# Existence of small ball constants for fractional Brownian motions

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**Abstract.**
Let $\{B_\gamma(t),\ 0 \le t \le 1\}$ be a fractional Brownian motion of order $\gamma \in (0, 2)$, and let $B(t) = B_1(t)$ be the standard Brownian motion. We show the existence of a $C_\gamma \in (0, \infty)$ such that:

$$\lim_{\varepsilon \to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left( \sup_{0 \le t \le 1} |B_\gamma(t)| \le \varepsilon \right) = \lim_{\varepsilon \to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left( \sup_{0 \le t \le 1} |W_\gamma(t)| \le \varepsilon/a_\gamma \right) = - C_\gamma.$$

where $a_\gamma$ is an explicit constant and

$$W_\gamma(t) = \frac{1}{\Gamma((\gamma + 1)/2)} \int_0^t (t - s)^{(\gamma - 1)/2} \, dB(s).$$

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### *L'existence de la limite pour l'asymptotique des petites boules du mouvement brownien fractionnaire*

**Résumé.**
*Soit* $\{B_\gamma(t),\ 0 \le t \le 1\}$ *le mouvement brownien fractionnaire d'ordre* $\gamma \in (0, 2)$ *et soit* $B(t) = B_1(t)$ *le mouvement brownien ordinaire. Nous montrons l'existence d'une constante* $C_\gamma \in (0, \infty)$ *telle que :*

$$\lim_{\varepsilon \to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left( \sup_{0 \le t \le 1} |B_\gamma(t)| \le \varepsilon \right) = \lim_{\varepsilon \to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left( \sup_{0 \le t \le 1} |W_\gamma(t)| \le \varepsilon/a_\gamma \right) = - C_\gamma.$$

*où* $a_\gamma$ *est une constante explicite et*

$$W_\gamma(t) = \frac{1}{\Gamma((\gamma + 1)/2)} \int_0^t (t - s)^{(\gamma - 1)/2} \, dB(s).$$

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## Version française abrégée

Soit $\{B_\gamma(t) : 0 \le t \le 1\}$ le mouvement brownien fractionnaire d'ordre $\gamma \in (0,2)$. Alors, $\{B_\gamma(t) : 0 \le t \le 1\}$ est le processus gaussien centré avec la fonction de covariance

$$\mathbb{E}\,(B_\gamma(t)B_\gamma(s)) = \frac{1}{2}(|s|^\gamma + |t|^\gamma - |s - t|^\gamma).$$

Dans [7] et [6] on a démontré que

$$-\infty < \liminf_{\varepsilon \to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left(\sup_{0 \le t \le 1} |B_\gamma(t)| \le \varepsilon\right) \le \limsup_{\varepsilon \to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left(\sup_{0 \le t \le 1} |B_\gamma(t)| \le \varepsilon\right) < 0.$$

Naturellement, on se demande si la limite ci-dessus existe. Notre résultat principal dit que c'est vrai. Plus précisément, nous montrons :

THÉORÈME. – *Soit* $\{B_\gamma(t) : 0 \le t \le 1\}$ *le mouvement brownien fractionnaire d'ordre* $\gamma \in (0,2)$ *et soit* $B(t) = B_1(t)$ *le mouvement brownien ordinaire. On a :*

$$\lim_{\varepsilon \to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left(\sup_{0 \le t \le 1} |B_\gamma(t)| \le \varepsilon\right) = \lim_{\varepsilon \to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left(\sup_{0 \le t \le 1} |W_\gamma(t)| \le \varepsilon/a_\gamma\right) = -C_\gamma,$$

*où* $0 < C_\gamma = -\inf_{\varepsilon > 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left(\sup_{0 \le t \le 1} |W_\gamma(t)| \le \varepsilon/a_\gamma\right) < \infty$,

$$a_\gamma = \Gamma((\gamma + 1)/2)\left(\gamma^{-1} + \int_{-\infty}^{0} ((1 - s)^{(\gamma-1)/2} - (-s)^{(\gamma-1)/2})^2 ds\right)^{-1/2}$$

*et*

$$W_\gamma(t) = \frac{1}{\Gamma((\gamma + 1)/2)} \int_0^t (t - s)^{(\gamma-1)/2} dB(s).$$

Une simple conséquence du théorème précédent et des résultats dans [9] est :

$$\liminf_{T \to \infty} \left(\frac{\log \log T}{T}\right)^{\gamma/2} \sup_{0 \le t \le T} |B_\gamma(t)| = \liminf_{T \to \infty} a_\gamma \left(\frac{\log \log T}{T}\right)^{\gamma/2} \sup_{0 \le t \le T} |W_\gamma(t)| = C_\gamma^{\gamma/2}.$$

## 1. Introduction

Let $\{B_\gamma(t) : t \ge 0\}$ denote the $\gamma$-fractional Brownian motion with $B_\gamma(0) = 0$ and $0 < \gamma < 2$. Then $\{B_\gamma(t) : t \ge 0\}$ is a Gaussian process with mean zero and covariance function

$$\mathbb{E}\,(B_\gamma(t)B_\gamma(s)) = \frac{1}{2}(|s|^\gamma + |t|^\gamma - |s - t|^\gamma).$$

It was proved in [7] and [6] that

$$-\infty < \liminf_{\varepsilon \to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left(\sup_{0 \le t \le 1} |B_\gamma(t)| \le \varepsilon\right) \le \limsup_{\varepsilon \to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left(\sup_{0 \le t \le 1} |B_\gamma(t)| \le \varepsilon\right) < 0. \quad (1.1)$$

A natural and important question is the existence of the above limit. It was shown in [4] that this limit exists if the Gaussian correlation conjecture holds. The main purpose of this Note is to show that

the limit in (1.1) exists and is related to a more attractable Gaussian process. During the preparation of this paper, Professor Q. Shao has informed us that he obtained a different proof of the existence of the constant for $B_\gamma(t)$ and his paper is in preparation.

THEOREM 1.1. - *Let* $\{B_\gamma(t),\ 0 \le t \le 1\}$ *be a fractional Brownian motion of order* $\gamma \in (0,2)$ *and let* $B(t) = B_1(t)$ *be the standard Brownian motion. Then*

$$\lim_{\varepsilon \to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left( \sup_{0 \le t \le 1} |B_\gamma(t)| \le \varepsilon \right) = \lim_{\varepsilon \to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left( \sup_{0 \le t \le 1} |W_\gamma(t)| \le \varepsilon/a_\gamma \right) = -C_\gamma,$$

*where* $0 < C_\gamma = -\inf_{\varepsilon > 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left( \sup_{0 \le t \le 1} |W_\gamma(t)| \le \varepsilon/a_\gamma \right) < \infty$,

$$a_\gamma = \Gamma((\gamma + 1)/2)\left( \gamma^{-1} + \int_{-\infty}^{0} ((1 - s)^{(\gamma-1)/2} - (-s)^{(\gamma-1)/2})^2 ds \right)^{-1/2} \tag{1.2}$$

*and*

$$W_\gamma(t) = \frac{1}{\Gamma((\gamma + 1)/2)} \int_{0}^{t} (t - s)^{(\gamma-1)/2} dB(s). \tag{1.3}$$

In the Brownian motion case, i.e. $\gamma = 1$, it is well known that $C_1 = \pi^2/8$ and $a_1 = 1$.
As a simple consequence of the above theorem and results in [9], we have

$$\liminf_{T \to \infty} \left( \frac{\log \log T}{T} \right)^{\gamma/2} \sup_{0 \le t \le T} |B_\gamma(t)| = \liminf_{T \to \infty} a_\gamma \left( \frac{\log \log T}{T} \right)^{\gamma/2} \sup_{0 \le t \le T} |W_\gamma(t)| = C_\gamma^{\gamma/2}.$$

One of the key step in our investigation of $B_\gamma(t)$ is the following useful representation when $\gamma \ne 1$ (*see* [5]),

$$B_\gamma(t) = a_\gamma(W_\gamma(t) + Z_\gamma(t)), \quad 0 \le t \le 1, \tag{1.4}$$

where $a_\gamma$ is given in (1.2), $W_\gamma(t)$ is given in (1.3), and

$$Z_\gamma(t) = \frac{1}{\Gamma((\gamma + 1)/2)} \int_{-\infty}^{0} \{(t - s)^{(\gamma-1)/2} - (-s)^{(\gamma-1)/2}\} dB(s).$$

Furthermore, $W_\gamma(t)$ is independent of $Z_\gamma(t)$. Observe that the centered Gaussian process $W_\beta(t)$ is defined for all $\beta > 0$ as a fractional Wiener integral. Thus $W_\beta(t)$ will be called *fractional Wiener process of order* $\beta$.

The remaining of the paper is organized as follows. In section 2, we estimate the small ball rate for $W_\beta(t)$ for any $\beta > 0$ and show the existence of the constant for $W_\beta(t)$. In section 3, we give the proof of Theorem 1.1.

## 2. Fractional Wiener processes

THEOREM 2.1. - *For any* $\beta > 0$,

$$\lim_{\varepsilon \to 0} \varepsilon^{2/\beta} \log \mathbb{P}\left( \sup_{0 \le t \le 1} |W_\beta(t)| \le \varepsilon \right) = -k_\beta,$$

*where* $0 < k_\beta = -\inf_{\varepsilon > 0} \varepsilon^{2/\beta} \log \mathbb{P}\left( \sup_{0 \le t \le 1} |W_\beta(t)| \le \varepsilon \right) < \infty$.

*Proof.* – The lower estimate for all $\beta > 0$

$$\liminf_{\varepsilon \to 0} \varepsilon^{2/\beta} \log \mathbb{P}\left( \sup_{0 \le t \le 1} |W_\beta(t)| \le \varepsilon \right) > -\infty$$

is given in [3]. But when $\beta = \gamma < 2$, the estimate follows easily from (1.1) and

$$\mathbb{P}\left( \sup_{0 \le t \le 1} |B_\gamma(t)| \le \varepsilon \right) \le \mathbb{P}\left( \sup_{0 \le t \le 1} |W_\gamma(t)| \le \varepsilon/a_\gamma \right). \tag{2.1}$$

which is a direct consequence of representation (1.4) and Anderson's inequality. Thus we only need to show the upper bound and the existence of the constant.

Let $\widehat{W}_\beta(t) = \Gamma((\beta+1)/2)W_\beta(t)$ for simplicity. Then for any $x > 0$ and $0 < \lambda < 1$, we have

$$\mathbb{P}\left( \sup_{0 \le t \le 1} \left|\widehat{W}_\beta(t)\right| \le x \right) = \mathbb{P}\left( \sup_{0 \le t \le 1} \left| \int_0^t (t-s)^{(\beta-1)/2} \mathrm{d}B(s) \right| \le x \right)$$

$$= \mathbb{P}\left( \sup_{0 \le t \le \lambda} \left|\widehat{W}_\beta(t)\right| \le x, \ \sup_{\lambda \le t \le 1} \left| Y_{\beta,\lambda}(t) + \int_\lambda^t (t-s)^{(\beta-1)/2} \mathrm{d}B(s) \right| \le x \right) \tag{2.2}$$

where $Y_{\beta,\lambda}(t) = \int_0^\lambda (t-s)^{(\beta-1)/2} \mathrm{d}B(s)$. Note that the processes $Y_{\beta,\lambda}(t)$, $0 \le t \le 1$, and $\widehat{W}_\beta(t) = \int_0^t (t-s)^{(\beta-1)/2}\mathrm{d}B(s)$, $0 \le t \le \lambda$, are jointly independent of the process $\widehat{W}_\beta(t) - Y_{\beta,\lambda}(t) = \int_\lambda^t (t-s)^{(\beta-1)/2} \mathrm{d}B(s)$, $\lambda \le t \le 1$. Furthermore, Anderson's inequality implies that for any real number $a$,

$$\mathbb{P}\left( \sup_{\lambda \le t \le 1} \left| a + \int_\lambda^t (t-s)^{(\beta-1)/2} \mathrm{d}B(s) \right| \le x \right) \le \mathbb{P}\left( \sup_{\lambda \le t \le 1} \left| \int_\lambda^t (t-s)^{(\beta-1)/2} \mathrm{d}B(s) \right| \le x \right). \tag{2.3}$$

Thus by first conditioning on $\mathrm{d}B(s)$, $0 \le s \le \lambda$, then using (2.3), we obtain from (2.2) that

$$\mathbb{P}\left( \sup_{0 \le t \le 1} \left|\widehat{W}_\beta(t)\right| \le x \right) \le \mathbb{P}\left( \sup_{0 \le t \le \lambda} \left|\widehat{W}_\beta(t)\right| \le x \right)\mathbb{P}\left( \sup_{\lambda \le t \le 1} \left| \int_\lambda^t (t-s)^{(\beta-1)/2}\mathrm{d}B(s) \right| \le x \right)$$

$$= \mathbb{P}\left( \sup_{0 \le t \le 1} \left|\widehat{W}_\beta(t)\right| \le x/\lambda^{\beta/2} \right)\mathbb{P}\left( \sup_{0 \le t \le 1-\lambda} \left|\widehat{W}_\beta(t)\right| \le x \right)$$

where the last equality follows from simple substitution and scaling. Taking $\lambda = 1/n$ and iterating the above procedure, we have for any $x > 0$ and any integer $n$

$$\mathbb{P}\left( \sup_{0 \le t \le 1} \left|\widehat{W}_\beta(t)\right| \le x \right) \le \mathbb{P}\left( \sup_{0 \le t \le 1} \left|\widehat{W}_\beta(t)\right| \le n^{\beta/2}x \right)^n$$

which finishes the proof by following exactly the same argument as the proof of Prop. 2.1 in [1].

## 3. Proof of Theorem 1.1

From (2.1) and Theorem 2.1, it is clear that

$$\limsup_{\varepsilon \to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left( \sup_{0 \le t \le 1} |B_\gamma(t)| \le \varepsilon \right) \le \lim_{\varepsilon \to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left( \sup_{0 \le t \le 1} |W_\gamma(t)| \le \varepsilon/a_\gamma \right) = -k_\gamma a_\gamma^{2/\gamma}.$$

So we only need to show the lower estimate. We need the following result which was established in [8] and reformulated in [2], page 257.

LEMMA 3.1. – *Let* $(X_t)_{t \in T}$ *be a centered Gaussian process. For every* $\varepsilon > 0$, *let* $N(T, d; \varepsilon)$ *denote the minimal number of balls of radius* $\varepsilon$, *under the (pseudo-)metric*

$$d(s, t) = (E|X_s - X_t|^2)^{1/2}.$$

*that are necessary to cover* $T$. *Assume that there is a nonnegative function* $\psi$ *on* $\mathbb{R}_+$ *such that* $N(T, d; \varepsilon) \le \psi(\varepsilon)$ *and such that* $c_1\psi(\varepsilon) \le \psi(\varepsilon/2) \le c_2\psi(\varepsilon)$ *for some constants* $1 < c_1 \le c_2 < \infty$. *Then, for some* $K > 0$ *and every* $\varepsilon > 0$ *we have* $\mathbb{P}\left(\sup_{s,t \in T} |X_s - X_t| \le \varepsilon\right) \ge \exp(-K\psi(\varepsilon))$.

LEMMA 3.2. – *For any* $0 < \gamma < 2$,

$$\lim_{\varepsilon \to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left(\sup_{0 \le t \le 1} |Z_\gamma(t)| \le \varepsilon\right) = 0.$$

*Proof.* – Note that for any $s, t \in (0, 1) = T$ and $s \le t$, with $X_\gamma(t) = \Gamma((\gamma + 1)/2)Z_\gamma(t)$,

$$d_\gamma^2(s, t) = E(X_\gamma(t) - X_\gamma(s))^2 = E\left(\int_{-\infty}^{0} ((t - u)^{(\gamma-1)/2} - (s - u)^{(\gamma-1)/2})dB(u)\right)^2$$

$$= \int_{-\infty}^{0} \left((t - u)^{(\gamma-1)/2} - (s - u)^{(\gamma-1)/2}\right)^2 du = \int_{s}^{\infty} \left((t - s + u)^{(\gamma-1)/2} - u^{(\gamma-1)/2}\right)^2 du.$$

Since by the mean value theorem

$$\left|(t - s + u)^{(\gamma-1)/2} - u^{(\gamma-1)/2}\right| \le |t - s|u^{(\gamma-3)/2}.$$

we have

$$d_\gamma(s, t) \le (2 - \gamma)^{-1/2}(t - s)\, s^{-(2-\gamma)/2}$$

for $0 < s \le t < 1$. When $s = 0$, it follows

$$d_\gamma^2(0, t) = t^\gamma \int_{0}^{\infty} \left((1 + u)^{(\gamma-1)/2} - u^{(\gamma-1)/2}\right)^2 du,$$

which implies $d_\gamma(0, t) \le Ct^{\gamma/2}$ with $C > 0$ only depending on $\gamma$. For any $\varepsilon > 0$ small, we define numbers $0 < t_0 < t_1 < \cdots$ by: $t_0 = (\varepsilon/C)^{2/\gamma}$, so that $d_\gamma(0, t_0) \le \varepsilon$, and for $i \ge 1$ by

$$(2 - \gamma)^{-1/2}(t_i - t_{i-1})t_{i-1}^{-(2-\gamma)/2} = \varepsilon.$$

Let $N(\varepsilon) = \min\{n : t_n > 1\}$. Then for $1 \le i \le N(\varepsilon)$ we obtain:

$$t_i = t_{i-1}(1 + (2 - \gamma)^{1/2}\varepsilon t_{i-1}^{-\gamma/2}) \ge t_{i-1}(1 + (2 - \gamma)^{1/2}\varepsilon).$$

thus by iterating:

$$1 \ge t_{N(\varepsilon)-1} \ge t_0(1 + (2 - \gamma)^{1/2}\varepsilon)^{N(\varepsilon)-1} = (\varepsilon/C)^{2/\gamma}(1 + (2 - \gamma)^{1/2}\varepsilon)^{N(\varepsilon)-1},$$

which implies $N(\varepsilon) \le c\varepsilon^{-1}\log(1/\varepsilon)$ for some $c > 0$. Hence, using $t_i$, $0 \le i \le N(\varepsilon) - 1$, as centers, we finally get $N(T, d_\gamma; \varepsilon) \le N(\varepsilon) \le c\varepsilon^{-1}\log(1/\varepsilon)$ which finishes the proof by Lemma 3.1.

To obtain the lower bound of Theorem 1.1, we have for any $0 < \delta < 1$:

$$\mathbb{P}\left(\sup_{0 \leq t \leq 1} |B_\gamma(t)| \leq \varepsilon\right) \geq \mathbb{P}\left(\sup_{0 \leq t \leq 1} |W_\gamma(t)| \leq (1-\delta)\varepsilon/a_\gamma\right)\mathbb{P}\left(\sup_{0 \leq t \leq 1} |Z_\gamma(t)| \leq \delta\varepsilon/a_\gamma\right)$$

since $W_\gamma(t)$ and $Z_\gamma(t)$ are independent of each other. Thus

$$\liminf_{\varepsilon \to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left(\sup_{0 \leq t \leq 1} |B_\gamma(t)| \leq \varepsilon\right) \geq -k_\gamma(1-\delta)^{-2/\gamma}a_\gamma^{2/\gamma}.$$

So we obtain the desired lower bound with $C_\gamma = k_\gamma a_\gamma^{2/\gamma}$ by taking $\delta \to 0$.

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