{\alpha}\left(\log\log\frac{1}{\rho_k}\right)^{-\alpha/N}\right\}$$

$$\geqslant 1-\exp(-Kc_2\log\lambda_{k_2})-\sum_{k=k_1}^{k_2}\exp\left(-\frac{U^{2\beta}}{K(\log\log 1/\rho_k)^{2\alpha/N}}\right)$$

$$\geqslant 1-\exp(-c\log\lambda_{k_2})$$

for suitably chosen $c_2>1$, using the first inequality in Eq. (3.15). This proves Eq. (3.13).

Remark. The above proof also yields the following result, which will be needed in proving Theorem 4.2 for the case of $N>\alpha d$. Under the conditions of Proposition 3.1, for any positive constant c there exists some constant $K>0$ such that for any integer $k_0\geqslant 1$ there exist integers $k_2\geqslant k_1\geqslant k_0$ with $\lambda_{k_2}\leqslant\lambda_{k_1}^2$ and

$$P\left\{\exists k \text{ such that } k_1\leqslant k\leqslant k_2 \text{ and } \sup_{|t|\leqslant\sqrt{N}2^{-\lambda_k}}|X(t)|\leqslant K2^{-\lambda_k\alpha}(\log\log 2^{\lambda_k})^{-\alpha/N}\right\}$$

$$\geqslant 1-\exp(-c\log\lambda_{k_2}). \tag{3.21}$$

4. Hausdorff-type measures for fractional brownian motion

In this section, we consider the (ϕ,Λ)-Hausdorff measure of the image and graph of fractional Brownian motion.

Let $X(t)$ $(t\in\mathbb{R}^N)$ be a d-dimensional fractional Brownian motion of index α $(0<\alpha<1)$. For any $0<r<1$ and $y\in\mathbb{R}^d$, let

$$T_y(r)=\int_{[0,1]^N}1_{B(y,r)}(X(t))\,\mathrm{d}t$$

be the sojourn time of $X(t)$ ($t \in [0,1]^N$) in the open ball $B(y,r)$. If $y=0$, we write $T(r)$ for $T_0(r)$. Similarly, for $t \in [0,1]^N$ and $y \in \mathbb{R}^d$, define

$$T_{t,y}(r,r) = \int_{B(t,r)} 1_{\{|X(s)-y|<r\}}(s)\,\mathrm{d}s.$$

If $(t,y)=(0,0)$, we denote $T_{t,y}(r,r)$ by $T(r,r)$.

We will need the following lemma, for the proof of Part (1) see Goldman (1988) or Xiao (1996), for Part (2) see Xiao (1997b).

Lemma 4.1. (1) *If $N<\alpha d$, then there exists a positive finite constant K, depending on N, α, and d only, such that for any $u>0$ we have*

$$P(T(r) \geqslant r^{N/\alpha} u) \leqslant \exp(-Ku). \tag{4.1}$$

(2) *If $N>\alpha d$, then there exists a positive finite constant K, depending on N, α, and d only, such that for any $u>0$ we have*

$$P(T(r,r) \geqslant r^{N+(1-\alpha)d} u^{\alpha d/N}) \leqslant \exp(-Ku). \tag{4.2}$$

Now we state and prove the main results of this section. Recall that $\phi_1(s)$ and $\phi_2(s)$ are defined by Eqs. (1.6) and (1.9) respectively.

Theorem 4.1. *Let $X(t)$ ($t \in \mathbb{R}^N$) be a d-dimensional fractional Brownian motion of index α ($0<\alpha<1$) and $N<\alpha d$. Let $\Lambda = \{\lambda_k\}$ be an increasing sequence of positive integers. Then*

$$0 < \phi_1\text{-}m_\Lambda(X([0,1]^N)) < \infty \quad a.s.$$

if and only if there exists $\gamma>0$ such that

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k^{\gamma}} = \infty.$$

Proof. We start the proof with the easy part. If for any $\gamma>0$

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k^{\gamma}} < \infty.$$

Then by Eq. (4.1), we have

$$P(T(2^{-\lambda_k}) \geqslant \gamma \phi_1(2^{-\lambda_k})) \leqslant \frac{K_5}{\lambda_k^{K\gamma}}.$$

It follows from the Borel–Cantelli lemma that with probability 1,

$$\limsup_{k\to\infty} \frac{T(2^{-\lambda_k})}{\phi_1(2^{-\lambda_k})} \leqslant \gamma.$$

Since $\gamma > 0$ is arbitrary, we have almost surely,

$$\limsup_{k\to\infty} \frac{T(2^{-\lambda_k})}{\phi_1(2^{-\lambda_k})} = 0.$$

Since $X(t)$ has stationary increments, we have that for every $t_0 \in [0,1]^N$, almost surely

$$\limsup_{k\to\infty} \frac{T_{X(t_0)}(2^{-\lambda_k})}{\phi_1(2^{-\lambda_k})} = 0. \tag{4.3}$$

Now we define a random Borel measure μ on $X([0,1]^N)$ as follows. For any Borel set $B \subseteq \mathbb{R}^d$, let

$$\mu(B) = L_N\{t \in [0,1]^N,\ X(t) \in B\},$$

where L_N is the Lebesgue measure in $\mathbb{R}^N$. Then $\mu(\mathbb{R}^d) = \mu(X([0,1]^N)) = 1$. By Eq. (4.3), for each fixed $t_0 \in [0,1]^N$, with probability 1

$$\limsup_{k\to\infty} \frac{\mu(B(X(t_0), 2^{-\lambda_k}))}{\phi_1(2^{-\lambda_k})} \leqslant \limsup_{k\to\infty} \frac{T_{X(t_0)}(2^{-\lambda_k})}{\phi_1(2^{-\lambda_k})} = 0. \tag{4.4}$$

Let $E(\omega) = \{X(t_0)\colon t_0 \in [0,1]^N$ and Eq. (4.4) holds$\}$. Then $E(\omega) \subseteq X([0,1]^N)$. A Fubini argument shows $\mu(E(\omega)) = 1$ almost surely. Hence by Lemma 2.2, we have

$$\phi_1\text{-}m_A(E(\omega)) = \infty.$$

Now we prove the sufficiency. By Lemma 2.1(ii) and Eq. (1.5), we see that it is sufficient to prove that there exists a constant $K > 0$ such that with probability 1

$$\phi_1\text{-}m_A(X([0,1]^N)) \leqslant K. \tag{4.5}$$

We will make use of Proposition 3.1. Let $c > 0$ be a constant whose value will be determined later. By Proposition 3.1, for each $n \geqslant 1$, there exist integers $l_n \geqslant k_n \geqslant n$ such that $\lambda_{l_n} \leqslant \lambda_{k_n}^2$ and Eq. (3.13) holds. For each such pair (k_n, l_n), consider the set

$$R_n = \left\{ t \in [0,1]^N\colon \exists k_n \leqslant k \leqslant l_n \text{ such that} \right.$$
$$\left. \sup_{|s-t| \leqslant \sqrt{N}\rho_k} |X(s) - X(t)| \leqslant K\rho_k^{\alpha} \left(\log\log \frac{1}{\rho_k}\right)^{-\alpha/N} \right\}.$$

Then by Eq. (3.13), we have

$$P(t \in R_n) \geqslant 1 - \exp(-c \log \lambda_{l_n}).$$

It follows from Fubini's theorem and Chebyshev's inequality that

$$\sum_{n=1}^{\infty} P(D_n^c) < \infty,$$

where D_n is the event $L_N(R_n) \geqslant 1 - \exp(-\frac{c}{2}\log \lambda_{l_n})$. Hence we have $P(\Omega_0) = 1$, where

$$\Omega_0 = \left\{ \omega\colon L_N(R_n) \geqslant 1 - \exp\left(-\frac{c}{2}\log \lambda_{l_n}\right) \text{ infinitely often} \right\}.$$

On the other hand, it is well known (see e.g. Kahane, 1985) that there exists an event Ω_1 such that $P(\Omega_1)=1$ and for all $\omega\in\Omega_1$, there exists $m_0=m_0(\omega)$ large enough such that for all dyadic cubes C of order $m\geqslant m_0$ that meets $[0,1]^N$, we have

$$\sup_{s,t\in C}|X(t)-X(s)|\leqslant K2^{-m\alpha}\sqrt{m}. \tag{4.6}$$

Now we fix an $\omega\in\Omega_0\cap\Omega_1$ and we will show Eq. (4.5) holds by constructing an economic covering for $X([0,1]^N)$. Consider $n\geqslant m_0$ such that

$$L_N(R_n)\geqslant 1-\exp\left(-\frac{c}{2}\log\lambda_{l_n}\right).$$

Then for any $t\in R_n$, there exists k such that $k_n\leqslant k\leqslant l_n$ and

$$\sup_{|s-t|\leqslant\sqrt{N}\rho_k}|X(s)-X(t)|\leqslant K\rho_k^{\alpha}\left(\log\log\frac{1}{\rho_k}\right)^{-\alpha N}\leqslant K2^{-\lambda_k}, \tag{4.7}$$

where the last inequality follows from Eq. (3.12). We denote by $C_k(t)$ the dyadic cube of side ρ_k that contains t. Then by Eq. (4.7) we have

$$\sup_{s\in C_k(t)}|X(s)-X(t)|\leqslant K2^{-\lambda_k}. \tag{4.8}$$

Hence

$$R_n\subseteq V=\bigcup_{k=k_n}^{l_n}V_k,$$

where each V_k is a union of dyadic cubes with side ρ_k for which Eq. (4.8) holds. For each dyadic cube C_{kj} in V_k, $X(C_{kj})$ can be covered by at most K (depending on d only) balls of radius $2^{-\lambda_k}$, and

$$\begin{aligned}\sum_{k=k_n}^{l_n}\sum_j K\phi_1(2^{-\lambda_k-1})&\leqslant K\sum_{k=k_n}^{l_n}\sum_j\rho_k^N\\&=KL_n(V)\leqslant K.\end{aligned} \tag{4.9}$$

Now notice that $[0,1]^N\setminus V$ can be covered by a union of dyadic cubes of side ρ_{l_n}, none of which meets R_n. There are at most

$$K\rho_{l_n}^{-N}\exp\left(-\frac{c}{2}\log\lambda_{l_n}\right)$$

such cubes $\{C_{nj}\}$. By Eq. (4.6), each $X(C_{nj})$ is contained in a ball of radius $K\rho_{l_n}^{\alpha}\sqrt{\log 1/\rho_{l_n}}$, and hence can be covered by at most

$$K\left(\log\frac{1}{\rho_{l_n}}\right)^{d/2}\left(\log\log\frac{1}{\rho_{l_n}}\right)^{\alpha d/N}\leqslant K\lambda_{l_n}^{d/2\alpha}\log\lambda_{l_n}$$

balls of radius $2^{-\lambda_{l_n}}$. Therefore $X([0,1]^N\setminus V)$ can be covered by a family of balls of radius $2^{-\lambda_{l_n}}$ and by taking $c>d/\alpha$ we have

$$\sum_j\phi_1(2^{-\lambda_{l_n}-1})\leqslant K\rho_{l_n}^{-N}\exp\left(-\frac{c}{2}\log\lambda_{l_n}\right)\lambda_{l_n}^{d/2\alpha}\log\lambda_{l_n}\cdot\rho_{l_n}^N\leqslant 1 \tag{4.10}$$

for n large enough. Combining Eqs. (4.9) and (4.10), we get Eq. (4.5). This finishes the proof. □

For the graph set $\mathrm{Gr}\,X([0,1]^N)$ of fractional Brownian motion, we have the following similar result.

Theorem 4.2. *Let $X(t)$ ($t \in \mathbb{R}^N$) be a d-dimensional fractional Brownian motion of index α ($0<\alpha<1$) and let $\Lambda = \{\lambda_k\}$ be an increasing sequence of positive integers. Then*

$$0 < \phi\text{-}m_\Lambda(\mathrm{Gr}\,X([0,1]^N)) < \infty \quad a.s.$$

if and only if there exists $\gamma > 0$ such that

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k^{\gamma}} = \infty$$

where $\phi(s) = \phi_1(s)$ if $N < \alpha d$ and $\phi(s) = \phi_2(s)$ if $N > \alpha d$.

Proof. In the case of $N < \alpha d$, the proof is very similar to the proof of Theorem 4.1 above. In the case of $N > \alpha d$, the proof which relies on Lemma 4.1(2) and Eq. (3.21) is similar to the proof of Theorem 3.1 in Xiao (1997b). We omit the details.

If the sequence $\Lambda = \{\lambda_k\}$ satisfies

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k^{\gamma}} < \infty$$

for every $\gamma > 0$, then by Theorem 4.1 we know that almost surely $\phi_1\text{-}m_\Lambda(X([0,1]^N)) = \infty$. It is natural to ask whether there is some smaller measure function ϕ such that the $0 < \phi\text{-}m_\Lambda(X([0,1]^N)) < \infty$. The following theorem gives an affirmative answer in certain cases.

Theorem 4.3. *Let $X(t)$ ($t \in \mathbb{R}^N$) be a d-dimensional fractional Brownian motion of index α ($0<\alpha<1$) and let $\Lambda = \{\lambda_k\}$ with $\lambda_k = 2^{[k^\eta]}$ and $0<\eta<1$. If $N < \alpha d$, then with probability 1,*

$$0 < \phi_3\text{-}m_\Lambda(X([0,1]^N)) < \infty, \tag{4.11}$$

where $\phi_3(s) = s^{N/\alpha} \log\log\log 1/s$, and

$$0 < \phi_3\text{-}m_\Lambda(\mathrm{Gr}\,X([0,1]^N)) < \infty. \tag{4.12}$$

If $N > \alpha d$, then with probability 1,

$$0 < \phi_4\text{-}m_\Lambda(\mathrm{Gr}\,X([0,1]^N)) < \infty. \tag{4.13}$$

where

$$\phi_4(s) = s^{N+(1-\alpha)d} (\log\log\log 1/s)^{\alpha d/N}.$$

The proof of the upper bounds in Theorem 4.3 depends on the following lemma, which can be proved in a way similar to the proof of Proposition 3.1.

Lemma 4.2. *Let* $\Lambda = \{\lambda_k\}$ *with* $\lambda_k = 2^{[k^\eta]}$ *and* $0<\eta<1$ *and let*

$$r_k = \sup\left\{ r = 2^{-m}:\ r^\alpha \left(\log\log\log \frac{1}{r}\right)^{-\alpha/N} \leqslant 2^{-\lambda_k}\right\}. \tag{4.14}$$

Then for any fixed constant $c>0$, *there exists a constant* $K>0$ *with the following property: for every integer* $k_0 \geqslant 1$

$$P\left\{ \exists k \text{ such that } k_0 \leqslant k \leqslant 2k_0 \text{ and } \sup_{|t|\leqslant \sqrt{N} r_k} |X(t)| \leqslant K r_k^\alpha \left(\log\log\log \frac{1}{r_k}\right)^{-\alpha/N} \right\}$$
$$\geqslant 1 - \exp(-c k_0^\eta). \tag{4.15}$$

Now we sketch the proof of Theorem 4.3. It is easy to verify that there exist positive and finite constants K_6, K_7 such that for k large enough

$$K_6 2^{-\lambda_k/\alpha}(\log\log\log 2^{\lambda_k})^{1/N} \leqslant r_k \leqslant K_7 2^{-\lambda_k/\alpha}(\log\log\log 2^{\lambda_k})^{1/N}. \tag{4.16}$$

Proof of Theorem 4.3. An argument similar to the first half of the proof of Theorem 4.1, using Lemmas 2.2 and 4.1, shows that with probability 1, $\phi_3\text{-}m_\Lambda(X([0,1]^N))>0$ if $N<\alpha d$, and $\phi_4\text{-}m_\Lambda(\mathrm{Gr}\,X([0,1]^N))>0$ if $N>\alpha d$. The proof of the upper bounds in Eqs. (4.11)–(4.13), using Eqs. (4.15), (4.16) and (4.6), is similar to the proof of Eq. (4.5) and the proof of Theorem 3.1 in Xiao (1997b) respectively. But this time, the assumption that $0<\eta<1$ is essential in obtaining inequalities similar to Eq. (4.10).

Remark. Another important example of Gaussian random fields is Brownian sheet or the N-parameter Wiener process $W(t)$ ($t \in \mathbb{R}_+^N$), see Orey and Pruitt (1973). The exact Hausdorff measure of the image and graph set of W were considered by Ehm (1981). It would be interesting to know whether similar results hold for the Brownian sheet.

The proof of Proposition 3.1, the small ball estimates (see Talagrand, 1993; Monrad and Rootzén, 1995) and a zero-one law of Pitt and Tran (1979) imply the following Chung type laws of iterated logarithm for fractional Brownian motion: Let $\Lambda = \{\lambda_k\}$ be an infinite increasing sequence of positive integers and consider the sequence $t_k = 2^{-\lambda_k}$ $(k\geqslant 1)$ (or $t_k = 2^{\lambda_k}$ $(k \geqslant 1)$). Then there exists a positive and finite constant K_Λ, depending only on Λ, N and α, such that with probability 1

$$\liminf_{k\to\infty} \frac{\sup_{|t|\leqslant t_k} |X(t)|}{t_k^\alpha (\log\log \frac{1}{t_k})^{-\alpha/N}} = K_\Lambda$$

if and only if there exists $\gamma>0$ such that

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k^\gamma} = \infty.$$

Under the conditions of Lemma 4.2, for sequences $\{t_k\}$ defined above or $\{r_k\}$ in Eq. (4.14), we have

$$\liminf_{k\to\infty} \frac{\sup_{|t|\leqslant t_k} |X(t)|}{t_k^{\alpha}(\log\log\log \frac{1}{t_k})^{-\alpha/N}} = K_\eta,$$

where K_η is a positive finite constant depending on η, N and α only.

More generally, let $f(t):\mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing (or non-increasing) function which tends to ∞ (or 0) as $t\to\infty$, it would be interesting to prove an integral test on f and ξ such that

$$P\left(\sup_{|t|\leqslant f(k)} |X(t)| \geqslant f(k)^{\alpha}\xi(f(k)) \text{ for all } k \text{ large enough}\right) = 1.$$

Such functions ξ are called lower functions of $X(t)$ $(t\in\mathbb{R}^N)$ on sequence $\{f(k)\}$. For fractional Brownian motion $X(t)$ $(t\in\mathbb{R})$, lower functions in the usual sense were characterized by Talagrand (1996a).

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